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ON THE EQUIVALENCE OF THE STATIC AND DYNAMIC POINT OF VIEW FOR DIFFUSIONS IN A RANDOM ENVIRONMENT

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Abstract. We study the equivalence of the static and dynamic point of view for diffusions in a random environment in dimension one. First we prove that the static and dynamic distributions are equivalent if and only if either the speed in the law of large numbers does not vanish, or b/a is a.s. the gradient of a stationary function, where a and b are the covariance coefficient resp. the local drift attached to the diffusion. We moreover show that the equivalence of the static and dynamic point of view is characterized by the existence of so-called "almost linear coordinates".

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1. Introduction

The process of the "environment viewed from the particle" is an important tool in the study of many models in random motions in random media. A key property is the existence of an invariant measure for this process, that is absolutely continuous with respect to the static law of the environment. It implies the equivalence of the static and dynamic distributions of the environment, and is the starting point in the analysis of the environment viewed from the particle. For applications of these techniques, we refer to Bolthausen and Sznitman [2], Kipnis and Varadhan [18], Kozlov [20], Lawler [22], de Masi et al. [9], Molchanov [24], Olla [25],[26], Papanicolaou and Varadhan [27], Rassoul-Agha [28] and also the overviews [32],[33],[35],[36] and the references therein. The main purpose of this work is to characterize the equivalence of the static and dynamic point of view in the specific setting of one-dimensional diffusions in random environment.

Before explaining our results in detail, let us define the setting. The *random environment* is described by a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that there exists a group $\{t_x : x \in \mathbb{R}\}$ of transformations on Ω, jointly measurable in x, ω, that preserve the probability $\mathbb{P}$, and that act ergodically on Ω, see the beginning of section 2 for details. On $(\Omega, \mathcal{A}, \mathbb{P})$ we consider random variables a and b and we write

$$a(x,\omega) = a(t_x\omega), \quad b(x,\omega) = b(t_x\omega)\,. \tag{1.1}$$

We assume that there are positive constants ν, β such that for all $\omega \in \Omega$, all $x \in \mathbb{R}$,

$$\frac{1}{\nu} \le a(x,\omega) \le \nu, \quad |b(x,\omega)| \le \beta\,. \tag{1.2}$$

We further assume that a and b are Lipschitz continuous, i.e. there is a constant K such that for all $\omega \in \Omega$, $x, y \in \mathbb{R}$,

$$|a(x,\omega) - a(y,\omega)| + |b(x,\omega) - b(y,\omega)| \le K|x-y|\,. \tag{1.3}$$

We denote by $(C(\mathbb{R}_+), \mathcal{F}, W)$ the canonical Wiener space, and with $(B_t)_{t\ge 0}$ Brownian motion (which is independent from $(\Omega, \mathcal{A}, \mathbb{P})$). The diffusion process in the random environment ω is described by the family of laws $(P_{x,\omega})_{x\in\mathbb{R}}$ (we call them the *quenched* laws) on $(C(\mathbb{R}_+), \mathcal{F})$ of the solution of the stochastic differential equation

$$\begin{cases} dX_t = \sigma(X_t,\omega)dB_t + b(X_t,\omega)dt, \\ X_0 = x, \quad x \in \mathbb{R},\ \omega \in \Omega, \end{cases} \tag{1.4}$$

where $\sigma = \sqrt{a}$. The attached differential operator is defined through

$$\mathcal{L}_\omega = \tfrac{1}{2} a(x,\omega)\partial_x^2 + b(x,\omega)\partial_x\,. \tag{1.5}$$

To restore some stationarity to the problem, it is convenient to introduce the *annealed* laws P_x, which are defined as the semi-direct products:

$$P_x = \mathbb{P} \times P_{x,\omega}, \text{ for } x \in \mathbb{R}. \tag{1.6}$$

Observe that the Markov property is typically lost under the annealed laws.
Our model can be regarded as a continuous space-time analogue of random walks in random environment, that has been extensively studied in dimension one, see section 3 in Sznitman [33] for an abundant list of references. Another one-dimensional model studied recently deals with diffusions in a Brownian environment; loosely speaking, the drift term b is given by white noise. We refer to Brox [6], and to Tanaka [34] and the references therein.

Let us now define the relevant objects. The "environment viewed from the particle" is the Ω-valued Markov process

$$\bar{\omega}_t = t_{X_t}\omega, \quad t \ge 0. \tag{1.7}$$

It describes the environment seen from an observer sitting on top of the diffusing particle. An invariant measure $\mathbb{Q}$ for this process satisfies $P_t\,\mathbb{Q} = \mathbb{Q}$, where P_t is the semigroup associated to $\bar{\omega}_t$. $\mathbb{Q}$ describes the *effective* environment that governs the asymptotic behaviour of the particle. A key assumption is the

(1.8) existence of an invariant measure $\mathbb{Q}$ that is *absolutely continuous* w.r.t. $\mathbb{P}$.

Under (1.8) holds the following well known fact (valid in all dimensions):
(1.9)
There is at most one measure $\mathbb{Q}$ that satisfies (1.8). Moreover $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, and the process $\bar{\omega}_t$ with initial distribution $\mathbb{Q}$ is stationary ergodic.

A proof of (1.9) is contained in the proof of Theorem 2.1 in Papanicolaou and Varadhan [27] when $b \equiv 0$. The same argument applies for a local drift b that satisfies (1.2) and (1.3). As a consequence of (1.9), roughly speaking, the "static" and the "dynamic" point of view are comparable, and moreover, (1.9) allows the use of ergodic theory in the study of the diffusion in random environment.

(1.10) We are thus naturally led to ask under when (1.8) holds?

The main object of this work is to answer this question in dimension one.
Let us incidentally mention that the answer to this question is still widely open in higher dimensions. A measure $\mathbb{Q}$ that satisfies (1.8) in higher dimensions is only known in few specific cases, see section 4 in Sznitman [33]. We also refer to Bolthausen and Sznitman [3] for examples of non-nestling walks in high dimensions that satisfy (1.8) in the presence of low disorder. Recent progress in higher dimensions is thus rather built up on new techniques, see again [33] for an overview.

In dimension one the situation is different. Here the method of the environment viewed from the particle applies for a large class of environments, and enables us to prove a strong law of large numbers, cf. Proposition 3.3. In particular the law of large numbers shows that a non-vanishing speed implies (1.8). Theorem 4.1 provides a complete answer to the question (1.10). It shows that when the speed vanishes, then (1.8) holds if and only if b/a is $\mathbb{P}$-a.s. the gradient of a stationary random variable. This result is compatible with its discrete counterpart proved by Conze and Guivarc'h [8] for random walk in a random environment on $\mathbb{Z}$ (see also Brémont [4], [5] for generalizations to finite range random walks on $\mathbb{Z}$), and our proof is inspired by the methods used in [8]. When $b/a = \nabla V$ $\mathbb{P}$-a.s., where ∇V is defined as the pointwise limit $\lim_{x\to 0}\frac{1}{x}(V\circ t_x - V)$, then $\mathbb{E}[b/a]=0$, and Proposition 3.1 shows that the diffusion in a random evironment is recurrent. In particular (1.10) has a negative answer for diffusions that are transient with vanishing speed.
It is interesting to notice that, if $\mathbb{E}[b/a]=0$ and suitable mixing assumptions hold, then b/a is $\mathbb{P}$-a.s. the gradient of a stationary random variable if and only if the $\mathbb{P}$-variance of the additive functional

$$A(x,\omega) \stackrel{\text{def}}{=} \int_0^x (b/a)(u,\omega)\,du,\ \ x\in\mathbb{R}, \tag{1.11}$$

is a bounded function of x, see Theorem 4.5. Loosely speaking, large fluctuations of A create powerful traps for the diffusing particle, and as a consequence, the dynamic distributions fail to be absolutely continuous w.r.t. the static distributions.

Moreover we provide a further characterization of (1.8), that has, up to our knowledge, no corresponding counterpart in the discrete setting. Assume that (1.8) holds, and that $\mathbb{P}$-a.s.,

$$X_t - vt = X(t,X_t,\omega) + \chi(X_t,\omega), \tag{1.12}$$

where v denotes the speed in the law of large numbers, $X(t,x,\omega)$ is a parabolic function so that the first term in the right-hand side of (1.12) is a martingale, and the second term is a corrector term. We show that (1.8) is equivalent to the existence of a corrector χ, defined through (1.12), such that $\partial_x\chi$ is stationary and $\mathbb{E}[\partial_x\chi]=0$ (under some additional assumption in the recurrent setting, see below). In particular *sublinear* growth of the corrector χ at infinity follows from the ergodic theorem, see remark 4.3. In

this case the coordinates X are sometimes called "almost linear coordinates", and when $v = 0$, they are also known as "almost linear harmonic coordinates". Theorem 4.2 covers the transient setting, whereas Theorem 4.5 deals with the recurrent setting under the additional assumption of an exponential mixing property with rate $\gamma > 2\beta\nu$.
If in addition one can prove that χ exhibits at most *diffusive* growth at infinity, then a central limit theorem for martingales with stationary ergodic increments can be applied to prove diffusive behavior for the diffusion in random environment. This explains the interest in almost linear coordinates. This approach can for instance be found in Kozlov[20], Kipnis and Varadhan[18], Molchanov[24], Kozlov and Molchanov[21], section 2.2 in Zeitouni[35] and section 4 in Brémont[4].

This article is organised as follows. After some preliminaries in Section 2, we prove a recurrence-transience dichotomy and a strong law of large numbers in Section 3. In Section 4 we show the dynamic and static point of view are equivalent if and only if the speed does not vanish, or $\mathbb{P}$-a.s., b/a is the gradient of a stationary random variable, see Theorem 4.1. We then provide an additional characterization of the equivalence of the dynamic and static distributions in terms of almost linear coordinates, see Theorem 4.2 for the transient case, and Theorem 4.5 for the recurrent case.

2. Preliminaries

We provide some details about the group $(t_x)_x$ and its generator. $t_x, x \in \mathbb{R}$, is a group of transformations $t_x : \Omega \to \Omega$, i.e. $t_0 = Id$ and $t_x \circ t_y = t_{x+y}$, $x, y \in \mathbb{R}$. The mapping $(x, \omega) \mapsto t_x\omega$ is $(\mathcal{B} \otimes \mathcal{A}, \mathcal{A})$-measurable, with $\mathcal{B}$ denoting the Borel σ-field on $\mathbb{R}$. For a measure μ on Ω, we write $t_x\mu(\cdot) = \mu(t_{-x}\cdot)$. We assume that t_x preserves the measure $\mathbb{P}$ and is ergodic, i.e. $t_x\mathbb{P} = \mathbb{P}$, $x \in \mathbb{R}$, and if $A \in \mathcal{A}$ is such that $\mathbb{P}$-a.s., for all x, $t_xA = A$, then $\mathbb{P}[A] = 0$ or 1. When there is no source of confusion, L^p always denotes $L^p(\Omega, \mathcal{A}, \mathbb{P})$, $1 \le p \le \infty$. t_x induces a group of operators on L^1:

$$(T_xf)(\omega) \stackrel{\text{def}}{=} f(t_x\omega), \quad f \in L^1. \tag{2.1}$$

$(T_x)_x$ is a strongly continuous group of isometric operators on L^1, see [15] p.223 (the proof given there works for all L^p spaces, $1 \le p < \infty$). We denote with D the infinitesimal generator of $(T_x)_x$ on L^1. D is closed and its domain $\mathcal{D}(D)$ is dense in L^1 ([12] p.10). For $f \in \mathcal{D}(D)$ it holds that

$$Df = \lim_{x\to 0} \tfrac{1}{x}(T_xf - f) \quad \text{in } L^1. \tag{2.2}$$

We define the pointwise limit $\nabla f(\omega) = \lim_{x\to 0} \frac{1}{x}(T_xf(\omega) - f(\omega))$ for all f and $\omega \in \Omega$ where this limit exists. The invariance of $\mathbb{P}$ implies that

$$\int_\Omega Df \, d\mathbb{P} = 0, \quad f \in \mathcal{D}(D). \tag{2.3}$$

It follows from the product rule for D, and from (2.3), that

$$\int_\Omega Df \cdot g \, d\mathbb{P} = -\int_\Omega f \cdot Dg \, d\mathbb{P}\,, \tag{2.4}$$

for suitable f, g such that both integrals are well-defined. To the environment process defined in (1.7) is attached a strongly continuous semigroup on L^∞ with generator

$$Lf(\omega) = \tfrac{1}{2}a(\omega)D^2 f(\omega) + b(\omega)\, Df(\omega), \quad f \in \mathcal{D}(L). \tag{2.5}$$

The martingale problem for L is well-posed, and hence proposition 9.2 in [12] shows that a measure $\mathbb{Q}$ is invariant for L if and only if

$$\int Lf\, d\mathbb{Q} = 0 \quad \text{for all } f \in \mathcal{D}(L). \tag{2.6}$$

We conclude this section by introducing some notation. For $m \in \mathbb{R}$, we define the $(\mathcal{F}_t)_{t\geq 0}$-stopping time ($(\mathcal{F}_t)_{t\geq 0}$ denotes the canonical right-continuous filtration on $(C(\mathbb{R}), \mathcal{F})$)

$$S_m = \inf\{t \geq 0 : X_t = m\}\,.$$

For $\omega \in \Omega$, $x_1, x_2 \in \mathbb{R}$, we define (recall A in (1.11))

$$s_{x_1,x_2}(\omega) = \int_{x_1}^{x_2} \exp(-2A(u,\omega))\, du\,. \tag{2.7}$$

For fixed x_1, $s_{x_1,\cdot}(\omega)$ is a scale function for the quenched diffusion in the environment ω, cf. [16] p.339.

3. Asymptotic Behavior

We start by characterizing recurrence and transience. The discrete counterpart of the following proposition is due to Solomon [31], see also [35] p.196.

Proposition 3.1. *There are three cases:*

(i) *If $\mathbb{E}[b/a] > 0$, then P_0-a.s., $\lim_{t\to\infty} X_t = +\infty$.*
(ii) *If $\mathbb{E}[b/a] = 0$, then P_0-a.s., $-\infty = \liminf_{t\to\infty} X_t < \limsup_{t\to\infty} X_t = +\infty$.*
(iii) *If $\mathbb{E}[b/a] < 0$, then P_0-a.s., $\lim_{t\to\infty} X_t = -\infty$.*

Remark 3.2. In the setting of finite range dependence, the result of Proposition 3.1 follows from Proposition 2.7 in Goergen [13] together with equation (2.77) therein.

Proof. We introduce the random variables (recall (1.11))

$$S_+(\omega) \stackrel{\text{def}}{=} \int_0^\infty e^{-2A(u,\omega)}\, du, \quad S_-(\omega) \stackrel{\text{def}}{=} \int_{-\infty}^0 e^{-2A(u,\omega)}\, du. \tag{3.1}$$

We start by showing that

$$\mathbb{P}-\text{a.s. } S_+ < \infty \Longleftrightarrow \mathbb{E}[b/a] > 0, \quad \text{and} \quad \mathbb{P}-\text{a.s. } S_- < \infty \Longleftrightarrow \mathbb{E}[b/a] < 0. \tag{3.2}$$

When $\mathbb{E}[b/a] > 0$, then the ergodic theorem implies that $\mathbb{P}-$ a.s. $S_+ < \infty$. On $\{S_+ < \infty\}$ one has that $\lim_{x\to\infty} A(x,\omega) = \infty$. From Lemma 5.3 in the Appendix applied to $f(\omega) = \int_0^1 b/a(u,\omega)\, du$, we deduce that $\mathbb{E}[b/a] > 0$ (alternatively, one might invoke Kesten's lemma, see [17], or [36] p.197, together with the ergodic theorem). The second equivalence in (3.2) is entirely similar. Further notice that for all x,

$$T_x S_+(\omega) = e^{2A(x,\omega)}\big(S_+(\omega) - \int_0^x e^{-2A(u,\omega)}\, du\big)\,. \tag{3.3}$$

Thus the event $\{S_+ < \infty\}$ is invariant under t_x, and by ergodicity, $\mathbb{P}[S_+ < \infty] = 0$ or 1. A similar argument applies to S_-, and then (3.2) can be refined to

$$\begin{aligned} &\mathbb{P} - \text{a.s. } S_+ < \infty,\ S_- = \infty \Longleftrightarrow \mathbb{E}[b/a] > 0, \\ &\mathbb{P} - \text{a.s. } S_- < \infty,\ S_+ = \infty \Longleftrightarrow \mathbb{E}[b/a] < 0. \end{aligned} \tag{3.4}$$

Let us now show statement (i) of the Proposition. The scale function $s_{0,\cdot}(\omega)$ is $\mathcal{L}_\omega$-harmonic, and hence $s_{0,X_t}(\omega)$ is a martingale under $P_{0,\omega}$. The equivalences (3.4), together with the martingale convergence theorem, imply that $P_{0,\omega}$-a.s. $s_{0,X_t}(\omega)$ converges to a finite limit, so that, necessarily, P_0-a.s., $\lim_{t\to\infty} X_t = +\infty$. The case (iii) is similar. Let us now turn to case (ii). When $\mathbb{E}[b/a] = 0$, the equivalences (3.2) show that $\mathbb{P}$-a.s., $S_+ = \infty, S_- = \infty$. But then, for $m < 0$,

$$\mathbb{P} - \text{a.e. } \omega, \quad \lim_{M\to\infty} P_{0,\omega}[S_m < S_M] = \lim_{M\to\infty} \frac{s_{0,M}(\omega)}{s_{0,M}(\omega) - s_{0,m}(\omega)} = 1, \tag{3.5}$$

since $S_+(\omega) = \lim_{M\to\infty} s_{0,M}(\omega)$. Similarly $\mathbb{P}$ − a.s. $\lim_{m\to-\infty} P_{0,\omega}[S_M < S_m] = 1$, and the claim follows. □

Throughout the remainder of this article, we use the following notation:

$$\phi_+(\omega) \stackrel{\text{def}}{=} \tfrac{1}{a(\omega)} \int_0^\infty e^{-2A(u,\omega)}\, du, \quad \phi_-(\omega) \stackrel{\text{def}}{=} \tfrac{1}{a(\omega)} \int_{-\infty}^0 e^{-2A(u,\omega)}\, du. \tag{3.6}$$

We now derive a strong law of large numbers with the help of the method of the "environment viewed from the particle". The same method has been applied in the discrete setting, cf. Kozlov [20], and also Molchanov [24] and Sznitman [32]. Our proof uses similar arguments as the exposition in [32].

Proposition 3.3.

$$P_0 - a.s. \ \lim_{t\to\infty} \frac{X_t}{t} = v = \begin{cases} \frac{1}{2\mathbb{E}[\phi_+]}, & \text{if } \mathbb{E}[\phi_+] < \infty, \\ -\frac{1}{2\mathbb{E}[\phi_-]}, & \text{if } \mathbb{E}[\phi_-] < \infty, \\ 0, & \text{if } \mathbb{E}[\phi_+] = \mathbb{E}[\phi_-] = \infty. \end{cases} \tag{3.7}$$

Proof. Assume that $\mathbb{E}[\phi_+] < \infty$. We show that $\mathbb{Q} = \phi\mathbb{P}$, where $\phi = \phi_+/\mathbb{E}[\phi_+]$, is invariant for L by checking (2.6). Since $\mathbb{E}[b/a] > 0$ (see (3.2)), by dominated convergence,

$$\nabla(a\phi) = 2b\phi - 1/\mathbb{E}[\phi_+]. \tag{3.8}$$

By Lemma 5.4, $a\phi \in \mathcal{D}(D)$ and $D(a\phi) = \nabla(a\phi)$ $\mathbb{P}$-a.s. (1.2) and (1.3) imply that for all $\omega \in \Omega$, $\partial_x(b/a)(x,\omega)$ exists and is bounded by K for a.e. x. Hence $\nabla(b/a)$ exists and is bounded by K $\mathbb{P}$-a.s. (by the ergodic theorem). It follows from $b\phi = b/a \cdot a\phi$ and from Lemma 5.4 that $b\phi \in \mathcal{D}(D)$ and $D(b\phi) = \nabla(b\phi)$ $\mathbb{P}$-a.s. Thus, we obtain that

$$\tfrac{1}{2} D^2(a\phi) - D(b\phi) = 0. \tag{3.9}$$

It follows from (2.4) that for $f \in \mathcal{D}(L)$,

$$\int Lf\, d\mathbb{Q} = \int Lf\, \phi\, d\mathbb{P} = \int f\, (\tfrac{1}{2} D^2(a\phi) - D(b\phi))\, d\mathbb{P} = 0\,, \tag{3.10}$$

which implies that $\mathbb{Q}$ is invariant. Then $\mathbb{Q}$ is also ergodic, see (1.9). With $M_t = \int_0^t \sigma(X_s,\omega)\,dB_s = X_t - X_0 - \int_0^t b(X_s,\omega)\,ds$, we have $\langle M\rangle_t = \int_0^t a(X_s,\omega)\,ds \le \nu t$, and we find with the help of Bernstein's inequality, cf. [29] p. 153:

$$\begin{aligned} P_{0,\omega}[\tfrac{1}{n}|M_n| > \varepsilon/2] &\le 2\exp(-\tfrac{\varepsilon^2}{8\nu}\, n)\,, n \ge 1, \\ \sup_x P_{x,\omega}[\tfrac{1}{t} \sup_{0\le s<1} |M_s - M_0| > \tfrac{\varepsilon}{2}] &\le 2\exp(-\tfrac{\varepsilon^2}{8\nu} t^2). \end{aligned} \tag{3.11}$$

We obtain from (3.11) and from the Markov property and the lemma of Borel-Cantelli that $P_{0,\omega}$-a.s. $\lim_{t\to\infty} M_t/t = 0$. The ergodic theorem implies that $\mathbb{Q}\times P_{0,\omega}$-a.s., and hence P_0-a.s.,

$$\frac{1}{t}\int_0^t b(X_s,\omega)\,ds = \frac{1}{t}\int_0^t b(\bar{\omega}_s)\,ds \stackrel{t\to\infty}{\longrightarrow} \int b\,d\mathbb{Q}\,. \tag{3.12}$$

By (3.8) and the invariance of $\mathbb{P}$ we find that

$$\int b\,d\mathbb{Q} = \int b\,\phi\,d\mathbb{P} = \tfrac{1}{2}\int \nabla(a\phi)\,d\mathbb{P} + \frac{1}{2\mathbb{E}[\phi_+]} = \frac{1}{2\mathbb{E}[\phi_+]}\,. \tag{3.13}$$

Collecting the above facts, we obtain the first claim in (3.7). The case $\mathbb{E}[\phi_-] < \infty$ is treated similarly. It thus remains to consider the case $\mathbb{E}[\phi_+] = \mathbb{E}[\phi_-] = \infty$. In a first step, we show that

$$\limsup_{t\to\infty} X_t/t \le 0. \tag{3.14}$$

With the help of Proposition 3.1, we see that (3.14) only needs a proof when $\mathbb{E}[b/a] > 0$. In the latter case, we will prove that (3.14) follows from $\mathbb{E}[\phi_+] = \infty$. We do a comparison argument. We define the auxiliary drift term

$$b^\eta(x,\omega) = (1-\eta)b(x,\omega) + \eta\beta, \quad \eta \ge 0\,, \tag{3.15}$$

where β was introduced in (1.2). Let X denote the diffusion from (2.3) attached to the local characteristics a, b, and X^η the diffusion, defined similarly as X in (2.3), but with a, b replaced by a, b^η. We assume that both diffusions are driven by the same Brownian motion, in the same environment ω, and that $P_{0,\omega}$-a.s. $X_0 = X_0^\eta = 0$. Since $b^\eta \ge b$, Proposition 2.18 p.293 in [16] shows that for all $\omega \in \Omega$,

$$P_{0,\omega}[X_t \le X_t^\eta \text{ for all } 0 \le t < \infty] = 1\,. \tag{3.16}$$

We define ϕ_+^η similarly as ϕ_+ in (3.6), with b replaced by b^η, and set $\eta_0 = \sup\{\eta \in [0,1] : \mathbb{E}[\phi_+^\eta] = \infty\}$. Since $b^1 = \beta > 0$, and $b^0 = b$, it holds that $0 \le \eta_0 \le 1$. For $\eta > \eta_0$, it holds that $\mathbb{E}[\phi_+^\eta] < \infty$, and hence, P_0-a.s.,

$$\limsup_{t\to\infty} \frac{X_t}{t} \le \lim_{t\to\infty} \frac{X_t^\eta}{t} = \frac{1}{\mathbb{E}[\phi_+^\eta]}. \tag{3.17}$$

Notice that the map $\eta \mapsto \phi_+^\eta$ is decreasing. By monotone convergence, $\lim_{\eta\downarrow\eta_0} \mathbb{E}[\phi_+^\eta] = \mathbb{E}[\phi_+^{\eta_0}]$. Hence, in view of proving (3.14), it is enough to show that

$$\mathbb{E}[\phi_+^{\eta_0}] = \infty\,. \tag{3.18}$$

It suffices to consider $\eta_0 > 0$, since otherwise (3.18) holds by assumption. We define for $\eta \geq 0$, and $x > 0$,

$$F_\omega(\eta, x) \overset{\text{def}}{=} \tfrac{1}{x}\int_0^x dy\, \phi_+^\eta(t_y\omega) = \int_0^\infty du\, \tfrac{1}{x}\int_0^x dy\, \tfrac{1}{a(t_y\omega)} e^{-2\int_y^{y+u} \frac{b^\eta}{a}(v,\omega)\,dv}. \tag{3.19}$$

Recall that we assumed $\mathbb{E}[b/a] > 0$, and notice that for $0 \leq \eta \leq 1$, $\mathbb{E}[b^\eta/a] \geq \mathbb{E}[b/a] > 0$. By the ergodic theorem, there is a constant $C(\omega)$ that is $\mathbb{P}$-a.s. finite such that

$$\tfrac{1}{x}\int_0^x dy\, \tfrac{1}{a(t_y\omega)} e^{-2\int_y^{y+u} \frac{b^\eta}{a}(v,\omega)\,dv} \leq C(\omega)\nu \tfrac{1}{x}\int_0^x dy\, e^{-\mathbb{E}[b^\eta/a]u} \leq C(\omega)\nu e^{-\mathbb{E}[b/a]u}. \tag{3.20}$$

Hence we find by dominated convergence, and by the ergodic theorem, that $\mathbb{P}$-a.s.,

$$\begin{aligned}
\lim_{\eta\uparrow\eta_0}\lim_{x\to\infty} F_\omega(\eta,x) &= \int_0^\infty du \lim_{\eta\uparrow\eta_0}\lim_{x\to\infty} \tfrac{1}{x}\int_0^x dy\, \tfrac{1}{a(t_y\omega)} e^{-2\int_y^{y+u}\frac{b^\eta}{a}(v,\omega)\,dv} \\
&= \int_0^\infty du \lim_{\eta\uparrow\eta_0} \mathbb{E}[\tfrac{1}{a(\omega)} e^{-2\int_0^u \frac{b^\eta}{a}(v,\omega)\,dv}] \\
&= \int_0^\infty du\, \mathbb{E}[\tfrac{1}{a(\omega)} e^{-2\int_0^u \frac{b^{\eta_0}}{a}(v,\omega)\,dv}] \\
&= \mathbb{E}[\phi_+^{\eta_0}]\,.
\end{aligned} \tag{3.21}$$

Observe that the ergodic theorem applied to $\varphi_n = \phi_+^\eta \wedge n$, implies $\mathbb{P}$-a.s.

$$\liminf_{x\to\infty} F_\omega(\eta,x) \geq \liminf_{x\to\infty} \tfrac{1}{x}\int_0^x dy\, \varphi_n(t_y\omega) = \mathbb{E}[\varphi_n]. \tag{3.22}$$

It follows from monotone convergence, and the definition of η_0, that $\lim_{x\to\infty} F_\omega(\eta,x) = \infty$ for $\eta < \eta_0$. Using (3.21), we finally obtain $\mathbb{E}[\phi_+^{\eta_0}] = \lim_{\eta\uparrow\eta_0}\lim_{x\to\infty} F_\omega(\eta,x) = \infty$. This shows (3.18), and thus (3.14) holds. A completely analogous argument shows that $\liminf_{t\to\infty} X_t/t \geq 0$. Hence we have shown that $\mathbb{E}[\phi_+] = \mathbb{E}[\phi_-] = \infty$ implies $\lim_{t\to\infty} X_t/t = 0$, and the proof of the proposition is finished. ☐

4. Invariant measures

Recall that $\mathbb{E}[\phi_+] < \infty$ resp. $\mathbb{E}[\phi_-] < \infty$ characterize positive resp. negative speed, see Proposition 3.3.

Theorem 4.1. *We have the equivalences:*

I. *There is an invariant probability measure $\mathbb{Q}$ that is absolutely continuous w.r.t. $\mathbb{P}$.*

II. *Exactly one of the following conditions hold:*
   - (i) $\mathbb{E}[\phi_+] < \infty$ *(and hence $\mathbb{E}[b/a] > 0$)*
   - (ii) $\mathbb{E}[\phi_-] < \infty$ *(and hence $\mathbb{E}[b/a] < 0$)*
   - (iii) *There is V stationary with $\mathbb{E}[e^{2V}] < \infty$ and such that $\mathbb{P}$-a.s., $x \mapsto V(t_x\omega)$ is absolutely continuous and $b/a = \nabla V$ (and hence $\mathbb{E}[b/a] = 0$).*

Proof of I.⇒ II.
The proof consists of several steps.
Step 1: Reduction argument
We start by showing that it suffices to prove the claim for the environment process $\bar{\omega}_t$ attached to $\tilde{L} = \frac{1}{2}D^2 + \frac{b}{a}D\,(= \frac{1}{a}L)$. The reason for this step becomes clear in (4.4).

Notice that condition II.(iii) remains unchanged. If II.(i) or (ii) hold, denote the density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$ with ϕ. Since $\mathbb{Q}$ is invariant, it holds that

$$\int \tilde{L}f\, a\phi\, d\mathbb{P} = \int Lf\, \phi\, d\mathbb{P} = \int Lf\, d\mathbb{Q} = 0\,, \quad f \in \mathcal{D}(L), \tag{4.1}$$

so that the measure $a\phi\,\mathbb{P}$ is invariant for $\tilde{L}$, see [12] p.239. We define $\tilde{\phi}_\pm$ as in (3.6), with the covariance coefficient being the identity, and the drift term b/a, and observe that $\tilde{\phi}_\pm = a\phi_\pm$. By (1.2), $\mathbb{E}[\phi_\pm] < \infty$ is equivalent to $\mathbb{E}[\tilde{\phi}_\pm] < \infty$. The claim I. $\Rightarrow$ II. thus follows once we have shown that the existence of an invariant measure for $\tilde{L}$ implies $\mathbb{E}[\tilde{\phi}_\pm] < \infty$. In the subsequent steps of the proof I. $\Rightarrow$ II., we thus assume that

$$\bar{\omega}_t \text{ is attached to the operator } \tfrac{1}{2}D^2 + \tfrac{b}{a}D. \tag{4.2}$$

Further one can interpret the continuous map $x \mapsto T_x(b/a)(\omega)$ as a realization of the environment ω. In the remainder of the proof of I. $\Rightarrow$ II. we therefore work with the sample space $\Omega = C(\mathbb{R})$ endowed with the canonical σ-field $\sigma(b/a)$.

Step 2: Discretization

We first introduce some further notation. For integers i, j and for $\delta > 0$, we define $s^\delta_{i,j}(\omega) = s_{i\delta,\, j\delta}(\omega)$. We write $t^\delta_k = t_{k\delta}$, $T^\delta_k = T_{k\delta}$, $k \in \mathbb{Z}$, for the induced group of transformations resp. operators on the discrete lattice $\delta\mathbb{Z}$. We introduce discrete jump probabilities attached to the lattice $\delta\mathbb{Z}$ in the following natural way:

$$p^\delta(\omega) = P_{0,\omega}[S_\delta < S_{-\delta}] = \frac{s^\delta_{-1,0}(\omega)}{s^\delta_{-1,1}(\omega)}, \text{ and } q^\delta(\omega) = P_{0,\omega}[S_\delta > S_{-\delta}] = \frac{s^\delta_{0,1}(\omega)}{s^\delta_{-1,1}(\omega)}. \tag{4.3}$$

We define the σ-field $\mathcal{A}_\delta = \sigma(p^\delta)$, and note that t^δ_1 is $\mathcal{A}_\delta$-measurable. Notice that $\lim_{\delta\to 0} \frac{1}{4\delta}(p^\delta/T^\delta q^\delta - 1) = b/a$. As a result,

$$\sigma(b/a) = \bigvee_{\delta>0} \mathcal{A}_\delta. \tag{4.4}$$

We now consider the discrete-time Markov chain with transition kernel

$$R_\delta f(\omega) = p^\delta(\omega)\, T^\delta_1 f(\omega) + q^\delta(\omega) T^\delta_{-1} f(\omega), \;\; f \text{ bounded and } \mathcal{A}_\delta\text{-measurable}. \tag{4.5}$$

We denote with $\mathbb{Q}_\delta$ the restriction of $\mathbb{Q}$ to $\mathcal{A}_\delta$.

Step 3: invariance and quasi-invariance of $\mathbb{Q}_\delta$

We denote with $\tilde{P}_\omega$ the canonical law of $\bar{\omega}_t$ started at ω, and with $\tilde{E}_\omega$ resp. $\tilde{\mathbb{E}}_\mathbb{Q}$ the expectation w.r.t. the measure $\tilde{P}_\omega$ resp. $\mathbb{Q} \times \tilde{P}_\omega$. We show that $\mathbb{Q}_\delta$ is invariant for R_δ,

$$\tilde{\mathbb{E}}_{\mathbb{Q}_\delta}[R_\delta f] = \mathbb{E}_{\mathbb{Q}_\delta}[f], \quad \text{for all } f \text{ bounded } \mathcal{A}_\delta\text{-measurable}. \tag{4.6}$$

Further we denote with $U_\delta = S_\delta \wedge S_{-\delta}$ the exit time of the interval $(-\delta, \delta)$. The invariance of $\mathbb{Q}_\delta$ for R_δ is equivalent to

$$\tilde{\mathbb{E}}_\mathbb{Q}[f(\bar{\omega}_{U_\delta})] = \mathbb{E}_\mathbb{Q}[f(\omega)]\,. \tag{4.7}$$

Applying the martingale problem at the time $t \wedge U_\delta$, we obtain for $f \in \mathcal{D}(L)$ that

$$\tilde{E}_\omega[f(\bar{\omega}_{t\wedge U_\delta})] - f(\omega) = \tilde{E}_\omega[\int_0^{t\wedge U_\delta} Lf(\bar{\omega}_s)\, ds]. \tag{4.8}$$

Since $E_{0,\omega}[U_\delta] < \infty$ (Lemma 7.4 p.365 in [16]), we apply dominated convergence and find, after $\mathbb{Q}$-integration:

$$\tilde{\mathbb{E}}_\mathbb{Q}[f(\bar{\omega}_{U_\delta})] - \mathbb{E}_\mathbb{Q}[f] = \tilde{\mathbb{E}}_\mathbb{Q}[\int_0^{U_\delta} Lf(\bar{\omega}_s)\, ds]. \tag{4.9}$$

Moreover, by Fubini's theorem,

$$\tilde{\mathbb{E}}_\mathbb{Q}[\int_0^{U_\delta} Lf(\bar{\omega}_s)\, ds] = \mathbb{E}_\mathbb{Q}\Big[\int_0^\infty \tilde{E}_\omega[Lf(\bar{\omega}_s), U_\delta > s]\, ds\Big] = \mathbb{E}_\mathbb{Q}[GLf], \tag{4.10}$$

in the notation introduced above Lemma 5.5 in the Appendix. Lemma 5.5 and the invariance of $\mathbb{Q}$ imply that $\mathbb{E}_\mathbb{Q}[GLf] = \mathbb{E}_\mathbb{Q}[LGf] = 0$. Now (4.7) follows from (4.9).
(4.6), and the fact that $p^\delta, q^\delta > 0$, imply that $\mathbb{Q}_\delta$ is quasi-invariant, i.e. $t_1^\delta \mathbb{Q}_\delta$ and $t_{-1}^\delta \mathbb{Q}_\delta$ are absolutely continuous w.r.t. $\mathbb{Q}_\delta$. In particular $t_{-1}^\delta \mathbb{Q}_\delta$ and $\mathbb{Q}_\delta$ are equivalent,

$$t_{-1}^\delta \mathbb{Q}_\delta = \beta_\delta\, \mathbb{Q}_\delta \ \text{ for some } \beta_\delta > 0 \ \ \mathbb{P}\text{-a.s.} \tag{4.11}$$

Combining (4.6) with (4.11), we obtain that

$$T_{-1}^\delta p^\delta\ T_{-1}^\delta \beta_\delta^{-1} + T_1^\delta q^\delta\ \beta_\delta = 1 \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.12}$$

With the notation $\gamma_\delta = \beta_\delta\, T_1^\delta q^\delta / p^\delta$, we obtain from (4.12) that

$$p^\delta\, \gamma_\delta \ + \ q^\delta\ T_{-1}^\delta \gamma_\delta^{-1} = 1 \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.13}$$

This implies that $\mathbb{Q}_\delta$-a.s. the sets $\{\gamma_\delta > 1\}$, $\{\gamma_\delta = 1\}$, $\{\gamma_\delta < 1\}$ are invariant under t_{-1}^δ. $\mathbb{Q}$ is ergodic under t_x since $\mathbb{P}$ is, and hence exactly one of the above sets has full $\mathbb{Q}_\delta$-measure. We accordingly distinguish between three cases.

Step 4: $\gamma_\delta = 1$ $\mathbb{Q}$-a.s. implies II.(iii)
First notice that there is ϕ such that $\mathbb{Q} = \phi\, \mathbb{P}$, and (1.9) implies that $\phi > 0$ $\mathbb{P}$-a.s. From the invariance of $\mathbb{P}$ we obtain

$$t_{-x}\mathbb{Q} \ = \ T_x\phi\, \mathbb{P} = \frac{T_x\phi}{\phi}\, \mathbb{Q}\,. \tag{4.14}$$

In particular $t_{-x}\mathbb{Q}$ and $\mathbb{Q}$ are equivalent. From the definition of γ_δ, see below (4.12), and from (4.11),

$$t_{-1}^\delta \mathbb{Q}_\delta = \frac{p^\delta}{T_1^\delta q^\delta}\, \mathbb{Q}_\delta \ \ \mathbb{Q}_\delta - \text{a.s.} \tag{4.15}$$

For $x > 0$ fixed we choose $\delta_n = 2^{-n}x$, and we observe that by (4.15) and (4.3), $\mathbb{Q}_{\delta_n}$-a.s.,

$$M_n^x \overset{\text{def}}{=} \frac{d\, t_{-x}\mathbb{Q}_{\delta_n}}{d\, \mathbb{Q}_{\delta_n}} = \frac{d\, t_{-2^n}^{\delta_n}\mathbb{Q}_{\delta_n}}{d\, \mathbb{Q}_{\delta_n}} = \prod_{k=0}^{2^n-1} \frac{T_k^{\delta_n}\, p^{\delta_n}}{T_{k+1}^{\delta_n}\, q^{\delta_n}} = \frac{s_{-1,0}^{\delta_n}\, s_{0,1}^{\delta_n}\, s_{2^n-1,2^n+1}^{\delta_n}}{s_{-1,1}^{\delta_n}\, s_{2^n-1,2^n}^{\delta_n}\, s_{2^n,2^n+1}^{\delta_n}}. \tag{4.16}$$

We compute the limit of the $(\mathcal{A}_{\delta_n})$-martingale M_n^x and find that $\mathbb{Q}$-a.s.,

$$M^x \stackrel{\text{def}}{=} \lim_{n\to\infty} M_n^x \;=\; \exp(\,2A(x,\cdot))\,. \tag{4.17}$$

Since $\sup_\omega M^x(\omega) < \infty$, it follows from (4.4) and from theorem 3.3 p.242 in [10] that $d\,t_{-x}\mathbb{Q}/d\,\mathbb{Q} = M^x$. Comparing with (4.14) we obtain that $\mathbb{Q}$-a.s., $T_x\phi = \exp(\,2A(x,\cdot))\,\phi$. The definition of A in (1.11) shows that $\mathbb{P}$-a.s.,

$$2\int_0^x (b/a)(t_v\omega)\,dv \;=\; \log\phi(t_x\omega) - \log\phi(\omega)\,. \tag{4.18}$$

By continuity, we obtain that (4.18) holds for all $x > 0$ outside a $\mathbb{P}$-null set. We obtain a similar result for $x < 0$ by using the equivalent reformulation of (4.11)

$$t_1^\delta \mathbb{Q}_\delta = t_{-1}^\delta \beta_\delta^{-1}\,\mathbb{Q}_\delta\,. \tag{4.19}$$

Hence, with $V = \frac{1}{2}\log\phi$, we have that $\mathbb{P}$-a.s., $x \mapsto V(t_x\omega)$ is absolutely continuous, and $b/a = \nabla V$. Clearly $\mathbb{E}[e^{2V}] = \mathbb{E}[\phi] = 1$. Hence condition II.(iii) holds.

Step 5: The case $\gamma_\delta < 1$ $\mathbb{Q}_\delta$-a.s. implies II.(i)
The lines between (4.20) and (4.22) are taken from the proof of Theorem 3.1 in Conze and Guivarc'h [8]. By assumption, the function $z_\delta = 1/(1-\gamma_\delta)$ satisfies $1 < z_\delta < \infty$ $\mathbb{Q}_\delta$-a.s. Applying T_1^δ to the equation (4.13), and substituting $\gamma_\delta = (z_\delta - 1)/z_\delta$, we obtain

$$z_\delta = 1 + \frac{T_1^\delta q}{T_1^\delta p}\; T_1^\delta z_\delta \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.20}$$

We introduce the notation $a_k^\delta = \prod_{i=1}^k (T_i^\delta q^\delta / T_i^\delta p^\delta)$, $k \ge 1$, and iterating (4.20), we obtain that for $n \ge 2$,

$$z_\delta = 1 + \sum_{k=1}^{n-1} a_k^\delta + a_n^\delta T_n^\delta z_\delta \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.21}$$

Since $z_\delta < \infty$ $\mathbb{Q}_\delta$-a.s., the sum $\sum_{k=1}^n a_k^\delta$, and $a_n^\delta\, T_n^\delta z$ converge $\mathbb{Q}_\delta$-a.s. to finite limits. Since t_1^δ acts ergodically on Ω under $\mathbb{Q}_\delta$, we can find M finite such that $\mathbb{Q}_\delta$-a.s., the orbit $\{t_n^\delta\omega\}_{n\ge 0}$ visits the set $\{z_\delta \le M\}$ i.o. In particular, along a suitable subsequence n_j tending to ∞, $\lim_j a_{n_j}^\delta T_{n_j}^\delta z_\delta = 0$ $\mathbb{Q}_\delta$-a.s., and hence, $\lim_n a_n^\delta\, T_n^\delta z_\delta = 0$ $\mathbb{Q}_\delta$-a.s. It follows that

$$z_\delta = 1 + \sum_{k=1}^{\infty} a_k^\delta \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.22}$$

The definition of p^δ, q^δ in (4.3) shows that $a_k^\delta = s_{k,k+1}^\delta / s_{0,1}^\delta$, and hence (recall (3.1))

$$z_\delta(\omega) = \frac{\int_0^\infty e^{-2A(u,\omega)}\,du}{s_{0,1}^\delta(\omega)} = \frac{\phi_+(\omega)}{s_{0,1}^\delta(\omega)} \quad \mathbb{Q}_\delta - \text{a.s.} \tag{4.23}$$

By (1.2), there is $c(\delta,\nu,\beta)$ such that $0 < 1/c \le s_{0,1}^\delta \le c < \infty$. Since $\mathbb{Q}$-a.s. $0 < z_\delta^{-1} < 1$, the measure $\hat{\mathbb{P}} = \phi_+^{-1}\,\mathbb{Q}$ is finite, and equivalent to $\mathbb{Q}$ and $\mathbb{P}$. The ergodic theorem implies that there is at most one invariant ergodic probability measure that is equivalent to $\mathbb{P}$.

$\hat{\mathbb{P}}$ is ergodic since it is equivalent to $\mathbb{P}$. Once we have shown that $\hat{\mathbb{P}}$ is invariant under t_x, then $\mathbb{E}[\phi_+] < \infty$ will follow from $\hat{\mathbb{E}}[\phi_+] = 1$, and hence condition II.(i) holds.
We now show that $\hat{\mathbb{P}}$ is invariant using a discretisation argument. Notice that the restriction of $\hat{\mathbb{P}}$ to $\mathcal{A}_\delta$ is given by $\hat{\mathbb{P}}_\delta = \mathbb{E}_{\mathbb{Q}}[\phi_+^{-1}|\mathcal{A}_\delta]\,\mathbb{Q}_\delta$. Using that $\phi_+ = s^\delta_{0,1} z_\delta$, and that $z_\delta, \beta_\delta, p^\delta, q^\delta$ are $\mathcal{A}_\delta$-measurable, we find that

$$t^\delta_{-1}\,\hat{\mathbb{P}}_\delta = \mathbb{E}_{\mathbb{Q}}[T^\delta_1 \phi_+^{-1}|\mathcal{A}_\delta] \cdot \beta_\delta\,\mathbb{Q}_\delta = T^\delta_1 z_\delta^{-1}\,\mathbb{E}_{\mathbb{Q}}[(T^\delta_1 s^\delta_{0,1})^{-1}|\mathcal{A}_\delta] \cdot \frac{p^\delta}{T^\delta_1 q^\delta}\,\frac{z_\delta - 1}{z_\delta}\,\mathbb{Q}_\delta\,.$$

With the help of (4.20), and with the definition of p^δ, q^δ, see (4.3), we rewrite the r.h.s. of the last line as
(4.24)

$$\frac{p^\delta}{T^\delta_1 p^\delta}\,\mathbb{E}_{\mathbb{Q}}[(T^\delta_1 s^\delta_{0,1})^{-1}|\mathcal{A}_\delta]\,\frac{1}{z_\delta}\,\mathbb{Q}_\delta = \frac{s^\delta_{-1,0}}{s^\delta_{-1,1}}\,\frac{s^\delta_{0,2}}{s^\delta_{1,2}}\,\mathbb{E}_{\mathbb{Q}}[\exp(-2A(\delta,\cdot))|\mathcal{A}_\delta]\,\hat{\mathbb{P}}_\delta \stackrel{\text{def}}{=} \alpha_\delta\,\hat{\mathbb{P}}_\delta\,,$$

so that $t^\delta_{-1}\,\hat{\mathbb{P}}_\delta = \alpha_\delta \hat{\mathbb{P}}_\delta$. As a next step, we show that

$$\hat{\mathbb{P}} - \text{a.s.} \quad \lim_{\delta\downarrow 0} \tfrac{1}{\delta}\big(\alpha_\delta - 1\big) = 0\,. \tag{4.25}$$

First we expand $f(x,\omega) = \int_0^x e^{-2A(u,\omega)}\,du$ in a Taylor polynomial around 0. Observe that the map $x \mapsto (b/a)(x,\omega)$ is Lipschitz continuous, and thus its derivative exists for a.e. x. By the ergodic theorem, $\nabla(b/a)(\omega)$ exists $\mathbb{P}$-a.s. Hence we find for $\mathbb{P}$-a.e. ω, and for x in a neigborhood of 0, that $f(x,\omega) = x - \frac{b}{a}(\omega) + R_\omega(x)$, with $\sup_\omega |R_\omega(x)| \le \frac{1}{3}\sup_\omega |\nabla(\frac{b}{a})(\omega)| x^3 \le c_1(\nu,\beta,K)x^3$, where we used (1.2) and (1.3) in the last inequality. We obtain the following expansions

$$\begin{aligned} &\tfrac{1}{\delta} s^\delta_{-1,0}(\omega) = 1 + \tfrac{b}{a}(\omega)\delta + R_{1,\omega}\delta^2, && \tfrac{1}{2\delta} s^\delta_{0,2}(\omega) = 1 - 2\tfrac{b}{a}(\omega)\delta + R_{2,\omega}\delta^2,\\ &\tfrac{1}{2\delta} s^\delta_{-1,1}(\omega) = 1 + R_{3,\omega}\delta^2, && \tfrac{1}{\delta} s^\delta_{1,2}(\omega) = 1 - 3\tfrac{b}{a}(\omega)\delta + R_{4,\omega}\delta^2, \end{aligned} \tag{4.26}$$

with $\sup_\omega R_{i,\omega} \le c_2(\nu,\beta,K) < \infty$, $1 \le i \le 4$. After inserting these expansions in the definition of α_δ, we see that

$$\Big|\frac{s^\delta_{-1,0}}{s^\delta_{-1,1}}\,\frac{s^\delta_{0,2}}{s^\delta_{1,2}}\exp(-2A(\delta,\omega)) - 1\Big| \le \frac{c_3(\nu,\beta,K)}{\frac{1}{2\delta}s^\delta_{-1,1}\,\frac{1}{\delta}s^\delta_{1,2}}\,\delta^2, \tag{4.27}$$

and (4.25) follows from dominated convergence for conditional expectations. We now use (4.25) to show that $\hat{\mathbb{P}}$ is invariant. The group $(T_x)_x$ is strongly continuous on $L^1(\hat{\mathbb{P}})$, see Lemma 5.6 in the Appendix. We write $\hat{D}$ for its generator and $\mathcal{D}(\hat{D})$ for the domain of $\hat{D}$. Proposition 1.5 p.9 in [12] shows that for $A \in \sigma(b/a)$, $\int_0^x T_y \mathbf{1}_A\,dy \in \mathcal{D}(\hat{D})$, and

$$\hat{\mathbb{P}}[t_x A] - \hat{\mathbb{P}}[A] = \hat{\mathbb{E}}[\,\hat{D}\int_0^x T_y \mathbf{1}_A\,dy\,], \tag{4.28}$$

where $\hat{\mathbb{E}}$ denotes expectation w.r.t. $\hat{\mathbb{P}}$. It follows from (4.4) that $\lim_{\delta\to 0}\hat{\mathbb{E}}[T_y\mathbf{1}_A|\mathcal{A}_\delta] = T_y\mathbf{1}_A$ $\hat{\mathbb{P}}$-a.s. and in $L^1(\hat{\mathbb{P}})$. Since $\hat{D}$ is closed in $L^1(\hat{\mathbb{P}})$ ([12] p.10), it follows that

$$\hat{\mathbb{E}}[\hat{D}\int_0^x T_y\mathbf{1}_A\,dy\,] = \lim_{\delta\to 0}\hat{\mathbb{E}}[\hat{D}\int_0^x \hat{\mathbb{E}}[T_y\mathbf{1}_A|\mathcal{A}_\delta]\,dy\,]. \tag{4.29}$$

Fix $\delta > 0$, and set $f = \int_0^x \hat{\mathbb{E}}[T_y \mathbf{1}_A | \mathcal{A}_\delta]\, dy$. Hence f is $\mathcal{A}_\delta$-measurable. By L^1-convergence, and by (4.25) and dominated convergence, we find that for $\delta_n = \delta 2^{-n}$, $n \geq 1$,

$$\begin{aligned}\hat{\mathbb{E}}[\hat{D}f] &= \lim_{n\to\infty} \int \tfrac{1}{\delta_n}(T_{\delta_n} f - f)\, d\hat{\mathbb{P}} = \lim_{n\to\infty} -\tfrac{1}{\delta_n}\Big(\int f \; d\, t_{-1}^{\delta_n}\, \hat{\mathbb{P}}_{\delta_n} - \int f\, d\hat{\mathbb{P}}\Big) \\ &= -\int f \cdot \lim_{n\to\infty} \tfrac{1}{\delta_n}\big(\alpha_{\delta_n} - 1\big)\, d\hat{\mathbb{P}} = 0.\end{aligned} \tag{4.30}$$

Hence the left-hand side of (4.29) vanishes, and the invariance of $\hat{\mathbb{P}}$ now follows from (4.28). Hence condition II.(ii) holds.

Step 6: The case $\gamma_\delta > 1$ $\mathbb{Q}$-a.s. implies II.(ii)
The proof is similar to step 5. We define $z_\delta = 1/(1 - T_{-1}^\delta \gamma^{-1})$, and apply T_{-1}^δ to (4.13). We then find that $\mathbb{Q}_\delta$-a.s.

$$z_\delta = 1 + \sum_{k=1}^{\infty} a_k^\delta, \text{ where } a_k^\delta = \prod_{i=1}^{k} (T_{-i}^\delta p^\delta / T_{-i}^\delta q^\delta),\ k \geq 1,$$

which yields $z_\delta = \phi_- / s_{-1,0}^\delta$ $\mathbb{Q}_\delta$-a.s. We then conclude by similar considerations as above. This finishes the proof of I. $\Rightarrow$ II. □

Proof of II. $\Rightarrow$ I.
Assume that II.(i) holds. From (3.9)-(3.10), we see that $\mathbb{Q} = \phi\, \mathbb{P}$ is invariant for L, with $\phi = \phi_+ / \mathbb{E}[\phi_+]$, and similarly when II.(ii) holds. When II.(iii) holds, then L can be written in divergence form: $L = \frac{1}{2} a e^{-2V} D(e^{2V} D)$. We define $\mathbb{Q} = \frac{1}{a} e^{2V}\mathbb{P}$, and observe that, by (1.2) and the assumption $\mathbb{E}[e^{2V}] < \infty$, the measure $\mathbb{Q}$ is finite. Using (2.3), we find for $f \in \mathcal{D}(L)$

$$\int Lf\, d\mathbb{Q} = \tfrac{1}{2} \int D(\exp(2V) Df)\, d\mathbb{P} = 0\,. \tag{4.31}$$

This shows that $\mathbb{Q}$ is invariant, cf. [12] p.239. □

4.1. **The transient case.** Recall ϕ_+, ϕ_- in (3.6), and recall that transience is characterized by $\mathbb{E}[b/a] \neq 0$, see Proposition 3.1.

Theorem 4.2. *(transient case)*
Under the assumption $\mathbb{E}[b/a] \neq 0$, we have the following equivalences:

I. *There is an invariant probability measure $\mathbb{Q}$ that is absolutely continuous w.r.t. $\mathbb{P}$.*
II. *The speed in the law of large numbers (3.7) does not vanish.*
III. *Either $\mathbb{E}[\phi_+] < \infty$ or $\mathbb{E}[\phi_-] < \infty$.*
IV. *(existence of "almost linear coordinates")*
There is a function X that $\mathbb{P}$-a.s. solves the heat equation

$$(\partial_t + \mathcal{L}_\omega)\, X(t,x,\omega) = 0, \quad t \text{ and } x \text{ real}, \tag{4.32}$$

and there is a constant v, and a function $\chi(x,\omega)$, such that for all $\omega \in \Omega$,
(i) $X(t,x,\omega) = x - vt + \chi(x,\omega)$

(ii) $\chi(0,\omega)=0$, *and there is* $\psi(\omega)$ *with* $\mathbb{E}[\psi]=0$ *such that* $\partial_x\chi(x,\omega)=\psi(t_x\omega)$ *(i.e.* $\partial_x\chi$ *is stationary)*

If any of the above statements holds true, then the measure $\mathbb{Q}$ *from condition I. is given, up to normalization, by* $\mathbb{Q}=\phi_\pm\,\mathbb{P}$*, depending on* $\mathbb{E}[\phi_+]<\infty$ *or* $\mathbb{E}[\phi_-]<\infty$*, and the constant* v *in condition IV. is equal to the speed in the law of large numbers (3.7).*

Remark 4.3. χ is usually called the corrector. Under condition IV. the ergodic theorem implies sublinear growth of χ at infinity:

$$\mathbb{P}-\text{a.s.}\quad \lim_{|x|\to\infty}\chi(x,\omega)/x=\lim_{|x|\to\infty}\tfrac{1}{x}\int_0^x\psi(t_y\omega)\,dy=\mathbb{E}[\psi]=0. \tag{4.33}$$

This explains the name "almost linear coordinates". It also follows from the stationarity of $\partial_x\chi$ that χ is additive:

$$\chi(x+y,\omega)-\chi(x,\omega)=\chi(y,t_x\omega),\;\; x,y\in\mathbb{R}. \tag{4.34}$$

□

Remark 4.4. Harmonic coordinates are unique: Assume that X_1,X_2 are harmonic coordinates, with χ_1 resp. χ_2 denoting the correctors. Then $\chi\overset{\text{def}}{=}\chi_1-\chi_2=X_1-X_2$ satisfies a.s. $\mathcal{L}_\omega\chi=0$ and $\chi(0,\omega)=0$ a.s. Hence $\chi(x,\omega)=cs_{0,x}(\omega)$ a.s. (recall (2.7)), and since $\lim_{|x|\to\infty}\chi(x,\omega)/x=0$ a.s., the ergodic theorem implies $\chi=0$ a.s.

Proof of Theorem 4.2. The equivalence of II. and III. follows from Proposition 3.3, and the equivalence of I. and II. follows from Theorem 4.1. It thus remains to prove III. ⇔ IV.

IV. ⇒ III.
Define $Z(\omega)=1+\psi(\omega)$, and set $Z(x,\omega)=Z(t_x\omega)$. It follows from (4.32) and IV.(i) that $\mathbb{P}$-a.s., $\frac{1}{2}a(x,\omega)\partial_xZ(x,\omega)+b(x,\omega)Z(x,\omega)=v$. Multiplying both sides with $\frac{1}{a(\cdot,\omega)}e^{2A(\cdot,\omega)}$, and integrating leads to

$$Z(\omega)=Z(x,\omega)e^{2A(x,\omega)}-2v\int_0^x\tfrac{1}{a(u,\omega)}e^{2A(u,\omega)}\,du. \tag{4.35}$$

First case: $v\neq 0$
We first assume that $v>0$. We define $A=\{Z\geq 0\}$, and observe that (4.35) implies $A\subseteq t_{-x}A$, $x>0$. Notice that the images of $t_{-x}A\setminus A$ under t_{-kx}, $k\geq 1$, are disjoint. The stationarity of $\mathbb{P}$ implies that $\mathbb{P}[t_{-x}A\setminus A]=0$, and hence, for $\mathbb{P}$-a.e. ω, and all positive rational x (and hence all rational x), $t_{-x}A=A$. Since the right-hand side of (4.35) is a continuous function of x, it holds that for $\mathbb{P}-$ a.e. ω, and all x, $t_{-x}A=A$. Since $\mathbb{E}[Z]=1$, we conclude by ergodicity that $\mathbb{P}[A]=1$. Taking expectations in (4.35), and using that $Z\geq 0$, we find that, for $x<0$,

$$0\leq 2\,v\,\mathbb{E}[\int_x^0\tfrac{1}{a(u,\omega)}e^{2A(u,\omega)}\,du]\leq 1. \tag{4.36}$$

By monotone convergence, and using stationarity, we obtain from (4.36) that

$$\tfrac{1}{2v}\geq\mathbb{E}[\int_{-\infty}^0\tfrac{1}{a(u,\omega)}e^{2A(u,\omega)}\,du]=\mathbb{E}[\tfrac{1}{a(\omega)}\int_0^\infty e^{-2A(u,\omega)}\,du]=\mathbb{E}[\phi_+]. \tag{4.37}$$

Hence $\mathbb{E}[\phi_+] < \infty$, and a similar argument shows that $v < 0$ implies $\mathbb{E}[\phi_-] < \infty$.

Second case: $v = 0$
(4.35) shows that the events $\{Z > 0\}$, $\{Z = 0\}$, $\{Z < 0\}$ are invariant under t_{-x} for all x, and since $\mathbb{E}[Z] = 1$, by ergodicity, $Z > 0\,\mathbb{P}$-a.s. It follows again from (4.35) that

$$2\int_0^x (b/a)(v,\omega)\,dv = -\log Z(t_x\omega) + \log Z(\omega) \;\; \mathbb{P}\text{-a.e. } \omega, \tag{4.38}$$

so that $\mathbb{P}$-a.s., b/a is the gradient of the stationary function $V = -\frac{1}{2}\log Z$. But then the stationarity of $\mathbb{P}$ implies $\mathbb{E}[b/a] = 0$, a contradiction. □

III. ⇒ IV.
We assume that $\mathbb{E}[\phi_+] < \infty$, we fix $v = \frac{1}{2\,\mathbb{E}[\phi_+]}$ (so that v is the speed in the law of large numbers (3.7)), and we define

$$Z(\omega) = 2v\int_{-\infty}^0 \tfrac{1}{a(u,\omega)} e^{2A(u,\omega)}\,du\,. \tag{4.39}$$

Notice that stationarity and the choice of v imply

$$\mathbb{E}[Z] = 2v\,\mathbb{E}[\tfrac{1}{a(\omega)}\int_0^\infty e^{-2A(u,\omega)}\,du] = 2v\,\mathbb{E}[\phi_+] = 1. \tag{4.40}$$

(1.1) shows that $A(u, t_x\omega) = A(u+x,\omega) - A(x,\omega)$. Hence, with $Z(x,\omega) = Z(t_x\omega)$, it holds that

$$Z(x,\omega) = 2v\,e^{-2A(x,\omega)}\int_{-\infty}^0 \tfrac{1}{a(u+x,\omega)} e^{2A(u+x,\omega)}\,du\,. \tag{4.41}$$

By (1.3) $\partial_x a(\cdot,\omega)$ exists a.e. Hence, for fixed x, we find for a.e. u,

$$\partial_x\big(\tfrac{1}{a(u+x,\omega)} e^{2A(u+x,\omega)}\big) = \tfrac{1}{a(u+x,\omega)} e^{2A(u+x,\omega)}\big(2\tfrac{b}{a} - \tfrac{\partial_x a}{a}\big)(u+x,\omega). \tag{4.42}$$

In particular, by (1.2),(1.3), and by the ergodic theorem, there is a constant $c(\nu,\beta,K)$ such that for $u + x < 0$, the modulus of right-hand side of (4.42) is $\mathbb{P}$-a.s. bounded by $c\,e^{-\mathbb{E}[b/a]|u+x|}$. We differentiate (4.41), and find, using first dominated convergence and (4.42), then partial integration, the $\mathbb{P}$-a.s. equalities

$$\begin{aligned}\partial_x Z(x,\omega) &= -\,2\tfrac{b}{a}(x,\omega)Z(x,\omega) + 2ve^{-2A(x,\omega)}\cdot \tfrac{e^{2A(\cdot+x,\omega)}}{a(\cdot+x,\omega)}\Big|_{-\infty}^0 \\ &= -\,2\tfrac{b}{a}(x,\omega)Z(x,\omega) + \tfrac{2v}{a(x,\omega)},\end{aligned} \tag{4.43}$$

where the last equality follows from the ergodic theorem. With the definition

$$X(t,x,\omega) = \int_0^x Z(t_u\omega)\,du - vt, \quad x \text{ and } t \text{ real}, \tag{4.44}$$

we see that (4.32) is $\mathbb{P}$-a.s. satisfied. Set $\chi(x,\omega) = \int_0^x Z(t_u\omega)\,du - x$. Clearly IV.(i) holds, and IV.(ii) follows from (4.40). The case $\mathbb{E}[\phi_-] < \infty$ is similar. This finishes the proof. □

4.2. **The recurrent case.** We recall that recurrence is characterized by $\mathbb{E}[b/a] = 0$, see Proposition 3.1. In this section we assume in addition to ergodicity that the dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, (t_x)_x)$ mixes exponentially. More precisely, for $-\infty \leq x_1 < x_2 \leq \infty$, we define the σ-fields $\mathcal{H}_{x_1}^{x_2} = \sigma\{t_x a, t_x b : x \in (x_1, x_2)\}$, and we assume that

$$\limsup_{y\to\infty} y^{-1} \log \sup_{A\in\mathcal{H}_{-\infty}^{0},\, B\in\mathcal{H}_y^{\infty}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| < -2\beta\nu. \tag{4.45}$$

Recall the functional $A(x, \omega)$ in (1.11).

Theorem 4.5. *(recurrent case)*
Under the assumption $\mathbb{E}[b/a] = 0$, and under (4.45), one has the following equivalences:

I. *There is an invariant probability measure $\mathbb{Q}$ that is absolutely continuous w.r.t. $\mathbb{P}$.*
II. $\limsup_{x\to\infty} \mathit{Var}[A(x, \omega)] < \infty$
III. *There is a stationary random variable V such that $\mathbb{P}$-a.s. $x \mapsto V(t_x\omega)$ is absolutely continuous and $b/a = \nabla V$.*
IV. *(existence of almost linear harmonic coordinates)*
There is a function X that is $\mathbb{P}$-a.s. $\mathcal{L}_\omega$-harmonic, i.e. for $\mathbb{P} -$ a.e. ω,

$$\mathcal{L}_\omega X(x, \omega) = 0, \tag{4.46}$$

and there is a function $\chi(x, \omega)$ such that for all $\omega \in \Omega$
 (i) $X(x, \omega) = x + \chi(x, \omega)$
 (ii) $\chi(0, \omega) = 0$*, and there is* $\psi(\omega)$ *with* $\mathbb{E}[\psi] = 0$ *such that* $\partial_x\chi(x, \omega) = \psi(t_x\omega)$ *(i.e. $\partial_x\chi$ is stationary)*

If any of the above equivalent statements holds true, then $\mathbb{Q}$ is, up to normalization, given by $\mathbb{Q} = \frac{1}{a}e^{2V}\mathbb{P}$.

Remark 4.6. (i) Of course the remarks 4.3 and 4.4 also apply here. To see the uniqueness of harmonic coordinates, proceed as in remark 4.4, and use condition III. before applying the ergodic theorem.
(ii) Condition II. is very interesting, since it directly relates the existence of an absolutely continuous invariant measure resp. the existence of almost linear harmonic coordiantes to the occurence of traps. The fluctuations of the functional $A(x, \omega)$ are responsible for the creation of traps, and condition II. roughly states that there are no strong traps.
(iii) The uniqueness of harmonic coordinates and (4.48) show that $\mathbb{E}[e^{-2V}] < \infty$ and $\mathbb{E}[e^{2V}] < \infty$ is a necessary condition for the existence of almost linear harmonic coordinates, resp. the existence of an absolutely continuous and invariant measure. This is the reason for the assumption (4.45), and remark 5.2 shows that (4.45) is nearly optimal.
(iv) Brémont [4] (see also Letchikov[23], Bulycheva and Molchanov [7]) characterises the existence of a quenched functional CLT for recurrent random walks (with finite range) in an ergodic random environment on $\mathbb{Z}$. We believe that a similar statement holds in our setting, which would read as: when $\mathbb{E}[b/a] = 0$, then a quenched CLT holds if and only if condition III. in Theorem 4.5 hold, and $\mathbb{E}[e^{2V}] < \infty$, $\mathbb{E}[e^{-2V}] < \infty$. Under the mixing assumption (4.45), it would follow

from Theorem 4.5 and Lemma 5.1 that a quenched functional CLT is characterised by either condition I.-IV. in Theorem 4.5. But we will not address this question here. □

Proof. I. ⇔ III. follows from Theorem 4.1 (in particular this equivalence holds without the assumption (4.45)).

III. ⇔ II.
It follows from (4.45) and from Theorem 17.2.1 p.306 in [14] that the autocovariance function satisfies

$$\lim_{x\to\infty} \mathbb{E}[(b/a)(\omega)(b/a)(t_x\omega)] = 0\,.$$

Theorem 18.3.2 p.331 in [14] then shows that II. holds if and only if there is a random variable V such that

$$b/a = \lim_{h\to 0} \tfrac{1}{h}(T_h V - V) \quad \text{in } L^2. \tag{4.47}$$

By applying Lemma 5.4 in the Appendix with $p = 2$, this is equivalent to $\mathbb{P}$-a.s. $x \mapsto V(t_x\omega)$ being absolutely continuous and $b/a = \nabla V$, which is condition III.

IV. ⇒ III.
The proof of this implication is contained in the proof of IV. ⇒ III. in Theorem 4.2, see in particular the case $v = 0$.

III. ⇒ IV.
We can write the operator $\mathcal{L}_\omega$ in divergence form, similarly to the expression above (4.31) (replace D by ∂_x). Lemma 5.1 shows that $\exp(-2V) \in L^1(\mathbb{P})$, and it is then immediate that for all $\omega \in \Omega$, the function

$$X(x,\omega) = \frac{1}{\mathbb{E}[\exp(-2V)]}\int_0^x \exp(-2V(u,\omega))\,du \text{ is } \mathcal{L}_\omega-\text{harmonic.} \tag{4.48}$$

We define $\chi(x,\omega) = X(x,\omega) - x$. Clearly all requirements on χ and $\partial_x\chi$ are satisfied. This finishes the proof of the theorem. □

5. Appendix

Lemma 5.1. *Assume (4.45), and that $\mathbb{P}$-a.s, $b/a = \nabla V$ for some random variable V on Ω. Then*

$$\mathbb{E}[e^{2|V|}] < \infty\,. \tag{5.1}$$

Remark 5.2. We will now show that the mixing condition in (4.45) is optimal, in the sense that if the left-hand side of (4.45) equals $-2\beta\nu$, then it is possible to construct V stationary with ∇V bounded and globally Lipschitz such that $\mathbb{E}[e^{2V}] = \infty$. We now provide an example.
The random environment is given by a Poisson point process on the line with intensity one, and we denote the Poisson cloud with $\omega = (\omega_i)_{i\in\mathbb{Z}}$. We define the stationary field $U(\omega) = \inf_i |\omega_i|$, the distance of the origin to the Poisson cloud. Then $T_x U(\omega)$ is the distance from the point x to the Poisson cloud, and hence $x \mapsto T_x U(\omega)$ is a sawtooth

function with slope 1 or -1. We choose a bounded Lipschitz continuous mollifier η that is compactly supported on $(-1,1)$, and define $V(\omega)=\int_{-1}^{1}U(t_{-y}\omega)\,\eta(y)\,dy$. Since $x\mapsto U(t_x\omega)$ is $\mathbb{P}$-a.s. globally Lipschitz with constant one, it is differentiable a.e. and $\|\nabla U\|_\infty=1$. Then for $\mathbb{P}$-a.e. ω and every x,

$$\nabla V(t_x\omega)=\int_{-1}^{1}\nabla U(t_{x-y}\omega)\,\eta(y)\,dy=\int_{-1}^{1}\nabla U(t_y\omega)\,\eta(x-y)\,dy, \tag{5.2}$$

which shows that $\|\nabla V\|_\infty=1$ and $x\mapsto\nabla V(t_x\omega)$ is $\mathbb{P}$-a.s. globally Lipschitz continuous with a Lipschitz constant that is independent of the environment ω. We then set $a=1$ and $b=\nabla V$, so that $\nu=\beta=1$, and hence the right-hand side of (4.45) equals -2. For an interval I, denote with N_I the number of Poisson points in the interval I. Then $\mathbb{P}[V>u]\geq\mathbb{P}[N_{(-u-1,u+1)}=0]=e^{-2u-2}$, so that

$$\mathbb{E}[e^{2V}]=1+\int_0^\infty 2e^{2u}\,\mathbb{P}[V>u]\,du=\infty.$$

We now show that the onedimensional distributions of U mix exponentially:

$$\limsup_{x\to\infty}x^{-1}\log\sup_{a,b>0}\mid\mathbb{P}(T_xU\leq a,U\leq b)-\mathbb{P}(T_xU\leq a)\,\mathbb{P}(U\leq b)\mid=-2. \tag{5.3}$$

This can be extended to the finite-dimensional distributions of U, so that the left-hand side of (4.45) is equal to -2. Fix $x>0$, choose $a,b>0$, and write $A=(0,a),B=(0,b)$. We distinguish three cases according to the relative position of $a,x-b,x+b$. Assume first that $a\geq x+b$. Then

$$\begin{aligned}&\mathbb{P}[U\in A,T_xU\in B]-\mathbb{P}[U\in A]\,\mathbb{P}[T_xU\in B]\\ =&\mathbb{P}[T_xU\in B](1-\mathbb{P}[U\in A])\\ =&(1-\mathbb{P}[N_{(x-b,x+b)}=0])\mathbb{P}[N_{(-a,a)}=0]\\ =&(1-e^{-2b})e^{-2a}.\end{aligned} \tag{5.4}$$

Using that $a>x+b$, we find that the left-hand side of (5.3) equals -2. In the case $x-b<a<x+b$, we find similarly that the left-hand side of (5.4) equals $1-e^{-2a}-e^{-2b}+e^{-x-a-b}$, and using that $a>x-b$, we find again that the left-hand side of (5.3) equals 2. Finally, in the case $a\leq x-b$, the left-hand side of (5.4) vanishes. Hence (5.3) holds. □

Proof of Lemma 5.1. Notice first that $\mathbb{P}$-a.s.,

$$|T_xV(\omega)-V(\omega)|\leq\int_0^x|\nabla V|(t_u\omega)\,du\leq\beta\nu x. \tag{5.5}$$

We define $\delta\overset{\text{def}}{=}\mathbb{P}[V<0]$, and w.l.o.g. we can assume that $0<\delta<1$ (the claim is clear for constant V, and otherwise look at $\tilde{V}=V\pm c$ for a suitable c). By (4.45), and using (5.5), we find for some constant $\gamma>2\beta\nu$,

$$\exp(-\gamma x)\geq\mathbb{P}[V<0]\,\mathbb{P}[T_xV>\beta\nu x]-\mathbb{P}[V<0,\,T_xV>\beta\nu x]=\delta\,\mathbb{P}[T_xV>\beta\nu x]\,,$$

and hence, using stationarity,

$$\mathbb{P}[V>u]\leq\delta^{-1}\exp(-\tfrac{\gamma}{\beta\nu}u).$$

Similarly, we obtain that $\mathbb{P}[V < -u] \leq (1-\delta)^{-1} \exp(-\frac{\gamma}{\beta\nu} u)$. Since, by Fubini's theorem,

$$\mathbb{E}[e^{2|V|}] = 1 + \int_0^\infty 2\exp(2u)\,\mathbb{P}[|V| > u]\,du\,,$$

the claim (5.1) follows from $\gamma > 2\beta\nu$. □

We cite the main result from Atkinson [1]:

Lemma 5.3. *Let $f : \Omega \to \mathbb{R}$ be integrable, and define (the empty sum is by convention zero)*

$$a_f(n,\omega) \overset{\text{def}}{=} \begin{cases} \sum_0^{n-1} f(t_i\omega), & n \geq 0, \\ -\sum_{-n+1}^0 f(t_i\omega), & n \leq 0. \end{cases} \tag{5.6}$$

Then a_f is recurrent if and only if $\mathbb{E}[f] = 0$. □

Recall the generator D of the translation group T_x and the gradient ∇ in (2.2) and below.

Lemma 5.4. *On $L^p(\mathbb{P})$, $1 \leq p < \infty$, it holds that $D = \nabla$ with domain $\mathcal{D}(D) = \{f \in L^p(\mathbb{P}) : x \mapsto f(t_x\omega)$ is absolutely continuous $\mathbb{P}$-a.s. and $\nabla f \in L^p(\mathbb{P})\}$.*

Proof. Choose $f \in \mathcal{D}(D) = \{h \in L^p(\mathbb{P}) : \lim_{x\to 0} \frac{1}{x}(T_x h - h)$ exists in $L^p(\mathbb{P})\}$, let $g = Df \in L^p(\mathbb{P})$, and define $A = \{h \in L^p(\mathbb{P}) : x \mapsto h(t_x\omega)$ is abs. cont. $\mathbb{P}$-a.s. and $\nabla h \in L^p(\mathbb{P})\}$. Following the methods exposed in the proof of Proposition 1 p.66 in [11], it suffices to show that $(D, \mathcal{D}(D))$ is a restriction of (∇, A).
For arbitrary $a, b \in \mathbb{R}$, we obtain by dominated convergence that

$$\int_a^b dy \int d\mathbb{P}\, |\tfrac{1}{x}(T_{x+y}f - T_y f) - DT_y f|^p \to 0 \quad \text{as } x \to 0.$$

By Fubini's theorem, we obtain that, $\mathbb{P}$-a.s., $(T_{x+y}f - T_y f)/x$ converges in $L^p_{loc}(\mathbb{R})$. In particular the right-hand side of

$$\frac{1}{x}\int_b^{b+x} T_y f\,dy - \frac{1}{x}\int_a^{a+x} T_y f\,dy = \int_a^b \frac{1}{x}(T_{x+y}f - T_y f)\,dy \tag{5.7}$$

converges $\mathbb{P}$-a.s. to $\int_a^b T_y g\,dy$ as $x \downarrow 0$. Since the left-hand side converges for almost all a, b to $T_b f - T_a f$, we obtain $\mathbb{P}$-a.s., by redefining f on a Lebesgue null set, for all a, b,

$$T_b f = T_a f + \int_a^b T_y g\,dy. \tag{5.8}$$

By the ergodic theorem we have redefined f on a $\mathbb{P}$-null set. Hence, for $\mathbb{P}$-a.e. ω, the map $y \mapsto T_y f$ is absolutely continuous. In particular, $\mathbb{P}$-a.s., $\partial_y f$ exists for Lebesgue-a.e. y. Again by the ergodic theorem, ∇f exists $\mathbb{P}$-a.s. Of course (5.8) implies that $Df = \nabla f$. This shows that $\mathcal{D}(D) \subseteq A$ and $\nabla|_{\mathcal{D}(D)} = D$, which is our claim. □

For $f \in L^\infty(\Omega)$, we define the Green operator $Gf = \int_0^\infty P_s^\delta f\,ds$, where P_s^δ is the semigroup attached to the diffusion stopped when exiting $(-\delta, \delta)$, i.e. for $\omega \in \Omega$,

$$P_s^\delta f(\omega) = \tilde{E}_\omega[f(\bar{\omega}_s), s < U_\delta] = E_{0,\omega}[f(X_s,\omega), s < U_\delta] \tag{5.9}$$

(recall U_δ below (4.6)). This semigroup is strongly continuous, and, with a slight abuse of notation, we denote its generator with L.

Lemma 5.5. *For $f \in \mathcal{D}(L)$, it holds that $Gf \in \mathcal{D}(L)$ and $GLf = LGf$.*

Proof. We define $f_n = \int_0^n P_s^\delta f\, ds$, and observe that

$$\|Gf - f_n\|_\infty \le \|f\|_\infty \sup_\omega \int_n^\infty P_{0,\omega}[U_\delta > s]\, ds\,. \tag{5.10}$$

Observe that $\sup_\omega E_{0,\omega}[U_\delta] < \infty$, as follows from Lemma 7.4 p.365 in [16], together with (1.2). We conclude that $\lim_n \|Gf - f_n\|_\infty = 0$. Further, combining point (a) and (c) of Proposition 1.5 p.9 in [12], we notice that $f_n \in \mathcal{D}(L)$ and

$$Lf_n = \int_0^n LP_s^\delta f\, ds = \int_0^n P_s^\delta Lf\, ds\,. \tag{5.11}$$

As in (5.10), this implies that $\lim_{n\to\infty} \|Lf_n - GLf\|_\infty = 0$. Since L is closed ([12] p.10), we conclude that $Gf \in \mathcal{D}(L)$ and $LGf = GLf$. □

Recall the measure $\hat{\mathbb{P}}$ below (4.23).

Lemma 5.6. *The group $(T_x)_x$ is strongly continuous on $L^1(C(\mathbb{R}), \sigma(b/a), \hat{\mathbb{P}})$.*

Proof. With a slight abuse of notation, we write $\Omega = C(\mathbb{R})$. $\sigma(b/a)$ is generated by the finite-dimensional cylinder sets, and is therefore contained in the Borel σ-field generated by the topology of uniform convergence on compacts. It follows that Ω is a Polish space with metric $d(\omega_1, \omega_2) = \sum_{n=1}^\infty 2^{-n} \sup_{-n \le t \le n}(|\omega_1(t) - \omega_2(t)| \wedge 1)$ (cf. [16] p.60, problem 4.1 and 4.2). Notice that t_x acts continuously on Ω. For $f \in C_b(\Omega)$, it follows from dominated convergence that

$$\lim_{x\to 0} T_x f = f \text{ in } L^1(\hat{\mathbb{P}}). \tag{5.12}$$

$\hat{\mathbb{P}}$ is regular ([30] p.48), and then it follows that $C_b(\Omega)$ is dense in $L^1(\hat{\mathbb{P}})$ ([30] p.69). The claim of the lemma now follows from Proposition 5.3 p.38 in [11] (see also Ex. 5.9(5) p.42), once we have shown that

$$\sup_{0\le x\le 1} \|T_x\| < \infty\,. \tag{5.13}$$

Choose $f \in L^1(\hat{\mathbb{P}})$ and $\sigma(b/a)$-measurable, and fix $0 \le x \le 1$. It follows from (4.4) that $f_n = \hat{\mathbb{E}}[f|\mathcal{A}_{x/n}]$ converges to f in $L^1(\hat{\mathbb{P}})$. Notice that (4.24) implies $C = \sup_{\omega,x} \alpha_x(\omega) < \infty$. This shows that $\|T_x f_n\|_1 = \int |f_n|\alpha_x\, d\hat{\mathbb{P}} \le C\|f_n\|_1$, and that $T_x f_n$ converges to $T_x f$ in $L^1(\hat{\mathbb{P}})$. This implies (5.13), which finishes the proof. □

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