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Viscous Flows in Domains with a Multiply Connected Boundary

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Abstract. In this paper we consider stationary Navier-Stokes equations in a bounded domain with a boundary, which has several connected components. The velocity vector is given on the boundary, where the fluxes differ from zero on its components. In general case, the solvability of this problem is an open question up to now. We provide a survey of previous results, which deal with partial versions of the problem. We construct an a priori estimate of the Dirichlet integral for velocity vector in the case, when the flow has an axis of symmetry and a plane of symmetry perpendicular to it, moreover this plane intersects each component of the boundary. Having available this estimate, we prove the existence theorem for axially symmetric problem in a domain with a multiply connected boundary. We consider also the problem in a curvilinear ring and formulate a conditional result concerning its solvability.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded domain with smooth boundary $\partial\Omega$ consisting from m disjoint components $\Sigma_1, \ldots, \Sigma_m$. Stationary problem for the Navier-Stokes equations in zero external force field

$$\mathbf{v} \cdot \nabla\mathbf{v} = -\nabla p + \nu\Delta\mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \quad x \in \Omega, \tag{1.1}$$

$$\mathbf{v} = \mathbf{a}_i(x), \quad x \in \Sigma_i \ (i = 1, \ldots, m) \tag{1.2}$$

is considered. Let introduce values

$$q_i = \int\limits_{\Sigma_i} \mathbf{n}_i \cdot \mathbf{a}_i d\Sigma_i \quad (i = 1, \ldots, m) \tag{1.3}$$

where $\mathbf{n}_i$ is a unit vector of an exterior normal to the surface Σ_i . In view of the continuity equation,

$$q_i + \ldots + q_m = 0. \tag{1.4}$$

Let there is fulfilled a stronger condition

$$q_i = 0 \ (i = 1, \ldots, m) \tag{1.5}$$

instead of (1.4). In this case, under corresponding smoothness conditions, the global existence theorem for the problem (1.1), (1.2) takes place (J.Leray, [1]). The proof is based on finiteness of the Dirichlet integral

$$I = \int\limits_{\Omega} \nabla\mathbf{v} : \nabla\mathbf{v}\, dx \tag{1.6}$$

for all possible solutions of problem (1.1), (1.2), (1.5). Leray demonstration used argument by contradiction and did not contain an *a priori estimate* of I in terms of problem data.

E.Hopf [2] first obtained an effective estimate of Dirichlet integral. His construction is based on the following lemma

Lemma 1. *Assume that* $\Sigma_i \in C^{2+\alpha}$, $0 < \alpha < 1$, *and* $\mathbf{a}_i \in C^{2+\alpha}(\Sigma_i)$, $i = 1, \dots, m$. *If condition (1.5) is satisfied, then for arbitrary* $\varepsilon > 0$ *there exists a solenoidal continuation* $\mathbf{b}_i(x) \in C^{2+\alpha}(\bar{\Omega})$ *of vector* $\mathbf{a}_i(x)$ *into domain* Ω *such that for any* $\mathbf{u}(x) \in \mathbf{H}(\Omega)$

$$\Big|\int\limits_{\Omega} \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dx \Big| \leq \varepsilon ||\nabla \mathbf{u}||^2_{L_2}, \quad i = 1, \dots, m. \tag{1.7}$$

Here $\mathbf{H}(\Omega)$ is the functional space introduced by O.A.Ladyzhenskaya [3]. Everywhere below the smoothness conditions formulated in Lemma 1 are assumed to be fulfilled. A.Takeshita proved [4] that the condition (1.5) is not only sufficient but also a necessary one for possibility of continuation of vector field $\mathbf{a}_i$ so that inequality (1.7) is satisfied for any $\varepsilon > 0$.

We will consider problem (1.1), (1.2) under general outflow conditions. It means that $q_i \neq 0$ at least for one $i \in 1, \dots, m$. It should be noted that violation of condition (1.5) does not lead to principal difficulties for the non-stationary problem for the Navier-Stokes equations [3]. As for stationary one, there are no general results about its solvability in case $q_i \neq 0$ up to now. On the other hand, there are a number of papers, in which the existence theorems are proved under some additional conditions on the problem data. Next section is devoted to description of results obtained in this direction.

2. Survey of Previous Results

As shown by R.Finn [5], the existence theorem for the problem (1.1), (1.2) remains valid if one assumes that $|q_i| < c_* \nu$, $i = 1, \dots, m$, and c_* is small enough. G.P.Galdi [6, 7] has given the bound c_* in terms of imbedding constants depending on the domain Ω and properties of solutions of non-uniform divergence equation. The constant was computed explicitly in the flow in an annulus. For special cases if domain Ω is confined by concentric spheres (or circles as $n = 2$) with radii R_1 and $R_2 > R_1$, W.Borchers and K.Pileckas [8] have obtained effective estimates of admissible $|q_i|$ bounds in terms of R_1, R_2 and ν.

C.J.Amick [9] showed how to relax condition (1.5) without the smallness assumption on $|q_i|$ quantities. He studied two-dimensional flow under certain symmetry assumption. Following [9], let us introduce

Definition 1. A bounded domain $\Omega \subset \mathbb{R}^2$ is said to be *admissible* if (a) $\partial\Omega$ is of class $C^{2+\alpha}$, (b) $\partial\Omega$ consists of $m \geq 2$ components Σ_i, (c) Ω is symmetric about the line $\{x_2 = 0\}$ and (d) each component Σ_i intersects the line $\{x_2 = 0\}$.

A function $\mathbf{h} = (h_1, h_2)$ mapping Ω or $\partial\Omega$ into $\mathbb{R}^2$ is said to be *symmetric about the line* $\{x_2 = 0\}$ if h_1 is an even function of x_2 while h_2 is an odd function of x_2.

Definition 2. A vector $\mathbf{a}$ is said to be *admissible data* if (a) $\mathbf{a} \in C^{2+\alpha}(\partial\Omega \to \mathbb{R}^2)$ and (b) $\mathbf{a}$ is symmetric about the line $\{x_2 = 0\}$.

It is well known that the Navier-Stokes equations are invariant with respect to reflection about the coordinate axis. This property allows us to seek symmetric solutions $(\mathbf{v}, p)$ of this system, with $\mathbf{v}$ being symmetric about $\{x_2 = 0\}$ and the corresponding pressure p being an even function of x_2 .

Theorem 1 [9]. *Let* $\Omega \subset \mathbb{R}^2$ *be an admissible domain and let* $(\mathbf{a_1}, \dots, \mathbf{a}_m)$ *be admissible data. Then for every* $\nu > 0$ *there exists a solution* $(\mathbf{v}, p) \in C^{2+\alpha}(\bar{\Omega} \to \mathbb{R}^2) \times C^{1+\alpha}(\bar{\Omega} \to \mathbb{R})$ *of (1.1), (1.2). The function* $\mathbf{v}$ *is symmetric about* $\{x_2 = 0\}$ *and the pressure* p *is an even function of* x_2.

Similarly to Leray's basic work [1], Amick's result was obtained via the proof by contradiction. The next important step was done by H.Fujita [10], who presented a constructive proof of an existence of symmetric solutions via an a priori estimate of Dirichlet integral (1.6). Fujita construction is based on the concept of a *virtual drain* introduced by him.

Definition 3. Vector field $\mathbf{c}_i(x)$ is said to be a virtual drain if (a) $\mathbf{c}_i \in C^{2+\alpha}(\bar{\Omega})$ is solenoidal and parallel to the x_1 - axis, (b) the outflow of $\mathbf{c}_i$ from each Σ_i coincides with that of $\mathbf{a}_i$; namely,

$$\int_{\Sigma_1} \mathbf{n}_i \cdot \mathbf{c}_i \, d\Sigma_i = \int_{\Sigma_1} \mathbf{n}_i \cdot \mathbf{a}_i \, d\Sigma_i \quad (i = 1, \ldots, m) \tag{2.1}$$

and (c) $\mathbf{c}_i$ contains a positive free parameter κ, and by choosing κ sufficiently small, we can make $\sup(|x_2||c_{i1}(x)|)$ arbitrarily small.

Another method for obtaining an a priori estimate of Dirichlet integral under conditions of Theorem 1 was proposed by H. Morimoto [11]. Her construction exploits the stream function ψ of plane flow defined by relations

$$v_1 = \frac{\partial \psi}{\partial x_2}, \quad v_2 = -\frac{\partial \psi}{\partial x_1}. \tag{2.2}$$

H. Fujita and H. Morimoto [12] studied problem (1.1), (1.2) in a domain Ω with two components of the boundary Σ_1 and Σ_2. Functions $\mathbf{a}_i$ in (1.2) were taken in the form $\mu\nabla\varphi + \tilde{\mathbf{a}}_i$ where $\mu \in \mathbb{R}$, φ is a fundamental solution of the Laplace operator and $\tilde{\mathbf{a}}_i$ $(i = 1, 2)$ satisfy the condition (1.5). Authors proved that there is a countable subset N of $\mathbb{R}$ such that if $\mu \notin N$ and $\tilde{\mathbf{a}}_i$ are small (in a suitable norm), then problem (1.1), (1.2) has a weak solution. Moreover, if $\Omega \in \mathbb{R}^2$ is an annulus, then N is empty.

In the conclusion of this section, we mention results of papers [13-15] dedicated to flows in an annular domain $\Omega = \{x \in \mathbb{R}^2;\ R_1 < |x| < R_2\}$ under boundary conditions with non-vanishing outflow. H.Morimoto [13] considered this problem in the case

$$\mathbf{a}_i = \mu R_i^{-1}\mathbf{e}_r + b_i\mathbf{e}_\theta \quad \text{on} \quad \Sigma_i = \{x \in \mathbb{R}^2;\ |x| = R_i\}, \quad i = 1, 2 \tag{2.3}$$

where μ, b_1, b_2 are constants and $\mathbf{e}_r,\ \mathbf{e}_\theta$ are the unit vectors in polar coordinates $\{r, \theta\}$. Problem (1.1), (2.3) has an exact rotationally symmetric solution, in which $\mathbf{v} = \mathbf{v}(r),\ p = p(r)$ are given by explicit formulae. As $\mu = 0$, this solution describes the well known Couette flow. In [13] it was proved that if $|\mu|,\ |b_1 - b_2|$ are sufficiently small and $\mu \neq -2\nu$ then the solution of problem (1.1), (2.3) is unique. The uniqueness theorem is valid also in case $\mu = -2\nu$ and $|\mu|,\ |b_1|,\ |b_2|$ are sufficiently small. Besides, for sufficiently large ν, the above exact solutions are exponentially stable.

Let now the boundary condition has the form

$$\mathbf{a}_i = \{\mu R_i^{-1} + \varphi_i(\theta)\}\mathbf{e}_r + \{\omega_i R_i + \psi_i(\theta)\}\mathbf{e}_\theta \quad \text{on} \quad \Sigma_i = \{x \in \mathbb{R}^2;\ |x| = R_i\}, \quad i = 1, 2 \tag{2.4}$$

where $\varphi_i(\theta),\ \psi_i(\theta)$ are smooth functions of θ with a zero mean value. Problem (1.1), (2.4) was studied by H.Morimoto and S.Ukai [14]. The main result of the paper [14] is

Theorem 2. *Suppose the inequality*

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \Big(\log\frac{R_2}{R_1}\Big)^2 < 2\nu \tag{2.5}$$

holds. Then there exists at most discrete countable set M such that for each $\mu \in \mathbb{R}\backslash M$ the boundary problem (1.1), (2.4) has a solution for sufficiently small $\varphi_i(\theta)$, $\psi_i(\theta)$ $(i = 1, 2)$.

We note that under condition of Theorem 2 the quantity $|\mu|$ can be large in comparison with ν. It is interesting to distinguish a class of conditions (2.4) as the set M is empty. H.Fujita, H.Morimoto and H.Okamoto [15] established that this is true as $\omega_1 = \omega_2$; in this case, inequality (2.5) is fulfilled automatically. The special case $b_1 = b_2 = 0$ in (2.3) corresponds to a radial flow with velocity field $v_r = \mu r^{-1}$, $v_\theta = 0$. As it is shown in [15], the radial flow in an annulus is stable to perturbation of steady state, whatever the Reynolds number μ/ν or the aspect ratio R_1/R_2 are. At the same time, the precise calculations carried out in [15] provide the numerical evidence that Hopf's bifurcations occur for the case $b_1R_1 = b_2R_2$. In this case, the solution of (1.1), (2.3) problem is self-similar; corresponding velocity field is $v_r = \mu r^{-1}$, $v_\theta = \lambda r^{-1}$ where $\lambda = b_1R_1$.

3. Axially symmetric flows

In this section we consider problem (1.1), (1.2) in the case, when domain $\Omega \in \mathbb{R}^3$ has an axis of symmetry and a plane of symmetry, which is perpendicular to this axis.

Definition 4. A bounded domain $\Omega \in \mathbb{R}^3$ is said to be *admissible* is (a) $\partial\Omega$ is of class $C^{2+\alpha}$, (b) $\partial\Omega$ consists of $m \geq 2$ simply connected components Σ_i, (c) Ω has the axis of symmetry $\{x_1 = x_2 = 0\}$ and the plane of symmetry $\{x_3 = 0\}$, and (d) each component Σ_i intersects the plane $\{x_3 = 0\}$.

Let us introduce cylindrical coordinates $r = (x_1^2 + x_2^2)^{1/2}$, $\theta = \mathrm{arctg}(x_2/x_1)$, $z = x_3$ and denote as v_r, v_θ, v_z projections of vector $\mathbf{v}$ on the axis r, θ, z.

A function $\mathbf{h} = (h_r, h_\theta, h_z)$ mapping Ω or $\partial\Omega$ is said to be *axially symmetric* if $h_\theta = 0$ while h_r and h_z do not depend on θ. A function $\mathbf{h} = (h_r, 0, h_z)$ mapping Ω or $\partial\Omega$ is said to be *symmetric about the planes* $\{z = 0\}$ if h_r is an even function of z while h_z is an odd function of z.

Definition 5. A vector $\mathbf{a}$ is said to be *admissible data* if (a) $\mathbf{a} \in C^{2+\alpha}(\partial\Omega \to \mathbb{R}^3)$ and (b) $\mathbf{a}$ is axially symmetric and symmetric about the plane $\{z = 0\}$.

Our purpose is to prove the existence theorem for problem (1.1), (1.2) in the class of axially symmetric flows. It means that the vector $\mathbf{v} = (v_r, 0, v_z)$ is axially symmetric and symmetric about plane $\{z = 0\}$, moreover the corresponding pressure p does not depend on θ. In consequence (1.1), functions v_r, v_z and p satisfy the following system:

$$
\begin{gathered}
v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z} = -\frac{\partial p}{\partial r} + v\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right), \\
v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right), \\
\frac{1}{r}\frac{\partial (rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0.
\end{gathered} \tag{3.1}
$$

Lemma 2. *Let $\Omega \to \mathbb{R}^3$ be an admissible domain and let$(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ be admissible data. Then the Dirichlet integral (1.6) is finite for all possible solutions of problem (1.1), (1.2).*

Proof. It is based on a special construction of the virtual drain, which generalizes the Fujita construction [9].

According to conditions of Lemma 2, boundary of domain Ω consists of $m \geq 2$ disjoint smooth simply connected components $\Sigma_1, \ldots, \Sigma_m$. We assume that surface Σ_m encloses the other components $\Sigma_1, \ldots, \Sigma_{m-1}$. Each of surfaces $\Sigma_1, \ldots, \Sigma_{m-1}$ is a surface of revolution. Let us denote as $S_1, \ldots, S_m$ plane domains, which are meridian sections of $\Sigma_1, \ldots, \Sigma_{m-1}$. Further, we notice that the ray $r > 0$ in semi-plane $\{r,\ z : r > 0,\ z \in \mathbb{R}\}$ and each S_i intersects orthogonally at two different points $P_i = (y_i, 0)$ and $P_i^* = (y_i^*, 0)$ of which we can assume that $y_i > y_i^*$ $(i = 1, \ldots, m)$. Moreover, we may assume that $y_m > y_{m-1} > \ldots > y_1 > 0$.

Now we define the domain $\Omega^+ = \{r,\ \theta,\ z : r > 0,\ z > 0,\ (r, \theta, z) \in \Omega\}$. In other words, Ω^+ is "a half of Ω ". It should mark that the domain Ω^+ has a simply connected boundary $\partial\Omega^+$, as domain Ω is admissible. Let designate $C_\delta = \{r,\ \theta,\ z : r \in \mathbb{R}^+,\ 0 < z < \delta\}$. Then we choose a small positive number δ so that the domain $\Omega^+ \cap C_\delta$ consists of m disjoint components K_i. Each of domains K_i will be support of the i-th component of the virtual drain.

Let us consider domain K_1. Its boundary consists of two basements, belonging to planes $z = 0,\ z = \delta$, and two lateral parts L_1^l and L_1^r, which are surfaces of revolution. Lower basement of K_1 is a ring $y_1 < r < y_2^*,\ z = 0$. First virtual drain $\mathbf{c_i}(\mathbf{x})$ we take in the form

$$\mathbf{c}_1 = -\frac{1}{4\pi r} q_1(\eta(z), 0, 0), \tag{3.2}$$

where $\eta(t) = \eta(t;\ \delta, \kappa)$ is the Hopf-type cutting function [16] and $\kappa \in (0, 1/2)$ is a free parameter. Here we use a small modification of function $\eta(t)$ construction given in [9], namely:

$$\eta(t) = \frac{1}{\gamma_\kappa \delta} \zeta_k(\frac{1}{\delta}),\ \zeta_\kappa \in C_0^\infty(\mathbb{R}),\ \zeta_\kappa(t) \geq 0\ (\forall t \geq 0),\ \zeta_\kappa(-t) = \zeta_\kappa(t),\ \zeta_\kappa(t) = 0\ (t \geq 1),$$

$$\zeta_\kappa(t) \leq \frac{1}{t}\ (0 < t < \infty),\ \zeta_\kappa(t) = \frac{1}{t}\ (\kappa \leq t \leq \frac{1}{2}),\ \gamma_k = \int\limits_{-1}^{1} \zeta_k(t) t\, dt,\ \gamma_k \geq 2 \int\limits_{k}^{1/2} \frac{dt}{t} \to \infty\ (\kappa \to 0),$$

$$\int\limits_{-\infty}^{\infty} \eta(t) t\, dt = \int\limits_{-\delta}^{\delta} \eta(t) t\, dt = 1. \tag{3.3}$$

If $m = 2$, the structure of a virtual drain is completed. In fact, vector $\mathbf{c_i}(\mathbf{x})$ is solenoidal and smooth. In view of (3.2), (3.3), the equality

$$\int_{L_2^l} \mathbf{n}_1 \cdot \mathbf{c}_1\, dL_2^l = \frac{1}{2} q_1 \tag{3.4}$$

holds (we remind that $\mathbf{n}_1$ is unit vector of an *exterior* normal to the surface Σ_1). Taking into account (3.4) and extending function $\mathbf{c_1}(\mathbf{x})$ on negative values of z as an even function of these variables, we guarantee fulfillment of equality (2.1) as $i = 1$. At last, choosing parameter small, we can provide $\sup(|z||c_1(z)|),\ x \in \bar{K}_1,\ k = 1, 2$, arbitrary small. In view of (1.4), $q_2 = -q_1$ if $m = 2$. Replacing y_2^* by y_2 in the case $m = 2$ and identifying $\mathbf{c}_2$ with $\mathbf{c}_1$, we arrive to relation

$$\int_{L_2^r} \mathbf{n}_2 \cdot \mathbf{c}_2\, dL_2^r = \frac{1}{2} q_2,$$

which ensures (2.1) for $i = 2$.

Let now $m = 3$. In this case, we consider domain K_2 connecting surfaces Σ_2 and Σ_3. We define as before function $\mathbf{c_1}(\mathbf{x})$ by formula (3.2). Further, let denote as L_2^l and L_2^r the left and right lateral sides of surface ∂K_2, which intersects orthogonally the plane $z = 0$, and set

$$\mathbf{c}_2 = -\frac{1}{4\pi r}(q_1 + q_2)(\eta(z), 0, 0).$$

By (3.4), the liquid surface L_1^l equals to $q_1/2$, hence the flux through surface $L_2^r \subset \Sigma_2$ is $-q_1/2$. At the same time, the flux through surface $L_2^l \subset \Sigma_2$ is $(q_1 + q_2)/2$. It implies equality (2.1) for $i = 2$. Thus, second component of virtual drain in the case $m = 3$ is constructed. The third component is defined by relation

$$\mathbf{c}_3 = \frac{1}{4\pi r} q_3(\eta(z), 0, 0) \tag{3.5}$$

in an annular layer K_2 and by continuation of function given by (3.5) on negative values of z. Relation (2.1) for $i = 3$ will be satisfied on account of equality $q_1 + q_2 + q_3 = 0$ (1.4). When $m > 3$, we continue described procedure until its completion for $m - 1$ steps.

Due to the symmetry condition, this is sufficient to estimate the following integral

$$I^+ = \int_{\Omega^+} \nabla \mathbf{v} : \nabla \mathbf{v}\, dx \tag{3.6}$$

to prove Lemma 2 since $I = 2I^+$ where I is Dirichlet integral (1.6). To this end, we represent $\mathbf{v}(x)$ in the form

$$\mathbf{v} = \mathbf{u} + \sum_{i=1}^{m} (\mathbf{b}_i + \mathbf{c}_i). \tag{3.7}$$

Here $\mathbf{u} \in \mathbf{H}(\Omega)$, $\{\mathbf{c}_i\}$, is the set of virtual drains, each of the solenoidal vector-functions $\mathbf{b}_i$ $(i = 1, \ldots, m)$ satisfies zero flux condition

$$\int_{\Sigma_i} \mathbf{n}_i \cdot \mathbf{b}_i \, d\Sigma_i = 0 \tag{3.8}$$

and

$$\mathbf{a}_i(x) = \mathbf{b}_i(x) + \mathbf{c}_i(x), \ x \in \Sigma_i, \ i = 1, \ldots, m. \tag{3.9}$$

Moreover, the support of each function $\mathbf{b}_i(x)$ is a narrow strip near the surface Σ_i. Functions $\mathbf{c}_i$ (virtual drains) are determined previously. Equality (3.9) means that vector $\mathbf{b}_i + \mathbf{c}_i$ is a solenoidal continuation of vector $\mathbf{a}_i$ into domain Ω. As far as $(\mathbf{a}_1, \ldots, \mathbf{a}_m)$ are admissible data and vectors $\mathbf{c}_i$ $(i = 1, \ldots, m)$ satisfy the symmetry condition we can consider that vectors $\mathbf{b}_i$ $(i = 1, \ldots, m)$ satisfy the same condition too. Then (3.7) implies that vector $\mathbf{u}$ is symmetric also. There is a freedom in the choice of these vectors. In view of (3.8) we can apply Lemma 1 and realize the construction of functions $\mathbf{b}_i$ so that inequalities (1.7) hold with a positive constant ε, which will be chosen lower.

To provide symmetry properties of vectors $\mathbf{b}_i$, we define their components in the form

$$b_{i,r} = -\frac{1}{r}\frac{\partial(\eta(n)\Psi_i)}{\partial z}, \quad b_{i,z} = -\frac{1}{r}\frac{\partial(\eta(n)\Psi_i)}{\partial r}, \quad (i = 1, \ldots, m).$$

Here n is the distance of current point $(r, z) \in \Omega$ from $\partial\Omega$, $\eta(n)$ is the cutting function defined by formulas (3.3) and $\Psi_i(r, z)$ is the stream function of an axially symmetric solenoidal vector field admitted given boundary values $\mathbf{b}_i = \mathbf{a}_i - \mathbf{c}_i$ on the surface Σ_i.

For any smooth axially symmetric vector $\mathbf{h}$ we can define its strain tensor $D = D(\mathbf{h})$ with elements

$$D_{rr} = \frac{\partial h_r}{\partial r}, \quad D_{\theta\theta} = \frac{h_r}{r}, \quad D_{zz} = \frac{\partial h_z}{\partial z}, \quad D_{rz} = \frac{1}{2}\left(\frac{\partial h_r}{\partial z} + \frac{\partial h_z}{\partial r}\right), \quad D_{r\theta} = D_{\theta z} = 0.$$

If function $\mathbf{h}$ is symmetric about the planes $\{z = 0\}$ then the following equalities take place:

$$h_z = 0, \quad D_{rz} = 0, \quad (r, z) \in \partial\Omega^+ \cap \{z = 0\}. \tag{3.10}$$

Next step of our consideration is obtaining an integral relation for the sought vector $\mathbf{u}$. To get it, we substitute representation (3.7) into system (3.1), multiply its first equation by $\mathbf{u}$ and integrate result over domain Ω^+. We note that each of functions in right side of (3.7) satisfies conditions (3.10). Having applied this conditions and well-known Green identity for the Stokes operator [3], we come after simple calculations to the required relation:

$$2\nu \int_{\Omega^+} D(\mathbf{u}) : D(\mathbf{u})\, dx - \sum_{i=1}^{m} \int_{\Omega^+} \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx - \sum_{i=1}^{m} \int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx =$$

$$= -2\nu \sum_{i=1}^{m} \int_{\Omega^+} D(\mathbf{b}_i + \mathbf{c}_i) : D(\mathbf{u})\, dx + \sum_{i=1}^{m} \int_{\Omega^+} \mathbf{u} \cdot (\mathbf{b}_i + \mathbf{c}_i) \cdot \nabla(\mathbf{b}_i + \mathbf{c}_i)\, dx. \tag{3.11}$$

Since function $\mathbf{u}$ is symmetric, the following equalities are valid:

$$\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u}\, dx = 2\int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\ dx, \quad \int_{\Omega} D(\mathbf{u}) : D(\mathbf{u})\, dx = 2\int_{\Omega^+} D(\mathbf{u}) : D(\mathbf{u})\ dx,$$

$$\int_{\Omega} |\mathbf{u}^2|\ dx = 2\int_{\Omega^+} |\mathbf{u}|^2\ dx. \tag{3.12}$$

For any $\mathbf{u} \in \mathbf{H}(\Omega)$, the Korn inequality [7, 17]

$$\int_{\Omega} D(\mathbf{u}) : D(\mathbf{u})\, dx \geq M_1 \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u}\, dx, \tag{3.13}$$

and the Poincare inequality [3, 7]

$$\int_{\Omega} |\mathbf{u}^2|\ dx \leq M_2 \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{u}\, dx \tag{3.14}$$

are true with positive constants $M_k = M_k(\Omega), \quad k = 1, 2$. Relations (3.12) allow us to replace the integration domain Ω in inequalities (3.13), (3.14) by domain Ω^+. This gives desired estimates of right side in relation (3.11) (upper estimate) and first term of its left side (lower estimate). To estimate second term in left side of (3.11) above, we apply Lemma 1 with $\varepsilon = \nu M_1/2m$ that gives

$$\left|\int_{\Omega^+} \mathbf{b}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx\right| \leq \frac{\nu M_1}{2m} \int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\ dx, \ i = 1, \ldots, m. \tag{3.15}$$

The most crucial point of our examination is derivation of the same estimate for third term in left side of (3.11),

$$\left|\int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u}\, dx\right| \leq \frac{\nu M_1}{2m} \int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u}\ dx, \ i = 1, \ldots, m. \tag{3.16}$$

Integral in left side of (3.16) can be written as

$$\int_{\Omega^+} \mathbf{c}_i \cdot \mathbf{u} \cdot \nabla \mathbf{u} \, dx = J_1 + J_2, \tag{3.17}$$

where

$$J_1 = 2\pi \int_{K_i} c_{i1} u_r \frac{\partial u_r}{\partial x_1} \, r dr dz,$$

$$J_2 = 2\pi \int_{K_i} c_{i1} u_z \frac{\partial u_r}{\partial z} \, r dr dz.$$

Here we took into account that vector $\mathbf{c}_i$ has only nonzero component c_{i1} and its support is K_i. Evaluating integrals J_2, we note that

$$\sup_t |t| \eta(t) \to 0 \quad \text{as} \quad \kappa \to 0 \tag{3.18}$$

as follows from (3.3). Next, component u_z of the symmetric vector $\mathbf{u}$ vanishes on the plane $z = 0$ in the sense of trace. Hence, we can apply the Hardy-type inequality [3, 7]

$$\int_{K_i} \frac{u_z^2}{z^2} \, dx \leq 4 \int_{K_i} |\nabla u_z|^2 \, dx. \tag{3.19}$$

Remembering the expressions for virtual drains (3.1), (3.4) and similar to them, we obtain inequality

$$|J_2| \leq \frac{1}{8\pi}(m-1) q_* \int_{K_i} \sup_{K_i}(z\eta(z)) \frac{|u_z|}{z} \left| \frac{\partial u_r}{\partial z} \right| dx,$$

where $q_* = \max|q_i|, \ i = 1, \ldots, m$. Choosing sufficiently small κ, we arrive from this inequality and (3.18), (3.19) to the desired estimate (3.16). As for integral J_1, it is equal to zero because rc_{i1} does not depend on r and function u_1 vanishes on the end-wall parts of K_i boundary, which belongs to $\partial\Omega$.

Combining equalities (3.12), inequalities (3.13), (3.14) and estimates (3.15), (3.16) we conclude from (3.11) that

$$\int_{\Omega^+} \nabla \mathbf{u} : \nabla \mathbf{u} \, dx \leq \frac{1}{\nu} M_3, \tag{3.20}$$

where $M_3 = M_3(\Omega, ||\mathbf{b}_i||_{H^1})$ is a positive constant. Inequality (3.20) and representation (3.7) lead to the required estimates of integral I^+ (3.6) and consequently of Dirichlet integral $I = 2I^+$,

$$\int_{\Omega^+} \nabla \mathbf{v} : \nabla \mathbf{v} \, dx \leq M_4 \tag{3.21}$$

with constant $M_4 = M_4(\Omega, \nu, ||\mathbf{b}_i||_{H^1}, \ ||\mathbf{c}_i||_{H^1}) > 0$. This completes the proof of Lemma 2.

Theorem 3. *Under assumptions as in Lemma 2, there exists a solution* $\mathbf{v}(x) \in C^{2+\alpha}(\bar{\Omega})$, $p(x) \in C^{1+\alpha}(\bar{\Omega})$ *of the problem (1.1), (1.2).*

The proof of Theorem 3 is omitted here. It is based on estimate (3.20) and follows the classical scheme given in [3] or [18]. Another important corollary of Lemma 2 is the existence theorem for an axially symmetric problem.

4. Flow in a curvilinear ring

Here we return to the problem mentioned in Section 2. Let $\Omega \in \mathbf{R}^2$ be a curvilinear ring bounded by smooth curves Σ_1 (interior boundary) and Σ_2 (exterior boundary). All previous results on the solvability of problem (1.1), (1.2) for this case had either a local character or dealt with an annular geometry of Ω [5-8, 12-15]. The point is to obtain an a priori estimate of the Dirichlet integral for an arbitrary large flux. This problem is still open. We postpone a discussion on the next section and now we propose some construction, which can be useful for the problem treatment.

This is well known that any plane flow of an incompressible liquid is characterized by the stream function ψ defined by relations (2.2). Level lines of function ψ coincide with trajectories of liquid particles in the case of steady-state flow. If domain Ω is simply connected, stream function is a single-valued one. In the opposite case, this property takes place only under condition (1.5). We consider situation when this condition is violated. If Ω is a curvilinear ring, it means that $q_1 = -q_2 \neq 0$. In this case, function $\psi(x_1, x_2)$ is a multi-valued one, which receives the increment q_1 after going around Σ_1. If $q_1 \neq 0$, there is at least one stream line l_1 which intersects both components Σ_1 and Σ_2 of $\partial\Omega$. This assertion can be proved by contradiction. We assume that the line about the mentioned above intersection is transversal and denote by P_1 and P_2 the points of intersection of l_1 with Σ_1 and Σ_2, respectively.

Further, we assume that there exists another stream line l_2 , which also intersects curves Σ_1 and Σ_2. Thus, both curves Σ_i $(i = 1, 2)$ are divided by lines l_i on two parts Σ_i^- and Σ_i^+. Respectively, the domain Ω is divided on two simply connected domains Ω^+ and Ω^-. We suppose additionally that the flux through components Σ_1^- and Σ_1^+ are equal to $q_1/2$. Choosing a single-valued branch of function ψ and denoting it by $\Psi(x_1, x_2)$ we can consider that $\Psi \to q_1/2$ when one tends to point P_1 along curve Σ_1^+ and $\Psi \to -q_1/2$ when one tends to point P_2 along curve Σ_1^-. Now we will construct a virtual drain for the flow in domain Ω. Idea of construction is close to the structure proposed in the paper [11] for symmetric flow, but we are not able to apply this structure word for word. Our idea consists in construction of a virtual drain with support in a narrow curvilinear strip near the line l_1. To this end, we pass in system (1.1) to curvilinear orthogonal coordinates following [19].

Let us denote the curvilinear orthogonal coordinates as s_1 and s_2, and the corresponding Lame coefficients as H_1 and H_2. We preserve notations v_1 and v_2 for projections of velocity vector on the axes s_1 and s_2 because this will not bring to misunderstanding. System (1.1) written in curvilinear coordinates takes the form

$$\frac{v_1}{H_1}\frac{\partial v_1}{\partial s_1} + \frac{v_2}{H_2}\frac{\partial v_1}{\partial s_2} + \frac{v_2}{H_1 H_2}\left(v_1\frac{\partial H_1}{\partial s_2} - v_2\frac{\partial H_2}{\partial s_1}\right) = -\frac{1}{H_1}\frac{\partial p}{\partial s_1} + \nu\left[\frac{1}{H_1^2}\frac{\partial^2 v_1}{\partial s_1^2} + \frac{1}{H_2^2}\frac{\partial^2 v_1}{\partial s_2^2} + \right.$$

$$+\frac{1}{H_1H_2}\frac{\partial(H_1^{-1}H_2)}{\partial s_1}\frac{\partial v_1}{\partial s_1} + +\frac{1}{H_1H_2}\frac{\partial(H_2^{-1}H_1)}{\partial s_2}\frac{\partial v_1}{\partial s_2} + \frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial s_2}\frac{\partial v_2}{\partial s_1} - \frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial s_1}\frac{\partial v_2}{\partial s_2} +$$

$$+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_1 + \frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_1 +$$

$$\left.+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_2 - \frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_2\right],$$

$$\frac{v_1}{H_1}\frac{\partial v_2}{\partial s_1}+\frac{v_2}{H_2}\frac{\partial v_2}{\partial s_2}-\frac{v_1}{H_1H_2}\left(v_1\frac{\partial H_1}{\partial s_2}-v_2\frac{\partial H_2}{\partial s_1}\right)=-\frac{1}{H_2}\frac{\partial p}{\partial s_2}+\nu\left[\left(\frac{1}{H_1^2}\frac{\partial^2 v_2}{\partial s_1^2}+\frac{1}{H_2^2}\frac{\partial^2 v_2}{\partial s_2^2}+\right.\right.$$
$$+\frac{1}{H_1H_2}\frac{\partial(H_1^{-1}H_2)}{\partial s_1}\frac{\partial v_2}{\partial s_1}+\frac{1}{H_1H_2}\frac{\partial(H_2^{-1}H_1)}{\partial s_2}\frac{\partial v_2}{\partial s_2}-\frac{2}{H_1^2H_2}\frac{\partial H_1}{\partial s_2}\frac{\partial v_1}{\partial s_1}+\frac{2}{H_1H_2^2}\frac{\partial H_2}{\partial s_1}\frac{\partial v_1}{\partial s_2}+$$
$$+\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_2+\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_2-$$
$$\left.-\frac{1}{H_1}\frac{\partial}{\partial s_1}\left(\frac{1}{H_1H_2}\frac{\partial H_1}{\partial s_2}\right)v_1+\frac{1}{H_2}\frac{\partial}{\partial s_2}\left(\frac{1}{H_1H_2}\frac{\partial H_2}{\partial s_1}\right)v_1\right],$$
$$\frac{\partial(H_2v_1)}{\partial s_1}+\frac{\partial(H_1v_2)}{\partial s_2}=0. \tag{4.1}$$

Let us choose now a special coordinates system proposed at first by R. von Mises in the paper [20] devoted to boundary layer theory (see also [19]). Taking point P_1 as the origin, we will determine position of point $P\in\Omega$ by coordinates $s_1=s$ and $s_2=n$, where s is the arc length of line l_1 and n is the length of normal to this line taken with an appropriate sign. Then the first quadratic form is written as

$$d\sigma^2=[1+\frac{n}{\rho(s)}]^2ds^2+dn^2$$

and therefore

$$H_1=1+\frac{n}{\rho(s)},\ H_2=1. \tag{4.2}$$

Here $\rho(s)$ is the curvature radius of curve l_1 in the point with coordinate s. We will suppose that the curve l_1 is smooth enough so that $\rho(s)\in C^1[0,L]$ where L is length of l_1.

Let us denote as S_δ the strip $S_\delta=\{s,n\,:\,s\in\mathbf{R},\ |n|<\delta\}$ and define the domain K_1 by relation $K_1=\Omega\cap S_\delta$. The virtual drain $\mathbf{c}_1$ is defined by formula

$$\mathbf{c}_1=-\lambda q_1(\eta(n),\ 0) \tag{4.3}$$

where $\eta(n)$ is the cutting function introduced in [10] and $\lambda=\lambda(\Omega,l_1,\delta)$ is a positive constant. In view of (4.2) and last equation of (4.1), vector $\mathbf{c}_1$ is smooth and solenoidal. Choosing a suitable correcting multiplier λ, we are able to provide the required flux $q_1/2$ through the curves Σ_1^+ and Σ_1^-.

Having available two simply connected domains $\Omega^+,\ \Omega^-$ and the virtual drain (4.3), we can repeat almost literally the procedure described in the proof of Lemma 2. It consists in obtaining the identity like (3.10) for vector $\mathbf{u}\in\mathbf{H}(\Omega)$ defined by an analogue of formula (3.6). As a result, we come to an a priory estimate of type (3.20). Unfortunately, now constant M_4 depends not only on $\Omega,v,||\mathbf{b}_1||_{H^1}$ and $||\mathbf{c}_1||_{H^1}$ but also on C^3 – norm of the function, which parametrizes curve l_1. Because of this reason, our result has a conditional character.

5. Discussion

a) An a priori estimate of the Dirichlet integral (1.6) for the solution of problem (1.1), (1.2) has not only a theoretical interest but also allows us to justify approximate methods, in particular, Galerkin method [18]. The result of Lemma 2 guarantees such justification for symmetric flows in $\mathbb{R}^3$.

b) Detailing the proof of Lemma 2, we may conclude that dependence of value M_4 in (3.20) on norms $||\mathbf{c}_i||_{H^1}$ $(i = 1, \ldots, m)$ is not more than a linear one. It means that norm $||\mathbf{v}||_{H^1}$ of has maximum linear growth in $q_* = \max|\mathrm{q_i}|$ $(i = 1, \ldots, m)$ because norms $||\mathbf{b}_i||_{H^1}$ $(i = 1, \ldots, m)$ and value M_3 in (3.19) do not depend on q_*. This assertion is compatible with results of article [13] where there are studied a number of exact solutions to the problem.

c) During our treatment of the problem, functions $\mathbf{a}_i$ in boundary condition (1.2) were supposed to be smooth. This is possible to relax this condition up to inclusion $\mathbf{a}_i \in H^{1/2}(\Sigma_i)$, $(i = 1, \ldots, m)$ as it was done in [10, 11]. The statement of Lemma 2 holds in this case.

d) We restrict our analysis by the case of absence of external body force acting on a liquid. The case of potential force is reduced to previous case with the help of pressure transform. Let us consider the general situation where an acceleration of body force is $\mathbf{f}(x)$, where $\mathbf{f}$ is a given admissible function. Following the arguments of [10, 11], we can prove an analogue of Lemma 2 if $\mathbf{f} \in L^2(\Omega)$ and analogues of Theorem 3 and Theorem 4 if $\mathbf{f} \in C^\alpha(\bar{\Omega})$.

e) The conditional result declared in Section 4 stimulates the study of stream lines structure in two-dimensional stationary incompressible viscous flows. For classical symmetric solutions of the problem (1.1), (1.2) in admissible domain $\Omega \in \mathbb{R}^2$, we can apply the Kronrod theorem [21]. In particular, this theorem implies the following conclusion.

Let us consider a set of level lines

$$\psi(x_1, x_2) = c \tag{5.1}$$

where $\psi \in C^2(\bar{\Omega})$ *and* $c \in \mathbb{R}$. *There exists a set* N *of zero measure such that for any* $c \in \mathbb{R} \setminus N$ *the corresponding level line (5.1) outcomes on* $\partial\Omega$ *or this line is closed.*

This illuminates situation with the structure of stream lines in a plane symmetric case. On the base of Theorem 4, a similar statement is true for axially symmetric flows. As for a general two-dimensional flow, we know almost nothing about the structure of stream lines set.

Let consider a flow in a curvilinear ring Ω under additional conditions

$$\mathbf{a}_1 \cdot \mathbf{n}_1 < 0,\ x \in \Sigma_1; \quad \mathbf{a}_2 \cdot \mathbf{n}_2 > 0,\ x \in \Sigma_2 \tag{5.2}$$

or

$$\mathbf{a}_1 \cdot \mathbf{n}_1 > 0,\ x \in \Sigma_1; \quad \mathbf{a}_2 \cdot \mathbf{n}_2 < 0,\ x \in \Sigma_2. \tag{5.3}$$

In other words, each point of curve Σ_1 is an input (output) point of a stream line inside (outside) domain Ω and the same property is valid for curve Σ_2. The following conjecture (C) seems to be likely.

Let $\mathbf{v}, p$ *be a solution to problem (1.1), (1.2), (5.2) or (1.1), (1.2), (5.3) in a curvilinear ring* Ω. *Then each stream line connects* Σ_1 *with* Σ_2 *and intersects transversally these curves.*

This conjecture is the most plausible if the Reynolds number $\mathrm{Re} = |q_1|/\nu$ is sufficiently large.

f) Let us consider the flow in a curvilinear ring assuming that the Reynolds number $\mathrm{Re} \to \infty$. In this case, a formal asymptotic solution of the problem (1.1), (1.2) solution can be constructed by a certain modification of the Vishik-Lyusternik method [22]. In contrast to a boundary layer near a solid wall, the thickness of boundary layer in problem (1.1), (1.2) has an order of Re^{-1}. This boundary layer is localized near the curve Σ_1 if $q_1 > 0$ and near the curve Σ_2 in the opposite case.

Unfortunately, we are not able to establish closeness of approximate and exact solutions of the problem as $\mathrm{Re} \to \infty$. A natural approach based on the linearization of the problem in the approximate solution and consequent application of the Kantorovich theorem on convergence of Newton method does not lead to success since we have no sufficient information concerning the linearized operator.

g) In the conclusion, we discuss briefly how to weaken symmetry assumptions in the solution to problem (1.1), (1.2). One of the reasonable ways is to preserve the symmetry of flow domain but to cancel the symmetry property of boundary conditions.

For simplicity, let us consider a curvilinear ring Ω, which is symmetric about the line $\{x_2 = 0\}$. Now we will not suppose the symmetry of functions $\mathbf{a}_i$ $(1 = 1, 2)$ in boundary condition (1.2). Let decompose functions $\mathbf{a}_i$ on symmetric and antisymmetric parts,

$$\mathbf{a}_i = \mathbf{h}_i + \mathbf{g}_i, \quad i = 1, 2. \tag{5.4}$$

Here $\mathbf{h}_i$ is a symmetric function with respect to the line $\{x_2 = 0\}$ in the sense of definition 1, while $\mathbf{g}_i$ is an antisymmetric one with respect to this line. The latter means that g_1 is an odd function of x_2 while g_2 is an even function of x_2.

The solution to problem (1.1), (1.2) is sought in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad p = p_s + p_a. \tag{5.5}$$

Here $\mathbf{u}$ is symmetric function, $\mathbf{w}$ is antisymmetric function, p_s is even in variable x_2 and p_a is odd in this variable. Substituting (5.4), (5.5) into the system (1.1) and boundary condition (1.2) we obtain as a result of decomposition procedure:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{w} \cdot \nabla \mathbf{w} = -\nabla p_s + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad x \in \Omega, \tag{5.6}$$

$$\mathbf{u} = \mathbf{h}_i(x), \quad x \in \Sigma, \ (i = 1, 2), \tag{5.7}$$

$$\mathbf{u} \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u} = -\nabla p_a + \nu \Delta \mathbf{w}, \quad \nabla \cdot \mathbf{w} = 0, \quad x \in \Omega, \tag{5.8}$$

$$\mathbf{w} = \mathbf{g}_i(x), \quad x \in \Sigma, \ (i = 1, 2). \tag{5.9}$$

At given $\mathbf{u}$, function $\mathbf{w}$ is determined as the solution of linear problem (5.8), (5.9). If the corresponding linear operator is convertible and an appropriate norm of $\mathbf{g}_i$ is small, we can prove the solvability of problem (5.6), (5.7). Unfortunately, there are no sufficient conditions for existence of the unique solvability to the problem (5.8), (5.9). It would be interesting to prove the following statement in view of the result, obtained in paper [12]:

Let Ω is a symmetric curvilinear ring with a smooth boundary $\Sigma_1 \cup \Sigma_2$. Let $\mathbf{h}_i \in C^{2+\alpha}(\Sigma_i)$ are symmetric functions while $\mathbf{g}_i \in C^{2+\alpha}(\Sigma_i)$ are antisymmetric functions $(i = 1, 2)$. There is a countable subset N of $\mathbf{R}$ such that if $q_1 \notin N$ and $||\mathbf{g}_i||_{C^{2+\alpha}}$ are small, then problem (5.6)-(5.9) has at least one classical solution.

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References

[1] Leray, J. Etude de diverses equations intergales nonlineaires et de quelques problemes que pose l'hydrodynamique, J. Math. Pure Appl., **12** (1933), pp. 1-82.

[2] Hopf, E., Ein allgemeiner Endlichkeitssatz der Hydrodynamik, Math. Ann., **117** (1941), pp. 764-775.

[3] Ladyzhenskaya, O.A., *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969.

[4] Takeshita, A., A remark on Leray's inequality, Pacific J. Math., **157** (1993), pp. 151-158.

[5] Finn, R., On steady-state solutions of the Navier-Stokes equations. III, Acta Math., **105** (1961), pp. 197-244.

[6] Galdi, G.P., On the existence of steady motions of a viscous flow with non-homogeneous conditions, Le Matematiche, **66** (1991), pp. 503-524.

[7] Galdi, G.P., *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. 2*, Springer, 1994.

[8] Borchers, W. and Pileckas, K., Note on the flux problem for stationary Navier-Stokes equations in domains with multiply connected boundary, Acta Appl. Math., **37** (1994), pp. 21-30.

[9] Amick, Ch. J., Existence of solutions to the nonhomogeneous steady Navier-Stokes equations, Indiana Univ. Math. J., **33** (1984), pp. 817-830.

[10] Fujita, H., On stationary solutions to Navier-Stokes equations in symmetric plane domains under general out-flow conditions, Proceedings of International Conference on Navier-Stokes Equations, Theory and Numerical Methods, June 1997, Varenna Italy, Pitman Research Notes in Mathematics, **388**, pp. 16-30.

[11] Morimoto, H., A remark on the existence of 2-D steady Navier-Stokes flow in bounded symmetric domain under general outflow condition, J. Math. Fluid Mech., **9** (2007), pp. 411-418.

[12] Fujita, H. and Morimoto, H., A remark on the existence of the Navier-Stokes flow with non-vanishing outflow conditions, GAKUTO Internat. Ser. Math. Sci. Appl., **10** (1997), pp. 53-61.

[13] Morimoto, H., Stationary Navier-Stokes equations under general outflow condition, Hokkaido Math. J., **24** (1995), pp. 641-648.

[14] Morimoto, H. and Ukai, S., Perturbation of the Navier-Stokes in an annular domain with non-vanishing outflow condition, J. Math. Sci., Univ. Tokyo, **3** (1996), pp. 73-82.

[15] Fujita, H., Morimoto, H. and Okamoto, H. Stability analysis of the Navier-Stokes equations flows in annuli, Math. Methods in the Appl. Sciences, **20** (1997), pp. 959-978.

[16] Hopf, E., On nonlinear partial differential equations, in *Lecture Series of the Symposium on Partial Differential Equations, Berkeley, 1955*, The University of Kansas, 1-29, 1957.

[17] Solonnikov, V.A. and Shchadilov, V.E., A certain boundary value problem for the stationary system of Navier-Stokes equations, Trudy Math. Inst. AN SSSR, **125** (1973), pp. 196-210 (in Russian).

[18] Fujita, H., On the existence and regularity of the steady-state solutions of the Navier-Stokes equations, J. Fac. Sci., Univ. Tokyo, Sec. 1, **9** (1961), pp. 59-102.

[19] Kochin, N.E., Kibel, I.A. and Roze, N.V. *Theoretical Hydrodynamics, Vol. 2*, Wiley, New York, 1964.

[20] Von Mises, R., Bemerkungen zur Hydrodynamik, Z. Angew. Math. Mech., **7** (1927), pp. 425-429.

[21] Kronrod, A.S., On functions of two variables, Uspekhi Matematicheskikh Nauk, **5.1** (1950), pp. 24-134 (in Russian).

[22] Vishik, M.I. and Lyusternik, L.A., Regular degeneration and boundary layer for linear differential equations with a small parameter, Uspekhi Matematicheskikh Nauk, **12.5** (1957), pp. 3-122 (in Russian).