# Quasiconvex relaxation of multidimensional control problems with integrands $f(t, \xi, v)$

by

*Marcus Wagner*

Preprint no.: 68 2008

# Quasiconvex relaxation of multidimensional control problems with integrands $f(t, \xi, v)$

*Marcus Wagner*

**Abstract.** We prove a general relaxation theorem for multidimensional control problems of Dieudonné-Rashevsky type with nonconvex integrands $f(t, \xi, v)$ in presence of a convex control restriction. The relaxed problem, wherein the integrand $f$ has been replaced by its lower semicontinuous quasiconvex envelope with respect to the gradient variable, possesses the same finite minimal value as the original problem, and admits a global minimizer. As an application, we provide existence theorems for the image registration problem with convex and polyconvex regularization terms.

**Keywords.** Quasiconvex functions with infinite values, lower semicontinuous quasiconvex envelope, multidimensional control problem, relaxation, existence of global minimizers, image registration, polyconvex regularization

**AMS Subject Classification (2000).** 26 B 05, 26 B 25, 49 J 20, 49 J 45, 68 U 10.

## 1. Introduction.

### a) Dieudonné-Rashevsky type problems with nonconvex integrands.

The present paper is concerned with the existence theory for multidimensional control problems with nonconvex integrands $f(t, \xi, v)$, which depend not only on $v$ but explicitely on $t$ and $\xi$ as well, while the control set is assumed to be convex. More precisely, we study problems of the type

$$(\mathrm{P}): \quad F(x) = \int_\Omega f(t, x(t), Jx(t))\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \tag{1.1}$$

$$Jx(t) = \begin{pmatrix} \dfrac{\partial x_1}{\partial t_1}(t) & \dots & \dfrac{\partial x_1}{\partial t_m}(t) \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial t_1}(t) & \dots & \dfrac{\partial x_n}{\partial t_m}(t) \end{pmatrix} \in \mathrm{K} \subset \mathbb{R}^{nm} \;\; (\forall)\, t \in \Omega \tag{1.2}$$

and choose $n \geqslant 1$, $m \geqslant 2$, $\Omega \subset \mathbb{R}^m$ as the closure of a bounded strongly Lipschitz domain with $\mathfrak{o} \in \mathrm{int}\,(\Omega)$ and the control set $\mathrm{K} \subset \mathbb{R}^{nm}$ as a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. The integrand $f(t, \xi, v)\colon\ \Omega \times \mathbb{R}^n \times \mathrm{K} \to \mathbb{R}$ is, in general, nonconvex with respect to $v$. The structure of (P) as an optimal control problem will become clear if one introduces formal control variables $u \in L^\infty(\Omega, \mathbb{R}^{nm})$ with $Jx(t) = u(t)$.

Problems of this kind, also called Dieudonné-Rashevsky type problems, arise e. g. in elasticity theory,[01] in population dynamics[02] and in the framework of mathematical image processing.[03] In order to motivate the necessity to treat nonconvex integrands, we mention the following problems from image processing: the image registration problem with polyconvex regularization,[04] the determination of the optical flow with nonconvex regularization[05] and the optimal control formulation of the Shape-from-Shading problem (multiple image

---

[01] [ Ting 69a ] , p. 531 f., [ Ting 69b ] and [ Wagner 96 ] , pp. 76 ff.

[02] [ Brokate 85 ] , [ Feichtinger/Tragler/Veliov 03 ] .

[03] [ Brune/Maurer/Wagner 08 ] , [ Franek/Franek/Maurer/Wagner 08 ] , [ Wagner 06a ] , pp. 108 ff., and [ Wagner 07a ] .

[04] See Section 4 below where this problem will be considered in detail.

[05] E. g. regularization terms of Perona-Malik type, cf. [ Aubert/Kornprobst 06 ] , pp. 90 – 93, and [ Wagner 06a ] , p. 114. Instead, in [ Hinterberger/Scherzer/Schnörr/Weickert 02 ] , p. 82, a polyconvex regularization term has been proposed.

method).$^{06)}$ All these problems must be formulated with dimensions $n = m = 2$, consequently, in analogy to the multidimensional Calculus of Variations we have to look for a *quasiconvex relaxation* instead of a convex one.

A significant difference between variational and optimal control problems results lies in the fact that the integrand in (P) is defined a priori on $v \in \mathrm{K}$ only. The examples from [ Wagner 06a ], pp. 16 ff., and [ Wagner 06b ], p. 241 f., show that, in order to conserve the minimal value of (P) in the process of relaxation, the integrand must be extended to $v \in \mathbb{R}^{nm} \setminus \mathrm{K}$ "in the best possible way", i. e. by $(+\infty)$. For this reason, the quasiconvex functions used in the forming of a possible envelope must be allowed to take the value $(+\infty)$ as well. We will consider the following classes of integrands:

**Definition 1.1.** *Let $\Omega \subset \mathbb{R}^m$ be the closure of a bounded strongly Lipschitz domain and $\mathrm{K} \subset \mathbb{R}^{nm}$ a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$.*

*1)* **(Function class $\mathcal{F}_\mathrm{K}$)** *We say that a function $f\colon\ \mathbb{R}^{nm} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ belongs to the class $\mathcal{F}_\mathrm{K}$ iff $f\,\big|\,\mathrm{K} \in C^0(\mathrm{K},\mathbb{R})$ and $f\,\big|\,(\,\mathbb{R}^{nm} \setminus \mathrm{K}\,) \equiv (+\infty)$.*

*2)* **(Function class $\widetilde{\mathcal{F}}_\mathrm{K}$)** *We say that a function $f(t,\xi,v)\colon\ \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\,(+\infty)\,\}$ belongs to the class $\widetilde{\mathcal{F}}_\mathrm{K}$ iff there exists a $m$-dimensional Lebesgue null set $\mathrm{N} \subset \Omega$ with:*

*a) $f(t,\xi,v) = (+\infty)$ for all $(t,\xi,v) \in \big(\,\Omega \setminus \mathrm{N}\,\big) \times \mathbb{R}^n \times \big(\mathbb{R}^{nm} \setminus \mathrm{K}\big)$,*

*b) $f(t,\xi,v) < (+\infty)$ for all $(t,\xi,v) \in \big(\,\Omega \setminus \mathrm{N}\,\big) \times \mathbb{R}^n \times \mathrm{K}$,*

*c) the restriction $f\,\big|\,\big(\big(\,\Omega \setminus \mathrm{N}\,\big) \times \mathbb{R}^n \times \mathrm{K}\big)$ is Borel measurable with respect to $t$ and continuous with respect to $(\xi,v)$,*

*d) $f$ satisfies a growth condition*$^{07)}$

$$\big|\,f(t,\xi,v)\,\big| \leqslant A(t) + B(\xi,v) \quad \forall\,(t,\xi,v) \in \Omega \times \mathbb{R}^n \times \mathrm{K} \tag{1.3}$$

*where $A \in L^1(\Omega,\mathbb{R})$, $A\,\big|\,\mathrm{int}\,(\Omega)$ is continuous, and $B$ is bounded on every bounded subset of $\mathbb{R}^n \times \mathrm{K}$.*

For the special case where the integrand in (P) resp. its extension to the whole space $\mathbb{R}^{nm}$ belongs to $\mathcal{F}_\mathrm{K}$ and, consequently, depends on $v$ only, a relaxation theorem has been proved in [ Wagner 07b ] (cited below as Theorem 1.3., 2) ). In this case, the appropriate envelope for the integrand turns out to be the so-called lower semicontinuous quasiconvex envelope (see Definition 2.6. below). The main result of the present paper is the generalization of this relaxation result to Dieudonné-Rashevsky type problems with integrands $f \in \widetilde{\mathcal{F}}_\mathrm{K}$. We will see that the known proof scheme from the multidimensional Calculus of Variations works in the case of control problems (P) as well: the general situation can be reduced to the case $f = f(v)$ where the theorem has been already established.$^{08)}$

### b) Relaxation of (P) by replacement of the integrand; main result.

Relaxation of a variational or optimal control problem means to define a new problem with the same minimal value as before, whose feasible domain contains the original one (eventually in the sense of an embedding), and whose objective is lower semicontinuous with respect to an appropriate topology.$^{09)}$ The fact that the

---

(06) Cf. [ Wagner 07a ], pp. 19 ff.

(07) Cf. [ Acerbi/Fusco 84 ], p. 132, Theorem [II.1], (II.4), and p. 134. We need the continuity of the majorant $A$ in the proof of Proposition 3.3., Step 1, below, in order to assure the openness of the level sets of $A$.

(08) [ Dacorogna 08 ], pp. 377 ff.

(09) Cf. [ Buttazzo 89 ], pp. 2 ff. and pp. 16 ff., as well as [ Roubíček 97 ], pp. vii ff.

relaxed problem admits global minimizers justifies the subsequent application of direct numerical methods. 10) In the present paper, the relaxation of (P) will be performed through the replacement of the integrand $f$ within the objective by an appropriate semiconvex envelope. 11) The conditions, which must be satisfied by this envelope, are summarized in the following theorem.

**Theorem 1.2. (Relaxation of the problem** (P)**)** *Consider the problem* (P) *under the assumptions from Section 1.a) and a function* $f^{\#}(t,\xi,v)\colon\ \Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ *with the following properties:*

*a) There exists a* $m$*-dimensional Lebesgue null set* $\mathrm{N}\subset\Omega$ *such that it holds for all* $(\hat{t},\hat{\xi})\in\big(\Omega\setminus\mathrm{N}\big)\times\mathbb{R}^n$*: The effective domain of the function* $f^{\#}(\hat{t},\hat{\xi},\,\cdot\,)\colon\ \mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ *is a Borel set with* $\mathrm{K}\subseteq\operatorname{dom}\big(\,f^{\#}(\hat{t},\hat{\xi},\,\cdot\,)\,\big)$*, and the restriction of* $f^{\#}(\hat{t},\hat{\xi},\,\cdot\,)$ *to its effective domain is a Borel measurable function which is bounded from below on every bounded subset of its domain.*

*b) It holds that* $f^{\#}(t,\xi,v)\leqslant f(t,\xi,v)$ *for all* $(t,\xi,v)\in\big(\Omega\setminus\mathrm{N}\big)\times\mathbb{R}^n\times\mathrm{K}$*, consequently,*

$$F^{\#}(x)=\int_{\Omega}f^{\#}(t,x(t),Jx(t))\,dt\;\leqslant\;\int_{\Omega}f(t,x(t),Jx(t))\,dt=F(x)\ \textit{for all admissible functions in}\ (\mathrm{P}).$$

*c) For every sequence* $\{\,x^N\,\}$ *of admissible functions in* (P) *with* $x^N\overset{*}{\longrightarrow}_{L^{\infty}(\Omega,\mathbb{R}^n)}\hat{x}$ *and* $Jx^N\overset{*}{\longrightarrow}_{L^{\infty}(\Omega,\mathbb{R}^{nm})}J\hat{x}$*, the lower semicontinuity relation* $F^{\#}(\hat{x})\leqslant\liminf_{N\to\infty}F^{\#}(x^N)$ *holds.*

*d) The minimal values of* (P) *and the following problem* $(\mathrm{P})^{\#}$ *coincide:*

$$(\mathrm{P})^{\#}\colon\quad F^{\#}(x)=\int_{\Omega}f^{\#}(t,x(t),Jx(t))\,dt\;\longrightarrow\;\inf!\,;\quad x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,;\quad Jx(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega \tag{1.4}$$

*Then the (finite) minimal values of the problems* (P) *and* $(\mathrm{P})^{\#}$ *are identical, and every minimzing sequence* $\{\,x^N\,\}$ *of* (P) *contains a subsequence* $\{\,x^{N'}\,\}$ *converging together with their generalized derivatives weakly* (in the sense of* $L^{\infty}(\Omega,\mathbb{R}^n)$ *resp.* $L^{\infty}(\Omega,\mathbb{R}^{nm})$*) to a global minimizer* $\hat{x}$ *of* $(\mathrm{P})^{\#}$*.*

Only a few relaxation results are known for problems of type (P). We mention the following theorems of Ekeland/Témam and Wagner, assuming that the integrands as members of the function classes $\mathcal{F}_{\mathrm{K}}$ resp. $\widetilde{\mathcal{F}}_{\mathrm{K}}$ are defined from the outset on the whole space $\mathbb{R}^{nm}$:

**Theorem 1.3.** *Consider the problem* (P) *under the assumptions from Section 1.a).*

*1)* 12) **(Convex relaxation of** (P)**, the integrand depends on** $t$ **and** $v$ **only,** $n=1$**)** *Assume further that* $m\geqslant 2$*,* $n=1$*, and* $\mathrm{K}=\mathrm{K}(\mathfrak{o},\varrho)\subset\mathbb{R}^{nm}$ *is a closed ball centered in the origin. The integrand* $f(t,v)\colon\ \Omega\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ *belongs to* $\widetilde{\mathcal{F}}_{\mathrm{K}}$ *but does not depend on* $\xi$*. Then the function* $f^{\#}(t,v)\colon\ \Omega\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$*, which is defined as the convex envelope of* $f$ *with respect to* $v$ *by*

$$f^{\#}(\hat{t},v)\;=\;f^{c}(\hat{t},v)\;=\;\sup\big\{\,g(v)\;\big|\;g\colon\ \mathbb{R}^{nm}\to\mathbb{R}\ \ \textit{convex},\ \ g(w)\leqslant f(\hat{t},w)\ \ \forall\,w\in\mathbb{R}^{nm}\,\big\} \tag{1.5}$$

*for all* $\hat{t}\in\big(\Omega\setminus\mathrm{N}\big)$ *and by zero for all* $\hat{t}\in\mathrm{N}$*, admits the properties a) – d) from Theorem 1.2.*

*2)* 13) **(Quasiconvex relaxation of** (P)**, the integrand depends on** $v$ **only,** $n\geqslant 1$**)** *Assume further that* $m\geqslant 2$*,* $n\geqslant 1$*, and* $\mathrm{K}\subset\mathbb{R}^{nm}$ *is an arbitrary convex body with* $\mathfrak{o}\in\operatorname{int}(\mathrm{K})$*. The integrand* $f(v)\colon\ \mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ *does not depend on* $t$ *and* $\xi$ *and belongs to* $\mathcal{F}_{\mathrm{K}}$*. Then the function*

---

10) Cf. [ Morrey 66 ] , pp. 15 ff., and [ Dacorogna 08 ] , pp. 3 ff.

11) Concerning relaxation of (P) by introduction of generalized controls (“Young measures”), see [ Wagner 08 ] .

12) [ Ekeland/Témam 99 ] , p. 327, Corollary 2.17., together with p. 334, Proposition 3.4., and p. 335 f., Proposition 3.6.

13) [ Wagner 07b ] , p. 3, Theorem 1.3.

*$f^{\#}(v)\colon\ \mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$, which is defined as the lower semicontinuous quasiconvex envelope of $f$ by*

$$f^{\#}(v)\;=\;f^{(qc)}(v)\;=\;\sup\bigl\{\,g(v)\;\big|\;g\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}\ \text{ quasiconvex, lower semicontinuous,}\\ g(w)\leqslant f(w)\ \ \forall\,w\in\mathbb{R}^{nm}\,\bigr\}\,, \tag{1.6}$$

*admits the properties a) – d) from Theorem 1.2.*

As the main result of the present paper, we prove the following generalization of Theorem 1.3.:

**Theorem 1.4. (Quasiconvex relaxation of (P) in the general case, $n\geqslant 1$)** *Consider the problem (P) under the assumptions form Section 1.a). In particular, we assume that $m\geqslant 2$, $n\geqslant 1$, $\mathrm{K}\subset\mathbb{R}^{nm}$ is an arbitrary convex body with $\mathfrak{o}\in\operatorname{int}(\mathrm{K})$, and the integrand $f(t,\xi,v)\colon\ \Omega\times\mathbb{R}^{n}\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ belongs to the function class $\widetilde{\mathfrak{F}}_{\mathrm{K}}$. Then the function $f^{\#}(t,\xi,v)\colon\ \Omega\times\mathbb{R}^{n}\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$, which is defined as the lower semicontinuous quasiconvex envelope of $f$ with respect to $v$ by*

$$f^{\#}(\hat{t},\hat{\xi},v)\;=\;f^{(qc)}(\hat{t},\hat{\xi},v)\;=\;\sup\bigl\{\,g(v)\;\big|\;g\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}\ \text{ quasiconvex and lower semicontinuous,}\\ g(w)\leqslant f(\hat{t},\hat{\xi},w)\ \ \forall\,w\in\mathbb{R}^{nm}\bigr\} \tag{1.7}$$

*for all fixed $(\hat{t},\hat{\xi})\in\bigl(\Omega\setminus\mathrm{N}\bigr)\times\mathbb{R}^{n}$ and by zero for all $(\hat{t},\hat{\xi})\in\mathrm{N}\times\mathbb{R}^{n}$, possesses the properties a) – d) from Theorem 1.2. Consequently, the problem*

$$(\mathrm{P})^{(qc)}:\quad F^{(qc)}(x)=\int_{\Omega}f^{(qc)}(t,x(t),Jx(t))\,dt\;\longrightarrow\;\inf!\,;\quad x\in W_0^{1,\infty}(\Omega,\mathbb{R}^{n})\,;\quad Jx(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega \tag{1.8}$$

*has the same finite minimal value as the problem (P), and every minimizing sequence $\{\,x^{N}\,\}$ of (P) contains a subsequence $\{\,x^{N'}\,\}$ converging weakly* (in the sense of $L^{\infty}(\Omega,\mathbb{R}^{n})$ resp. $L^{\infty}(\Omega,\mathbb{R}^{nm})$ ) together with their generalized derivatives to a global minimizer $\hat{x}$ of $(\mathrm{P})^{(qc)}$.*

As a consequence of Theorem 1.4., we obtain the following existence result for problems (P) with polyconvex integrands:

**Theorem 1.5. (Existence theorem for (P) with polyconvex integrand)** *Consider the problem (P) under the assumptions of Section 1.a). In particular, we assume that $m\geqslant 2$, $n\geqslant 1$, $\mathrm{K}\subset\mathbb{R}^{nm}$ is an arbitrary convex body with $\mathfrak{o}\in\operatorname{int}(\mathrm{K})$, and the integrand $f(t,\xi,v)\colon\ \Omega\times\mathbb{R}^{n}\times\mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty\,)\,\}$ belongs to $\widetilde{\mathfrak{F}}_{\mathrm{K}}$. Furthermore, for all fixed $(\hat{t},\hat{\xi})\in\bigl(\Omega\setminus\mathrm{N}\bigr)\times\mathbb{R}^{n}$, let $f(\hat{t},\hat{\xi},v)\colon\ \mathbb{R}^{nm}\to\mathbb{R}\cup\{\,(+\infty)\,\}$ be polyconvex as a function of $v$ (see Definition 3.9. below) where $\mathrm{N}\subset\Omega$ is the $m$-dimensional Lebesgue null set from Definition 1.1., 2). Then the problem (P) admits a global minimizer $\hat{x}\in W_0^{1,\infty}(\Omega,\mathbb{R}^{n})$.*

### c) Outline of the paper.

We close this section with a collection of notations and a short recall of some auxiliary facts from measure theory. In *Section 2*, we start with the definition of quasiconvexity for functions, which may take the value $(+\infty)$, and summarize the properties of the lower semicontinuous quasiconvex envelope $f^{(qc)}$ for integrands $f=f(v)\in\mathfrak{F}_{\mathrm{K}}$. Then we turn to the closer investigation of the lower semicontinuous quasiconvex envelope for integrands $f=f(t,\xi,v)\in\widetilde{\mathfrak{F}}_{\mathrm{K}}$, which is formed with respect to the variable $v$. In this case, we prove a number of estimates (Theorems 2.11., 2.12. and 2.14.) as well as an growth condition for $f^{(qc)}$ (Theorem 2.13.). *Section 3* contains the proofs of Theorems 1.2., 1.4. and 1.5. Finally, in *Section 4*, applying our general theorems to a basic problem from mathematical image processing, we obtain existence results for the image registration problem in the presence of convex and polyconvex regularization terms.

### d) Notations and abbreviations.

Let $k \in \{\, 0,\, 1,\, \dots\, ,\, \infty \,\}$ and $1 \leqslant p \leqslant \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of $r$-dimensional vector functions whose components are $k$-times continuously differentiable, belong to $L^p(\Omega, \mathbb{R})$ or to the Sobolev space of $L^p(\Omega, \mathbb{R})$-functions with weak derivatives up to $k$th order in $L^p(\Omega, \mathbb{R})$, respectively. In addition, functions within the subspaces $C_0^k(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ and $W_0^{1,p}(\Omega, \mathbb{R}^r) \subset W^{1,p}(\Omega, \mathbb{R}^r)$, $1 \leqslant p < \infty$, are compactly supported while the components of $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ possess Lipschitz continuous representatives[14] with zero boundary values. The symbols $x_{t_j}$ and $\partial x / \partial t_j$ may denote the classical as well as the weak partial derivative of $x$ by $t_j$. $Jx$ denotes the Jacobi matrix of the function $x$.

We denote by $\operatorname{int}(\mathrm{A})$, $\operatorname{ri}(\mathrm{A})$, $\partial \mathrm{A}$, $\operatorname{rb}(\mathrm{A})$, $\operatorname{cl}(\mathrm{A})$, $\operatorname{co}(\mathrm{A})$ and $|\,\mathrm{A}\,|$ the interior, relative interior, boundary, relative boundary, closure, the convex hull and the $r$-dimensional Lebesgue measure of a set $\mathrm{A} \subseteq \mathbb{R}^r$, respectively. $\mathbb{1}_{\mathrm{A}} \colon \mathbb{R}^r \to \mathbb{R}$ with $\mathbb{1}_{\mathrm{A}}(t) = 1 \iff t \in \mathrm{A}$ and $\mathbb{1}_{\mathrm{A}}(t) = 0 \iff t \notin \mathrm{A}$ is the characteristic function of the set A. Defining $\overline{\mathbb{R}} = \mathbb{R} \cup \{\, (+\infty) \,\}$, we equip $\overline{\mathbb{R}}$ with the natural topological and order structures where $(+\infty)$ is the greatest element.

Throughout the whole paper, we consider only *proper functions* $f \colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, assuming that the effective domain $\operatorname{dom}(f) = \{\, v \in \mathbb{R}^{nm} \mid f(v) < (+\infty) \,\}$ is always nonempty. The restriction of the function $f$ to the subset A of its range of definition is denoted by $f \,\big|\, \mathrm{A}$. If a function $f \colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to the function class $\mathcal{F}_{\mathrm{K}}$ defined above then its restriction $f \,\big|\, \mathrm{K}$ is bounded and uniformly continuous. Consequently, the class $\mathcal{F}_{\mathrm{K}}$ and the Banach space $C^0(\mathrm{K}, \mathbb{R})$ are isomorphic and isometric.

A convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ will be understood as a convex, compact set with nonempty interior.[15] A point $v \in \mathrm{K}$ is called extremal point of K iff from $v = \lambda'\, v' + \lambda''\, v''$, $\lambda'$, $\lambda'' > 0$, $\lambda' + \lambda'' = 1$, $v'$, $v'' \in \mathrm{K}$ it follows that $v' = v'' = v$. The set of all extremal points of K is denoted by $\operatorname{ext}(\mathrm{K})$. Every convex body possesses at least one extremal point. A convex subset $\varPhi \subseteq \mathrm{K}$ is called a face of K iff from $v \in \varPhi$ and $v = \lambda'\, v' + \lambda''\, v''$, $\lambda'$, $\lambda'' > 0$, $\lambda' + \lambda'' = 1$, $v'$, $v'' \in \mathrm{K}$ it follows that $v'$, $v'' \in \varPhi$.[16] The body K itself as well as $\emptyset$ will be regarded as improper faces. All nonempty faces of a convex body form compact sets. The dimension $k$ of a face is that of its affine hull; we define $\operatorname{Dim}(\emptyset) = (-1)$. Thus the null-dimensional faces of K are precisely the singletons $\{\, x \,\}$, $x \in \operatorname{ext}(\mathrm{K})$.

Finally, we introduce *three nonstandard notations.* "$\{\, x^N \,\}$, A" denotes a sequence $\{\, x^N \,\}$ with members $x^N \in \mathrm{A}$. If $\mathrm{A} \subseteq \mathbb{R}^r$ then the abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $t \in \mathrm{A}$" resp. "for all $t \in \mathrm{A}$ except a $r$-dimensional Lebesgue null set". The symbol $\mathfrak{o}$ denotes, depending on the context, the zero element resp. the zero function of the underlying space.

### e) Auxiliary facts from measure theory.

**Definition 1.6. (Carathéodory functions)** *Let* $\mathrm{K} \subseteq \mathbb{R}^{nm}$ *be a Borel set. Then a function* $f(t, \xi, v) \colon \Omega \times \mathbb{R}^n \times \mathrm{K} \to \mathbb{R}$ *is called a Carathéodory function iff there exists a* $m$*-dimensional Lebesgue null set* $\mathrm{N} \subset \Omega$ *such that the restriction* $f \,\big|\, \big(\, (\Omega \setminus \mathrm{N}) \times \mathbb{R}^n \times \mathrm{K} \,\big)$ *is Borel measurable with respect to* $t$ *and continuous with respect to* $(\xi, v)$*.*

From Definition 1.1., 2) it is clear that the restrictions of the functions $f \in \widetilde{\mathcal{F}}_{\mathrm{K}}$ to $\Omega \times \mathbb{R}^n \times \mathrm{K}$ are Carathéodory functions.

---

[14] [Evans/Gariepy 92], p. 131, Theorem 5.

[15] Cf. [Brøndsted 83] and [Schneider 93].

[16] We dispense with the distiction between "facets" and "faces", cf. [Brøndsted 83], p. 30.

**Theorem 1.7. (Scorza-Dragoni theorem)** [17] *Let* $\mathrm{K} \subseteq \mathbb{R}^{nm}$ *be a Borel set. Then the function* $f(t, \xi, v):$ $\Omega \times \mathbb{R}^n \times \mathrm{K} \to \mathbb{R}$ *is a Carathéodory function iff the following holds: For every compact subset* $\Omega_0 \subseteq \Omega$ *and arbitrary* $\varepsilon > 0$ *there exists a compact subset* $\Omega_c \subseteq \Omega_0$ *with* $|\,\Omega_0 \setminus \Omega_c\,| \leqslant \varepsilon$ *such that the restriction* $f\,\big|\,\big(\,\Omega_c \times \mathbb{R}^n \times \mathrm{K}\big)$ *is a continuous function with respect to* $(t, \xi, v)$.

**Lemma 1.8.** [18] *Given an open set* $\Omega \subset \mathbb{R}^m$ *and a function* $x \in L^1(\Omega, \mathbb{R}^n)$*, then for arbitrary values* $\eta > 0$ *and* $\delta > 0$*, one can find finitely many mutually disjoint closed cubes* $\mathrm{Q}_s \subseteq \Omega$*,* $1 \leqslant s \leqslant r$*, with edge length* $0 < \eta_s \leqslant \eta$*, with the following properties:*

*1)* $\big|\,\Omega \setminus \bigcup_{s=1}^{r} \mathrm{Q}_s\,\big| \leqslant \delta\,;$ (1.9)

*2)* $\Big|\,x_i(t) - \dfrac{1}{|\,\mathrm{Q}_s\,|} \displaystyle\int_{\mathrm{Q}_s} x_i(\tau)\,d\tau\,\Big| \leqslant \delta \quad (\forall)\, t \in \mathrm{Q}_s\,,\ 1 \leqslant s \leqslant r\,,\ 1 \leqslant i \leqslant n\,.$ (1.10)

## 2. The lower semicontinuous quasiconvex envelope.

### a) Quasiconvex functions which can take the value $(+\infty)$.

**Definition 2.1. (Quasiconvex functions with values in $\overline{\mathbb{R}}$)** [19] *A function* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *with the following properties is said to be quasiconvex:*

*a)* $\mathrm{dom}\,(f) \subseteq \mathbb{R}^{nm}$ *is a nonempty Borel set;*

*b)* $f\,\big|\,\mathrm{dom}\,(f)$ *is Borel measurable and bounded from below on every bounded subset of* $\mathrm{dom}\,(f)$*;*

*c) for all* $v \in \mathbb{R}^{nm}$*,* $f$ *satisfies Morrey's integral inequality*

$$f(v) \leqslant \frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \quad \forall\, x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,, \tag{2.1}$$

*or equivalently*

$$f(v) = \inf\Big\{\,\frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,,\ v + Jx(t) \in \mathbb{R}^{nm}\ (\forall)\, t \in \Omega\,\Big\}\,. \tag{2.2}$$

*Here* $\Omega \subset \mathbb{R}^m$ *is the closure of a bounded strongly Lipschitz domain.*

We adopt the convention that the integral $\int_{\mathrm{A}} (+\infty)\,dt$ takes the values zero or $(+\infty)$ if either $\mathrm{A} \subseteq \mathbb{R}^m$ is a $m$-dimensional Lebesgue null set or has positive measure. Note that the values of the integrand $f$ cannot be changed even on a Lebesgue null set of $\mathbb{R}^{nm}$. If $\mathrm{dom}\,(f)$ is a convex body then the set of "test functions" within Morrey's integral inequality can be restricted as follows:

**Theorem 2.2. (Morrey's integral inequality for functions with $\mathrm{dom}\,(f) = \mathrm{K}$)** [20] *Let a convex body* $\mathrm{K} \subset \mathbb{R}^{nm}$ *and a function* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *with* $\mathrm{dom}\,(f) = \mathrm{K}$ *be given. Assume that* $f\,\big|\,\mathrm{K}$ *is measurable and bounded. Then* $f$ *satisfies Morrey's integral inequality in a point* $v \in \mathrm{K}$ *iff*

$$f(v) = \inf\Big\{\,\frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,,\ v + Jx(t) \in \mathrm{K}\ (\forall)\, t \in \Omega\,\Big\}\,. \tag{2.3}$$

---

[17] [ Ekeland/Témam 99 ] , p. 235, Scorza-Dragoni Theorem.

[18] Slightly modified from [ Wagner 07b ] , p. 10, Lemma 3.4. The proof remains unchanged.

[19] [ Wagner 06c ] , p. 6, Definition 2.9., as a specification of [ Ball/Murat 84 ] , p. 228, Definition 2.1., in the case $p = (+\infty)$. Cf. also [ Conti 08 ] , p. 16.

[20] [ Wagner 06c ] , p. 7, Theorem 2.11., 2).

### b) The envelope $f^*$ related to $\mathrm{K}$.

In this subsection, we fix a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$ and the quantities $c_{\mathrm{K}} = \operatorname{Dist}(\mathfrak{o}\,,\partial\mathrm{K})$ and $C_{\mathrm{K}} = \operatorname{Max}\big(1\,,\operatorname{Max}_{v\in\mathrm{K}}|\,v\,|\big)$, thus $0 < c_{\mathrm{K}} \leqslant C_{\mathrm{K}}$ and $\operatorname{Diam}(\mathrm{K}) \leqslant 2\,C_{\mathrm{K}}$.

**Definition 2.3. (Envelope $f^*$ related to $\mathrm{K}$)** [21] *Consider the convex body* $\mathrm{K} \subset \mathbb{R}^{nm}$ *mentioned above and a function* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *with the following properties: the set* $\operatorname{dom}(f)$ *is measurable,* $f\,\big|\operatorname{dom}(f)$ *is a measurable function, and* $f$ *is bounded from below on* $\mathbb{R}^{nm}$*. Then we define for* $v \in \mathbb{R}^{nm}$*:*

$$f^*(v) \;=\; \inf\Big\{\,\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\; v + Jx(t) \in \mathrm{K}\;\;(\forall)\,t \in \Omega\,\Big\}\,. \tag{2.4}$$

In the following, we will make use of two particular properties of $f^*$:

**Theorem 2.4. (Definition of $f^*$ does not depend on $\Omega$)** [22] *Let* $\mathrm{K} \subset \mathbb{R}^{nm}$ *and* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *be given as in Definition 2.3. If both sets* $\Omega,\,\widetilde{\Omega} \subset \mathbb{R}^m$ *are closures of bounded strongly Lipschitz domains then*

$$\begin{aligned} f^*(v) \;&=\; \inf\Big\{\,\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\; v + Jx(t) \in \mathrm{K}\;\;(\forall)\,t \in \Omega\,\Big\} && (2.5)\\ &=\; \inf\Big\{\,\frac{1}{|\,\widetilde{\Omega}\,|}\int_{\widetilde{\Omega}} f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\widetilde{\Omega}\,,\,\mathbb{R}^n)\,,\; v + Jx(t) \in \mathrm{K}\;\;(\forall)\,t \in \widetilde{\Omega}\,\Big\}\,. && (2.6) \end{aligned}$$

**Theorem 2.5. (Special sequence $\{\,x^N\,\}$ realizing the infimum in Definition 2.3.)** [23] *Let* $\mathrm{K} \subset \mathbb{R}^{nm}$ *and* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *be given as in Definition 2.3. Assume that* $\Omega \subset \mathbb{R}^m$ *is a closed cube. Then for every* $v \in \mathbb{R}^{nm}$ *there exists a sequence* $\{\,x^N\,\}\,, W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *with*

$$\begin{aligned} f^*(v) \;=\; \lim_{N\to\infty}\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,v + Jx^N(t)\,)\,dt\,,&\\ v + Jx^N(t) \in \mathrm{K}\;\;(\forall)\,t \in \Omega\;\;\forall\,N \in \mathbb{N}\,,\quad x^N \overset{*}{\rightharpoondown}{}_{L^{\infty}(\Omega,\mathbb{R}^n)}\,\mathfrak{o}\quad \textit{and}\quad Jx^N \overset{*}{\rightharpoondown}{}_{L^{\infty}(\Omega,\mathbb{R}^{nm})}\,\mathfrak{o}\,.& \qquad (2.7) \end{aligned}$$

### c) The lower semicontinuous quasiconvex envelope $f^{(qc)}(v)$ for $f \in \mathcal{F}_{\mathrm{K}}$.

**Definition 2.6. (Lower semicontinuous quasiconvex envelope $f^{(qc)}$ for functions with values in $\overline{\mathbb{R}}$)** [24] *To a function* $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *bounded from below, we define the lower semicontinuous quasiconvex envelope* $f^{(qc)}\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *through*

$$\begin{aligned} f^{(qc)}(v) \;=\; \sup\Big\{\,g(v)\;\Big|\; g\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}\;\; &\textit{quasiconvex and lower semicontinuous,}\\ &g(w) \leqslant f(w)\;\;\forall\,w \in \mathbb{R}^{nm}\,\Big\}\,. \qquad (2.8) \end{aligned}$$

**Remarks.** a) Definition 2.6. is motivated by the observation that any finite, quasiconvex function $g\colon \mathbb{R}^{nm} \to \mathbb{R}$ is from the outset continuous. [25] If a measurable function $f$ is bounded from below and takes only values

---

[21] The function $f^*$ has been introduced in [ Kinderlehrer/Pedregal 91 ], p. 356, in the special case $\mathrm{K} = \mathrm{K}(\mathfrak{o},\varrho)$ and in [ Dacorogna/Marcellini 97 ], p. 27, Theorem 7.2., for arbitrary convex bodies K. In both cases it was assumed that $f \in C^0(\mathrm{K}\,,\mathbb{R})$. We follow [ Wagner 06c ], p. 14, Definition 3.1., and formulate the definition from the outset for functions $f\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$.

[22] [ Dacorogna/Marcellini 97 ], p. 28 f., Step 1.

[23] Ibid., p. 35, Step 6.

[24] [ Wagner 06c ], p. 9, Definition 2.14., 2).

[25] [ Dacorogna 08 ], p. 159, Theorem 5.3., (iv).

in $\mathbb{R}$ then Definition 2.6. coincides with the usual definition of the quasiconvex envelope,[26] and the function $f^{(qc)}$ is quasiconvex and continuous as well.

b) If two functions $f_1$, $f_2 : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ are bounded from below then the implication $f_1(v) \leqslant f_2(v)\ \forall\, v \in \mathbb{R}^{nm} \Longrightarrow {f_1}^{(qc)}(v) \leqslant {f_2}^{(qc)}(v)\ \forall\, v \in \mathbb{R}^{nm}$ holds.[27]

c) For $f \in \mathfrak{F}_{\mathrm{K}}$, $f^{(qc)}$ satisfies the inequality $f^c(v) \leqslant f^{(qc)}(v) \leqslant f(v)$ for all $v \in \mathbb{R}^{nm}$, which implies particularly $f^{(qc)}(v) = (+\infty)$ for all $v \in \mathbb{R}^{nm} \setminus \mathrm{K}$ and $f^{(qc)}(v) = f(v)$ for all $v \in \mathrm{ext}\,(\mathrm{K})$. Furthermore, $f^{(qc)}$ itself is a lower semicontinuous and quasiconvex function and is, consequently, admissible in the process of its own forming.[28] Thus it follows that $f^{(qc)}$ is the largest quasiconvex, lower semicontinuous function below $f$ in this case.[29]

The structure of the lower semicontinuous quasiconvex envelope for an integrand $f \in \mathfrak{F}_{\mathrm{K}}$ will be described by the following representation theorem:

**Theorem 2.7. (Representation theorem for $f^{(qc)}$)**[30] *For arbitrary $f \in \mathfrak{F}_{\mathrm{K}}$, the lower semicontinuous quasiconvex envelope $f^{(qc)}$ can be represented as*

$$f^{(qc)}(v_0) = \begin{cases} \phantom{\lim\limits_{v \to v_0\,,\, v\, \in\, \mathrm{R}\, \cap\, \mathrm{int}\,(\mathrm{K})}} f^*(v_0) \;\big|\; v_0 \in \mathrm{int}\,(\mathrm{K})\,; \\ \lim\limits_{v \to v_0\,,\, v\, \in\, \mathrm{R}\, \cap\, \mathrm{int}\,(\mathrm{K})} f^*(v) \;\big|\; v_0 \in \partial\mathrm{K}\,; \\ \phantom{\lim\limits_{v \to v_0\,,\, v\, \in\, \mathrm{R}\, \cap\, \mathrm{int}\,(\mathrm{K})}} (+\infty) \;\big|\; v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K} \end{cases} \tag{2.9}$$

*where $f^*(v)$ is defined by* (2.4) *(see Definition 2.3. above).*

In the following theorems, the relation between the uniform continuity of the restriction of $f \in \mathfrak{F}_{\mathrm{K}}$ to K and the continuity resp. semicontinuity of $f^{(qc)}$ will be quantified. We will relate to a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ with the quantities $c_{\mathrm{K}}$ and $C_{\mathrm{K}}$ introduced in Section 2.b) above.

**Theorem 2.8. ($\varepsilon$-$\delta$ relation for the restriction of $f^{(qc)}$ to faces of K)**[31] *Let $f \in \mathfrak{F}_{\mathrm{K}}$ and a $k$-dimensional face $\varPhi \subseteq \mathrm{K}$, $0 \leqslant k \leqslant nm$, be given. Assume that the uniform continuity of $f$ on K is described through the $\varepsilon$-$\delta$ relation*

$$\big|\, v' - v'' \,\big| \leqslant \delta(\varepsilon) < 1 \Longrightarrow \big|\, f(v') - f(v'') \,\big| \leqslant \varepsilon \quad \forall\, v',\, v'' \in \mathrm{K}\,. \tag{2.10}$$

*Then $f^{(qc)} \,\big|\, \varPhi$ obeys the following $\varepsilon$-$\delta$ relation:*

$$\begin{aligned} \big|\, v' - v'' \,\big| \leqslant \frac{\delta(\varepsilon)}{4\, C_{\mathrm{K}}} \cdot \mathrm{Min}\,\big(\, 1\,,\, \mathrm{Dist}\,(v',\, \mathrm{rb}\,(\varPhi))\,,\, \mathrm{Dist}\,(v'',\, \mathrm{rb}\,(\varPhi))\,\big) \Longrightarrow \\ \big|\, f^{(qc)}(v') - f^{(qc)}(v'') \,\big| \leqslant 2\,\varepsilon \quad \forall\, v',\, v'' \in \mathrm{ri}\,(\varPhi) \end{aligned} \tag{2.11}$$

*where $C_{\mathrm{K}}$ is the quantity defined in the beginning of Section 2.b).*

As a particular consequence of this theorem, the restriction $f^{(qc)} \,\big|\, \mathrm{int}\,(\mathrm{K})$ is continuous.

---

[26] Cf. [DACOROGNA 08], p. 156 f., Definition 5.1., ii).

[27] [WAGNER 06C], p. 10, Lemma 2.15., 3).

[28] Ibid., p. 10, Theorem 2.17.

[29] Ibid., p. 10, Theorem 2.18.

[30] Ibid., p. 29, Theorem 4.1.

[31] Ibid., p. 16, Theorem 3.5., together with Theorem 2.7. above.

**Theorem 2.9.** **($\varepsilon$-$\delta$ relation for $f^{(qc)}$ along rays starting from $\mathfrak{o}$)**[32] *Let $f \in \mathfrak{F}_{\mathrm{K}}$ be given. Assume that the uniform continuity of $f$ on* K *is described again through the $\varepsilon$-$\delta$ relation*

$$\big|\, v' - v'' \,\big| \;\leqslant\; \delta(\varepsilon) \;<\; 1 \;\Longrightarrow\; \big|\, f(v') - f(v'') \,\big| \leqslant \varepsilon \quad \forall\, v',\, v'' \in \mathrm{K}\,. \tag{2.12}$$

*Consider two points $v,\ w\ \in$ K which a) are situated on the same ray* R *starting from $\mathfrak{o}$ and b) satisfy* $0 \leqslant \operatorname{Dist}(w\,,\,\partial\mathrm{K}) \leqslant \operatorname{Dist}(v\,,\,\partial\mathrm{K}) < \frac{1}{2}\,c_{\mathrm{K}}$. *Then $f^{(qc)}$ obeys the following $\varepsilon$-$\delta$ estimate, which holds uniformly for all rays starting from the origin:*

$$\operatorname{Dist}(w\,,\,v) \;\leqslant\; \delta(\varepsilon)\cdot\frac{c_{\mathrm{K}}}{6\,C_{\mathrm{K}}} \;\;\Longrightarrow\;\; f^{(qc)}(w) - f^{(qc)}(v) \;\geqslant\; -2\,\varepsilon\,. \tag{2.13}$$

$c_{\mathrm{K}}$ *and $C_{\mathrm{K}}$ are the quantities defined in the beginning of Section 2.b).*

### d) The lower semicontinuous quasiconvex envelope $f^{(qc)}(t,\xi,v)$ for $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$.

**Theorem 2.10.** **(Properties of $f^{(qc)}$ for $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$)** *Let $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$ be given. Then for every fixed $(\hat t,\hat\xi) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$ it holds that*

*1) $f^{c}(\hat t,\hat\xi,v) \;\leqslant\; f^{(qc)}(\hat t,\hat\xi,v) \;\leqslant\; f(\hat t,\hat\xi,v)$ for all $v \in \mathbb{R}^{nm}$, which implies particularly $f^{(qc)}(\hat t,\hat\xi,v) = (+\infty)$ for all $v \in \mathbb{R}^{nm} \setminus \mathrm{K}$ and $f^{(qc)}(\hat t,\hat\xi,v) = f(\hat t,\hat\xi,v)$ for all $v \in \operatorname{ext}(\mathrm{K})$.*

*2) $f^{(qc)}(\hat t,\hat\xi,v)\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is the largest lower semicontinuous, quasiconvex function below $f(\hat t,\hat\xi,v)$.*

*3) $f^{(qc)}(\hat t,\hat\xi,v)$ admits the representation*

$$f^{(qc)}(\hat t,\hat\xi,v_0) = \begin{cases} \hfill f^{*}(\hat t,\hat\xi,v_0) \;\big|\; v_0 \in \operatorname{int}(\mathrm{K})\,; \\[1ex] \displaystyle\lim_{v \to v_0\,,\; v \in \mathrm{R} \cap \operatorname{int}(\mathrm{K})} f^{*}(\hat t,\hat\xi,v) \;\big|\; v_0 \in \partial\mathrm{K}\,; \\[1ex] \hfill (+\infty) \;\big|\; v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K} \end{cases} \tag{2.14}$$

*where $f^{*}(\hat t,\hat\xi,v)$ is defined through*

$$f^{*}(\hat t,\hat\xi,v) \;=\; \inf\Big\{\,\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\hat t,\hat\xi,v + Jx(t))\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\;\; v + Jx(t) \in \mathrm{K}\;\;(\forall)\,t \in \Omega\,\Big\}\,. \tag{2.15}$$

*4) Let a $k$-dimensional face $\varPhi \subseteq$ K, $0 \leqslant k \leqslant nm$, be given. Assume that the uniform continuity of $f(\hat t,\hat\xi,v)$ on* K *is described through the $\varepsilon$-$\delta$ relation*

$$\big|\, v' - v'' \,\big| \;\leqslant\; \delta(\varepsilon) \;<\; 1 \;\Longrightarrow\; \big|\, f(\hat t,\hat\xi,v') - f(\hat t,\hat\xi,v'') \,\big| \leqslant \varepsilon \quad \forall\, v',\, v'' \in \mathrm{K}\,. \tag{2.16}$$

*Then $f^{(qc)}(\hat t,\hat\xi,v)\,\big|\,\varPhi$ obeys the following $\varepsilon$-$\delta$ estimate:*

$$\begin{aligned} \big|\, v' - v'' \,\big| \;\leqslant\; \frac{\delta(\varepsilon)}{4\,C_{\mathrm{K}}}\cdot \operatorname{Min}\big(\,1\,,\,\operatorname{Dist}(v',\,\operatorname{rb}(\varPhi))\,,\,\operatorname{Dist}(v'',\,\operatorname{rb}(\varPhi))\,\big) \;\Longrightarrow& \\ \big|\, f^{(qc)}(\hat t,\hat\xi,v') - f^{(qc)}(\hat t,\hat\xi,v'') \,\big| \;\leqslant\; 2\,\varepsilon \quad \forall\, v',\, v'' \in \operatorname{ri}(\varPhi)& \end{aligned} \tag{2.17}$$

*with $C_{\mathrm{K}}$ from Section 2.b). In particular, $f^{(qc)}(\hat t,\hat\xi,v)\,\big|\operatorname{int}(\mathrm{K})$ is continuous, and $f^{(qc)}(\hat t,\hat\xi,v)\,\big|\,(1-\gamma)\,\mathrm{K}$ is uniformly continuous for every $0 < \gamma < 1$.*

---

[32] [Wagner 06c], p. 22, Theorem 3.12., together with Theorem 2.7. above.

*5) Assume that the uniform continuity of* $f(\hat{t},\hat{\xi},v)$ *on* K *is described by the* $\varepsilon$*-*$\delta$ *relation from Part 4). If two points* $v,\ w \in \mathrm{K}$ *a) are situated on the same ray* R *starting from the origin and b) satisfy* $0 \leqslant \mathrm{Dist}\,(w\,,\,\partial\mathrm{K}) \leqslant \mathrm{Dist}\,(v\,,\,\partial\mathrm{K}) < \frac{1}{2}\,c_{\mathrm{K}}$ *then the* $\varepsilon$*-*$\delta$ *estimate*

$$\mathrm{Dist}\,(w\,,\,v) \;\leqslant\; \delta(\varepsilon)\cdot\frac{c_{\mathrm{K}}}{6\,C_{\mathrm{K}}} \quad\Longrightarrow\quad f^{(qc)}(\hat{t},\hat{\xi},w) - f^{(qc)}(\hat{t},\hat{\xi},v) \;\geqslant\; -2\,\varepsilon \tag{2.18}$$

*holds. Here* $c_{\mathrm{K}}$ *and* $C_{\mathrm{K}}$ *are the quantities from Section 2.b), and the estimate is the same for all rays* R *starting form the origin.*

**Proof. 1) – 3)** If a function $f(t,\xi,v) \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$ is given then, in consequence of Definition 1.1., 2), the function $f(\hat{t},\hat{\xi},v)\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to $\mathfrak{F}_{\mathrm{K}}$ for every fixed $(\hat{t},\hat{\xi}) \in \big(\,\Omega\setminus \mathrm{N}\,\big)\times\mathbb{R}^n$. Thus Parts 1) – 3) result from the remarks after Definition 2.6. and the theorems from [ Wagner 06c ] cited there.

**4) – 5)** For every fixed $(\hat{t},\hat{\xi}) \in \big(\,\Omega\setminus \mathrm{N}\,\big)\times\mathbb{R}^n$, the function $f(\hat{t},\hat{\xi},v)\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is uniformly continuous on K. Consequently, 4) and 5) will be implied by Theorems 2.7., 2.8. and 2.9. ∎

**Theorem 2.11. (Generalization of Theorem 2.8. for** $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$**)** *Let a function* $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$ *and compact subsets* $\Omega_c \subseteq \Omega$ *and* $\mathrm{A}_c \subset \mathbb{R}^n$ *be given such that the restriction* $f\,\big|\,\big(\,\Omega_c\times\mathrm{A}_c\times\mathrm{K}\big)$ *is continuous with respect to* $(t,\xi,v)$*. Assume that this (uniform) continuity may be described by the* $\varepsilon$*-*$\delta$ *relation*

$$\begin{aligned}&\big|\,t'-t''\,\big| + \big|\,\xi'-\xi''\,\big| + \big|\,v'-v''\,\big| \;\leqslant\; \delta_0(\varepsilon) \;<\; 1 \\ &\qquad\Longrightarrow\ \ \big|\,f(t',\xi',v') - f(t'',\xi'',v'')\,\big| \;\leqslant\; \varepsilon \quad \forall\,(t',\xi',v'),\ (t'',\xi'',v'') \in \big(\,\Omega_c\,\times\mathrm{A}_c\,\times\,\mathrm{K}\,\big)\,.\end{aligned} \tag{2.19}$$

*1) Then the restriction* $f^{(qc)}(t,\xi,v)\,\big|\,\big(\,\Omega_c\times\mathrm{A}_c\times\mathrm{int}\,(\mathrm{K})\,\big)$ *obeys the following continuity relation with respect to* $(t,\xi,v)$*:*

$$\begin{aligned}&\big|\,t'-t''\,\big| + \big|\,\xi'-\xi''\,\big| + \big|\,v'-v''\,\big| \;\leqslant\; \frac{\delta_0(\varepsilon)}{4\,C_{\mathrm{K}}}\cdot\mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(v',\,\partial\mathrm{K})\,,\,\mathrm{Dist}\,(v'',\,\partial\mathrm{K})\,\big) \;\Longrightarrow \\ &\qquad \big|\,f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi'',v'')\,\big| \;\leqslant\; 6\,\varepsilon \quad \forall\,(t',\xi',v'),\ (t'',\xi'',v'') \in \big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times\,\mathrm{int}\,(\mathrm{K})\,\big)\,.\end{aligned} \tag{2.20}$$

*2) For every* $0<\gamma<1$*, the restriction* $f^{(qc)}(t,\xi,v)\,\big|\,\big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times(1-\gamma)\,\mathrm{K}\big)$ *is uniformly continuous with respect to* $(t,\xi,v)$*.*

**Proof. 1)** For arbitrary $(t',\xi',v'),\ (t'',\xi'',v'') \in \big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times\,\mathrm{int}\,(\mathrm{K})\,\big)$, it holds that

$$\big|\,f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi'',v'')\,\big| \;\leqslant\; D_1 \,+\, D_2 \,+\, D_3 \quad \text{with} \tag{2.21}$$

$$D_1 \;=\; \big|\,f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi',v')\,\big|\,; \tag{2.22}$$

$$D_2 \;=\; \big|\,f^{(qc)}(t'',\xi',v') - f^{(qc)}(t'',\xi'',v')\,\big|\,; \tag{2.23}$$

$$D_3 \;=\; \big|\,f^{(qc)}(t'',\xi'',v') - f^{(qc)}(t'',\xi'',v'')\,\big|\,. \tag{2.24}$$

Fixing now $\varepsilon>0$, we find $x_1 \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ with (2.25)

$$f^{(qc)}(t',\xi',v') \;\leqslant\; \frac{1}{|\,\Omega\,|}\int_\Omega f(t',\xi',v'+Jx_1(t)\,)\,dt \;\leqslant\; f^{(qc)}(t',\xi',v')+\varepsilon \ \ \text{and}\ \ v'+Jx_1(t) \in \mathrm{int}\,(\mathrm{K})\ \ (\forall)\,t\in\Omega$$

(cf. [ Wagner 06b ] , p. 21, Theorem 3.4., 2), and [ Wagner 06c ] , p. 15, Theorem 3.4., 2) ). Then from the continuity relation (2.19) it follows that

$$\begin{aligned}&\big|\,t'-t''\,\big| \;\leqslant\; \delta_0(\varepsilon) \;\Longrightarrow \\ &\frac{1}{|\,\Omega\,|}\int_\Omega\Big(\,f(t',\xi',v'+Jx_1(t)\,) - f(t'',\xi',v'+Jx_1(t)\,)\Big)\,dt \,+\, \frac{1}{|\,\Omega\,|}\int_\Omega f(t'',\xi',v'+Jx_1(t)\,)\,dt \\ &\qquad\qquad\qquad\qquad\qquad\qquad \leqslant\; f^{(qc)}(t',\xi',v')+\varepsilon \;\Longrightarrow\end{aligned} \tag{2.26}$$

$$-\,\varepsilon + f^{(qc)}(t'',\xi',v') \;\leqslant\; -\varepsilon \,+\, \frac{1}{|\,\Omega\,|}\int_\Omega f(t'',\xi',v'+Jx_1(t)\,)\,dt \;\leqslant\; f^{(qc)}(t',\xi',v')+\varepsilon \;\Longrightarrow \tag{2.27}$$

$$f^{(qc)}(t'',\xi',v') - f^{(qc)}(t',\xi',v') \;\leqslant\; 2\,\varepsilon\,. \tag{2.28}$$

After exchanging the roles of $t'$ and $t''$, we get analogously

$$f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi',v') \;\leqslant\; 2\,\varepsilon \tag{2.29}$$

and together

$$D_1 \;=\; \big|\, f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi',v') \,\big| \;\leqslant\; 2\,\varepsilon\,. \tag{2.30}$$

Further, we may choose $x_2 \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ with

$$f^{(qc)}(t'',\xi',v') \;\leqslant\; \frac{1}{|\,\Omega\,|}\int_\Omega f(t'',\xi',v'+Jx_2(t)\,)\,dt \;\leqslant\; f^{(qc)}(t'',\xi',v')+\varepsilon \tag{2.31}$$
$$\text{and}\quad v'+Jx_2(t)\in \operatorname{int}(\mathrm{K})\;\;(\forall)\,t\in\Omega\,,$$

which implies together with the continuity relation (2.19):

$$\big|\,\xi'-\xi''\,\big| \;\leqslant\; \delta_0(\varepsilon) \;\Longrightarrow$$
$$\frac{1}{|\,\Omega\,|}\int_\Omega\Big(f(t'',\xi',v'+Jx_2(t)\,)-f(t'',\xi'',v'+Jx_2(t)\,)\Big)\,dt \;+\; \frac{1}{|\,\Omega\,|}\int_\Omega f(t'',\xi'',v'+Jx_2(t)\,)\,dt$$
$$\leqslant\; f^{(qc)}(t'',\xi',v')+\varepsilon \;\Longrightarrow \tag{2.32}$$
$$-\,\varepsilon+f^{(qc)}(t'',\xi'',v') \;\leqslant\; -\varepsilon\,+\,\frac{1}{|\,\Omega\,|}\int_\Omega f(t'',\xi'',v'+Jx_2(t)\,)\,dt \;\leqslant\; f^{(qc)}(t'',\xi',v')+\varepsilon \;\Longrightarrow \tag{2.33}$$
$$f^{(qc)}(t'',\xi'',v') - f^{(qc)}(t'',\xi',v') \;\leqslant\; 2\,\varepsilon\,. \tag{2.34}$$

After exchanging the roles of $\xi'$ and $\xi''$, we get

$$f^{(qc)}(t'',\xi',v') - f^{(qc)}(t'',\xi'',v') \;\leqslant\; 2\,\varepsilon \tag{2.35}$$

as well. Together we find

$$D_2 \;=\; \big|\, f^{(qc)}(t'',\xi',v') - f^{(qc)}(t'',\xi'',v') \,\big| \;\leqslant\; 2\,\varepsilon\,. \tag{2.36}$$

In order to estimate $D_3$, we apply Theorem 2.10., 4). Summing up, we arrive at the following $\varepsilon$-$\delta$ relation:

$$\big|\,t'-t''\,\big| \,+\, \big|\,\xi'-\xi''\,\big| \,+\, \big|\,v'-v''\,\big| \;\leqslant\; \frac{\delta_0(\varepsilon)}{4\,C_{\mathrm{K}}}\cdot \operatorname{Min}\big(\,1\,,\,\operatorname{Dist}(v',\,\partial\mathrm{K})\,,\,\operatorname{Dist}(v'',\,\partial\mathrm{K})\,\big) \;\Longrightarrow \tag{2.37}$$
$$\big|\, f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi'',v'') \,\big| \;\leqslant\; 6\,\varepsilon \quad \forall\,(t',\xi',v'),\,(t'',\xi'',v'') \in \big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times\,\operatorname{int}(\mathrm{K})\,\big)\,.$$

In analogy to [Wagner 06c], p. 17, Theorem 3.6., 1), (2.37) implies the continuity of $f^{(qc)}(t,\xi,v)$ with respect to $(t,\xi,v)$ on $\big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times\,\operatorname{int}(\mathrm{K})\,\big)$.

**2)** Let $0<\gamma<1$ be fixed. On $\big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times(1-\gamma)\,\mathrm{K}\big)$, we have

$$\operatorname{Min}\big(\,1\,,\,\operatorname{Dist}(v',\,\partial\mathrm{K})\,,\,\operatorname{Dist}(v'',\,\partial\mathrm{K})\,\big) \;\geqslant\; \operatorname{Min}\big(\,1\,,\,\operatorname{Dist}(\,(1-\gamma)\,\mathrm{K}\,,\,\partial\mathrm{K})\,\big)\,, \tag{2.38}$$

and (2.20) becomes a uniform continuity relation on this set. ■

**Theorem 2.12. (Generalization of Theorem 2.9. for $f\in\widetilde{\mathfrak{F}}_{\mathrm{K}}$)** *Let a function $f\in\widetilde{\mathfrak{F}}_{\mathrm{K}}$ and compact subsets $\Omega_c\subseteq\Omega$ and $\mathrm{A}_c\subset\mathbb{R}^n$ be given such that the restriction $f\,\big|\,\big(\,\Omega_c\,\times\,\mathrm{A}_c\,\times\,\mathrm{K}\,\big)$ is continuous with respect to $(t,\xi,v)$. Assume that the uniform continuity relation (2.19) holds. If the points $v,\,w\in\mathrm{K}$ a) are* *situated on the same ray* R *starting from* $\mathfrak{o}$ *and b) satisfy* $0 \leqslant \mathrm{Dist}\,(w\,,\,\partial\mathrm{K}) \leqslant \mathrm{Dist}\,(v\,,\,\partial\mathrm{K}) < \frac{1}{2}\,c_{\mathrm{K}}$ *then the* $\varepsilon$-$\delta$ *estimate*

$$\mathrm{Dist}\,(w\,,\,v) \;\leqslant\; \delta_0(\varepsilon)\cdot\frac{c_{\mathrm{K}}}{6\,C_{\mathrm{K}}} \quad\Longrightarrow\quad f^{(qc)}(\hat{t},\hat{\xi},w) - f^{(qc)}(\hat{t},\hat{\xi},v) \;\geqslant\; -2\,\varepsilon \tag{2.39}$$

*holds. In particular, the estimate is the same for all rays* R *starting form the origin and all* $(\hat{t},\hat{\xi}) \in \Omega_c \times \mathrm{A}_c$.

**Proof.** The estimate from Theorem 2.10., 5) does not depend on the choice of $(\hat{t},\hat{\xi}) \in \Omega_c \,\times\, \mathrm{A}_c$. ■

**Theorem 2.13. (Growth condition for** $f^{(qc)}$**)** *Let a function* $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$ *and a compact subset* $\mathrm{A}_c \subset \mathbb{R}^n$ *be given. The the function* $f^{(qc)}$*, which is formed with respect to the variable* $v$*, satisfies the growth condition*

$$\big|\,f^{(qc)}(t,\xi,v)\,\big| \;\leqslant\; A(t) \,+\, C_2 \tag{2.40}$$

*for all* $(t,\xi,v) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathrm{A}_c \times \mathrm{K}$. *A is the same function as in the growth condition for f from Definition 1.1., 2).*

**Proof.** From the growth condition in Definition 1.1., 2), Theorem 2.10., 1) and the representation theorem for the convex envelope (cf. [DACOROGNA 08], p. 52, Theorem 2.35.), we deduce for arbitrary $(\hat{t},\hat{\xi},v) \in \big(\Omega \,\setminus\, \mathrm{N}\big) \,\times\, \mathrm{A}_c \,\times\, \mathrm{K}$:

$$\begin{aligned}
&A(\hat{t}) + C_2 \;\geqslant\; A(\hat{t}) + B(\hat{\xi},v) \;\geqslant\; f(\hat{t},\hat{\xi},v) \;\geqslant\; f^{(qc)}(\hat{t},\hat{\xi},v) \;\geqslant\; f^{c}(\hat{t},\hat{\xi},v)\\
&= \inf\big\{\sum_{s=1}^{nm+1}\lambda_s\,f(\hat{t},\hat{\xi},v_s)\;\big|\;\sum_s\lambda_s = 1\,,\;\sum_s\lambda_s\,v_s = v\,,\;0 \leqslant \lambda_s \leqslant 1\,,\;v_s \in \mathrm{K}\,,\;1 \leqslant s \leqslant nm+1\big\}\\
&\geqslant \inf\big\{-\sum_{s=1}^{nm+1}\lambda_s\cdot|\,f(\hat{t},\hat{\xi},v_s)\,|\;\big|\;...\,\big\} \;\geqslant\; \inf\big\{\sum_{s=1}^{nm+1}\lambda_s\,\big(-A(\hat{t}) - B(\hat{\xi},v_s)\big)\;\big|\;...\,\big\} \;\geqslant\; -A(\hat{t}) - C_2
\end{aligned} \tag{2.41}$$

and, consequently, $|\,f^{(qc)}(t,\xi,v)\,| \;\leqslant\; A(t) \,+\, C_2$ for all $(t,\xi,v) \in \big(\Omega \,\setminus\, \mathrm{N}\big) \,\times\, \mathrm{A}_c \,\times\, \mathrm{K}$. ■

**Theorem 2.14.**[33] *Consider a function* $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$ *and compact subsets* $\Omega_c \subseteq \Omega$ *and* $\mathrm{A}_c \subset \mathbb{R}^n$ *such that the restriction* $f\,\big|\,\big(\Omega_c \,\times\, \mathrm{A}_c \,\times\, \mathrm{K}\big)$ *is continuous with respect to* $(t,\xi,v)$. *Assume further that* $\Omega_a \subset \Omega$ *is open.*

*1) Let functions* $x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *with* $x(t) \in \mathrm{A}_c \;\forall\, t \in \Omega_c$ *and* $u \in L^\infty(\Omega,\mathbb{R}^{nm})$ *with* $u(t) \in \mathrm{K}\;(\forall)\,t \in \Omega$ *be given. Then for every* $\varepsilon > 0$*, we may find an index* $K_0 \in \mathbb{N}$ *with*

$$\Big|\int_{\Omega_a\,\cap\,\Omega_c}\Big(f^{(qc)}(t,x(t),u(t)) \,-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt\,\Big| \;\leqslant\; 7\,|\,\Omega_a \,\cap\, \Omega_c\,|\,\varepsilon \quad \forall\,K \geqslant K_0(\varepsilon)\,. \tag{2.42}$$

*2) For every* $\varepsilon > 0$*, we may find an index* $K_1 \in \mathbb{N}$ *such that for arbitrary functions* $x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *with* $x(t) \in \mathrm{A}_c \;\forall\, t \in \Omega_c$ *and* $u \in L^\infty(\Omega,\mathbb{R}^{nm})$ *with* $u(t) \in \mathrm{K}\;(\forall)\,t \in \Omega$ *the following estimate holds:*

$$\int_{\Omega_c}\Big(f^{(qc)}(t,x(t),u(t)) \,-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt \;\geqslant\; -8\,|\,\Omega_c\,|\,\varepsilon \quad \forall\,K \geqslant K_1(\varepsilon)\,. \tag{2.43}$$

$K_1$ *does not depend on* $x$ *and* $u$ *but on* $\Omega_c$ *only.*

**Proof. 1)** Obviously, it holds that

$$\begin{aligned}
&\Big|\int_{\Omega_a\,\cap\,\Omega_c}\Big(f^{(qc)}(t,x(t),u(t)) \,-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt\,\Big|\\
&\leqslant \Big|\int_{\Omega_a\,\cap\,\Omega_c}\Big(f^{(qc)}(t,x(t),u(t)) \,-\, f^{(qc)}(t,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt\,\Big| + \Big|\int_{\Omega_a\,\cap\,\Omega_c}\Big(f^{(qc)}(t,x(t),\frac{K-1}{K}\,u(t))\\
&-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt\,\Big|\,.
\end{aligned} \tag{2.44}$$

---

[33] Generalization of [WAGNER 07B], p. 12, Lemma 3.6.

In consequence of Theorem Satz 2.13., we may apply Lebesgue's dominated convergence theorem to the first member, which results in

$$\Big| \int_{\Omega_a \,\cap\, \Omega_c} \Big( f^{(qc)}(t, x(t), u(t)) \,-\, f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt \Big| \,\leqslant\, |\,\Omega_a \,\cap\, \Omega_c\,|\, \varepsilon \tag{2.45}$$

if $K \geqslant K_0'(\varepsilon)$. Assume that the uniform continuity of the function $f(t,\xi,v)$ on the compact set $\big(\,\Omega_c \times \mathrm{A}_c \times \mathrm{K}\,\big)$ is described again by the $\varepsilon$-$\delta$ relation (2.19). By Theorem 2.11., 2), we get from this relation a uniform continuity relation for für $f^{(qc)}(t,\xi,v)\,\big|\,\big(\,\Omega_c \,\times\, \mathrm{A}_c \,\times\, \frac{K-1}{K}\,\mathrm{K}\,\big)$:

$$\begin{gathered} |\,t'-t''\,| \,+\, |\,\xi'-\xi''\,| \,+\, |\,v'-v''\,| \,\leqslant\, \delta_1(\varepsilon) \,=\, \frac{\delta_0(\varepsilon)}{4\,C_{\mathrm{K}}} \cdot \mathrm{Min}\,\big(\,1\,,\, \frac{c_{\mathrm{K}}}{K}\,\big) \;\Longrightarrow \\ \big|\, f^{(qc)}(t',\xi',v') - f^{(qc)}(t'',\xi'',v'')\,\big| \,\leqslant\, 6\,\varepsilon \quad \forall\,(t',\xi',v'),\,(t'',\xi'',v'') \in \big(\,\Omega_c \,\times\, \mathrm{A}_c \,\times\, \frac{K-1}{K}\,\mathrm{K}\,\big)\,. \end{gathered} \tag{2.46}$$

If we choose $K \geqslant K_0''(\varepsilon)$ with $\mathrm{Diam}\,(\mathrm{A}_c)/K_0''(\varepsilon) \leqslant \delta_1(\varepsilon)$ then (2.46) implies the following estimate for the second member in (2.44):

$$\begin{aligned} &\Big| \int_{\Omega_a \,\cap\, \Omega_c} \Big( f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \,-\, f^{(qc)}(t, \frac{K-1}{K}\, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt \Big| \\ &\leqslant \int_{\Omega_a \,\cap\, \Omega_c} \Big| f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \,-\, f^{(qc)}(t, \frac{K-1}{K}\, x(t), \frac{K-1}{K}\, u(t)) \Big|\, dt \,\leqslant\, 6\,|\,\Omega_a \,\cap\, \Omega_c\,|\,\varepsilon\,. \end{aligned} \tag{2.47}$$

For $K_0(\varepsilon) = \mathrm{Max}\,\big(\,K_0'(\varepsilon), K_0''(\varepsilon)\,\big)$, the claimed inequality results from (2.45) and (2.47).

**2)** Let us decompose:

$$\begin{aligned} &\int_{\Omega_c} \Big( f^{(qc)}(t, x(t), u(t)) \,-\, f^{(qc)}(t, \frac{K-1}{K}\, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt \\ &= \int_{\Omega_c} \Big( f^{(qc)}(t, x(t), u(t)) \,-\, f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt \,+\, \int_{\Omega_c} \Big( f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \\ &\quad -\, f^{(qc)}(t, \frac{K-1}{K}\, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt\,. \end{aligned} \tag{2.48}$$

From the uniform continuity relation (2.19) for $f(t,\xi,v)\,\big|\,\big(\,\Omega_c \,\times\, \mathrm{A}_c \,\times\, \mathrm{K}\,\big)$ and Theorem 2.12. we deduce that for

$$\mathrm{Dist}\,\big(\,u(t)\,,\, \frac{K-1}{K}\, u(t)\,\big) \,=\, \frac{|\,u(t)\,|}{K} \,\leqslant\, \frac{C_{\mathrm{K}}}{K} \,\leqslant\, \delta_0(\varepsilon) \cdot \frac{c_{\mathrm{K}}}{6\,C_{\mathrm{K}}}\,, \tag{2.49}$$

i. e., for all $K \in \mathbb{N}$ with

$$\frac{1}{K} \,\leqslant\, \frac{1}{K_1'(\varepsilon)} \,\leqslant\, \delta_0(\varepsilon) \cdot \frac{c_{\mathrm{K}}}{6\,(C_{\mathrm{K}})^2}\,, \tag{2.50}$$

the estimate

$$f^{(qc)}(t, x(t), u(t)) \,-\, f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \,\geqslant\, -2\,\varepsilon \tag{2.51}$$

holds for all $t \in \Omega_c$. From (2.51), we obtain

$$\int_{\Omega_c} \Big( f^{(qc)}(t, x(t), u(t)) \,-\, f^{(qc)}(t, x(t), \frac{K-1}{K}\, u(t)) \Big)\, dt \,\geqslant\, -2\,|\,\Omega_c\,|\,\varepsilon\,. \tag{2.52}$$

If $K \geqslant K_1''(\varepsilon)$ with $\mathrm{Diam}\,(\mathrm{A}_c)/K_1''(\varepsilon) \leqslant \delta_1(\varepsilon)$ then we get from the uniform continuity relation (2.46) for $f^{(qc)}(t,\xi,v)\,\big|\,\big(\Omega_c \times \mathrm{A}_c \times \frac{K-1}{K}\,\mathrm{K}\big)$:

$$\begin{aligned}&\int_{\Omega_c}\Big(f^{(qc)}(t,x(t),\frac{K-1}{K}\,u(t)) \,-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\Big)\,dt\\ &\geqslant -\int_{\Omega_c}\Big|\,f^{(qc)}(t,x(t),\frac{K-1}{K}\,u(t)) \,-\, f^{(qc)}(t,\frac{K-1}{K}\,x(t),\frac{K-1}{K}\,u(t))\,\Big|\,dt \,\geqslant\, -6\,|\,\Omega_c\,|\,\varepsilon\,.\end{aligned} \tag{2.53}$$

Combining (2.52) and (2.53), we arrive at the claimed inequality with $K \geqslant K_1(\varepsilon) = \mathrm{Max}\,\big(K_1'(\varepsilon), K_1''(\varepsilon)\big)$. ■

## 3. The relaxation theorem for problems (P) with integrands $f(t,\xi,v)$.

### a) Proof of Theorem 1.2.

We start with the following lemma:

**Lemma 3.1.**[34] *The feasible domain $\mathcal{B}$ of* (P) *is convex and bounded in $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$-norm.*

**Proof.** Together with K, $\mathcal{B}$ is convex. The boundedness of $\mathcal{B}$ follows from the equivalence of the norms $\|\,x\,\|_1 = \|\,x\,\|_{L^\infty(\Omega,\mathbb{R}^n)} + \|\,Jx\,\|_{L^\infty(\Omega,\mathbb{R}^{nm})}$ and $\|\,x\,\|_2 = \|\,Jx\,\|_{L^\infty(\Omega,\mathbb{R}^{nm})}$ on $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ (cf. [DACOROGNA 04], p. 37, Theorem 1.47). ■

Together with the growth condition d) from Definition 1.1., 2), Lemma 3.1. implies the boundedness of $F$ on $\mathcal{B}$. Consequently, (P) admits a finite minimal value $m$. Consider a minimizing sequence $\{\,x^N\,\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ of (P). The $L^\infty$-norm bounded sequences $\{\,x^N\,\}$ and $\{\,Jx^N\,\}$ must contain weakly$^*$-convergent subsequences $\{\,x^{N'}\,\} \overset{*}{\rightharpoonup}_{L^\infty(\Omega,\mathbb{R}^n)} \hat{x}$ resp. $\{\,Jx^{N'}\,\} \overset{*}{\rightharpoonup}_{L^\infty(\Omega,\mathbb{R}^{nm})} \hat{y}$ with $\hat{y} = J\hat{x}$. [DACOROGNA 04], p. 36, Corollary 1.45, implies the norm convergence $x^{N'} \rightarrow_{L^\infty(\Omega,\mathbb{R}^n)} \hat{x}$ and even the uniform convergence $x^{N'} \rightarrow_{C^0(\Omega,\mathbb{R}^n)} \hat{x}$ since the functions are continuous. By [DUNFORD/SCHWARTZ 88], p. 429, Theorem 7, the convex, bounded, norm-closed set $\{\,z \in L^\infty(\Omega,\mathbb{R}^{nm})\;\big|\; z(t) \in \mathrm{K}\;\;(\forall)\,t \in \Omega\,\}$ is weak$^*$-closed as well, and the feasibility of $\hat{x} \in \mathcal{B}$ results. From assumption b) it follows that

$$F^{\#}(x^{N'}) \,\leqslant\, F(x^{N'}) \quad \forall\, N' \in \mathbb{N}, \tag{3.1}$$

and with c) we obtain

$$F^{\#}(\hat{x}) \,\leqslant\, \liminf_{N'\to\infty} F^{\#}(x^{N'}) \,\leqslant\, \liminf_{N'\to\infty} F(x^{N'}) \,=\, \lim_{N\to\infty} F(x^N) \,=\, m\,. \tag{3.2}$$

Finally, if we denote the minimal value of $(\mathrm{P})^{\#}$ by $m^{\#}$ then d) implies

$$m^{\#} \,\leqslant\, F^{\#}(\hat{x}) \,\leqslant\, m \,=\, m^{\#}\,, \tag{3.3}$$

and $\hat{x}$ turns out to be a global minimizer of $(\mathrm{P})^{\#}$. The proof of Theorem 1.2. is complete. ■

### b) Proof of the relaxation theorem 1.4.

**Sketch of the proof.** We have to prove that the lower semicontinuous quasiconvex envelope $f^{(qc)}$ of $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$, which is formed with respect to the variable $v$, obeys the conditions a) – d) from Theorem 1.2. We prove

---

[34] Cf. [PICKENHAIN/WAGNER 00], p. 222, Lemma 2.1.

the fulfillment of a) and b) in Proposition 3.2., the lower semicontinuity of the relaxed objective functional $F^{(qc)}$ in Proposition 3.3. and the coincidence of the minimal values of (P) and $(\mathrm{P})^{(qc)}$ in Proposition 3.8.

**Proposition 3.2. ($f^{(qc)}$ satisfies the conditions a) and b) from Theorem 1.2.)** *Consider the problem* (P) *under the assumptions of Theorem 1.4. Then the function* $f^{(qc)}$*, which is defined for* $(\hat{t},\hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$ *as the lower semicontinuous quasiconvex envelope of* $f$ *with respect to* $v$ *and for* $(\hat{t},\hat{\xi}) \in \mathrm{N} \times \mathbb{R}^n$ *by zero, possesses the properties a) and b) from Theorem 1.2.*

**Proof.** For fixed $(\hat{t},\hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$, $f^{(qc)}(\hat{t},\hat{\xi},\,\cdot\,)$ possesses the effective domain K by Theorem 2.10., 1). Due to Theorem 2.10., 2), the restriction $f^{(qc)}(\hat{t},\hat{\xi},\,\cdot\,)\,\big|\,\mathrm{K}$ to the compact set K is lower semicontinuous and, consequently, measurable. The boundedness from below on K can be confirmed analogously to the proof of Theorem 2.13., and condition a) is satisfied. In consequence of the inequality

$$f^{(qc)}(\hat{t},\hat{\xi},v) \leqslant f(\hat{t},\hat{\xi},v) \quad \forall\, v \in \mathbb{R}^{nm} \tag{3.4}$$

from Theorem 2.10., 1), condition b) is satisfied as well. ■

**Proposition 3.3. (Lower semicontinuity of the functional $F^{(qc)}(\,\cdot\,)$)** *Consider again the problem* (P) *under the assumptions of Theorem 1.4. Then for every sequence* $\{\, x^N \,\}$ *of admissible functions for* (P)*, from* $x^N \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^n)} \hat{x}$ *and* $Jx^N \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^{nm})} J\hat{x}$ *it follows that*

$$F^{(qc)}(\hat{x}) \;=\; \int_\Omega f^{(qc)}(t,\hat{x}(t),J\hat{x}(t))\,dt \;\leqslant\; \liminf_{N\to\infty} \int_\Omega f^{(qc)}(t,x^N(t),Jx^N(t))\,dt \;=\; \liminf_{N\to\infty} F^{(qc)}(x^N)\,. \tag{3.5}$$

**Proof.** The proof of Proposition 3.3. will be divided into eight steps. .

• **Step 1.** *Application of the Scorza-Dragoni theorem to* $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$. $\mathcal{B}$ denotes again the feasible domain of (P). From Lemma 3.1. we deduce that

$$C_1 \;=\; \sup\big\{\, |\,x(t)\,| \;\big|\; x \in \mathcal{B}\,\big\} \;<\; (+\infty)\,. \tag{3.6}$$

Then from the growth condition d) in Definition 1.1., 2) it follows that

$$C_2 \;=\; \sup\big\{\, B(\xi,v) \;\big|\; |\,\xi\,| \leqslant C_1\,,\; |\,v\,| \leqslant C_{\mathrm{K}}\,\big\} \;<\; (+\infty)\,. \tag{3.7}$$

Now we fix $\varepsilon > 0$. Then, in relation to the integrable function $A$ from the growth condition, we may choose a sufficiently large number $C_3 \geqslant 1$ such that the set

$$\Omega_a \;=\; \big\{\, t \in \mathrm{int}\,(\Omega) \;\big|\; A(t) < C_3\,\big\} \tag{3.8}$$

satisfies

$$\big|\,\Omega \setminus \Omega_a\,\big| \;\leqslant\; \varepsilon/C_2 \quad \text{as well as} \quad \int_{\Omega\,\setminus\,\Omega_a} A(t)\,dt \;\leqslant\; \varepsilon\,. \tag{3.9}$$

In view of Lemma 3.1., for the proof of the lower semicontinuity of the cost functional it suffices to deal with the restriction of the integrand $f$ to the set $\Omega \times \mathrm{A}_c \times \mathrm{K}$ where $\mathrm{A}_c = \mathrm{K}(\mathfrak{o},C_1) \subset \mathbb{R}^n$.[35] Thus we apply the Scorza-Dragoni theorem (Theorem 1.7.) to the restriction $f\,\big|\,\big(\Omega \times \mathrm{A}_c \times \mathrm{K}\big)$ and find a compact subset $\Omega_c \subseteq \Omega$ with

---

[35] Cf. also [Marcellini/Sbordone 80], p. 251, Corollary 3.12.

$$\big|\,\Omega\setminus\Omega_c\,\big| \leqslant \varepsilon/(C_2+C_3) \tag{3.10}$$

such that the further restriction $f\,\big|\,\big(\Omega_c\times\mathrm{A}_c\times\mathrm{K}\big)$ is continuous with respect to $(t,\xi,v)$. Since $\big(\Omega_c\times\mathrm{A}_c\times\mathrm{K}\big)\subset\Omega\times\mathbb{R}^n\times\mathrm{K}$ is compact, this restriction obeys a uniform continuity relation, which may be stated as

$$\begin{aligned}&\big|\,t'-t''\,\big| + \big|\,\xi'-\xi''\,\big| + \big|\,v'-v''\,\big| \leqslant \delta_2(\varepsilon) \leqslant \delta_0(\varepsilon)\cdot\mathrm{Min}\Big(1\,,\ \frac{1}{3\,(C_1+C_{\mathrm{K}})}\Big) \quad\Longrightarrow \\ &\qquad\big|\,f(t',\xi',v')-f(t'',\xi'',v'')\,\big| \leqslant \varepsilon \quad \forall\,(t',\xi',v'),\,(t'',\xi'',v'')\in\big(\Omega_c\times\mathrm{A}_c\times\mathrm{K}\big)\,.\end{aligned} \tag{3.11}$$

In addition, the continuity of $A\,\big|\,\mathrm{int}\,(\Omega)$ implies that the level set $\Omega_a$ is open.

- **Step 2.** *Restriction of $F^{(qc)}(x)$ to $\Omega_a\cap\Omega_c$.*

**Lemma 3.4.** *Let functions $x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ with $x(t)\in A_c\ \forall\,t\in\Omega_c$ and $u\in L^\infty(\Omega,\mathbb{R}^{nm})$ with $u(t)\in\mathrm{K}$ $(\forall)\,t\in\Omega$ be given. Then it holds that*

$$\Big|\int_\Omega f^{(qc)}(t,x(t),u(t))\,dt - \int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,x(t),u(t))\,dt\Big| \leqslant 3\,\varepsilon\,. \tag{3.12}$$

**Proof.** By Theorem 2.13., we obtain

$$\Big|\int_\Omega f^{(qc)}(t,x(t),u(t))\,dt - \int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,x(t),u(t))\,dt\Big|$$

$$= \Big|\int_{\Omega_a\,\cap\,(\,\Omega\,\setminus\,\Omega_c\,)} f^{(qc)}(t,x(t),u(t))\,dt + \int_{\Omega\,\setminus\,\Omega_a} f^{(qc)}(t,x(t),u(t))\,dt\Big| \tag{3.13}$$

$$\leqslant \int_{\Omega_a\,\cap\,(\,\Omega\,\setminus\,\Omega_c\,)}\big|\,f^{(qc)}(\,\dots\,)\,\big|\,dt + \int_{\Omega\,\setminus\,\Omega_a}\big|\,f^{(qc)}(\,\dots\,)\,\big|\,dt \tag{3.14}$$

$$\leqslant \int_{\Omega_a\,\cap\,(\,\Omega\,\setminus\,\Omega_c\,)}\big(\,A(t)+C_2\,\big)\,dt + \int_{\Omega\,\setminus\,\Omega_a}\big(\,A(t)+C_2\,\big)\,dt \tag{3.15}$$

$$\leqslant \varepsilon + 2\,\varepsilon \tag{3.16}$$

by definition of $\Omega_a$ and $\Omega_c$. ■

- **Step 3.** *Decomposition of the integrals.* Consider a sequence of admissible functions $\{\,x^N\,\}\,,\,\mathcal{B}$ with $\{\,x^N\,\}\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^n)}\,\hat{x}$ and $\{\,Jx^N\,\}\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})}\,J\hat{x}$. As in the proof of Theorem 1.2., this implies the uniform convergence $x^N\to^{C^0(\Omega,\mathbb{R}^n)}\hat{x}$ and the feasibility of the limit element $\hat{x}$. We define:

$$y^N(t) = x^N(t)-\hat{x}(t) \quad\Longrightarrow\quad Jy^N(t) = Jx^N(t)-J\hat{x}(t)\,; \tag{3.17}$$

$$x^N\to^{C^0(\Omega,\mathbb{R}^n)}\hat{x} \quad\Longrightarrow\quad y^N\to^{C^0(\Omega,\mathbb{R}^n)}\mathfrak{o}\,; \tag{3.18}$$

$$Jx^N\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})}J\hat{x} \quad\Longrightarrow\quad Jy^N\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})}\mathfrak{o}; \tag{3.19}$$

$$Jx^N(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega\ \ \forall\,N\in\mathbb{N} \quad\Longrightarrow\quad J\hat{x}(t)+Jy^N(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega\ \ \forall\,N\in\mathbb{N}\,. \tag{3.20}$$

Using an index $K\in\mathbb{N}$ to be qualified in Step 4 below, we define further

$$z^N(t) = \frac{K-1}{K}\,y^N(t) \quad\Longrightarrow\quad Jz^N(t) = \frac{K-1}{K}\,Jy^N(t)\,; \tag{3.21}$$

$$\hat{z}(t) = \frac{K-1}{K}\,\hat{x}(t) \quad\Longrightarrow\quad J\hat{z}(t) = \frac{K-1}{K}\,J\hat{x}(t)\,; \tag{3.22}$$

$$y^N\to^{C^0(\Omega,\mathbb{R}^n)}\mathfrak{o} \quad\Longrightarrow\quad z^N\to^{C^0(\Omega,\mathbb{R}^n)}\mathfrak{o}\,; \tag{3.23}$$

$$Jy^N\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})}\mathfrak{o} \quad\Longrightarrow\quad Jz^N\xrightarrow{\ *\ }{}^{L^\infty(\Omega,\mathbb{R}^{nm})}\mathfrak{o}\,; \tag{3.24}$$

$$J\hat{x}(t)+Jy^N(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega\ \ \forall\,N\in\mathbb{N} \quad\Longrightarrow\quad J\hat{z}(t)+Jz^N(t)\in\frac{K-1}{K}\,\mathrm{K}\ \ (\forall)\,t\in\Omega\ \ \forall\,N\in\mathbb{N}\,; \tag{3.25}$$

$$J\hat{x}(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega \quad\Longrightarrow\quad J\hat{z}(t)\in\frac{K-1}{K}\,\mathrm{K}\ \ (\forall)\,t\in\Omega\,. \tag{3.26}$$

Now we decompose the integrals as follows:

$$\int_{\Omega_a \cap \Omega_c} f^{(qc)}(t, x^N(t), Jx^N(t))\, dt \;=\; \int_{\Omega_a \cap \Omega_c} f^{(qc)}(t, \hat{x}(t) + y^N(t), J\hat{x}(t) + Jy^N(t))\, dt$$

$$= \; J_1(N) \,+\, J_2(N) \,+\, J_3(N) \,+\, J_4(N) \,+\, J_5(N) \quad \text{with} \tag{3.27}$$

$$J_1(N) \;=\; \int_{\Omega_a \cap \Omega_c} \Big( f^{(qc)}(t, \hat{x}(t) + y^N(t), J\hat{x}(t) + Jy^N(t)) - f^{(qc)}(t, \hat{z}(t) + y^N(t), J\hat{z}(t) + Jz^N(t)) \Big)\, dt\,; \tag{3.28}$$

$$J_2(N) \;=\; \int_{\Omega_a \cap \Omega_c} \Big( f^{(qc)}(t, \hat{z}(t) + y^N(t)), J\hat{z}(t) + Jz^N(t)) - f^{(qc)}(t, \hat{z}(t), J\hat{z}(t) + Jz^N(t)) \Big)\, dt\,; \tag{3.29}$$

$$J_3(N) \;=\; \int_{\Omega_a \cap \Omega_c} f^{(qc)}(t, \hat{z}(t), J\hat{z}(t) + Jz^N(t))\, dt \,-\, \sum_s \int_{\Omega_a \cap \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jz^N(t))\, dt\,; \tag{3.30}$$

$$J_4(N) \;=\; \sum_s \int_{\Omega_a \cap \mathrm{Q}_s} \Big( f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jz^N(t))\, dt \,-\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + J\big(\varphi_s(t) \cdot z^N(t)\,\big)\,) \Big)\, dt \tag{3.31}$$

$$J_5(N) \;=\; \sum_s \int_{\Omega_a \cap \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + \big(\varphi_s(t) \cdot Jz^N(t) + \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t)\,\big)\,)\, dt\,. \tag{3.32}$$

The precise choice of $K \in \mathbb{N}$, $\mathrm{Q}_s \subset \Omega_a$, $t_s \in \mathrm{Q}_s$, $[\,\hat{z}\,]_s \in \mathbb{R}^n$, $[\,J\hat{z}\,]_s \in \mathbb{R}^{nm}$ and $\varphi_s \in C_0^\infty(\mathrm{Q}_s, \mathbb{R}^n)$ will be explained in the following steps.

• **Step 4.** *Investigation of* $J_1(N)$ *and* $J_2(N)$. Applying Theorem 2.14., we find, in relation to $\varepsilon > 0$ fixed above, two indices $K_0(\varepsilon)$ and $K_1(\varepsilon) \in \mathbb{N}$ with

$$\Big| \int_{\Omega_a \cap \Omega_c} \Big( f^{(qc)}(t, \frac{K-1}{K}\, \hat{x}(t), \frac{K-1}{K}\, J\hat{x}(t)) \,-\, f^{(qc)}(t, \hat{x}(t), J\hat{x}(t)) \Big)\, dt \Big| \;\leqslant\; 7\,|\,\Omega_a \,\cap\, \Omega_c\,|\,\varepsilon \;\; \forall\, K \geqslant K_0(\varepsilon) \tag{3.33}$$

and

$$\int_{\Omega_c} \Big( f^{(qc)}(t, x^N(t), Jx^N(t)) \,-\, f^{(qc)}(t, \frac{K-1}{K}\, x^N(t), \frac{K-1}{K}\, Jx^N(t)) \Big)\, dt \tag{3.34}$$

$$= \int_{\Omega_c} \Big( f^{(qc)}(t, \hat{x}(t) + y^N(t), J\hat{x}(t) + Jy^N(t)) \,-\, f^{(qc)}(t, \hat{z}(t) + y^N(t), J\hat{z}(t) +\, Jz^N(t)) \Big)\, dt \;\geqslant\; -8\,\varepsilon\,|\,\Omega_c\,|$$

for all $K \geqslant K_1(\varepsilon)$ and all $N \in \mathbb{N}$. We choose $K \geqslant \mathrm{Max}\big(K_0(\varepsilon)\,,\, K_1(\varepsilon)\big)$. Then from Theorem 2.13., for arbitrary $N \in \mathbb{N}$ it follows that

$$\Big| \int_{\Omega_c \setminus \Omega_a} \Big( f^{(qc)}(t, \hat{x}(t) + y^N(t), J\hat{x}(t) + Jy^N(t)) - f^{(qc)}(t, \hat{z}(t) + y^N(t), J\hat{z}(t) + Jy^N(t)) \Big)\, dt \Big|$$

$$\leqslant \int_{\Omega_c \setminus \Omega_a} |\, f^{(qc)}(t, \hat{x}(t) + y^N(t), J\hat{x}(t) + Jy^N(t))\,|\, dt \,+\, \int_{\Omega_c \setminus \Omega_a} |\, f^{(qc)}(t, \hat{z}(t) + y^N(t), J\hat{z}(t) + Jz^N(t))\,|\, dt$$

$$\leqslant 2 \int_{\Omega_c \setminus \Omega_a} \big(\, A(t) + C_2\,\big)\, dt \;\leqslant\; 2 \int_{\Omega \setminus \Omega_a} \big(\, A(t) + C_2\,\big)\, dt \;\leqslant\; 4\,\varepsilon\,. \tag{3.35}$$

Together we arrive at

$$J_1(N) \;=\; \int_{\Omega_a \cap \Omega_c} \Big(\; ... \;\Big)\, dt \;=\; \int_{\Omega_c} \Big(\; ... \;\Big)\, dt \;-\; \int_{\Omega_c \setminus \Omega_a} \Big(\; ... \;\Big)\, dt \;\geqslant\; \int_{\Omega_c} \Big(\; ... \;\Big)\, dt \;-\; \Big| \int_{\Omega_c \setminus \Omega_a} \Big(\; ... \;\Big)\, dt \Big| \quad \Longrightarrow \tag{3.36}$$

$$\liminf_{N \to \infty} J_1(N) \;\geqslant\; -\big(\, 8\,|\,\Omega_c\,| + 4\,\big)\,\varepsilon\,. \tag{3.37}$$

By Theorem 2.11., 2), the function $f^{(qc)}(t,\xi,v)\,\big|\,\big(\Omega_c\,\times\,\mathrm{A}_c\,\times\,\frac{K-1}{K}\,\mathrm{K}\big)$ is uniformly continuous with respect to $(t,\xi,v)$. Then from the uniform convergence $y^N \rightarrow^{C^0(\Omega,\mathbb{R}^n)} \mathfrak{o}$ it follows that

$$\liminf_{N\to\infty}\, J_2(N)\;=\;\lim_{N\to\infty}\, J_2(N)\;=\;0\,. \tag{3.38}$$

• **Step 5.** *Investigation of* $J_3(N)$. Due to (2.37), (2.38) and

$$\mathrm{Min}\,\big(\,1\,,\,\mathrm{Dist}\,(\,(1-\frac{1}{2K})\,\mathrm{K}\,,\,\partial\mathrm{K})\,\big)\;=\;\mathrm{Min}\,\big(\,1\,,\,\frac{c_{\mathrm{K}}}{2K}\,\big)\;\geqslant\;\mathrm{Min}\,\big(\,\varepsilon\,,\,1\,,\,\frac{c_{\mathrm{K}}}{2K}\,,\,\frac{\mathrm{Diam}\,(\mathrm{A}_c)}{2K}\,\big)\,, \tag{3.39}$$

the uniform continuity of $f^{(qc)}(t,\xi,v)$ on $\big(\Omega_c\,\times\,\mathrm{A}_c\,\times(1-\frac{1}{2K})\,\mathrm{K}\big)$ may be described by the relation

$$\begin{aligned}&\big|\,t'-t''\,\big|\,+\,\big|\,\xi'-\xi''\,\big|\,+\,\big|\,v'-v''\,\big|\;\leqslant\;\delta_3(\varepsilon)\;=\;\frac{\delta_2(\varepsilon)}{4\,C_{\mathrm{K}}}\cdot\mathrm{Min}\,\big(\,\varepsilon\,,\,1\,,\,\frac{c_{\mathrm{K}}}{2K}\,,\,\frac{\mathrm{Diam}\,(\mathrm{A}_c)}{2\,K}\,\big)\;\Longrightarrow\\ &\qquad\big|\,f^{(qc)}(t',\xi',v')-f^{(qc)}(t'',\xi'',v'')\,\big|\;\leqslant\;6\,\varepsilon\quad\forall\,(t',\xi',v'),\,(t'',\xi'',v'')\in\big(\Omega_c\,\times\,\mathrm{A}_c\,\times\,(1-\frac{1}{2K})\,\mathrm{K}\big)\,.\end{aligned} \tag{3.40}$$

In view to the proof of Proposition 3.8. below, we choose

$$\delta_4(\varepsilon)\;=\;\mathrm{Min}\,\Big(\,\big(\,\delta_2(\varepsilon)\,\big)^2\,,\,\delta_3(\varepsilon)\,\Big) \tag{3.41}$$

and apply Lemma 1.8. to the open set $\Omega_a\subset\mathbb{R}^m$, the functions $\hat{z}$ and $J\hat{z}$ and the numbers

$$\eta\;=\;\delta\;=\;\mathrm{Min}\,\Big(\,\varepsilon\,,\,\frac{\delta_4(\varepsilon)}{3\,\sqrt{m}}\,,\,\frac{\delta_4(\varepsilon)}{3\,\sqrt{n}}\,,\,\frac{\delta_4(\varepsilon)}{3\,\sqrt{nm}}\,,\,\frac{c_{\mathrm{K}}}{2\,nm\,K}\,\Big)\,. \tag{3.42}$$

We find a finite number of mutually disjoint closed cubes $\mathrm{Q}_s\subset\Omega_a$ with edge length less or equal than $\frac{1}{3\sqrt{m}}\,\delta_4(\varepsilon)$ and

$$\big|\,\Omega_a\,\setminus\,\bigcup_{s=1}^{r}\,\mathrm{Q}_s\,\big|\;\leqslant\;\varepsilon\,; \tag{3.43}$$

$$\Big|\,\hat{z}_i(t)\;-\;\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\hat{z}_i(\tau)\,d\tau\,\Big|\;\leqslant\;\frac{\delta_4(\varepsilon)}{3\sqrt{n}}\quad(\forall)\,t\in\mathrm{Q}_s\,,\;1\leqslant s\leqslant r\,,\;1\leqslant i\leqslant n\,; \tag{3.44}$$

$$\Big|\,\frac{\partial\hat{z}_i(t)}{\partial t_j}\;-\;\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\frac{\partial\hat{z}_i(\tau)}{\partial t_j}\,d\tau\,\Big|\;\leqslant\;\frac{\delta_4(\varepsilon)}{3\sqrt{nm}}\quad(\forall)\,t\in\mathrm{Q}_s\,,\;1\leqslant s\leqslant r\,,\;1\leqslant i\leqslant n\,,\;1\leqslant j\leqslant m\,. \tag{3.45}$$

Let us choose now points $t_s\in\mathrm{Q}_s\setminus\mathrm{N}$ in such a way that $\big(\mathrm{Q}_s\cap\Omega_c\big)\setminus\mathrm{N}\neq\emptyset$ implies $t_s\in\big(\mathrm{Q}_s\cap\Omega_c\big)\setminus\mathrm{N}$. From the convexity of the integral (cf. [ Bourbaki 52 ], Chap. IV, § 6, p. 204, Corollaire) it follows that

$$[\,\hat{z}\,]_s\;=\;\big(\,\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\hat{z}_1(\tau)\,d\tau\,,\,\dots\,,\,\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\hat{z}_n(\tau)\,d\tau\,\big)^{\mathrm{T}}\;\in\;\frac{K-1}{K}\,\mathrm{A}_c\quad\text{and} \tag{3.46}$$

$$[\,J\hat{z}\,]_s\;=\;\begin{pmatrix}\displaystyle\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\frac{\partial\hat{z}_1}{\partial t_1}(t)\,dt & \dots & \displaystyle\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\frac{\partial\hat{z}_1}{\partial t_m}(t)\,dt\\ \vdots & & \vdots\\ \displaystyle\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\frac{\partial\hat{z}_n}{\partial t_1}(t)\,dt & \dots & \displaystyle\frac{1}{|\,\mathrm{Q}_s\,|}\int_{\mathrm{Q}_s}\frac{\partial\hat{z}_n}{\partial t_m}(t)\,dt\end{pmatrix}\;\in\;\frac{K-1}{K}\,\mathrm{K} \tag{3.47}$$

for all $1\leqslant s\leqslant r$. We deduce further that

$$\big|\,t-t_s\,\big|\;\leqslant\;\frac{\delta_4(\varepsilon)}{3}\quad\forall\,t\in\mathrm{Q}_s\,;\quad\big|\,\hat{z}(t)-[\,\hat{z}\,]_s\,\big|\;\leqslant\;\frac{\delta_4(\varepsilon)}{3}\quad\forall\,t\in\mathrm{Q}_s\,; \tag{3.48}$$

$$\big|\,J\hat{z}(t)-[\,J\hat{z}\,]_s\,\big|\;\leqslant\;\mathrm{Min}\,\Big(\,\frac{\delta_4(\varepsilon)}{3}\,,\,\frac{c_{\mathrm{K}}}{2K}\,\big)\quad(\forall)\,t\in\mathrm{Q}_s \tag{3.49}$$

as well as

$$[\,J\hat{z}\,]_s + Jz^N(t) \;=\; J\hat{z}(t) + Jz^N(t) + \big(\,[\,J\hat{z}\,]_s - J\hat{z}(t)\,\big) \;\in\; \frac{K-1}{K}\,\mathrm{K} + \mathrm{K}(\mathfrak{o}_{nm}, \frac{c_{\mathrm{K}}}{2K}) \;\subseteq\; \frac{2K-1}{2K}\,\mathrm{K} \quad (\forall)\, t \in \mathrm{Q}_s\,, \tag{3.50}$$

which implies, in particular, $f^{(qc)}(\,t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jz^N(t)\,) < (+\infty)$ for almost all $t \in \mathrm{Q}_s$. Then for almost all $t \in \mathrm{Q}_s$ and $1 \leqslant s \leqslant r$ it holds that

$$\big|\,t - t_s\,\big| + \big|\,\hat{z}(t) - [\,\hat{z}\,]_s\,\big| + \big|\,J\hat{z}(t) - [\,J\hat{z}\,]_s\,\big| \;\leqslant\; \delta_4(\varepsilon)\,, \tag{3.51}$$

and we obtain

$$\begin{aligned} J_3(N) \;=\; & \int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r \mathrm{Q}_s} f^{(qc)}(t, \hat{z}(t), J\hat{z}(t) + Jz^N(t))\,dt \\ & + \sum_s \int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s} \Big(\, f^{(qc)}(t, \hat{z}(t), J\hat{z}(t) + Jz^N(t)) \,-\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jz^N(t))\,\Big)\,dt \\ & - \sum_s \int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jz^N(t))\,dt \end{aligned} \tag{3.52}$$

$$\geqslant -\int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r \mathrm{Q}_s} \big|\,f^{(qc)}(\,\dots\,)\,\big|\,dt \,-\, \sum_s \int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s} \Big|\;\dots\;\Big|\,dt \,-\, \sum_s \int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s} \big|\,f^{(qc)}(\,\dots\,)\,\big|\,dt \tag{3.53}$$

$$\geqslant -\int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r \mathrm{Q}_s} \big(\,A(t) + C_2\,\big)\,dt \,-\, \sum_s \int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s} \Big|\;\dots\;\Big|\,dt \,-\, \sum_s \int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s} \big(\,A(t) + C_2\,\big)\,dt \tag{3.54}$$

$$\geqslant -\int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r \mathrm{Q}_s} \big(\,C_2 + C_3\,\big)\,dt \,-\, \sum_s \int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s} \Big|\;\dots\;\Big|\,dt \,-\, \sum_s \int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s} \big(\,C_2 + C_3\,\big) \tag{3.55}$$

$$\geqslant -\int_{\Omega_a\,\setminus\,\cup_{s=1}^r \mathrm{Q}_s} \big(\,C_2 + C_3\,\big)\,dt \,-\, \sum_s \int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s} \Big|\;\dots\;\Big|\,dt \,-\, \sum_s \int_{\Omega\,\setminus\,\Omega_c} \big(\,C_2 + C_3\,\big)\,dt \tag{3.56}$$

$$\geqslant -\,\varepsilon\,\big(\,C_2 + C_3\,\big) \,-\, 6\,\varepsilon\,\big|\,\Omega_a\,\big| \,-\, \varepsilon \quad \Longrightarrow \tag{3.57}$$

$$\liminf_{N\to\infty}\; J_3(N) \;\geqslant\; -\big(\,6\,|\,\Omega_a\,| + C_2 + C_3 + 1\,\big)\,\varepsilon\,. \tag{3.58}$$

• **Step 6.** *Investigation of* $J_4(N)$ *and* $J_5(N)$. Before we can exploit the quasiconvexity of $f^{(qc)}$, the values of $z^N$ on the boundaries $\partial\mathrm{Q}_s$ of the cubes must be altered to zero. We proceed in the following way. First, we choose closed cubes $\mathrm{Q}_s^0 \subset \mathrm{int}\,(\mathrm{Q}_s)$ with the same center as $\mathrm{Q}_s$ and $|\,\mathrm{Q}_s \;\setminus\; \mathrm{Q}_s^0\,| \;\leqslant\; \varepsilon \cdot |\,\mathrm{Q}_s\,|$. Let $\mathrm{Dist}\,(\partial\mathrm{Q}_s^0\,,\, \partial\mathrm{Q}_s) = \kappa_s$. Then we define functions $\varphi_s \in C^\infty(\mathrm{Q}_s, \mathbb{R})$ with

$$\varphi_s(t) \begin{cases} = 1 & \big|\; t \in \mathrm{Q}_s^0; \\ \in [\,0\,,\,1\,] & \big|\; t \in \mathrm{Q}_s \;\setminus\; \mathrm{Q}_s^0; \\ = 0 & \big|\; t \in \partial\mathrm{Q}_s \end{cases} \tag{3.59}$$

and $|\,\nabla\varphi_s(t)\,| \;\leqslant\; C_6/\kappa_s \;\leqslant\; \mathrm{Max}_{\,1\,\leqslant\,s\,\leqslant\,r}\big(\,C_6/\kappa_s\,\big)$ with a constant $C_6 \;>\; 0$. Let us investigate now the arguments

$$[\,J\hat{z}\,]_s \,+\, \varphi_s(t)\cdot Jz^N(t) \,+\, \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t)\,. \tag{3.60}$$

By Step 5, $[\,J\hat{z}\,]_s$ as well as $[\,J\hat{z}\,]_s + Jz^N(t)$ belong to $\frac{2K-1}{2K}\,\mathrm{K}$ for almost all $t \in \Omega$. Since $0 \leqslant \varphi_s(t) \leqslant 1$ it follows that

$$[\,J\hat{z}\,]_s \,+\, \varphi_s(t)\cdot Jz^N(t) \;\in\; \big[\,[\,J\hat{z}\,]_s\,,\, [\,J\hat{z}\,]_s + Jz^N(t)\,\big] \;\subset\; \frac{2K-1}{2K}\,\mathrm{K} \tag{3.61}$$

for almost all $t \in \Omega$. With a further constant $C_7 > 0$, we may estimate

$$\big|\, \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t) \,\big| \;\leqslant\; C_7 \cdot \big|\, \nabla\varphi_s(t) \,\big| \cdot \|\, z^N \,\|_{C^0(\Omega,\mathbb{R}^n)} \;\leqslant\; \operatorname*{Max}_{1 \leqslant s \leqslant r} \frac{C_6\, C_7}{\kappa_s} \cdot \|\, z^N \,\|_{C^0(\Omega,\mathbb{R}^n)}\,. \tag{3.62}$$

The convergence $z^N \to^{C^0(\Omega,\mathbb{R}^n)} \mathfrak{o}$ implies for all sufficiently large $N \geqslant N_0(\varepsilon)$:

$$\big|\, \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t) \,\big| \;\leqslant\; \frac{c_{\mathrm{K}}}{4K} \tag{3.63}$$

and

$$[\, J\hat{z}\,]_s \;+\; \varphi_s(t) \cdot Jz^N(t) \;+\; \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t) \;\in\; \frac{2K-1}{2K}\,\mathrm{K} \;+\; \mathrm{K}(\mathfrak{o}_{nm}, \frac{c_{\mathrm{K}}}{4K}) \subseteq \frac{4K-1}{4K}\,\mathrm{K}\,. \tag{3.64}$$

Consequently, for all $N \geqslant N_0(\varepsilon)$ and all $1 \leqslant s \leqslant r$ and almost all $t \in \Omega$ it results that

$$f^{(qc)}\big(\, t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s \;+\; \varphi_s(t) \cdot Jz^N(t) \;+\; \nabla\varphi_s(t)^{\mathrm{T}}\, z^N(t) \big) \;<\; (+\infty)\,. \tag{3.65}$$

We obtain

$$\begin{aligned} &\int_{\Omega_a \,\cap\, \mathrm{Q}_s} \Big(\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + Jz^N(t)) \;-\; f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + J\big(\, \varphi_s(t) \cdot z^N(t)\,\big)\,)\Big)\, dt \\ &= \int_{\Omega_a \,\cap\, (\,\mathrm{Q}_s \,\setminus\, \mathrm{Q}_s^0\,)} \Big(\; ... \;\Big) \;\geqslant\; -\int_{\mathrm{Q}_s \,\setminus\, \mathrm{Q}_s^0} \Big|\; ... \;\Big| \;\geqslant\; -2 \int_{\mathrm{Q}_s \,\setminus\, \mathrm{Q}_s^0} \big(\, A(t) + C_2 \,\big)\, dt \;\geqslant\; -2\,\varepsilon\, \big|\, \mathrm{Q}_s \,\big|\, \big(\, C_2 + C_3 \,\big) \end{aligned} \tag{3.66}$$

for all $1 \leqslant s \leqslant r$. Summing up, we arrive at (3.67)

$$\begin{aligned} J_4(N) \;=\; \sum_s \int_{\Omega_a \,\cap\, \mathrm{Q}_s} &\Big(\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + Jz^N(t)) \;-\; f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + J\big(\, \varphi_s(t) \cdot z^N(t)\,\big)\,)\Big)\, dt \\ &\geqslant\; -2\,\varepsilon \sum_s \big|\, \mathrm{Q}_s \,\big|\, \big(\, C_2 + C_3 \,\big) \;\geqslant\; -2\,\varepsilon\, \big|\, \Omega_a \,\big|\, \big(\, C_2 + C_3 \,\big) \quad\Longrightarrow \end{aligned}$$

$$\liminf_{N \to \infty}\; J_4(N) \;\geqslant\; -2\,\varepsilon\, \big|\, \Omega_a \,\big|\, \big(\, C_2 + C_3 \,\big)\,. \tag{3.68}$$

Now from the quasiconvexity of the functions $f^{(qc)}(t_s, [\,\hat{z}\,]_s, \,\cdot\,)$ (Theorem 2.10., 2) ) it follows for all $1 \leqslant s \leqslant r$:

$$\begin{aligned} &\frac{1}{|\,\Omega\,|} \int_{\Omega} \Big(\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + J\big(\, \varphi_s(t) \cdot z^N(t)\,\big)\,) \;-\; f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s\,)\Big)\, dt \\ &\qquad = \frac{1}{|\,\Omega\,|} \int_{\Omega_a \,\cap\, \mathrm{Q}_s} \Big(\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + J\big(\, \varphi_s(t) \cdot z^N(t)\,\big)\,) \;-\; f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s\,)\Big)\, dt \;\geqslant\; 0\,, \end{aligned} \tag{3.69}$$

which gives finally

$$J_5(N) \;=\; \sum_s \int_{\Omega_a \,\cap\, \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s + J\big(\, \varphi_s(t) \cdot z^N(t)\,\big)\,)\, dt \tag{3.70}$$

$$\geqslant\; \sum_s \int_{\Omega_a \,\cap\, \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s)\, dt \tag{3.71}$$

and

$$\liminf_{N \to \infty}\; J_5(N) \;\geqslant\; \sum_s \int_{\Omega_a \,\cap\, \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\, J\hat{z}\,]_s)\, dt\,. \tag{3.72}$$

• **Step 7.** *Synopsis of the previous Steps 2 – 6.*

**Lemma 3.5.** *It holds that*

$$\liminf_{N\to\infty}\int_\Omega f^{(qc)}(t,x^N(t),Jx^N(t))\,dt \;\geqslant\; \sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\;-\;C_4\,\varepsilon \tag{3.73}$$

*with* $C_4 \;=\; (2\,C_2+2\,C_3+6)\,|\,\Omega_a\,|+8\,|\,\Omega_c\,|+C_2+C_3+5.$

**Proof.** From Lemma 3.4. and (3.27), it follows that

$$\begin{aligned}\liminf_{N\to\infty}\int_\Omega f^{(qc)}(t,x^N(t),Jx^N(t))\,dt \;&\geqslant\; \liminf_{N\to\infty}\int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,x^N(t),Jx^N(t))\,dt\;-\;3\,\varepsilon\\ &\geqslant\;\liminf_{N\to\infty}\,J_1(N)\;+\;\liminf_{N\to\infty}\,J_2(N)\;+\;\liminf_{N\to\infty}\,J_3(N)\;+\;\liminf_{N\to\infty}\,J_4(N)\;+\;\liminf_{N\to\infty}\,J_5(N)\;-\;3\,\varepsilon\,. \end{aligned}\tag{3.74}$$

From Steps 4 – 6, we conclude with (3.37), (3.38), (3.58), (3.68) and (3.71):

$$\begin{aligned}&\liminf_{N\to\infty}\,J_1(N)\;+\;\liminf_{N\to\infty}\,J_2(N)\;+\;\liminf_{N\to\infty}\,J_3(N)\;+\;\liminf_{N\to\infty}\,J_4(N)\;+\;\liminf_{N\to\infty}\,J_5(N)\\ &\geqslant\;-\big(8\,|\,\Omega_c\,|+4\big)\,\varepsilon\;-\;\big(6\,|\,\Omega_a\,|+C_2+C_3+1\big)\,\varepsilon\;-2\,\varepsilon\,\big|\,\Omega_a\,\big|\,\big(\,C_3+C_2\,\big)\;+\;\sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\,,\end{aligned}\tag{3.75}$$

which gives together

$$\liminf_{N\to\infty}\int_\Omega f^{(qc)}(t,x^N(t),Jx^N(t))\,dt \;\geqslant\; \sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\;-\;C_4\,\varepsilon\,.\quad\blacksquare \tag{3.76}$$

• **Step 8.** *Conclusion of the proof.*

**Lemma 3.6.** *It holds that*

$$\Big|\int_\Omega f^{(qc)}(t,\hat x(t),J\hat x(t))\,dt\;-\;\sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\Big|\;\leqslant\;C_5\,\varepsilon \tag{3.77}$$

*with* $C_5 = 6\,|\,\Omega_a\,|+7\,|\,\Omega_a\,\cap\,\Omega_c\,|+C_2+C_3+4.$

**Proof.** Let us decompose

$$\int_\Omega f^{(qc)}(t,\hat x(t),J\hat x(t))\,dt\;-\;\sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\;=\;J_6\;+\;J_7\;+\;J_8\quad\text{with} \tag{3.78}$$

$$J_6\;=\;\int_\Omega f^{(qc)}(t,\hat x(t),J\hat x(t))\,dt\;-\;\int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,\hat x(t),J\hat x(t))\,dt\,; \tag{3.79}$$

$$J_7\;=\;\int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,\hat x(t),J\hat x(t))\,dt\;-\;\int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,\hat z(t),J\hat z(t))\,dt\,; \tag{3.80}$$

$$J_8\;=\;\int_{\Omega_a\,\cap\,\Omega_c} f^{(qc)}(t,\hat z(t),J\hat z(t))\;-\;\sum_s\int_{\Omega_a\,\cap\,Q_s} f^{(qc)}(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s)\,dt\,. \tag{3.81}$$

From Lemma 3.4. it follows that $|\,J_6\,|\;\leqslant\;3\,\varepsilon$, and the index $K$ has been chosen in the definition of $\hat z$ in such a way that the inequality (3.33) holds. Consequently, we find $|\,J_7\,|\;\leqslant\;7\,|\,\Omega_a\,\cap\,\Omega_c\,|\,\varepsilon$. When estimating $J_8$, by (3.51) we obtain analogously to (3.58):

$$\begin{aligned} J_8 \,=\, & \int_{(\,\Omega_a \cap \Omega_c\,) \setminus \cup_{s=1}^r \mathrm{Q}_s} f^{(qc)}(t, \hat{z}(t), J\hat{z}(t))\, dt \\ & + \sum_s \int_{\Omega_a \cap \Omega_c \cap \mathrm{Q}_s} \Big( f^{(qc)}(t, \hat{z}(t), J\hat{z}(t)) \,-\, f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s) \Big)\, dt \\ & - \sum_s \int_{(\,\Omega_a \setminus \Omega_c\,) \cap \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s)\, dt \quad \Longrightarrow \end{aligned} \tag{3.82}$$

$$\big|\, J_8 \,\big| \,\leqslant\, \int_{(\,\Omega_a \cap \Omega_c\,) \setminus \cup_{s=1}^r \mathrm{Q}_s} \big|\, f^{(qc)}(\,\dots\,) \,\big|\, dt \,+\, \sum_s \int_{\Omega_a \cap \Omega_c \cap \mathrm{Q}_s} \Big|\,\dots\,\Big|\, dt \,+\, \sum_s \int_{(\,\Omega_a \setminus \Omega_c\,) \cap \mathrm{Q}_s} \big|\, f^{(qc)}(\,\dots\,) \,\big|\, dt \tag{3.83}$$

$$\leqslant\, \int_{(\,\Omega_a \cap \Omega_c\,) \setminus \cup_{s=1}^r \mathrm{Q}_s} \big(\, A(t) + C_2 \,\big)\, dt \,+\, \sum_s \int_{\Omega_a \cap \Omega_c \cap \mathrm{Q}_s} \Big|\,\dots\,\Big|\, dt \,+\, \sum_s \int_{(\,\Omega_a \setminus \Omega_c\,) \cap \mathrm{Q}_s} \big(\, A(t) + C_2 \,\big)\, dt \tag{3.84}$$

$$\leqslant\, \int_{(\,\Omega_a \cap \Omega_c\,) \setminus \cup_{s=1}^r \mathrm{Q}_s} \big(\, C_2 + C_3 \,\big)\, dt \,+\, \sum_s \int_{\Omega_a \cap \Omega_c \cap \mathrm{Q}_s} \Big|\,\dots\,\Big|\, dt \,+\, \sum_s \int_{(\,\Omega_a \setminus \Omega_c\,) \cap \mathrm{Q}_s} \big(\, C_2 + C_3 \,\big)\, dt \tag{3.85}$$

$$\leqslant\, \int_{\Omega_a \setminus \cup_{s=1}^r \mathrm{Q}_s} \big(\, C_2 + C_3 \,\big)\, dt \,+\, \sum_s \int_{\Omega_a \cap \Omega_c \cap \mathrm{Q}_s} \Big|\,\dots\,\Big|\, dt \,+\, \sum_s \int_{\Omega \setminus \Omega_c} \big(\, C_2 + C_3 \,\big)\, dt \tag{3.86}$$

$$\leqslant\, (C_2 + C_3)\,\varepsilon \,+\, 6\,|\,\Omega_a\,|\,\varepsilon \,+\, \varepsilon\,. \tag{3.87}$$

We arrive at

$$\begin{aligned} & \Big|\int_\Omega f^{(qc)}(t, \hat{x}(t), J\hat{x}(t))\, dt \,-\, \sum_s \int_{\Omega_a \cap \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s)\, dt \Big| \,\leqslant\, \big|\, J_6 \,\big| \,+\, \big|\, J_7 \,\big| \,+\, \big|\, J_8 \,\big| \\ & \leqslant\, 3\,\varepsilon \,+\, |\,\Omega_a \,\cap\, \Omega_c\,|\,\varepsilon \,+\, \big(\, 6\,|\,\Omega_a\,| \,+\, C_2 \,+\, C_3 \,+\, 1 \,\big)\,\varepsilon\,. \quad \blacksquare \end{aligned} \tag{3.88}$$

Finally, we deduce from Lemma 3.5. and 3.6.:

$$\liminf_{N \to \infty} \int_\Omega f^{(qc)}(t, x^N(t), Jx^N(t))\, dt \,\geqslant\, \int_\Omega f^{(qc)}(t, \hat{x}(t), J\hat{x}(t))\, dt \,-\, \big(\, C_4 + C_5 \,\big)\,\varepsilon\,. \tag{3.89}$$

Since neither $C_4$ nor $C_5$ depends on $\varepsilon$, (3.89) implies the claimed lower semicontinuity relation

$$\int_\Omega f^{(qc)}(t, \hat{x}(t), J\hat{x}(t))\, dt \,\leqslant\, \liminf_{N \to \infty} \int_\Omega f^{(qc)}(t, x^N(t), Jx^N(t))\, dt\,, \tag{3.90}$$

and the proof of Proposition 3.3. is complete. $\blacksquare$

**Corollary 3.7.** *The problem* $(\mathrm{P})^{(qc)}$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$*.*

**Proof.** The feasible domain of the problem $(\mathrm{P})^{(qc)}$ is identical with the feasible domain $\mathcal{B}$ of (P). Consequently, Lemma 3.1. together with Theorem 2.13. implies the boundedness of $F^{(qc)}$ on $\mathcal{B}$:

$$\big|\, F^{(qc)}(x) \,\big| \,\leqslant\, \int_\Omega \big|\, f^{(qc)}(t, x(t), Jx(t)) \,\big|\, dt \,\leqslant\, \big\|\, A \,\big\|_{L^1(\Omega, \mathbb{R})} \,+\, C_2 \cdot |\,\Omega\,| \,<\, (+\infty)\,, \tag{3.91}$$

and $(\mathrm{P})^{(qc)}$ admits a minimizing sequence $\{\, x^N \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$. Analogously to the proof of Theorem 1.2., we may assume from the outset that $\{\, x^N \,\} \xrightarrow{\;*\;}_{L^\infty(\Omega, \mathbb{R}^n)} \hat{x}$ and $\{\, Jx^N \,\} \xrightarrow{\;*\;}_{L^\infty(\Omega, \mathbb{R}^{nm})} J\hat{x}$ with $\hat{x} \,\in\, \mathcal{B}$. Denoting the (finite) minimal value of $(\mathrm{P})^{(qc)}$ by $m^{(qc)}$, we conclude from Proposition 3.3.:

$$m^{(qc)} \,\leqslant\, F^{(qc)}(\hat{x}) \,\leqslant\, \liminf_{N \to \infty} F^{(qc)}(x^N) \,=\, \lim_{N \to \infty} F^{(qc)}(x^N) \,=\, m^{(qc)}\,, \tag{3.92}$$

and $\hat{x}$ is a global minimizer of $(\mathrm{P})^{(qc)}$. $\blacksquare$

**Proposition 3.8. (Coincidence of the minimal values of** (P) **and** $(\mathrm{P})^{(qc)}$**)** *The problems* (P) *and* $(\mathrm{P})^{(qc)}$ *possess global minimizers, and its minimal values are identical.*

**Proof.** Let $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ be a global minimizer of $(\mathrm{P})^{(qc)}$ (its existence is assured by Corollary 3.7.). We have to prove that

$$F^{(qc)}(\hat{x}) \,=\, \int_\Omega f^{(qc)}(t, \hat{x}(t), J\hat{x}(t))\, dt \tag{3.93}$$

can be approximated arbitrarily close with terms

$$F(x) \,=\, \int_\Omega f(t, x(t), Jx(t))\, dt \tag{3.94}$$

where the functions $x \in \mathcal{B}$ are admissible in (P). Let us fix $\varepsilon > 0$. For $1 \leqslant s \leqslant r$, we may write in accordance with Theorem 2.5.:

$$\int_{\Omega_a \,\cap\, \mathrm{Q}_s} f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s)\, dt \,=\, |\,\mathrm{Q}_s\,| \cdot f^{(qc)}(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s) \,=\, \lim_{N\to\infty} \int_{\mathrm{Q}_s} f(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jw_s^N(t))\, dt\,, \tag{3.95}$$

assuming that $w_s^N \in W_0^{1,\infty}(\mathrm{Q}_s, \mathbb{R}^n)$, $[\,J\hat{z}\,]_s + Jw_s^N(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$ and $\lim_{N\to\infty} \|\, w_s^N \,\|_{C^0(\mathrm{Q}_s, \mathbb{R}^n)} \,=\, 0$ (cf. the proof of Lemma 3.1.). Consequently, there exist functions $w_s \in W_0^{1,\infty}(\mathrm{Q}_s, \mathbb{R}^n)$ with the following properties:

$$[\,J\hat{z}\,]_s + Jw_s(t) \in \mathrm{K} \quad (\forall)\, t \in \Omega\,; \tag{3.96}$$

$$\|\, w_s \,\|_{C^0(\mathrm{Q}_s, \mathbb{R}^n)} \,\leqslant\, \frac{\delta_4(\varepsilon)}{3}\,; \tag{3.97}$$

$$\Big|\int_{\mathrm{Q}_s} \Big( f(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s) \,-\, f(t_s, [\,\hat{z}\,]_s, [\,J\hat{z}\,]_s + Jw_s(t)) \Big)\, dt\,\Big| \,\leqslant\, \varepsilon\,. \tag{3.98}$$

Since $\delta_4(\varepsilon) \leqslant \mathrm{Diam}\,(\mathrm{A}_c)/(2K)$, from $|\,\hat{z}(t) - [\,\hat{z}\,]_s\,| \leqslant \delta_4(\varepsilon)/3 \;\forall\, t \in \mathrm{Q}_s$ it follows that

$$\hat{z}(t) \,+\, w_s(t) \,\in\, \frac{K-1}{K}\,\mathrm{A}_c \,+\, \mathrm{K}(\mathfrak{o}, \frac{\delta_4(\varepsilon)}{3}) \,+\, \mathrm{K}(\mathfrak{o}, \frac{\delta_4(\varepsilon)}{3}) \quad\Longrightarrow\quad \hat{z}(t) \,+\, w_s(t) \,\in\, \mathrm{A}_c\,. \tag{3.99}$$

Further, from $|\,J\hat{z}(t) - [\,J\hat{z}\,]_s\,| \leqslant \delta_4(\varepsilon)/3 \,\leqslant\, \big(\,\delta_2(\varepsilon)\,\big)^2$ $(\forall)\, t \in \mathrm{Q}_s$ we conclude that

$$J\hat{z}(t) \,+\, Jw_s(t) \,\in\, \frac{c_{\mathrm{K}} + \big(\,\delta_2(\varepsilon)\,\big)^2}{c_{\mathrm{K}}}\,\mathrm{K} \tag{3.100}$$

for almost all $t \in \mathrm{Q}_s$ and all $1 \leqslant s \leqslant r$, thus

$$\frac{c_{\mathrm{K}}}{c_{\mathrm{K}} + \big(\,\delta_2(\varepsilon)\,\big)^2}\big(\,J\hat{z}(t) \,+\, Jw_s(t)\,\big) \,\in\, \mathrm{K} \tag{3.101}$$

for almost all $t \in \mathrm{Q}_s$ and all $1 \leqslant s \leqslant r$. We gather all functions $w_s$ into a single function $w \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ defined by

$$w(t) \,=\, \sum_{s=1}^{r} \mathbb{1}_{\mathrm{Q}_s}(t)\, w_s(t) \tag{3.102}$$

and study the difference

$$\begin{aligned}
&\Big|\int_{\Omega_a\,\cap\,\Omega_c} f\big(t, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\hat z(t)+w(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(J\hat z(t)+Jw(t))\big)\,dt\\
&\qquad -\sum_s\int_{\Omega_a\,\cap\,\mathrm{Q}_s} f(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s+Jw_s(t))\Big)\,dt\Big| \leqslant J_9\,+\,J_{10}\,+\,J_{11}\quad\text{with}
\end{aligned}\tag{3.103}$$

$$J_9\,=\,\Big|\int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r\,\mathrm{Q}_s} f\big(t, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\hat z(t)+w(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(J\hat z(t)+Jw(t))\big)\,dt\Big|\,;\tag{3.104}$$

$$\begin{aligned}
J_{10}\,=\,\Big|\sum_s\int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s}\Big(f\big(t, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\hat z(t)+w(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(J\hat z(t)+Jw(t))\big)&\\
-f(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s+Jw_s(t))\Big)\,dt\Big|\,;&
\end{aligned}\tag{3.105}$$

$$J_{11}\,=\,\Big|\sum_s\int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s} f(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s+Jw_s(t))\Big)\,dt\Big|\,.\tag{3.106}$$

In view of the growth condition for $f$ and the definitions of $\Omega_a$, $\Omega_c$ and $\bigcup_s\,\mathrm{Q}_s$, we arrive at

$$\begin{aligned}
J_9 &\leqslant \int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r\,\mathrm{Q}_s}\big|\,f(\,\dots\,)\,\big|\,dt \leqslant \int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r\,\mathrm{Q}_s}\big(\,A(t)+C_2\,\big)\,dt \leqslant \int_{(\,\Omega_a\,\cap\,\Omega_c\,)\,\setminus\,\cup_{s=1}^r\,\mathrm{Q}_s}\big(\,C_2+C_3\,\big)\,dt\\
&\leqslant \int_{\Omega_a\,\setminus\,\cup_{s=1}^r\,\mathrm{Q}_s}\big(\,C_2+C_3\,\big)\,dt\,\leqslant\,(C_2+C_3)\,\varepsilon\,;
\end{aligned}\tag{3.107}$$

$$\begin{aligned}
J_{11} &\leqslant \sum_s\int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s}\big|\,f(\,\dots\,)\,\big|\,dt \leqslant \sum_s\int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s}\big(\,A(t)+C_2\,\big)\,dt\\
&\leqslant \sum_s\int_{(\,\Omega_a\,\setminus\,\Omega_c\,)\,\cap\,\mathrm{Q}_s}\big(\,C_3+C_2\,\big)\,dt \leqslant \sum_s\int_{\Omega\,\setminus\,\Omega_c}\big(\,C_3+C_2\,\big)\,dt\,\leqslant\,\varepsilon\,.
\end{aligned}\tag{3.108}$$

For $J_{10}$, we obtain:

$$\begin{aligned}
J_{10} \leqslant \sum_s\int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s}\Big|\,f\big(t, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\hat z(t)+w_s(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(J\hat z(t)+Jw_s(t))\big)&\\
-f\big(t_s, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\,[\,\hat z\,]_s+w_s(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\,[\,J\hat z\,]_s+Jw_s(t))\,\big)\,\Big|\,dt&\\
+\sum_s\int_{\Omega_a\,\cap\,\Omega_c\,\cap\,\mathrm{Q}_s}\Big|\,f\big(t_s, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\,[\,\hat z\,]_s+w_s(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(\,[\,J\hat z\,]_s+Jw_s(t))\,\big)&\\
-f(t_s,[\,\hat z\,]_s,[\,J\hat z\,]_s+Jw_s(t))\,\Big|\,dt&
\end{aligned}\tag{3.109}$$

By (3.41) and (3.51), the difference of the arguments within the first member can be estimated as follows:

$$\big|\,t-t_s\,\big|\,+\,\frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,\big|\,\hat z(t)-[\,\hat z\,]_s\,\big|\,+\,\frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,\big|\,J\hat z(t)-[\,J\hat z\,]_s\,\big|\,\leqslant\,\delta_3(\varepsilon)\,\leqslant\delta_2(\varepsilon)\,.\tag{3.110}$$

For the second member, the following estimate holds:

$$\begin{aligned}
&\Big|\,\frac{\big(\delta_2(\varepsilon)\big)^2}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,[\,\hat z\,]_s\,+\,\frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,w_s(t)\,\Big|\,+\,\frac{\big(\delta_2(\varepsilon)\big)^2}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,\big|\,[\,J\hat z\,]_s+Jw_s(t)\,\big|\\
&\leqslant\,\frac{\big(\delta_2(\varepsilon)\big)^2}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\,(C_1+C_{\mathrm{K}})\,+\,\frac{c_{\mathrm{K}}}{c_{\mathrm{K}}+\big(\delta_2(\varepsilon)\big)^2}\cdot\frac{\delta_4(\varepsilon)}{3}
\end{aligned}\tag{3.111}$$

$$\leqslant \frac{c_{\mathrm{K}}}{3\,\big(c_{\mathrm{K}} + \big(\delta_2(\varepsilon)\big)^2\,\big)} \cdot \delta_2(\varepsilon) \leqslant \delta_2(\varepsilon)\,. \tag{3.112}$$

(3.110) and (3.112) give together

$$J_{10} \leqslant 2 \sum_s |\,\mathrm{Q}_s\,|\,\varepsilon \leqslant 2\,|\,\Omega_a\,|\,\varepsilon\,. \tag{3.113}$$

Finally, we apply Lemma 3.6. in order to summarize

$$\Big|\,F\Big(\frac{c_{\mathrm{K}}}{c_{\mathrm{K}} + \big(\delta_2(\varepsilon)\big)^2}\,(\hat{z} + w)\,\Big) - F^{(qc)}(\hat{x})\,\Big|$$

$$= \Big|\int_{\Omega_a\,\cap\,\Omega_c} f\Big(t, \frac{c_{\mathrm{K}}}{c_{\mathrm{K}} + \big(\delta_2(\varepsilon)\big)^2}\,(\hat{z}(t) + w(t)), \frac{c_{\mathrm{K}}}{c_{\mathrm{K}} + \big(\delta_2(\varepsilon)\big)^2}\,(J\hat{z}(t) + Jw(t))\,\Big)\,dt$$

$$- \int_{\Omega} f^{(qc)}(t, \hat{x}(t), J\hat{x}(t))\,dt\,\Big| \tag{3.114}$$

$$\leqslant C_5\,\varepsilon + J_9 + J_{10} + J_{11} \leqslant \big(C_5 + 1 + C_2 + C_3 + 2\,|\,\Omega_a\,|\,\big)\,\varepsilon\,. \tag{3.115}$$

The function

$$\frac{c_{\mathrm{K}}}{c_{\mathrm{K}} + \big(\delta_2(\varepsilon)\big)^2}\,(\hat{z} + w) \tag{3.116}$$

is admissible in (P), and the proof of Proposition 3.8. is complete. ■

This completes the proof of Theorem 1.4. ■

### c) Proof of the existence theorem 1.5.

The notion of polyconvexity is defined as follows:

**Definition 3.9. (Polyconvex function with values in $\overline{\mathbb{R}}$)** [36] *A function $r(v)\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is said to be polyconvex iff it can be represented as a composition $r(v) = h\big(g(v)\big)$ of a convex function $h$ with those mapping $g$, which assigns to every $(n, m)$-matrix $v \in \mathbb{R}^{nm}$ the vector of all its subdeterminants.*

Since (P) and $f$ satisfy all assumptions of the relaxation theorem 1.4., we have to prove that, for all fixed $(\hat{t}, \hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$, the polyconvex function $f(\hat{t}, \hat{\xi}, v)$ coincides with its lower semicontinuous quasiconvex envelope $f^{(qc)}(\hat{t}, \hat{\xi}, v)$ on the whole space $\mathbb{R}^{nm}$. The lower semicontinuity of $f(\hat{t}, \hat{\xi}, \cdot\,)$ results from Definition 1.1., 2), Part c), and by Remark c) after Definition 2.6., it holds that $f(\hat{t}, \hat{\xi}, v) = f^{(qc)}(\hat{t}, \hat{\xi}, v) = (+\infty)$ for $v \in \big(\mathbb{R}^{nm} \setminus \mathrm{K}\big)$. It remains to confirm that $f(\hat{t}, \hat{\xi}, v)$ satisfies Morrey's integral inequality where $\mathrm{dom}\,\big(\,f(\hat{t}, \hat{\xi}, \cdot\,)\,\big) = \mathrm{K}$. For $v \in \big(\mathbb{R}^{nm} \setminus \mathrm{K}\big)$, this will be implied by [ Wagner 06b ] , p. 238, Theorem 2, i); for $v \in \mathrm{K}$, we may take over the proof from [ Dacorogna 08 ] , p. 161, Proof of Theorem 5.3., Part 2. In this case, however, we may restrict ourselves to test functions $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with $v + Jx(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$ (Theorem 2.2.), and the integrals within the proof remain finite. ■

---

[36] [ Dacorogna 08 ] , p. 157, Definition 5.1., (iii).

## 4. Existence of global minimizers for the image registration problem with a polyconvex regularization term.

### a) Elastic image registration resp. elastic image matching.

Consider a rectangular domain $\Omega \subset \mathbb{R}^2$ with edges $a$ and $b$, containing the origin as the point of intersection of its diagonals.[37] Assume that two greyscale images $I_0(t)$, $I_1(t): \ \Omega \to [\,0\,,\,1\,]$ are given where $I_0$ is considered as the reference image. Then we search for a deformation $x(t): \ \Omega \to \mathbb{R}^2$, which satisfies $I_1(\,t - x(t)\,) \approx I_0(t)$, thus bringing $I_1$ in the best possible correspondence with $I_0$. The knowledge about $x$ will be further exploited e.g. in order to decide whether the objects captured in $I_1$ and $I_0$ are identical or to gain information about its possible evolution. In view of the numerous applications of imaging in modern science, engineering and medicine, this problem has to be considered as a basic problem in mathematical image processing.[38]
The determination of $x$ leads, however, to an ill-posed problem. For its solution, variational methods have been proposed, which are based on the minimization of the defect of the greyscale values[39]

$$\big(I_1(\,t - x(t)\,) \, - \, I_0(t)\,\big)^2 \tag{4.1}$$

or the difference of the normal directions to the isophotes[40]

$$\|\,\nabla I_1(\,t - x(t)\,)\,\|^2 \cdot \|\,\nabla I_0(t)\,\|^2 \, - \, \big(\nabla I_1(\,t - x(t)\,)^{\mathrm{T}}\,\nabla I_0(t)\,\big)^2 \tag{4.2}$$

together with a regularization term involving the first-order generalized partial derivatives of $x$. The corresponding variational problems can be stated within Sobolev spaces as follows:

$$(\mathrm{V})_1: \quad F(x) = \int_\Omega \Big(I_1(\,t - x(t)\,) - I_0(t)\Big)^2\,dt \, + \, \mu \cdot \int_\Omega r\big(\,Jx(t)\,\big)\,dt \, \longrightarrow \, \inf!\,; \ x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \tag{4.3}$$

resp.

$$\begin{aligned}(\mathrm{V})_2: \quad F(x) = \int_\Omega \Big(\,\|\,\nabla I_1(\,t - x(t)\,)\,\|^2\,\|\,\nabla I_0(t)\,\|^2 \, - \, \big(\nabla I_1(\,t - x(t)\,)^{\mathrm{T}}\,\nabla I_0(t)\,\big)^2\,\Big)\,dt \\ + \mu \cdot \int_\Omega r\big(\,Jx(t)\,\big)\,dt \, \longrightarrow \, \inf!\,; \ x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\end{aligned} \tag{4.4}$$

with (sufficiently regular, if necessary presmoothed) image data $I_0(t)$, $I_1(t): \ \Omega \to [\,0\,,\,1\,]$,[41] $2 \leqslant p < \infty$, a regularization parameter $\mu > 0$ and integrands $r(v)$ originating from models of elasticity theory as convex or polyconvex functions.[42] The comprehension of a convex gradient constraint

$$Jx(t) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \ \ (\forall)\,t \in \Omega \tag{4.5}$$

---

[37] In the literature, the image registration problem has been studied on a rectangular parallelepiped $\Omega \subset \mathbb{R}^3$ as well. Here we confine ourselves to the two-dimensional case.

[38] Cf. the introduction in [ Modersitzki 04 ] , pp. 1 ff. and pp. 21 ff.

[39] See e. g. [ Hermosillo/Chefd'hotel/Faugeras 02 ] , p. 331, [ Henn/Witsch 00 ] , [ Henn/Witsch 01 ] , [ Modersitzki 04 ] , pp. 77 ff. [ Alvarez/Weickert/Sánchez 00 ] aims for the determination of a “optical flow field”, which is, in fact, a deformation $x$ as well. Cf. also [ Wagner 07a ] , p. 16.

[40] If one cannot expect a correspondence between the intensities of $I_0$ and $I_1$ ( “multimodal matching” ) then this approach leads to a matching of the edge structures within the images. See [ Droske/Rumpf 04 ] , [ Gallardo/Meju 03 ] , [ Haber/Modersitzki 07 ] .

[41] In order to guarantee the existence of the integrals within the objectives, it should be demanded that additionally $t - x(t) \in \Omega$ holds for almost all $t \in \Omega$. This condition, however, can be eliminated if the image data $I_0$ and $I_1$ are embedded into a sufficiently large black frame, i. e. they will be extended by zero to $\mathbb{R}^2 \setminus \Omega$ (cf. [ Henn/Witsch 01 ] , p. 1078).

[42] Examples will be treated in detail in the following subsections.

with a convex body $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ converts (V) into a multidimensional control problem of the type (P). Then in analogy to [BRUNE/MAURER/WAGNER 08] and [FRANEK/FRANEK/MAURER/WAGNER 08] the simultaneous detection of the "discontinuities" of $x$ (i. e. regions with large gradients $\nabla x_1$, $\nabla x_2$) will be made possible where the indicator corresponds to the distance $\mathrm{Dist}\,\big(\, Jx(t)\,,\, \partial\mathrm{K}\,\big)$.

**b) Image registration as a multidimensional control problem with convex regularization.**

Since human tissue behaves approximately linear-elastic, within registration problems from medical imaging the use of convex regularization terms from linear elasticity is highly reasonable.[43] In this case, the addition of a convex gradient restriction is mandatory since the modulus of the resulting shear stress, which is proportional to $\|\, Jx \,\|$, must be uniformly bounded. Then from $(\mathrm{V})_1$ and $(\mathrm{V})_2$, we obtain the following optimal control problems:

$$(\mathrm{P})_1: \quad F(x) = \int_\Omega \Big( I_1(\, t - x(t)\,) - I_0(t) \Big)^2 \, dt \, + \, \mu \cdot \int_\Omega \sum_{i,j=1}^{2} \Big( \frac{\partial x_i(t)}{\partial t_j} \, + \, \frac{\partial x_j(t)}{\partial t_i} \Big)^2 \, dt \, \longrightarrow \, \inf!\,; \tag{4.6}$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,; \;\; Jx(t) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \;\; (\forall)\, t \in \Omega$$

resp.

$$(\mathrm{P})_2: \quad F(x) = \int_\Omega \Big( \big\|\, \nabla I_1(\, t - x(t)\,)\,\big\|^2 \, \big\|\, \nabla I_0(t)\,\big\|^2 \, - \, \big( \nabla I_1(\, t - x(t)\,)^{\mathrm{T}} \, \nabla I_0(t)\,\big)^2 \Big) \, dt \tag{4.7}$$

$$+ \, \mu \cdot \int_\Omega \sum_{i,j=1}^{2} \Big( \frac{\partial x_i(t)}{\partial t_j} \, + \, \frac{\partial x_j(t)}{\partial t_i} \Big)^2 \, dt \, \longrightarrow \, \inf!\,; \; x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,; \; Jx(t) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \;\; (\forall)\, t \in \Omega$$

with $2 \leqslant p < \infty$ and $\mu > 0$. $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ is a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$; the properties of the image data $I_0,\, I_1: \, \Omega \to [\,0\,,\,1\,]$ will be made precise in the following theorem.

**Theorem 4.1. (Existence theorem for $(\mathrm{P})_1$ and $(\mathrm{P})_2$)**

*1) Consider the problem* $(\mathrm{P})_1$ *with the above mentioned assumptions about the data. Assume further that* $I_0 \in L^\infty(\Omega, \mathbb{R})$ *and* $I_1 \in C_0^0(\Omega, \mathbb{R})$. *Then* $(\mathrm{P})_1$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$.

*2) Consider the problem* $(\mathrm{P})_2$ *with the above mentioned assumptions about the data. Assume further that* $I_0 \in W_0^{1,\infty}(\Omega, \mathbb{R})$ *and* $I_1 \in C_0^1(\Omega, \mathbb{R})$. *Then* $(\mathrm{P})_2$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ *as well.*

**Proof. 1)** The assumed zero boundary condition allows us to extend the image data $I_0$, $I_1$ by zero to $\mathbb{R}^2 \setminus \Omega$. With the convex body K, we associate the convex indicator function $\varrho_{\mathrm{K}}(v): \, \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}$ defined by

$$\varrho_{\mathrm{K}}(v) \; = \; \begin{cases} \;\; 0 & \big|\; v \in \mathrm{K}\,; \\ (+\infty) & \big|\; v \in \big( \mathbb{R}^{2\times 2} \setminus \mathrm{K}\,\big)\,. \end{cases} \tag{4.8}$$

On $\Omega \, \times \, \mathbb{R}^2 \, \times \, \mathbb{R}^{2\times 2}$, we define the function

$$f_1(t, \xi, v) \; = \; \big( I_1(t - \xi) - I_0(t)\,\big)^2 \, + \, \mu \cdot \sum_{i,j=1}^{2} \big( v_{ij} + v_{ji} \,\big)^2 \, + \, \varrho_{\mathrm{K}}(v) \tag{4.9}$$

with the properties a) – c) from Definition 1.1., 2). Since $I_0(t)$, $I_1(t - \xi) \in [\,0\,,\,1\,]$ it holds that

$$\big|\, f_1(t, \xi, v)\,\big| \, \leqslant \, I_1(t - \xi)^2 \, + \, I_0(t)^2 \, + \, 2\, I_0(t)\, I_1(t - \xi) \, + \, \mu \cdot \sum_{i,j=1}^{2} \big( \, v_{ij} + v_{ji} \,\big)^2 \tag{4.10}$$

$$\leqslant \, 4 \, + \, \mu \cdot \sum_{i,j=1}^{2} \big( \, v_{ij} + v_{ji} \,\big)^2 \quad \forall\, (t, \xi, v) \in \big( \, \Omega \, \setminus \, \mathrm{N} \,\big) \, \times \, \mathbb{R}^2 \, \times \mathrm{K}\,, \tag{4.11}$$

---

[43] We follow [HENN/WITSCH 01], p. 1079 f.

and $f_1$ satisfies the growth condition d) from Definition 1.1., 2) with $A(t) \equiv 4$ and $B(\xi, v) = \mu \cdot \sum_{i,j=1}^{2} (\, v_{ij} + v_{ji}\,)^2$. Finally, $f_1(\hat{t}, \hat{\xi}, v)$ is convex with respect to $v$ for all fixed $(\hat{t}, \hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^2$, and by Remark c) after Definition 2.6., it follows for all $v \in \mathbb{R}^{2\times 2}$:

$$f_1^c(\hat{t}, \hat{\xi}, v) \leqslant f_1^{(qc)}(\hat{t}, \hat{\xi}, v) \leqslant f_1(\hat{t}, \hat{\xi}, v) \leqslant f_1^c(\hat{t}, \hat{\xi}, v)\,. \tag{4.12}$$

Now the claim results from Theorem 1.4.

**2)** On $\Omega \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$, we define the function

$$f_2(t, \xi, v) = \big\| \nabla I_1(t-\xi) \big\|^2 \cdot \big\| \nabla I_0(t) \big\|^2 - \big( \nabla I_1(t-\xi)^{\mathrm{T}} \nabla I_0(t) \big)^2 + \mu \cdot \sum_{i,j=1}^{2} \big( v_{ij} + v_{ji} \big)^2 + \varrho_{\mathrm{K}}(v)\,, \tag{4.13}$$

admitting the properties a) – c) from Definition 1.1., 2). In consequence of our assumptions, $\| \nabla I_0 \|$ is bounded almost everywhere and $\| \nabla I_1 \|$ is bounded everywhere by a constant $C > 0$, and we obtain the estimate

$$\big| f_2(t, \xi, v) \big| \leqslant C^4 \Big( 1 + \big| \cos \sphericalangle (\, \nabla I_1(t-\xi)\,,\, \nabla I_0(t)\,) \big| \Big) + \mu \cdot \sum_{i,j=1}^{2} \big( v_{ij} + v_{ji} \big)^2 \qquad \forall\, (t, \xi, v) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^2 \times \mathrm{K}\,. \tag{4.14}$$

Consequently, $f_2$ satisfies the growth condition d) with $A(t) \equiv 2\,C^4$ and $B(\xi, v) = \mu \cdot \sum_{i,j=1}^{2} (\, v_{ij} + v_{ji}\,)^2$. Again $f_2(\hat{t}, \hat{\xi}, v)$ is a convex function with respect to $v$ for all fixed $(\hat{t}, \hat{\xi}) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^2$, and the proof can be finished as in Part 1). ■

**c) Image registration as a multidimensional control problem with polyconvex regularization.**

As an alternative approach, the image registration problem has been considered with polyconvex regularization terms instead of convex ones. Terms of this kind correspond with hyperelastic material laws. Additionally, the further restriction to orientation-preserving, bijective deformations (i. e. $\mathrm{Det}\,(Jx) > 0$) has been proposed.[44] Leaving aside the latter condition for the moment, we arrive at the following optimal control problems:

$$(\mathrm{P})_3: \quad F(x) = \int_\Omega \Big( I_1(\,t - x(t)\,) - I_0(t) \Big)^2 dt + \mu \cdot \int_\Omega \Big( c_1 \, \| \, Jx(t) \, \|^{\,p} + c_2 \, \big( \mathrm{Det}\, Jx(t) \big)^2 \Big) \, dt \longrightarrow \inf!\,; \\ x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,;\; Jx(t) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \;\; (\forall)\, t \in \Omega \tag{4.15}$$

resp.

$$(\mathrm{P})_4: \quad F(x) = \int_\Omega \Big( \big\| \nabla I_1(\,t - x(t)\,) \big\|^2 \big\| \nabla I_0(t) \big\|^2 - \big( \nabla I_1(\,t - x(t)\,)^{\mathrm{T}} \nabla I_0(t) \big)^2 \Big) \, dt \tag{4.16}$$

$$+ \mu \cdot \int_\Omega \Big( c_1 \, \| \, Jx(t) \, \|^{\,p} + c_2 \, \big( \mathrm{Det}\, Jx(t) \big)^2 \Big) \, dt \longrightarrow \inf!\,;\; x \in W_0^{1,p}(\Omega, \mathbb{R}^2)\,;\; Jx(t) \in \mathrm{K} \subset \mathbb{R}^{2\times 2} \;\; (\forall)\, t \in \Omega$$

with $2 \leqslant p < \infty$, $\mu > 0$ and weigths $c_1$, $c_2 > 0$. $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ is again a convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$. We will use the matrix norm $\| M \| = \mathrm{trace}\,(\, M^{\mathrm{T}} M\,)$. The properties of the image data $I_0$, $I_1 :\, \Omega \to [\,0\,,\,1\,]$ will be described in the followning theorem.

---

[44] [DROSKE/RUMPF 04], p. 673 f.

**Theorem 4.2. (Existence theorem for $(\mathrm{P})_3$ and $(\mathrm{P})_4$)**

*1) Consider the problem* $(\mathrm{P})_3$ *with the above mentioned assumptions about the data. Assume further that* $I_0 \in L^\infty(\Omega, \mathbb{R})$ *and* $I_1 \in C_0^0(\Omega, \mathbb{R})$. *Then* $(\mathrm{P})_3$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$.

*2) Consider the problem* $(\mathrm{P})_4$ *with the above mentioned assumptions about the data. Assume further that* $I_0 \in W_0^{1,\infty}(\Omega, \mathbb{R})$ *and* $I_1 \in C_0^1(\Omega, \mathbb{R})$. *Then* $(\mathrm{P})_4$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$ *as well.*

**Proof. 1)** Again we may assume that the image data $I_0$, $I_1$ have been extended by zero to $\mathbb{R}^2 \setminus \Omega$. On $\Omega \times \mathbb{R}^2 \times \mathbb{R}^{2\times 2}$, we define the function

$$f_3(t, \xi, v) = \big( I_1(t-\xi) - I_0(t) \big)^2 + \mu \cdot \Big( c_1 \, \| \, v \, \|^p + c_2 \, \big( \mathrm{Det}\, v \big)^2 \Big) + \varrho_{\mathrm{K}}(v) \,, \tag{4.17}$$

which satisfies a) – c) from Definition 1.1., 2). Analogously to the proof of Theorem 4.1., 1), since

$$\big|\, f_3(t, \xi, v) \,\big| \leqslant 4 + \mu \cdot \Big( c_1 \, \| \, v \, \|^p + c_2 \, \big( \mathrm{Det}\, v \big)^2 \Big) \quad \forall \, (t, \xi, v) \in \big( \Omega \setminus \mathrm{N} \big) \times \mathbb{R}^2 \times \mathrm{K} \,, \tag{4.18}$$

the growth condition d) is satisfied as well with $A(t) \equiv 4$ and $B(\xi, v) = \mu \, \big( c_1 \, \| \, v \, \|^p + c_2 \, (\mathrm{Det}\, v \,)^2 \, \big)$. Note that, for every fixed $(\hat{t}, \hat{\xi}) \in \big( \Omega \setminus \mathrm{N} \big)$, the function $f_3(\hat{t}, \hat{\xi}, v)$ is polyconvex with respect to $v$ as the sum of the polyconvex functions $\big( I_1(\hat{t} - \hat{\xi}) - I_0(\hat{t}) \big)^2 + \mu \cdot \big( c_1 \, \| \, v \, \|^p + c_2 \, (\mathrm{Det}\, v \,)^2 \big)$ and $\varrho_{\mathrm{K}}(v)$. Consequently, Theorem 1.5. can be applied, and $(\mathrm{P})_3$ admits a global minimizer.

**2)** We may argue in analogy to Part 1) and the proof of Theorem 4.1., noting that, for all $(t, \xi, v) \in \big( \Omega \setminus \mathrm{N} \big) \times \mathbb{R}^2 \times \mathrm{K}$, the integrand

$$f_4(t, \xi, v) = \big\| \, \nabla I_1(t-\xi) \, \big\|^2 \cdot \big\| \, \nabla I_0(t) \, \big\|^2 - \big( \, \nabla I_1(t-\xi)^{\mathrm{T}} \, \nabla I_0(t) \, \big)^2 + \mu \cdot \Big( c_1 \, \big\| \, v \, \big\|^p + c_2 \, \big( \, \mathrm{Det}\, v \, \big)^2 \Big) + \varrho_{\mathrm{K}}(v) \tag{4.19}$$

obeys the estimate

$$\big|\, f_4(t, \xi, v) \,\big| \leqslant C^4 \Big( 1 + \big| \cos \sphericalangle \big( \, \nabla I_1(t-\xi) \, , \, \nabla I_0(t) \, \big) \, \big| \Big) + \mu \cdot \Big( c_1 \, \big\| \, v \, \big\|^p + c_2 \, \big( \, \mathrm{Det}\, v \, \big)^2 \Big) + \varrho_{\mathrm{K}}(v) \,. \; \blacksquare \tag{4.20}$$

**d) Image registration as a multidimensional control problem with the constraint $\mathrm{Det}\,(Jx) > 0$ and polyconvex regularization.**

We consider $(\mathrm{P})_3$ together with the additional restriction $\mathrm{Det}\,(Jx) > 0$ and the polyconvex penalty term [45]

$$-c_3 \cdot \ln\big( \, \mathrm{Det}\, Jx(t) \, \big) \tag{4.21}$$

with $c_3 > 0$ within the objective. This leads to the problem

$$(\mathrm{P})_5 : \quad F(x) = \int_\Omega \Big( I_1(\, t - x(t) \,) - I_0(t) \Big)^2 \, dt + \mu \cdot \int_\Omega \Big( c_1 \, \big\| \, Jx(t) \, \big\|^{\,p} + c_2 \, \big( \, \mathrm{Det}\, Jx(t) \, \big)^2 \tag{4.22}$$

$$- \, c_3 \cdot \ln\big( \, \mathrm{Det}\, Jx(t) \, \big) \Big) \, dt \longrightarrow \inf ! \,;$$

$$x \in W_0^{1,p}(\Omega, \mathbb{R}^2) \,; \; Jx(t) \in \mathrm{K} \cap \big\{ \, v \in \mathbb{R}^{2\times 2} \; \big| \; \mathrm{Det}\,(v) > 0 \, \big\} \subset \mathbb{R}^{2\times 2} \;\; (\forall) \, t \in \Omega \,, \tag{4.23}$$

which does not match the analytical situation described in Section 1.a) since the compact control domain K has been intersected with an open set. Nevertheless, an existence theorem for $(\mathrm{P})_5$ can be easily derived from Theorem 4.2., 1).

---

[45] [Droske/Rumpf 04], p. 674, (3.2).

**Theorem 4.3. (Existence theorem for** $(\mathrm{P})_5$ **)** *Consider the problem* $(\mathrm{P})_5$ *under the following assumptions about the data:* $2 \leqslant p < \infty$, $I_0 \in L^\infty(\Omega, \mathbb{R})$, $I_1 \in C_0^0(\Omega, \mathbb{R})$, $\mu > 0$, $c_1$, $c_2$, $c_3 > 0$, *and* $\mathrm{K} \subset \mathbb{R}^{2\times 2}$ *is a convex body with* $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$. *Then* $(\mathrm{P})_5$ *admits a global minimizer* $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^2)$.

**Proof.** The assumptions about $(\mathrm{P})_5$ guarantee the existence of feasible solutions, e. g.

$$x(t) \;=\; \varepsilon \cdot \mathrm{Min}\,\Big(\mathrm{Dist}\,(t, \partial\Omega)\,,\, \frac{a}{4}\,,\, \frac{b}{4}\Big) \cdot \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \tag{4.24}$$

for sufficiently small $\alpha > 0$ and $\varepsilon > 0$. Since $Jx(t) \in \mathrm{K}\ (\forall)\, t \in \Omega$, the objective is bounded from below. Consequently, $(\mathrm{P})_5$ admits a minimizing sequence $\{\, x^N \,\}$, $W_0^{1,p}(\Omega, \mathbb{R}^2) \cap W_0^{1,\infty}(\Omega, \mathbb{R}^2)$, whose members are feasible in $(\mathrm{P})_3$ as well. Along a subsequence $\{\, x^{N'} \,\} \subseteq \{\, x^N \,\}$ with $x^{N'} \overset{*}{\rightharpoonup}_{L^\infty(\Omega,\mathbb{R}^2)} \hat{x}$ and $Jx^{N'} \overset{*}{\rightharpoonup}_{L^\infty(\Omega,\mathbb{R}^{2\times 2})} J\hat{x}$, we observe by Theorems 4.2., 1) and 1.4.

$$\begin{aligned} &\int_\Omega \Big( I_1(\, t - \hat{x}(t)\,) - I_0(t) \Big)^2 \, dt \,+\, \mu \cdot \int_\Omega \Big( c_1 \, \|\, J\hat{x}(t) \,\|^{\,p} \,+\, c_2 \,\big(\mathrm{Det}\, J\hat{x}(t)\,\big)^2 \Big)\, dt \\ &\leqslant \liminf_{N'\to\infty} \int_\Omega \Big( I_1(\, t - x^{N'}(t)\,) - I_0(t) \Big)^2 \, dt \,+\, \mu \cdot \int_\Omega \Big( c_1 \, \|\, Jx^{N'}(t) \,\|^{\,p} \,+\, c_2 \,\big(\mathrm{Det}\, Jx^{N'}(t)\,\big)^2 \Big)\, dt\,. \end{aligned} \tag{4.25}$$

To the polyconvex integrand $f_5 :\ \mathbb{R}^{2\times 2} \to \overline{\mathbb{R}}$ defined by

$$f_5(v) \;=\; \begin{cases} -\mu\, c_3\, \ln\,(\,\mathrm{Det}\, v\,) & \big|\ \mathrm{Det}\, v > 0\,; \\ \quad (+\infty) & \big|\ \mathrm{Det}\, v \leqslant 0\,, \end{cases} \tag{4.26}$$

we may apply [ Dacorogna 08 ], p. 391 f., Theorem 8.16, together with the Remark ibid., p. 392: After choosing $m = n = 2$ and $p = 2$, the convex function $h(v, \delta) :\ \mathbb{R}^5 \to \overline{\mathbb{R}}$ defined by

$$h(v, \delta) \;=\; \begin{cases} -\mu\, c_3\, \ln\delta & \big|\ \delta > 0\,; \\ \quad (+\infty) & \big|\ \delta \leqslant 0 \end{cases} \tag{4.27}$$

is bounded from below by $h(v, \delta) \geqslant -\mu\, c_3\, \delta$ where the constant function $(-\mu\, c_3)$ belongs to $L^2(\Omega, \mathbb{R})$. For the subsequence $\{\, x^{N'} \,\}$, it holds $Jx^{N'} \rightharpoonup_{L^2(\Omega,\mathbb{R}^{2\times 2})} J\hat{x}$ as well, and from the cited theorem we conclude that

$$-\,\mu \int_\Omega c_3\, \ln\big(\,\mathrm{Det}\, J\hat{x}(t)\,\big)\, dt \;\leqslant\; \liminf_{N'\to\infty} \Big( -\,\mu \int_\Omega c_3\, \ln\big(\,\mathrm{Det}\, Jx^{N'}(t)\,\big)\, dt \Big)\,. \tag{4.28}$$

(4.25) and (4.28) give together the existence of a global minimizer of $(\mathrm{P})_5$. ∎

The existence of a global minimizer for the modified problem $(\mathrm{P})_4$ can be confirmed in a completely analogous way if the assumptions about the data are carried over from Theorem 4.2., 2).

## Acknowledgement.

I wish to express my gratitude to Prof. B. Dacorogna and Prof. S. Luckhaus for their invitations to stay at the EPF Lausanne and the Max Planck Institute for Mathematics in the Sciences, Leipzig, where the present paper has been finished in autumn 2008.

## References.

1. [ACERBI/FUSCO 84] Acerbi, E.; Fusco, N.: *Semicontinuity problems in the calculus of variations.* Arch. Rat. Mech. Anal. **86** (1984), 125 – 145
2. [ALVAREZ/WEICKERT/SÁNCHEZ 00] Alvarez, L.; Weickert, J.; Sánchez, J.: *Reliable estimation of dense optical flow fields with large displacements.* Int. J. Computer Vision **39** (2000), 41 – 56
3. [AUBERT/KORNPROBST 06] Aubert, G.; Kornprobst, P.: *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations.* Springer; New York etc. 2006, 2nd ed.
4. [BALL/MURAT 84] Ball, J. M.; Murat, F.: *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals.* J. Funct. Anal. **58** (1984), 225 – 253
5. [BOURBAKI 52] Bourbaki, N.: *Éléments de Mathématique. Livre VI: Intégration, Chap. I – IV.* Hermann; Paris 1952
6. [BROKATE 85] Brokate, M.: *Pontryagin's principle for control problems in age-dependent population dynamics.* J. Math. Biology **23** (1985), 75 – 101
7. [BRØNDSTED 83] Brøndsted, A.: *An Introduction to Convex Polytopes.* Springer; New York - Heidelberg - Berlin 1983
8. [BRUNE/MAURER/WAGNER 08] Brune, C.; Maurer, H.; Wagner, M.: *Edge detection within optical flow via multidimensional control.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-02/2008. Submitted: SIAM Journal on Imaging Sciences
9. [BUTTAZZO 89] Buttazzo, G.: *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Longman; Harlow 1989 (Pitman Research Notes in Mathematics, Vol. 207)
10. [CONTI 08] Conti, S.: *Quasiconvex functions incorporating volumetric constraints are rank-one convex.* J. Math. Pures Appl. **90** (2008), 15 – 30
11. [DACOROGNA 04] Dacorogna, B.: *Introduction to the Calculus of Variations.* Imperial College Press; London 2004
12. [DACOROGNA 08] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Springer; New York etc. 2008, 2nd ed.
13. [DACOROGNA/MARCELLINI 97] Dacorogna, B.; Marcellini, P.: *General existence theorems for Hamilton-Jacobi equations in the scalar and vectorial case.* Acta Math. **178** (1997), 1 – 37
14. [DROSKE/RUMPF 04] Droske, M.; Rumpf, M.: *A variational approach to nonrigid morphological image registration.* SIAM J. Appl. Math. **64** (2004), 668 – 687
15. [DUNFORD/SCHWARTZ 88] Dunford, N.; Schwartz, J. T.: *Linear Operators. Part I: General Theory.* Wiley-Interscience; New York etc. 1988
16. [EKELAND/TÉMAM 99] Ekeland, I.; Témam, R.: *Convex Analysis and Variational Problems.* SIAM; Philadelphia 1999, 2nd ed.
17. [EVANS/GARIEPY 92] Evans, L. C.; Gariepy, R. F.: *Measure Theory and Fine Properties of Functions.* CRC Press; Boca Raton etc. 1992
18. [FEICHTINGER/TRAGLER/VELIOV 03] Feichtinger, G.; Tragler, G.; Veliov, V. M.: *Optimality conditions for age-structured control systems.* J. Math. Anal. Appl. **288** (2003), 47 – 68
19. [FRANEK/FRANEK/MAURER/WAGNER 08] Franek, L.; Franek, M.; Maurer, H.; Wagner, M.: *Image restoration and simultaneous edge detection by optimal control methods.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-05/2008. Submitted: Optim. Contr. Appl. Meth.
20. [GALLARDO/MEJU 03] Gallardo, L. A.; Meju, M. A.: *Characterization of heterogeneous near-surface materials by joint 2D inversion of dc resistivity and seismic data.* Geophysical Research Letters **30** (2003) 13, 1658, 1 - 1 – 1 - 4
21. [HABER/MODERSITZKI 07] Haber, E.; Modersitzki, J.: *Intensity gradient based registration and fusion of multi-modal images.* Methods of Information in Medicine **46** (2007), 292 – 299
22. [HENN/WITSCH 00] Henn, S.; Witsch, K.: *A multigrid approach for minimizing a nonlinear functional for digital image matching.* Computing **64** (2000), 339 – 348

23. [Henn/Witsch 01] Henn, S.; Witsch, K.: *Iterative multigrid regularization techniques for image matching.* SIAM J. Sci. Comput. **23** (2001), 1077 – 1093
24. [Hermosillo/Chefd'hotel/Faugeras 02] Hermosillo, G.; Chefd'hotel, C.; Faugeras, O.: *Variational methods for multimodal image matching.* Int. J. Computer Vision **50** (2002), 329 – 343
25. [Hinterberger/Scherzer/Schnörr/Weickert 02] Hinterberger, W.; Scherzer, O.; Schnörr, C.; Weickert, J.: *Analysis of optical flow models in the framework of the calculus of variations.* Num. Funct. Anal. Optim. **23** (2002), 69 – 89
26. [Kinderlehrer/Pedregal 91] Kinderlehrer, D.; Pedregal, P.: *Characterizations of Young measures generated by gradients.* Arch. Rat. Mech. Anal. **115** (1991), 329 – 365
27. [Marcellini/Sbordone 80] Marcellini, P.; Sbordone, C.: *Semicontinuity problems in the calculus of variations.* Nonlinear Analysis **4** (1980), 241 – 257
28. [Modersitzki 04] Modersitzki, J.: *Numerical Methods for Image Registration.* Oxford University Press; Oxford 2004
29. [Morrey 66] Morrey, C. B.: *Multiple Integrals in the Calculus of Variations.* Springer; Berlin - Heidelberg - New York 1966 (Grundlehren 130)
30. [Pickenhain/Wagner 00] Pickenhain, S.; Wagner, M.: *Critical points in relaxed deposit problems.* In: Ioffe, A.; Reich, S.; Shafrir, I. (Eds.): Calculus of Variations and Optimal Control, Technion 98, Vol. II (Research Notes in Mathematics, Vol. 411). Chapman & Hall / CRC Press; Boca Raton etc. 2000, 217 – 236
31. [Roubíček 97] Roubíček, T.: *Relaxation in Optimization Theory and Variational Calculus.* De Gruyter; Berlin - New York 1997
32. [Schneider 93] Schneider, R.: *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press; Cambridge 1993
33. [Ting 69a] Ting, T. W.: *Elastic-plastic torsion of convex cylindrical bars.* J. Math. Mech. **19** (1969), 531 – 551
34. [Ting 69b] Ting, T. W.: *Elastic-plastic torsion problem III.* Arch. Rat. Mech. Anal. **34** (1969), 228 – 244
35. [Wagner 96] Wagner, M.: *Erweiterungen des mehrdimensionalen Pontrjaginschen Maximumprinzips auf meßbare und beschränkte sowie distributionelle Steuerungen.* PhD thesis; University of Leipzig 1996
36. [Wagner 06a] Wagner, M.: *Mehrdimensionale Steuerungsprobleme mit quasikonvexen Integranden.* Habilitation thesis. BTU Cottbus 2006
37. [Wagner 06b] Wagner, M.: *Nonconvex relaxation properties of multidimensional control problems.* In: Seeger, A. (Ed.): Recent Advances in Optimization. Springer; Berlin etc. 2006 (Lecture Notes in Economics and Mathematical Systems 563), 233 – 250
38. [Wagner 06c] Wagner, M.: *On the lower semicontinuous quasiconvex envelope for unbounded integrands (I).* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-04/2006. To appear in: ESAIM: Control, Optimisation and Calculus of Variations
39. [Wagner 07a] Wagner, M.: *Pontryagin's maximum principle for multidimensional control problems in image processing.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-10/2007. To appear in: J. Optim. Theory Appl.
40. [Wagner 07b] Wagner, M.: *Quasiconvex relaxation of multidimensional control problems.* BTU Cottbus, Preprint-Reihe Mathematik, Preprint Nr. M-11/2007. To appear in: Adv. Math. Sci. Appl.
41. [Wagner 08] Wagner, M.: *Jensen's inequality for the lower semicontinuous quasiconvex envelope and relaxation of multidimensional control problems.* EPF Lausanne, Publications de l'Institut d'analyse et calcul scientifique, Préimpression No. 01.2008. Submitted: J. Math. Anal. Appl.

**Last modification.** 17. 10. 2008

**Author's address.** Marcus Wagner, Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, D-04103 Leipzig, Germany.

**Homepage / e-mail.** `www.thecitytocome.de` / `marcus.wagner@mis.mpg.de`