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Higher Asymptotics of Laplace's Approximation

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Abstract

We present an explicit asymptotic series for multiple integrals of Laplace type (the first term of which is known as Laplace's approximation) in terms of asymptotic series of the functions in the integrand.

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MSC: 41A60 (Primary); 41A63, 44A10 (Secondary)

Let $R \subset \mathbb{R}^d$ be a measurable set with 0 in the interior. Suppose f and g are measurable functions on R such that f has a unique minimum at 0 and such that both f and g admit asymptotic expansions near 0. Laplace's approximation is concerned with the asymptotic behavior of integrals of the form

$$\int_R e^{-kf} g dx$$

as $k \rightarrow \infty$. There is a vast literature devoted to this subject; we refer the reader to the books [BH75], [dB81], and [Erd56] for an introduction and further references.

It is a result of Fulks and Sather [FS61] that if $\int_R e^{-kf} g d^d x$ exists for some k_0 , then it exists for all $k \geq k_0$ and moreover, for some $\nu, \lambda > 0$ (depending on the expansions of f and g) there exists an asymptotic expansion

$$\int_R e^{-kf} g d^d x = \sum_{j=0}^N \zeta_j k^{-(\lambda+j)/\nu} + o(k^{-(N+\lambda)/\nu}), \quad k \rightarrow \infty.$$

Fulks and Sather also find the first coefficient. We will give an explicit expression for the coefficients ζ_j , $j \geq 0$ in terms of the coefficients of the expansions of f and g . In the 1-dimensional case, our results reproduce recent results of Wojdylo [Woj06a], [Woj06b].

1 Main results.

Let $\{x^1, \dots, x^d\}$ be coordinates on \mathbb{R}^d . Denote by $S^{d-1} = \{|x| = 1\} \subset \mathbb{R}^d$ the unit sphere and introduce polar coordinates $\rho := \sqrt{(x^1)^2 + \dots + (x^d)^2}$ and $\Omega = x/|x| \in S^{d-1}$.

We can assume without loss of generality that $f(0) = 0$. We adopt the hypotheses of Fulks–Sather [FS61]: let R , f , and g be as above and assume that there is an $N > 0$ and

1. $N + 1$ continuous functions $f_j(\Omega)$, $j = 0, \dots, N$ with $f_0 > 0$ such that for some $\nu > 0$

$$f(\rho, \Omega) = \rho^\nu \sum_{j=0}^N f_j(\Omega) \rho^j + o(\rho^{N+\nu}) \text{ as } \rho \rightarrow 0, \text{ and} \quad (1.1)$$

2. $N + 1$ functions $g_j(\Omega)$, $j = 0, \dots, N$ such that for some $\lambda > 0$

$$g(\rho, \Omega) = \rho^{\lambda-d} \sum_{j=0}^N g_j(\Omega) \rho^j + o(\rho^{N+\lambda-d}) \text{ as } \rho \rightarrow 0 \quad (1.2)$$

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We will often omit the Ω -dependence of the coefficients, writing for example $f_j = f_j(\Omega)$, when the meaning is clear.

Theorem 1 *With the hypotheses above, there exists an asymptotic expansion*

$$\int_B e^{-kf} g d^d x = \sum_{j=0}^N \zeta_j k^{-(\lambda+j)/\nu} + o(k^{-(N+\lambda)/\nu}) \quad (1.3)$$

where the coefficients are given by

$$\zeta_j = \frac{1}{\nu} \Gamma\left(\frac{j+\lambda}{\nu}\right) \int_{S^{d-1}} \left[f_0^{-(j+\lambda)/\nu} \sum_{m=0}^j g_{j-m} \sum_{r=1}^m \binom{-\frac{j+\lambda}{\nu}}{r} \frac{f_m^{(r)}}{f_0^r} \right] d\Omega,$$

where $f_m^{(r)}$ is the sum of all ordered products¹ of r elements of $\{f_1, f_2, \dots, f_N\}$ such that the subscripts add to m , and $\binom{\alpha}{r} := \alpha(\alpha-1)\cdots(\alpha-r+1)/r!$. Empty sums are understood to be 1.

The coefficients $f_m^{(r)}$ arise also in the 1-dimensional case studied by Wojdylo in [Woj06a], [Woj06b], where he calls them “partial ordinary Bell polynomials”. They are relatively simple combinatorial objects to compute; for example, $f_n^{(r)}$ may be alternatively defined as the coefficient of x^n appearing in $(f_1 x + f_2 x^2 + f_3 x^3 + \cdots)^r$. The following result describes a simple recursive algorithm for their computation. Explicit algorithms suitable for computer implementation may be found in [Woj06a], [Woj06b].

Proposition 1 *For $n > 0$ we have $f_n^{(1)} = f_n$, and for $1 < r \leq n$, the coefficients $f_n^{(r)}$ can be computed recursively via $f_n^{(r)} = \sum_{j=r-1}^{n-1} f_{n-j} f_j^{(r-1)}$.*

For example, the first few coefficients in the asymptotic expansion are

$$\begin{aligned} \zeta_0 &= \frac{1}{\nu} \Gamma\left(\frac{\lambda}{\nu}\right) \int_{S^{d-1}} g_0 f_0^{-\lambda/\nu} d\Omega, \\ \zeta_1 &= \frac{1}{\nu} \Gamma\left(\frac{\lambda+1}{\nu}\right) \int_{S^{d-1}} f_0^{-(\lambda+1)/\nu} \left[g_1 - \frac{\lambda+1}{\nu} g_0 f_1 f_0^{-1} \right] d\Omega, \\ \zeta_2 &= \frac{1}{\nu} \Gamma\left(\frac{\lambda+2}{\nu}\right) \int_{S^{d-1}} f_0^{-(\lambda+2)/\nu} \left[g_2 - \frac{\lambda+2}{\nu} (g_1 f_1 + g_0 f_2) f_0^{-1} + \binom{-\frac{\lambda+2}{\nu}}{2} g_0 f_1^2 f_0^{-2} \right] d\Omega, \\ \zeta_3 &= \frac{1}{\nu} \Gamma\left(\frac{\lambda+3}{\nu}\right) \int_{S^{d-1}} f_0^{-(\lambda+3)/\nu} \left[g_3 - \frac{\lambda+3}{\nu} (g_2 f_1 + g_1 f_2 + g_0 f_3) f_0^{-1} \right. \\ &\quad \left. + \binom{-\frac{\lambda+3}{\nu}}{2} (g_1 f_1^2 + 2g_0 f_1 f_2) f_0^{-2} + \binom{-\frac{\lambda+3}{\nu}}{3} g_0 f_1^3 f_0^{-3} \right] d\Omega. \end{aligned}$$

In the case that $\lambda = d$, so that g is continuous at the origin, and that $\nu = 2$ and f is twice differentiable at the origin with positive definite Hessian, there is a more common expression for the first term in the expansion in terms of the determinant $H_f(0)$ of the Hessian $h = (\partial_{x^j} \partial_{x^k} f(0))$ of f :

$$\int_B e^{-kf} g d^d x \sim \left(\frac{2\pi}{k}\right)^{d/2} \frac{g(0)}{\sqrt{H_f(0)}}.$$

Indeed, this follows from the following strange-looking (though elementary) identity which we prove in Section 2.1.

Proposition 2 *Let $A_d := 2\pi^{d/2}/\Gamma(\frac{d}{2})$ denote the area of the unit S^{d-1} sphere in \mathbb{R}^d . Then*

$$\int_{S^{d-1}} \frac{d\Omega}{f_2(\Omega)^{d/2}} = \frac{2^{d/2} A_d}{\sqrt{H_f(0)}},$$

¹For example, $f_6^{(3)} = 6f_1 f_2 f_3 + 3f_1^2 f_4 + f_2^3$.

Remarks.

1. We use results of Frame [Fra57] regarding power series of inverse functions to prove Theorem 1. There are several results in the literature equivalent to those of Frame in [Fra57], for example [Kam46] and [PS45]. We have chosen to use Frame's results because they are particularly simple to apply in the present situation.
2. With more restrictive hypotheses (smoothness and local form of f near 0), Bleistein and Handelsman [BH75, Sec 8.3] have given a complete asymptotic series for $\int_R e^{-kf} g d^d x$. The coefficients, though, are computed in terms of a Jacobian of a coordinate transformation that in general cannot be computed explicitly. Nevertheless, since the result is eventually evaluated at $x = 0$, it is possible to proceed term by term, obtaining explicit formulas in terms of the derivatives of f and g .

The method of the proof of Proposition 2, applied to the coefficients ζ_j , $j > 0$, should yield identities, in the same vein as Proposition 2, relating the coefficients f_j and g_j to the explicit formulas arising from the approach of Bleistein and Handelsman, though we do not do this.

3. In dimension one, Wojdylo has recently derived expressions for the coefficients the complete asymptotic expansion of Laplace type integrals in terms of the coefficients of expansions of f and g [Woj06a], [Woj06b]. In this case, our Theorem 1 reduces to his results; we demonstrate this in more detail in the next section.

1.1 The 1-dimensional case.

We compute the coefficients of Theorem 1 in the 1-dimensional case, where several simplifications occur, thus rederiving the recent results of Wojdylo [Woj06a], [Woj06b, Thm 1.1]. As an application, we mention a closed form expression for the Stirling series (*cf.* [AS64, 6.1.37], [Sti30]).

Corollary 1 *Suppose that for $x \rightarrow 0$, we have $f(x) = x^\nu \sum_{j=0}^N a_j x^j + o(x^{N+\nu})$ for some $\nu > 0$ and $g(x) = \sum_{j=0}^N b_j x^j + o(x^N)$. Then*

$$\int_{-a}^b e^{-kf} g dx = k^{-1/\nu} \sum_{j=0}^{\lfloor N/2 \rfloor} \zeta_{2j} k^{-2j/\nu} + o(k^{-(2\lfloor N/2 \rfloor + 1)/\nu}),$$

where the coefficients are given by

$$\zeta_{2j} = \frac{2}{\nu} \Gamma\left(\frac{2j+1}{\nu}\right) a_0^{-(2j+1)/\nu} \left(\sum_{m=0}^{2j} b_{2j-m} \sum_{r=1}^m \binom{-\frac{2j+1}{\nu}}{r} a_0^{-r} a_m^{(r)} \right)$$

in which $a_m^{(r)}$ is the sum of all ordered products of r terms of the set $\{a_1, a_2, \dots\}$ such that the subscripts add to m , $\lfloor N/2 \rfloor$ denotes the largest integer less than $N/2$ and empty sums are understood to be 1.

Proof. Introduce ‘‘polar coordinates’’ $\rho(x) = |x|$ and $\Omega(x) = x/|x| = \pm 1$ on \mathbb{R} . Note that $S^0 = \{\pm 1\}$ and since $\Omega^j = 1$ if j is even and $= \Omega$ if j is odd,

$$\int_{S^0} \Omega^j d\Omega = 2\varepsilon_j \tag{1.4}$$

where $\varepsilon_j = 1$ if j is even and 0 otherwise.

The series for f in polar coordinates is $f(\rho, \Omega) = \rho^\nu \sum_{j=0}^N a_j \Omega^j \rho^j$, so that $f_j(\Omega) = a_j \Omega^j$, which is equal to a_j if j is even and $a_j \Omega$ if j is odd. Similarly, $g_j(\Omega) = b_j$ if j is even and $b_j \Omega$ if j is odd. Using these facts, and also the observation that the power of Ω in $f_m^{(r)}$ is m , we obtain from Theorem 1 that

$$\zeta_j = \frac{1}{\nu} \Gamma\left(\frac{j}{\nu}\right) a_0^{-(j+1)/\nu} 2\varepsilon_j \left(\sum_{m=0}^j b_{j-m} \sum_{r=1}^m \binom{-\frac{j+1}{\nu}}{r} a_0^{-r} a_m^{(r)} \right)$$

from which it is clear that $\zeta_j = 0$ for j odd, whence we obtain the desired result. ■

A nice application, which already appeared in a similar form in [Woj06a], results by applying the corollary to the Γ -function, thus obtaining a closed expression for the Stirling series [AS64, 6.1.37]

$$k! \sim k^k e^{-k} \sqrt{2\pi k} \left(1 + \frac{1}{12k} + \frac{1}{288k^2} - \frac{139}{51840k^3} - \frac{571}{2488320k^4} + \frac{163879}{209018880k^5} + \dots \right), \quad k \rightarrow \infty.$$

Corollary 2 *The Stirling series is given by*

$$k! \sim k^k e^{-k} \sqrt{2\pi k} \sum_{j=0}^{\infty} k^{-j} \sum_{r=0}^{2j} \frac{(-1)^r}{r!} (2j + 2r - 1)!! a_{2j}^{(r)}$$

where $a_m^{(r)}$ is the sum of all ordered products of r terms of the set $\{a_1, a_2, \dots\}$ in which the subscripts add to m , where $a_j = \frac{1}{j+2}$, and where $(-1)!!$ is understood to be 0.

Proof. Write $k!$ in terms of the Γ -function as

$$k! = \Gamma(k+1) = \int_0^{\infty} e^{-t} t^k dt = k^{k+1} e^{-k} \int_{-1}^{\infty} e^{-k(x-\ln(x+1))} dx,$$

where we have applied the change of variables $t = k(x+1)$ in the last equality. With $f := x - \ln(x+1)$ and $g = 1$, the last integral above satisfies the hypotheses of Theorem 1 with $d = 1$, $\lambda = 1$ and $\nu = 2$. A short computation using Corollary 1 then yields the result. ■

2 Proof of Theorem 1.

We follow the existence proof of Fulks–Sather [FS61], applying the results of Frame to make it constructive. The required result of Frame [Fra57] is the following theorem.²

Theorem 2 [Fra57] *For any nonzero integers ν and q , let $u = f(\rho)$ and $\rho = f^{-1}(u)$ be inverse functions defined for ρ near 0 by the convergent power series*

$$u^\nu = \rho^\nu \sum_{j=0}^{\infty} a_j \rho^j, \quad \text{with } a_0 > 0, \quad \text{and} \quad (2.1)$$

$$\rho^q = u^q \sum_{j=0}^{\infty} b_j u^j. \quad (2.2)$$

Then the coefficients b_j in the inverted power series (2.2) are given explicitly in terms of the coefficients a_j of (2.1) by the inversion formula

$$b_0 = a_0^{-q/\nu}, \quad \text{and} \quad (2.3)$$

$$b_j = \frac{q}{j+q} a_0^{-(j+q)/\nu} \sum_{r=1}^j \binom{-\frac{j+q}{\nu}}{r} a_0^{-r} a_k^{(r)} \quad \text{for } j > 0,$$

where $\binom{\alpha}{r} := \alpha(\alpha-1)\dots(\alpha-r+1)/r!$ and $a_k^{(r)}$ is the sum of all ordered products of r terms of the set $\{a_1, a_2, a_3, \dots\}$ in which the sum of subscripts is k .

Proof of Theorem 1. Our proof closely follows that of Fulks–Sather [FS61], but for completeness we provide an outline of the parts of the main argument which remain unchanged. Fulks and Sather show that it is enough to consider the case that $g \geq 0$, so we henceforth assume as much.³

²The result we quote is a bit less than what Frame actually proves. Also, the version we give is slightly modified since Frame implicitly normalizes the first term of the original series to 1, which will not generally be the case for our application.

³Indeed, since the coefficients of the desired asymptotic expansion (Theorem 1) depend linearly on the coefficients of the expansion (1.2) for g , one may split g and its expansion into positive and negative parts, obtain the result for each part separately, and then subtract, thus obtaining the general case.

Let

$$I(k) := \int_R e^{-kf} g d^d x.$$

Denote by $B(r)$ the open ball of radius r centered at $0 \in \mathbb{R}^d$. Then Fulks and Sather show that there exist $A, \rho_0 > 0$ such that $B(\rho_0) \subset R$ and

$$I(k) = \int_{B(\rho_0)} e^{-kf} g d^d x + o(e^{-kA}) =: I_1(k) + o(e^{-kA}). \quad (2.4)$$

Moreover, they show that for $\rho < \rho_0$

$$\left| f(\rho, \Omega) - \rho^\nu \sum_{k=0}^N f_k \rho^k \right| < \varepsilon \rho^{N+\nu}, \quad \left| g(\rho, \Omega) - \rho^{\lambda-d} \sum_{k=0}^N g_k \rho^k \right| < \varepsilon \rho^{N+\lambda-d},$$

and the two functions

$$f_\pm(\rho, \Omega) := \rho^\nu \sum_{k=0}^N f_k \rho^k \pm \varepsilon \rho^{N+\nu}$$

are increasing in ρ for $0 \leq \rho \leq \rho_0$.

Let

$$I_\pm(k) := \int_{B(\rho_0)} e^{-kf_\pm} g d^d x.$$

Then since $g \geq 0$,

$$I_+(k) \leq I_1(k) \leq I_-(k). \quad (2.5)$$

We concentrate on $I_+(k)$. Fulks and Sather show that for any $a > 0$,

$$I_+(k) = k \int_0^a e^{-kt} G(t) dt + o(k^{-\infty})$$

where

$$G(t) := \int_{R_t} g d^d x, \quad \text{with } R_t := \{x : f_+(x) \leq t\}.$$

Choose a so small that $R_a \subset B(\rho_0)$. Then for each t with $0 \leq t \leq a$, the equation $t = f_+(\rho, \Omega)$ has a unique solution for ρ which is continuous in Ω (since f_+ is increasing in ρ). Substituting the series (1.2) for g and performing the integration over ρ yields

$$G(t) = \int_{S^{d-1}} \int_0^{\rho(t, \Omega)} g(\rho, \Omega) \rho^{d-1} d\rho d\Omega = \int_{S^{d-1}} \left[\sum_{j=0}^N \frac{g_j(\Omega)}{j + \lambda} \rho^{j+\lambda}(t, \Omega) + o(\rho^{N+\lambda}) \right] d\Omega$$

It is at this point that our proof deviates from that of Fulks and Sather; they show that the inverse function $\rho(t, \Omega)$ is of a form which is sufficient to allow them to conclude existence of the desired asymptotic expansion. We will rather use Theorem 2 to explicitly estimate the powers of the inverse function $\rho(t, \Omega)$, thus obtaining explicit estimates for $G(t)$ and hence for $I(k)$.

Let

$$a_j := \begin{cases} f_j(\Omega) & \text{for } j < N, \\ f_N(\Omega) + \varepsilon & \text{for } j = N, \text{ and} \\ 0 & \text{for } j > N. \end{cases}$$

Then $t := f_+ = \rho^\nu \sum_{j=0}^{\infty} a_j \rho^j$. With $u = t^{1/\nu}$, Theorem 2 yields $\rho^q = u^q \sum_{j=0}^{\infty} b_j^{(q)} u^j$, where the coefficients $b_j^{(q)}$ are given by (2.3). (We have added the superscript label (q) to the coefficients because we will need several instances of the theorem for various values of q .)

Clearly, $b_j^{(q)}$ depends only on $\{a_0, a_1, \dots, a_j\}$. In particular, $b_j^{(q)} = b_j^{(q)}(\Omega)$ is independent of ε for $j < N$. Moreover,

$$b_N^{(q)} = b_N^{(q)}(\Omega, \varepsilon) = \left[\frac{q}{N+q} \sum_{r=1}^j f_0(\Omega)^{-r-(j+q)/\nu} \binom{-j+q}{r} f_j^{(r)}(\Omega) \right] - \varepsilon \left[\frac{q}{\nu} f_0(\Omega)^{1+(N+q)/\nu} \right]$$

where we define $f_j^{(r)}(\Omega)$ to be the sum of ordered products of r terms from the set $\{f_1(\Omega), \dots, f_N(\Omega)\}$ such that the subscripts add to j . Notice that $b_N^{(q)}(\varepsilon, \Omega)$ is linear in ε and the constant term (with respect to ε) is of the same form as the $j < N$ terms.

Moreover, the remainder after N terms, $u^q \sum_{N+1}^{\infty} b_j^{(q)}(\Omega, \varepsilon) u^j$, is uniformly bounded for $\Omega \in S^{d-1}$, $0 \leq \varepsilon \leq 1$ and $0 \leq u \leq a^{1/\nu}$; that is,

$$u^q \sum_{N+1}^{\infty} b_j^{(q)}(\Omega, \varepsilon) u^j = O(u^{N+1+q}) = o(u^{N+q}).$$

Putting this all together yields

$$\rho^q = t^{q/\nu} \sum_{j=0}^N \gamma_j^{(q)}(\Omega) t^{j/\nu} - \varepsilon \left[\frac{q}{\nu} f_0(\Omega)^{1+(N+q)/\nu} \right] + o(t^{(N+q)/\nu}) \quad (2.6)$$

where the $\gamma_j^{(q)}$ are the coefficients, given by Theorem 2, of the q -th power of the inverse of the finite series $\rho^\nu \sum_{j=0}^N f_j(\Omega) \rho^j$ (that is, $\gamma_j^{(q)} = b_j^{(q)}$ for $\varepsilon = 0$).

Substituting (2.6) into the expression (2.6) for $G(t)$, we obtain

$$G(t) = \sum_{j=0}^N \sum_{l=0}^N t^{\frac{l+j+\lambda}{\nu}} \left[\frac{1}{j+\lambda} \int_{S^{d-1}} g_j(\Omega) \gamma_l^{(j+\lambda)}(\Omega) d\Omega \right] \quad (2.7)$$

$$- \varepsilon \left[\sum_{j=0}^N \frac{1}{\nu} \int_{S^{d-1}} g_j(\Omega) f_0(\Omega)^{-1-(N+j+\lambda)/\nu} d\Omega \right] t^{\frac{N+j+\lambda}{\nu}} + o(t^{\frac{N+\lambda}{\nu}}). \quad (2.8)$$

After some rearrangement,⁴ one obtains

$$G(t) = \sum_{j=0}^N \eta_j t^{\frac{j+\lambda}{\nu}} - \varepsilon \eta'_N t^{\frac{N+\lambda}{\nu}} + o(t^{\frac{N+\lambda}{\nu}}) \quad (2.9)$$

where

$$\eta_j = \sum_{l=0}^j \frac{1}{j-l+\lambda} \int_{S^{d-1}} g_{j-l}(\Omega) \gamma_l^{(j-l+\lambda)}(\Omega) d\Omega$$

and

$$\eta'_N = \frac{1}{\nu} \int_{S^{d-1}} g_0(\Omega) f_0(\Omega)^{-1-(N+\lambda)/\nu} d\Omega.$$

Finally, recall that for $s > -1$

$$k \int_0^a e^{-kt} t^s dt = k^{-s} \Gamma(s+1) + o(k^{-\infty}).$$

We multiply $G(t)$ by e^{-kt} and integrate termwise to get

$$\begin{aligned} I_+(k) &= k \int_0^a e^{-kt} \left(\sum_{j=0}^N \eta_j t^{\frac{j+\lambda}{\nu}} - \varepsilon \eta'_N t^{\frac{N+\lambda}{\nu}} + o(t^{\frac{N+\lambda}{\nu}}) \right) dt + o(k^{-\infty}) \\ &= \sum_{j=0}^N \zeta_j k^{-(j+\lambda)/\nu} - \varepsilon \zeta'_N k^{-(N+\lambda)/\nu} + o(k^{-(N+\lambda)/\nu}) \end{aligned} \quad (2.10)$$

⁴To obtain (2.9), define $h_{j,l} := \frac{1}{j+\lambda} \int_{S^{d-1}} g_{j-l}(\Omega) \gamma_l^{(j-l+\lambda)}(\Omega) d\Omega$. Then the double sum in (2.7) is $\sum_{j=0}^N \sum_{l=0}^N h_{j,l} t^{\frac{j+l+\lambda}{\nu}}$. Of course, if both j and l are large, then the corresponding term is absorbed into the error term. So in fact we only need consider

$$\sum_{j=0}^N \sum_{l+j \leq N} h_{j,l} t^{\frac{j+l+\lambda}{\nu}} = \sum_{j=0}^N \sum_{l=0}^{N-j} h_{j,l} t^{\frac{j+l+\lambda}{\nu}} = \sum_{m=0}^N \sum_{l+j=m} h_{j,l} t^{\frac{m+\lambda}{\nu}} = \sum_{m=0}^N \sum_{l=0}^m h_{m-l,l} t^{\frac{m+\lambda}{\nu}}.$$

Putting $h_{m-l,l}$ into this expression then yields the first sum of (2.9).

The h'_N term arises more simply: all of the $j > 0$ terms in the coefficient of ε in (2.7) are absorbed into the error term.

where, since $\Gamma(s+1) = s\Gamma(s)$,

$$\zeta_j = \frac{1}{\nu} \Gamma\left(\frac{j+\lambda}{\nu}\right) \int_{S^{d-1}} \left[f_0(\Omega)^{-(j+\lambda)/\nu} \sum_{m=0}^j g_{j-m}(\Omega) \sum_{r=1}^m \binom{-\frac{j+\lambda}{\nu}}{r} \frac{f_m^{(r)}(\Omega)}{f_0(\Omega)^r} \right] d\Omega$$

and

$$\zeta'_N = \frac{1}{\nu} \Gamma\left(\frac{N+\lambda}{\nu}\right) \int_{S^{d-1}} g_0(\Omega) f_0(\Omega)^{-1-(N+\lambda)/\nu} d\Omega.$$

The proof now follows again the proof of Fulks and Sather. The only difference between f_+ and f_- is the change in sign of ε , whence a similar argument yields the same estimate of $I_-(k)$ with ε replaced by $-\varepsilon$. Combining (2.5), (2.10) and its analogue for f_- , multiplying through by $k^{(N+\lambda)/\nu}$, and using (2.4) we obtain for each $\varepsilon > 0$ that

$$-\varepsilon \zeta'_N \leq \limsup_{k \rightarrow \infty} \left(I(k) - \sum_{j=0}^N \zeta_j k^{-(j+\lambda)/\nu} \right) k^{(N+\lambda)/\nu} \leq \varepsilon \zeta'_N$$

and similarly for $\liminf_{k \rightarrow \infty}$. ■

2.1 The first coefficient in terms of the Hessian.

We conclude with a proof of Proposition 2, from which one recovers the usual formula for the first term in the asymptotic expansion (1.3). Note that our assumption that f attains a unique minimum at 0 implies that the Hessian of f at 0 is positive definite.

Proof of Proposition 2. A short computation using $\rho = \sqrt{(x^1)^2 + \dots + (x^d)^2}$ and the Leibniz rule yields

$$\frac{\partial^2}{\partial \rho^2} = \sum_{j,k=1}^d \frac{x^j x^k}{\rho^2} \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^k}.$$

Since h is symmetric, there exists an orthonormal change of coordinates $y^j = P_k^j x^k$ such that PhP^{-1} is diagonal. Note that the quantity y^j/ρ is independent of ρ , so

$$f_2(\Omega) = \frac{1}{2} \sum_{j=1}^d \eta_j \frac{(y^j)^2}{\rho^2}, \quad (2.11)$$

where η^j are the eigenvalues of h .

Introduce spherical coordinates $(\rho, \phi_1, \phi_2, \dots, \phi_{d-1})$ on \mathbb{R}^d ; these are related to the $\{y^j\}$ via

$$\begin{aligned} y^1 &= \rho \cos \phi_1, & y^2 &= \rho \sin \phi_1 \cos \phi_2, & \dots \\ y^{d-1} &= \rho \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{d-2} \cos \phi_{d-1}, \\ y^d &= \rho \sin \phi_1 \sin \phi_2 \cdots \sin \phi_{d-2} \sin \phi_{d-1}. \end{aligned} \quad (2.12)$$

The coordinate ranges are $0 \leq \rho < \infty$, $0 \leq \phi_j \leq \pi$ for $j = 1, \dots, d-2$ and $0 \leq \phi_{d-1} \leq 2\pi$. The solid angle element in these spherical coordinates is $d\Omega = \sin^{d-2} \phi_1 \sin^{d-3} \phi_2 \cdots \sin \phi_{d-2} d\phi_1 \cdots d\phi_{d-1}$.

Substituting (2.11) and (2.12) into the integral $\int_{S^{d-1}} f_2(\Omega)^{-d/2} d\Omega$, one may then use the identity (assuming $a, b > 0$)

$$\int_0^\pi \frac{\sin^{m-2} \phi d\phi}{(a \cos^2 \phi + b \sin^2 \phi)^{m/2}} = \sqrt{\frac{\pi}{a}} \frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)} \frac{1}{b^{(m-1)/2}}. \quad (2.13)$$

to integrate with respect to ϕ_1 , then again with respect to ϕ_2 , etc... At each step, our assumption that h is positive definite insures that (2.13) is valid. After $d-2$ integrations, one is left with an integral over ϕ_{d-1} which may be evaluated directly to yield the desired formula. ■

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