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Evolution by mean curvature flow in
sub-Riemannian geometries: a stochastic
approach

by

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EVOLUTION BY MEAN CURVATURE FLOW IN SUB-RIEMANNIAN GEOMETRIES: A STOCHASTIC APPROACH.

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ABSTRACT. We study the phenomenon of evolution by horizontal mean curvature flow in sub-Riemannian geometries. We use a stochastic approach to prove the existence of a generalized evolution in these spaces. In particular we show that the value function of suitable family of stochastic control problems solves in the viscosity sense the level set equation for the evolution by horizontal mean curvature flow.

1. INTRODUCTION.

In Euclidean spaces, the motion by mean curvature flow of a hypersurface is a geometrical evolution such that the normal velocity at each point of the hypersurface is equal to mean curvature at that point. Unfortunately, even smooth surfaces can develop singularities in finite time, so a weak notion of evolution is necessary.

A well-known example in \mathbb{R}^3 is the surface given by two huge spheres, smoothly connected by a long straight cylinder. When that surface evolves by mean curvature, it splits, in finite time, in two connected components which are topological spheres. At the moment when this happens, the surface intersects itself. In the points of the self-intersection, the normal and so the mean curvature are not defined (e.g. [Gr]).

Different notions of generalized evolutions have been introduced in order to study the evolution of surfaces beyond the formation of singularities. Brakke defined in [Br] a varifold-based concept of weak solution which provides existence but no uniqueness, while a variational approach was developed by Almgren, Taylor and Wang [ATW] and Luckhaus and Sturzenhecker [LS]. Another approach, the so-called barrier solutions, has been introduced by E. De Giorgi [DG] and developed by G. Bellettini, M. Paolini and M. Novaga [BP2, BN], while a more recent approach was found by G. Barles and P.E. Souganidis, [BS].

The notion that we are going to use follows a nonlinear-PDE-approach, found for the first time in 1991, independently, by Chen-Giga-Goto ([CGG]) and Evans- Spruck ([ES]). Roughly speaking, the idea consists in associating a PDE to a smooth hypersurface evolving by mean curvature flow such that the function (of space and time) which solves this PDE has level sets which evolve by mean curvature flow. Then one can define the solutions of the “generalized evolution by mean curvature flow” as the hypersurface given by the zero-level sets of the viscosity solution of this PDE. In this paper we study the corresponding evolution in sub-Riemannian geometries with the help of stochastic control methods.

Sub-Riemannian geometries are degenerate Riemannian spaces where the Riemannian inner product is defined just on a sub-bundle of the tangent bundle. To be more

precise, we will consider X_1, \dots, X_m smooth vector fields on \mathbb{R}^n and a Riemannian inner product defined on the distribution \mathcal{H} generated by such vector fields. We assume that the vector fields verify the Hörmander condition which means that the associated Lie algebra is equal to \mathbb{R}^n at any point. This condition has many important consequences. The main one is that we can always connect two points by an admissible path. i.e. an absolutely continuous curve such that the velocity belongs to the distribution for almost any time (Chow's Theorem). Therefore we can always define an associated distance $d(x, y)$ on the whole \mathbb{R}^n . These spaces are topologically equivalent to the Euclidean \mathbb{R}^n but the metric is not. In fact, instead of the equivalence, we have only the following inequality $c|x - y| \leq d(x, y) \leq C|x - y|^{\frac{1}{k}}$, for some $c, C > 0$ constants and $k > 1$. For some recent applications of these geometries we refer to the papers by Citti and Sarti on the visual cortex ([CS]). In a sub-Riemannian space it is possible to define intrinsic derivatives of any order taking the derivatives along the vector fields X_1, \dots, X_m . That allows us to write differential operators like Laplacian, infinite-Laplacian etc, using intrinsic derivatives. We call these operators "horizontal". In particular we can define a notion of horizontal mean curvature flow.

While there are many results for evolution by mean curvature flow in the Euclidean setting, only little is known in these degenerate spaces. This evolution in a sub-Riemannian manifold is very different from the corresponding Euclidean motion, in particular because of the existence of the so-called characteristic points, which are points where the Euclidean normal is perpendicular to the horizontal space and so not admissible. Even if their geometrical meaning is very different, at the level of the level-set PDE, they look almost like the Euclidean "singularities," i.e. like the points where the level-set function has vanishing gradient. While in the Euclidean case the only "problematic" value of the gradient is zero, the corresponding set of gradients in our case is space-dependent and has, at each point, nonzero dimension. The different nature of these degeneracies creates serious difficulties in applying most of those techniques which are known to work for the euclidean setting. To avoid the problems created from the presence of these singularities, we will use a stochastic approach for showing existence of solutions.

A connection between a certain stochastic control problem and a large class of geometric evolution equations, including the (Euclidean) evolution by mean curvature flow, has been found by Buckdahn, Cardaliaguet and Quincampoix (in [BCQ]) and Soner and Touzi (in [ST2, ST3]). The control, loosely speaking, constrains the increments of the stochastic process to a lower dimensional subspace of \mathbb{R}^n , while the cost functional consists only of the terminal cost but involves an essential supremum over the probability space. It turns out that the value function solves the level set equation associated with the geometric evolution. Moreover, one can show that the set of points from which the initial hypersurface can be reached almost surely in a given time by choosing an appropriate control coincides with the set evolving by mean curvature flow. This stochastic approach generalizes very naturally to sub-Riemannian geometries. Instead of constraining the Euclidean Brownian motion, we use an intrinsic Brownian motion associated with the sub-Riemannian geometry. This allows us to obtain certain existence results in general sub-Riemannian manifolds which were previously unknown.

Our construction of controlled paths yields an analogue to the processes considered

for the Euclidean case in [BCQ, ST2, ST3], which could be called *locally codimension one constrained Brownian motion*. In the Euclidean case, any control $\nu(s)$, taking values in the space of co-rank-one orthogonal projections induces a *locally codimension one constrained Brownian motion* B^ν as solution of the following Itô SDE $dB^\nu = \nu(s)dB$. In the present sub-Riemannian case, in order to define the locally codimension one constrained or unconstrained Brownian motion, some extra care has to be taken due to the geometry. We define an “horizontal Brownian motion” as the stochastic process whose generator is the horizontal Laplacian operator $\Delta_0 = \sum_{i=1}^m X_i^2$. We would like to remark that, unlike the Euclidean case, the horizontal Laplacian has in general also a first order part, coming from taking the derivatives of the vector fields. The construction of the associated *unconstrained horizontal Brownian motion* by means of the following Stratonovich SDE, is natural: $d\xi(s) = \sum_{i=1}^m X_i(\xi(s)) \circ dB^i(s)$, where $B = (B^1, \dots, B^m)$ is a standard Brownian motion in \mathbb{R}^m .

Replacing in previous Stratonovich SDE the unconstrained Brownian motion B by a locally codimension one constrained euclidean Brownian motion B^ν in \mathbb{R}^m , we get the *locally constrained codimension one horizontal Brownian motion* $\xi^\nu(s)$ associated to Δ_0 and $\nu(s)$, which constitutes a controlled path for our problem.

Note that, in the sub-Riemannian case, the process defined by the Stratonovich SDE above differs from the one defined by the corresponding Itô integral just in the “vertical direction” (a consequence of [DGN]), so that their horizontal projections coincide. Moreover, in the Heisenberg group, it is possible to show, by an explicit calculation, that they are the same also in the “vertical” direction.

The value function associated to this stochastic control problem is defined as the infimum, over the admissible controls, of the essential supremum of the final cost g (at some fixed terminal time $T > t$), for the controlled path ξ^ν starting from x at the time t . We can show that this value function is a representation formula for the generalized evolution by horizontal mean curvature flow, i.e. solves in the (discontinuous) viscosity sense the equation $-v_t + H(x, Dv, D^2v) = 0$ where $H(x, Dv, D^2v) = \Delta_0 v - \Delta_{0,\infty}$, with terminal condition g . That means that $u(t, x) := v(T - t, x)$ is a viscosity solution of $u_t + H(x, Du, D^2u) = 0$ with initial condition g , which is exactly the level set equation of the evolution by horizontal mean curvature flow. So $\Gamma(t) = \{x \in \mathbb{R}^n \mid u(t, x) = 0\}$ is a (discontinuous) generalized evolution by horizontal mean curvature flow in general sub-Riemannian manifolds. Whenever comparison principles hold, we are able to show that such an evolution is also continuous. Unfortunately, comparison principles are still an open problem in sub-Riemannian geometries. As far as we know, there exists just a recent preprint by Capogna and Citti ([CC]) where the authors obtain comparison principles in Carnot groups but only for a particular class of initial data.

The organization of the paper is the following.

In Section 2 we introduce sub-Riemannian geometries. We recall the definitions of the Hörmander condition and the special case of Carnot groups. The main example for a Carnot group is the Heisenberg group hence, from time to time, we will focus on such a particular geometry. Then we give the definition of horizontal mean curvature and of the characteristic points. Note that, there are very few interesting hypersurfaces without characteristic points (for example in the Heisenberg group,

any compact surface topologically equivalent to a sphere, has at least one characteristic point). Therefore existence results for short times with “smooth” initial data are not as helpful as in the Euclidean case.

We conclude the section looking closer at the case of the Heisenberg group and giving several explicit examples in this geometry.

In Section 3 we introduce a notion of generalized evolution by mean curvature flow, following the level set formulation introduced by Chen-Giga-Goto in [CGG] for the corresponding Euclidean evolution. So we associate to the evolving surface a degenerate parabolic equation, where the degeneracy arises where the horizontal gradient vanishes, as it happens at the characteristic points. At these points, the differential operator as a function of first and second derivatives at the point, is not continuous any more. In the Euclidean case this situation arises only when the gradient is zero, i.e. a set of dimension zero in $\mathbb{R}^{n \times n} \times \mathbb{R}^n$. In the sub-Riemannian case, however, this set is both of non-zero dimension and depend on space, which makes the analysis far more difficult.

In Section 4 we define and study a stochastic control problem, whose associated value function solves in the viscosity sense the level set equation for the evolution by horizontal mean curvature flow, introduced in section 3. We introduce a family of (Stratonovich) stochastic ODEs driven by a “horizontal constrained Brownian motion” and we will show that the associated generator is exactly the horizontal Laplacian. Moreover we study some properties of value function. In particular, we show a property which implies that the time evolution of its zero level set depends on the terminal value only through the zero level set, and, moreover, that in Carnot groups the value function is bounded and continuous in space whenever the terminal cost is bounded and uniformly continuous and has a limit as $|x| \rightarrow \infty$.

In Section 5 we show that the value function is a bounded and lower semicontinuous viscosity solution of the level set equation for the evolution by horizontal mean curvature flow in the sub-Riemannian case. We first derive the PDE solved by the value function, assuming more regularities for the solution. This proof gives also a justification on why the optimal control is, at any point, the projection on the horizontal tangent space of the level set. We give several examples of sub-Riemannian geometries covered by our existence result. We conclude dealing with the continuity of the stochastic representation formula found. We show that if there exist comparison principles for the degenerate parabolic PDE introduced in section 3, the value function is continuous in any sub-Riemannian geometry and, therefore, it is a classic viscosity solution in the sense of Crandall and Lions. Nevertheless, as we already remarked, comparisons are known just in very few cases.

In the Appendix we give some of the technical proofs omitted in Section 5. In particular we will give a rigorous viscosity proof of the main existence result.

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2. MEAN CURVATURE IN SUB-RIEMANNIAN GEOMETRIES.

2.1. Sub-Riemannian geometries and Carnot groups.

In this section we recall briefly what sub-Riemannian geometries and Carnot groups are (for more information, see [Be, He, M]).

Let $X_1(x), \dots, X_m(x)$ a family of smooth vector fields on \mathbb{R}^n and

$$\mathcal{H}_x = \text{Span}(X_1(x), \dots, X_m(x))$$

then $\mathcal{H} = \{(x, v) \mid x \in \mathbb{R}^n, v \in \mathcal{H}_x\}$ is a *distribution* on \mathbb{R}^n .

Definition 2.1. A sub-Riemannian metric in \mathbb{R}^n is a Riemannian metric $\langle \cdot, \cdot \rangle_g$ defined on the fibers of a distribution \mathcal{H} .

An absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbb{R}^n$ is called *horizontal*, if and only if, $\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}$, a.e. $t \in [0, T]$, i.e.

$$\dot{\gamma}(t) = \sum_{i=1}^m \alpha_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in [0, T]. \quad (1)$$

For any horizontal curve, we defined a length-functional as

$$l(\gamma) = \int_0^T |\dot{\gamma}(t)|_g dt,$$

with $|\dot{\gamma}(t)|_g = \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g^{\frac{1}{2}}$. From now to on, we choose the Riemannian metric $\langle \cdot, \cdot \rangle_g$ such that the vector fields X_1, \dots, X_m are orthonormal, that means, by (1)

$$l(\gamma) = \int_0^T \sqrt{\alpha_1^2(t) + \dots + \alpha_m^2(t)} dt,$$

Once defined the length-functional we can introduce the following distance

$$d(x, y) := \inf\{l(\gamma) \mid \gamma \text{ horizontal curve joining } x \text{ to } y\}. \quad (2)$$

Let $m(x) = \dim(\text{Span}(\mathcal{H}_x))$. If $m(x) = n$, at any point $x \in \mathbb{R}^n$, the distribution generates the whole tangent bundle so the Riemannian metric $\langle \cdot, \cdot \rangle_g$ induces by (2) a Riemannian distance on \mathbb{R}^n . Otherwise, if $m < n$ at some point, this is not possible and (2) can be infinite for some pairs (x, y) . The so-called Hörmander condition, which we will explain below, guarantes that $d(x, y)$ remains finite.

Recall that the Lie bracket between two vector fields X and Y is defined as the vector field which acts on smooth real functions by $[X, Y]f = X(Yf) - Y(Xf)$. Let $\mathcal{L}^1 = \{X_1, \dots, X_m\}$, $\mathcal{L}^2 = \{[X_i, X_j] \mid i, j = 1, \dots, m\}$ and, for $k > 2$, $\mathcal{L}^k = \{[Y_i, Y_j] \mid Y_i \in \mathcal{L}^h, Y_j \in \mathcal{L}^l, h, l = 1, \dots, k-1\} \setminus \bigcup_{i=1}^{k-1} \mathcal{L}^i$, then the *Lie algebra* associated to the distribution \mathcal{H} is the set $\mathcal{L} = \bigcup_{i \in \mathbb{N}} \mathcal{L}^i$.

Definition 2.2. We say that the distribution \mathcal{H} satisfies the Hörmander condition if there exists $r \geq 1$ so that $\mathcal{L} = \bigcup_{i=1}^r \mathcal{L}^i$ and $\text{span}(\mathcal{L}) = \mathbb{R}^n$ at any point. The number r is called *step of the distribution*. In such a case, we call the function (2), induced by the distribution \mathcal{H} , a sub-Riemannian (or Carnot-Carathéodory) distance on \mathbb{R}^n , and the triple $(\mathbb{R}^n, \mathcal{H}, \langle \cdot, \cdot \rangle_g)$ is called a sub-Riemannian geometry.

The main consequence of the Hörmander condition is that the associated sub-Riemannian distance is finite (Chow's Theorem). Moreover, the Hörmander condition implies that the associated distance is continuous in \mathbb{R}^n with respect to the Euclidean topology, and it implies the existence of minimizing geodesics. The geodesics in these geometries are usually not locally unique.

Carnot groups are particular sub-Riemannian geometries, where a structure of Lie group is defined. We recall briefly the main definitions. For more details we refer to [CDPT, DGN, He]. Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ a Lie group and \mathfrak{g} the stratified Lie algebra of the left-invariant vector fields.

Definition 2.3. *A Carnot group is a Lie group $\mathbb{G} = (\mathbb{R}^n, \cdot)$, nilpotent and simply connected, whose Lie algebra \mathfrak{g} admits a stratification, i.e. it can be written as $\mathfrak{g} = \oplus_{i=1}^r V_i$, where V_i satisfy the property*

$$V_{i+1} = [V_1, V_i] \neq \emptyset \text{ for } i = 1, \dots, r-1 \quad \text{and} \quad [V_1, V_r] = \{0\}$$

\mathbb{R}^n endowed with the distribution $\mathcal{H} = V_1$ and the Euclidean metric on it, is a sub-Riemannian geometry with step r .

On a Carnot group we can define a family of dilations $\delta_\lambda(x) = \lambda^i x$, whenever $x \in V_i$. Note that $\|\delta_\lambda(x)\|_{CC} = \lambda \|x\|_{CC}$, where $\|x\|_{CC} = d(x, 0)$ is the Carnot-Carathéodory norm defined on \mathbb{G} . Moreover, in any Carnot group, it is possible to define an homogenous norm, that is

$$\|x\|_0 := \left(\sum_{i=1}^r |x_i|^{\frac{2r!}{i}} \right)^{\frac{1}{2r!}}$$

with $x = (x_1, \dots, x_k)$ and $x_i \in V_i$. This norm is indeed homogeneous w.r.t. the dilations defined on the Carnot group and it is equivalent to but more useful than the Carnot-Carathéodory norm. In fact, explicit calculations are easy and, if we define the homogenous distance, setting

$$d_0(x, y) = \|y^{-1} \cdot x\|_0$$

where y^{-1} is the inverse of y w.r.t. the group multiplication \cdot , then d_0^α is smooth for $\alpha = 2r!$, where r is the step of the distribution. By using the dilations and the group operation one can get far stronger results than in the general sub-Riemannian case. In particular, it is possible to define intrinsic regularizations by inf- and sup-convolutions (see [W], Definition 3.1 and Proposition 3.3) which imply many uniqueness results for viscosity solutions of nonlinear equations which are still open in the general case of Hörmander vector fields.

We would like to conclude the section with some examples.

Example 2.1 (Heisenberg group). *The most important sub-Riemannian geometry is the (1-dimensional) Heisenberg group \mathbb{H}^1 which is the Carnot group (\mathbb{R}^3, \cdot) , with law*

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \frac{xy' - yx'}{2} \right)$$

and $(x, y, z)^{-1} = (-x, -y, -z)$. Set $p = (x, y, z) \in \mathbb{R}^3$, the associated Lie algebra given by the left-invariant vector fields is $X(p) = (1, 0, -y/2)^t$ and $Y(p) = (0, 1, x/2)^t$. The bracket relations are $[X, Y] = T = (0, 0, 1)^t$ and $[X, T] = [Y, T] =$

0. Hence \mathbb{H}^1 is a sub-Riemannian geometry with step $r = 2$.

The differential of the left-translations is $L_p : \mathbb{H}^1 \rightarrow \mathbb{H}^1$, $q \rightarrow p \cdot q$ in \mathbb{H}^1

$$dL_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{y}{2} & \frac{x}{2} & 1 \end{pmatrix} = -dL_p^{-1}$$

The family of dilations is given by $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$ and the homogeneous norm is $\|(x, y, z)\|_0 = ((x^2 + y^2)^2 + z^2)^{\frac{1}{4}}$.

In general the n -dimensional Heisenberg group \mathbb{H}^n is defined on \mathbb{R}^{2n+1} and the group law, for any point $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, is given by $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(x \cdot y' - y \cdot x'))$ and all previous notions are still true.

Example 2.2 (Grušin plane). The Grušin plane is the sub-Riemannian geometry defined on \mathbb{R}^2 , by the distribution spanned by the two vector fields $X_1(x, y) = (1, 0)^t$ and $X_2(x, y) = (0, x)^t$. In this case the step of the distribution is 2 (as in the Heisenberg group) but the dimension of the distribution is not constant since it is $m = 1$ at the origin and $m = 2$, otherwise. Even if the Grušin plane is not a Carnot group, its structure is not so different from the structure of \mathbb{H}^1 . In fact, it is possible to define dilations and an homogeneous norm, which are $\delta_\lambda(x, y) = (\lambda x, \lambda^2 y)$ and $\|(x, y)\|_0 = |x| + |y|^{\frac{1}{2}}$.

Example 2.3 (Roto-translation geometry). The roto-translations geometry is generated on \mathbb{R}^3 by $X_1(x, y, \theta) = (\cos \theta, \sin \theta, 0)^t$ and $X_2(x, y, \theta) = (0, 0, 1)^t$. It is a 2-step sub-Riemannian geometry which was introduced by Citti and Sarti in [CS], in order to study the modal and amodal perceptual completion of the visual cortex.

2.2. Horizontal mean curvature.

We introduce the notion of horizontal mean curvature in sub-Riemannian manifolds. There are various ways how to define mean curvature in such spaces. For more details and the links with the variation of the area and the approximation by corresponding Riemannian objects, we refer to [CDPT] in the Heisenberg group and [CC, DGN, HP] in a more general context.

Given X_1, \dots, X_m smooth vector fields on \mathbb{R}^n , satisfying the Hörmander condition, we indicate with M the associated sub-Riemannian manifold $(\mathbb{R}^n, \mathcal{H}, \langle \cdot, \cdot \rangle_g)$ and we recall that $\langle \cdot, \cdot \rangle_g$ is built in such a way that X_1, \dots, X_m are orthonormal, i.e. for v, w horizontal vectors,

$$\langle v, w \rangle_g = \langle \alpha, \beta \rangle$$

where α and β are the coordinate-vectors of v and w w.r.t. X_1, \dots, X_m , and $\langle \cdot, \cdot \rangle$ is the usual inner product in \mathbb{R}^m .

First we recall that the *horizontal gradient* of a function $u : M \rightarrow \mathbb{R}$ is the horizontal vector field defined as

$$\Upsilon u(x) = (X_1 u)X_1(x) + \dots + (X_m u)X_m(x) \in \mathbb{R}^n$$

From now on, we will often omit the dependency on the point x and use the coordinate-vector field of Υu w.r.t. X_1, \dots, X_m , that is

$$\mathcal{X}u = (X_1 u, \dots, X_m u)^t \in \mathbb{R}^m$$

Note that

$$|\Upsilon u|_g^2 = \sum_{i=1}^m (X_i u)^2 = |\mathcal{X}u|^2$$

where $|\cdot|$ is the euclidean norm in \mathbb{R}^m .

Before giving the main definitions we want also to point out the following notation. We are interested in the study of hypersurface on sub-Riemannian manifolds so we have to treat two different kinds of tangent spaces: the tangent space of the manifold and the tangent space to the hypersurface.

Fix a point $x \in M$. In order to avoid confusion, we call *horizontal space* the tangent space of the sub-Riemannian manifold, denoted by $H_x M$, while the *tangent space* and *horizontal tangent space* are, respectively, the Euclidean tangent space of the hypersurface $\Sigma \subset M$ and the intersection of the Euclidean tangent space with the horizontal space. We indicate the latter two objects by $T_x \Sigma$ and $HT_x \Sigma$.

Definition 2.4. *Let $\Sigma = \{u = 0\}$ a hypersurface in M , we call horizontal normal of Σ the renormalized projection of the Euclidean normal on the horizontal space, which is*

$$n_0(x) = \frac{\Upsilon u}{|\Upsilon u|_g}$$

We introduce the horizontal mean curvature as the horizontal divergence of the horizontal normal:

$$k_0(x) := \sum_{i=1}^m X_i \left(\frac{X_i u}{|\mathcal{X}u|} \right) \quad (3)$$

Unlike in the Euclidean case, the horizontal normal to a smooth hypersurface is not always well defined. In fact, whenever the Euclidean normal is “vertical”, which means that its projection on the horizontal space vanishes, then n_0 and hence k_0 are not defined.

Definition 2.5. *Given a hypersurface $\Sigma = \{u = 0\} \subset M$, we call set of the characteristic points the set of the points where the Euclidean normal is perpendicular to the horizontal space, that is*

$$\text{char}(\Sigma) = \{x \in M \mid H_x M \subset T_x \Sigma\} = \{x \in M \mid HT_x \Sigma = H_x M\} = \{x \in M \mid |\mathcal{X}u| = 0\} \quad (4)$$

As we will see later, the existence of characteristic points make the evolution by horizontal mean curvature flow much different from the corresponding Euclidean or Riemannian evolution.

Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. In the Euclidean setting it is easy to show that the mean curvature of $\Sigma = \{x \in \mathbb{R}^n : u(x) = 0\}$ is equal to the Laplacian minus the infinite-Laplacian, both divided by the modulus of the gradient. We next recall the definition of the corresponding horizontal operators and we show that this fact is still true at the non characteristic points. First we recall that the symmetrized matrix of second derivatives is a $m \times m$ matrix defined as

$$(\mathcal{X}^2 u)_{i,j}^* = \frac{X_i(X_j u) + X_j(X_i u)}{2}$$

We call *horizontal Laplacian* and *horizontal infinite-Laplacian*, respectively, the following second order operators:

$$\Delta_0 u = \sum_{i=1}^m X_i(X_i u), \quad \Delta_{0,\infty} u = \left\langle (\mathcal{X}^2 u)^* \frac{\mathcal{X}u}{|\mathcal{X}u|}, \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle$$

Then, as in the Euclidean and Riemannian case, it is immediate to show that

$$k_0(x) = |\mathcal{X}u|^{-1} (\Delta_0 u - \Delta_{0,\infty} u) \quad (5)$$

For later use, we express all the previous objects by the matrix associated to the sub-Riemannian geometry, the Euclidean gradient Du and the Euclidean Hessian D^2u . So let $\sigma(x)$ be the smooth $m \times n$ matrix defined as $\sigma(x) = [X_1(x), \dots, X_m(x)]^t$, then the coordinate-vector of the horizontal gradient can be expressed as

$$\mathcal{X}u(x) = \sigma(x) Du(x).$$

The main point is to express the symmetrized matrix of horizontal second derivatives, using $\sigma(x)$. In fact, the matrix $(\mathcal{X}^2 u)^*$ does not depend on just second order derivatives like the corresponding Euclidean one but also on first order derivatives coming from the derivatives of the vector fields. To be more precise, one can write

$$(\mathcal{X}^2 u)^* = \sigma(x) D^2 u \sigma^t(x) + A(X_1, \dots, X_m, Du) \quad (6)$$

where the matrix A is a symmetric $m \times m$ matrix defined as

$$A_{i,j}(X_1, \dots, X_m, Du) = \frac{1}{2} \langle \nabla_{X_i} X_j + \nabla_{X_j} X_i, Du \rangle, \quad \text{for } i, j = 1, \dots, m \quad (7)$$

and $\nabla_{X_i} X_j$ is the covariant derivative of the vector field X_j w.r.t. the vector field X_i . Hence, it is possible to rewrite previous horizontal second order operators as

$$\Delta_0 u = \text{Tr}(\sigma(x)(D^2 u)\sigma^t(x)) + \sum_{i=1}^m \langle \nabla_{X_i} X_j, Du \rangle \quad (8)$$

and

$$\Delta_{0,\infty} u = \left\langle (\sigma(x)(D^2 u)\sigma^t(x)) \frac{\sigma(x) Du}{|\sigma(x) Du|}, \frac{\sigma(x) Du}{|\sigma(x) Du|} \right\rangle \quad (9)$$

$$+ \left\langle A(X_1, \dots, X_m, Du) \frac{\sigma(x) Du}{|\sigma(x) Du|}, \frac{\sigma(x) Du}{|\sigma(x) Du|} \right\rangle \quad (10)$$

This paves the way for studying the horizontal mean curvature flow by the techniques from stochastic control theory which we explain later.

We conclude this section looking at the particular case of the Heisenberg group. It is known that in general $\nabla_{X_i} X_j$ is perpendicular to the horizontal space (see [DGN] for a proof in Carnot groups). An easy calculation shows that in the Heisenberg group $\nabla_{X_i} X_j = 0$, for every $i, j = 1, 2$. So

$$\Delta_0 u = \text{Tr}(\sigma(x)(D^2 u)\sigma^t(x)) \text{ and } \Delta_{0,\infty} u = \left\langle (\sigma(x)(D^2 u)\sigma^t(x)) \frac{\sigma(x) Du}{|\sigma(x) Du|}, \frac{\sigma(x) Du}{|\sigma(x) Du|} \right\rangle.$$

That makes it easier to study explicit examples in the Heisenberg group. Hence, let $\Sigma = \{(x, y, z) \in \mathbb{H}^1 | u(x, y, z) = 0\}$ be a surface in \mathbb{H}^1 , then we can explicitly

calculate all previous quantities and, in particular, the set of characteristic points becomes:

$$\text{char}(\Sigma) = \left\{ (x, y, z) \in \Sigma \mid \left(u_x - \frac{y}{2}u_z\right)^2 + \left(u_y + \frac{x}{2}u_z\right)^2 = 0 \right\} \quad (11)$$

Whenever Σ is invariant by rotation around the z -axis, i.e. $u(x) = |z| - f(r)$, with $r = \sqrt{x^2 + y^2}$, we get

$$k_0(x, y, z) = \pm \frac{\frac{1}{4}r^2 f''(r) + \frac{(f'(r))^3}{r}}{\left((f'(r))^2 + \frac{1}{4}r^2\right)^{\frac{3}{2}}} \quad (12)$$

(depending if $z > 0$ or $z < 0$). In such a case the set of the characteristic points is

$$\text{char}(\Sigma) = \left\{ (x, y, z) \in \Sigma \mid 4(f'(r))^2 + r^2 = 0 \right\} \quad (13)$$

Obviously the only possible solutions are $r = 0$ and $f'(0) = 0$.

That means that the unique possible characteristic points are the flat intersection with the z -axis, i.e. $(0, 0, \pm f(0))$, with Neuman boundary condition $f'(0) = 0$. Using (12), it is easy to calculate the horizontal mean curvature in the following examples.

Example 2.4 ([CDPT]).

(1): Euclidean ball. If $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = R^2\}$,

$$k_0 = \frac{2(4 + R^2)}{\sqrt{x^2 + y^2}(4 + z^2)^{\frac{3}{2}}}.$$

The characteristic points are $(0, 0, \pm R)$.

(2): Korányi ball. If $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)^2 + 16z^2 = R^4\}$,

$$k_0 = \frac{3\sqrt{x^2 + y^2}}{R^2}.$$

The characteristic points are $(0, 0, \pm \frac{R^2}{4})$.

(3): Heisenberg ball. Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid d((x, y, z), (0, 0, 0)) = R^2\}$,

then, using the explicit formula for the Heisenberg geodesics, we have

$r = \sqrt{x^2 + y^2} = \frac{2}{c} \sin(cR/2)$ and $z = \frac{cR - \sin(cR)}{2c^2}$, therefore

$$k_0 = \frac{1}{2} \frac{c/2}{\sin(cR/2)} \frac{\sin(cR) - cR \cos(cR)}{\sin(cR/2) - (cR/2) \cos(cR/2)}.$$

The characteristic points are $(0, 0, \pm \frac{R^2}{4\pi})$.

The situation is particularly easy when there are no characteristic points.

Definition 2.6. We call regular hypersurface any C^1 hypersurface such that all the points are not characteristic.

In Riemannian geometries any C^1 hypersurface is regular, while in the sub-Riemannian case very few C^1 hypersurfaces are. We quote the following remark due by Roberto Monti.

Lemma 2.1. Any C^1 compact surface $\Sigma \subset \mathbb{H}^1$, topologically equivalent to the sphere, is not regular.

Proof. The “hairy ball theorem” from algebraic topology states that, given a vector field tangent to a surface in \mathbb{R}^3 , topologically equivalent to the sphere, there exists at least one point where the vector field vanishes. Let us consider as vector field the horizontal normal vector. Assuming that the surface is regular, such a vector field is different from zero at any point. Now we can (e.g. using the complex interpretation of the Heisenberg group) just rotate such a vector by $\frac{\pi}{2}$. This new vector field is still not vanishing at any point but it is tangent to the surface, which contradicts the topological theorem. \square

However it is possible to find some examples of (non-compact) regular surfaces among the rotational surfaces $\{|z| = f(\sqrt{x^2 + y^2}) \mid x^2 + y^2 \in [a, b]\}$ with $a, b \in \mathbb{R}$. Whenever $a > 0$ (i.e. $r \neq 0$ in whole the surface), there are no characteristic points. So rotational surfaces around the z -axis are regular, provided they do not intersect the z -axis. That remark leads to the following examples.

Example 2.5. *Regular surfaces are:*

- (1) any vertical plane $ax + by = d$,
- (2) any cylinder around the z -axis,
- (3) any torus around the z -axis.

Let us point out that non-regular surfaces are the far more interesting examples, because all sphere-type surfaces are not regular (Lemma 2.1) and, moreover, the characteristic points are what really makes this geometry so geometrically different from the analogous Euclidean one.

3. GENERALIZED EVOLUTION BY HORIZONTAL MEAN CURVATURE.

In Euclidean spaces, the motion by mean curvature flow of a manifold of codimension 1 is the geometrical evolution defined by requiring the normal velocity at each point of the manifold. Only few results are known for mean curvature flow in sub-Riemannian manifolds, i.e. for the evolution obtained by replacing all the geometrical objects by the corresponding horizontal quantities. In these degenerate spaces, such a kind of evolution is very different from the corresponding Euclidean motion, especially because of the existence of characteristic points, i.e. points where motion in the (euclidean) normal direction is not “admissible”.

Let us define rigorously the evolution (or motion) by mean curvature flow (MCF) in a sub-Riemannian geometry. We give first a notion assuming that the hypersurface is *regular* (i.e. smooth without characteristic points) and then we derive a weak notion holding for every hypersurface.

Definition 3.1. *For $t > 0$, let $\Gamma(t)$ a family of regular hypersurfaces in a sub-Riemannian geometry $(\mathbb{R}^n, \mathcal{H}, \langle \cdot, \cdot \rangle_g)$. We say that $\Gamma(t)$ is an evolution by horizontal mean curvature flow of the hypersurface Γ_0 if and only if the following holds: $\Gamma(0) = \Gamma_0$, and for any smooth horizontal curve $x(t) : [0, T] \rightarrow \mathbb{R}^n$ such that $x(t) \in \Gamma(t)$ for all $t \in [0, T]$, the “horizontal normal velocity” is equal to minus the horizontal mean curvature, i.e.*

$$v_0(x(t)) := \langle \dot{x}(t), n_0(x(t)) \rangle_g = -k_0(x(t)) \quad (14)$$

where $n_0(x(t))$ and $k_0(x(t))$ are the horizontal external normal and the horizontal curvature of $\Gamma(t)$, calculated at the point $x(t) \in \Gamma(t)$.

Note that $\langle \dot{x}(t), n_0(x(t)) \rangle_g$ is well defined since $x(t)$ is horizontal and smooth and we assume that $\Gamma(t)$ is a regular hypersurface.

(14) is not sufficient to describe the evolution since, like in the Euclidean case, it is not defined whenever the hypersurface develops singularities (which can happen in the Euclidean case starting from a smooth hypersurface) and it is not defined at the characteristic points, which are a specific feature of the sub-Riemannian MCF. We introduce a weak notion of evolution by mean curvature flow, using the level set approach. Such a definition was given first by Chen, Giga and Goto [CGG] and, independently, by Evans and Spruck [ES]. It is based on the idea of defining the evolution of a function $u(t, x)$ by a degenerate parabolic PDE in such a way that each level set $\{x \in \mathbb{R}^n : u(t, x) = c\}$ evolves by mean curvature as long as it is a smooth manifold, see e.g. [ES]. Exploiting the fact that this PDE is degenerate parabolic, one can define a generalized solution, called viscosity solution.

Next we derive this degenerate PDE for regular hypersurfaces.

Let $\Gamma(t) = \{u(t, x) = c\}$, then the horizontal normal of $\Gamma(t)$ at $x(t)$ is given by $n_0(x(t)) = \Upsilon u / |\Upsilon u|$. Since $x(t)$ is horizontal and smooth,

$$\dot{x}(t) = \sum_{i=1}^m \alpha_i(t) X_i(x(t)) = \sigma^t(x(t)) \boldsymbol{\alpha}(t), \quad \text{for every } t \in [0, T].$$

Set $\boldsymbol{\alpha}(t) = (\alpha_1(t), \dots, \alpha_m(t))^t$ and recalling that $\left\langle \dot{x}(t), \frac{\Upsilon u}{|\Upsilon u|} \right\rangle_g = \left\langle \boldsymbol{\alpha}(t), \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle_m$, the horizontal normal velocity can be written as

$$\langle \dot{x}(t), n_0(x(t)) \rangle_g = \left\langle \boldsymbol{\alpha}(t), \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle_m = \left\langle \boldsymbol{\alpha}(t), \frac{\sigma(x) Du}{|\sigma(x) Du|} \right\rangle_m = |\mathcal{X}u|^{-1} \langle \dot{x}(t), Du \rangle_n$$

with $\langle \cdot, \cdot \rangle_m$ and $\langle \cdot, \cdot \rangle_n$ denoting the inner product in \mathbb{R}^m and \mathbb{R}^n respectively.

From now to on, we can procede similarly to the Euclidean case. In fact, $x(t) \in \Gamma(t)$ if and only if $u(t, x(t)) = c$. Taking the derivative in time and using (14), yields

$$u_t(t, x(t)) = - \langle \dot{x}(t), Du(t, x(t)) \rangle_n = - |\mathcal{X}u| \langle \dot{x}(t), n_0(x(t)) \rangle_g = |\mathcal{X}u| k_0(x(t)) \quad (15)$$

It remains to use $k_0(x(t)) = \sum_{i=1}^m X_i \left(\frac{(\mathcal{X}u)_i}{|\mathcal{X}u|} \right)$, which gives

$$u_t = \text{Tr}((\mathcal{X}^2 u)^*) - \left\langle (\mathcal{X}^2 u)^* \frac{\mathcal{X}u}{|\mathcal{X}u|}, \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle = \Delta_0 u - \Delta_{0, \infty} u. \quad (16)$$

We want to point out that equation (16) is parabolic degenerate whenever $\mathcal{X}u = \sigma(x) Du = 0$. We call the points where the horizontal gradient vanishes *singularities*. In the Euclidean case it is known that singularities can lead to the so-called fattening of level sets. We say fattening occurs when the level set has non-empty interior, that means in particular that the gradient vanishes in an open subset, i.e. the co-dimension of the level set is locally zero (see [AAG, BP2, Gi], for more information). In the sub-Riemannian geometry, singularities are related to the vanishing of the horizontal gradient, which happens at characteristic points.

Note that the co-dimension of the horizontal tangent space is not zero at a characteristic point. Therefore, singularities of the level set equation in sub-Riemannian geometries describe very different geometrical phenomena in spite of the formal analogy with the Euclidean case.

In order to introduce a generalized motion by horizontal mean curvature, we follow the definition introduced by Chen, Giga and Goto in [CGG] for the Euclidean

evolution and by Giga in [Gi] for generic degenerate parabolic equations.

Recall that the structure of (16) is that of a degenerate parabolic equation

$$u_t + F(x, Du, D^2u) = 0 \quad (17)$$

with

$$F(x, p, S) = -\text{Tr}(\sigma(x)S\sigma^t(x) + A(x, p)) + \left\langle (\sigma(x)S\sigma^t(x) + A(x, p)) \frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle$$

with $A(x, p)$ defined in (7). Set

$$\tilde{S} = \sigma(x)S\sigma^t(x) + A(x, p),$$

then we can easily calculate that the upper and lower semicontinuous envelopes of equation (17) are

$$\begin{cases} -\text{Tr}(\tilde{S}) + \left\langle \tilde{S} \frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle, & |\sigma(x)p| \neq 0 \\ -\text{tr}(\tilde{S}) + \lambda_{\max}(\tilde{S}), & |\sigma(x)p| = 0 \end{cases}$$

and

$$\begin{cases} -\text{Tr}(\tilde{S}) + \left\langle \tilde{S} \frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle, & |\sigma(x)p| \neq 0 \\ -\text{tr}(\tilde{S}) + \lambda_{\min}(\tilde{S}), & |\sigma(x)p| = 0 \end{cases}$$

where $\lambda_{\max}(S)$ and $\lambda_{\min}(S)$ are the maximal and minimal eigenvalues of S .

Taking $\tilde{S} = (\mathcal{X}^2u)^*$ and $\sigma(x)p = \mathcal{X}u$, we can give the following definition for the generalized motion by horizontal mean curvature flow.

Definition 3.2. *Let $\Gamma_0 = \{x \in \mathbb{R}^n | u_0(x) = 0\}$ hypersurface in \mathbb{R}^n . We say that $\Gamma(t) = \{x \in \mathbb{R}^n | u(t, x) = 0\}$ is generalized evolution by horizontal mean curvature flow if u satisfies the initial condition $u(0, x) = u_0(x)$ and it is a viscosity solution of (16) in the sense of [Gi], that means u is a continuous function and*

- (1) *for any $\varphi \in \mathbb{C}^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local minimum at (t_0, x_0) , then*

$$\begin{cases} \varphi_t - \Delta_0\varphi + \Delta_{0,\infty}\varphi \geq 0, & \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) \neq 0 \\ \varphi_t - \Delta_0\varphi + \lambda_{\max}((\mathcal{X}^2\varphi)^*) \geq 0, & \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) = 0 \end{cases} \quad (18)$$

- (2) *for any $\varphi \in \mathbb{C}^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local maximum at (t_0, x_0) , then*

$$\begin{cases} \varphi_t - \Delta_0\varphi + \Delta_{0,\infty}\varphi \leq 0, & \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) \neq 0 \\ \varphi_t - \Delta_0\varphi + \lambda_{\min}((\mathcal{X}^2\varphi)^*) \leq 0, & \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) = 0 \end{cases} \quad (19)$$

As we will see later, we can give the same definition for discontinuous functions, requiring the subsolution condition (Def. 3.2, (2)) for the upper semicontinuous envelope of u and, the supersolution condition (Def. 3.2, (1)) for the lower semicontinuous envelope of u .

We would like to point out that the level set approach gives a well-posed notion of evolution, provided that the set $\Gamma(t)$ does not depend on the chosen parametrization u_0 but just on the level set Γ_0 . This is the case if whenever

$$U_0^+ := \{x \in \mathbb{R}^n \mid u_0(x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid v_0(x) \leq 0\} =: V_0^+,$$

indicating by $u(t, x)$ and $v(t, x)$, respectively, the viscosity solutions of equation (16) with initial conditions u_0 and v_0 , then

$$\{x \in \mathbb{R}^n \mid u(t, x) \leq 0\} \subset \{x \in \mathbb{R}^n \mid v(t, x) \leq 0\}$$

(similarly for the nonnegative level sets). In the Euclidean case, it is possible to prove this, using comparison principles for the level set equation (and a suitable reparametrization for the initial data such that we obtain $u_0 \geq v_0$ see [Gi], Theorem 4.2.8. for a complete proof).

The proof holds for equations with F strong geometric, provided a comparison principle for viscosity solutions holds. In our case F is strong geometric, hence the well-posedness of Definition 3.2 depends mainly on the existence of comparison principles. Unfortunately, very little is known about comparison principles (and hence uniqueness in the case of the evolution by horizontal mean curvature. In [CC], Capogna and Citti prove comparison principles in Carnot groups for special classes of initial data:

Theorem 3.1 ([CC], Theorems 3.1). *Let G be a Carnot group and u and v bounded viscosity subsolution and supersolution of (16), respectively, with initial datum u_0 and v_0 . Suppose u_0 or v_0 are uniformly continuous and for any $(x_H, x_V), (x_H, y_V) \in \mathbb{G} = \oplus_{i=1}^k V_i$, where we indicate by $z_H \in V_1$ the horizontal part of a point and by z_V the vertical part (i.e. $z_V \in \oplus_{i=2}^k V_i$), then $u_0(x_H, x_V) \leq v_0(x_H, y_V)$. Then $u(t, x) \leq v(t, x)$, for any $t \geq 0$ and $x \in \mathbb{G}$.*

The assumption on the initial datum is used to avoid the problems created by the characteristic points. They can circumvent the degeneracy of the equation in the spirit of the Euclidean proof, using the method of doubling variables and showing that the horizontal gradient of the chosen test function does not vanish at the maximum point. Spheres, tori and any compact surfaces (see Example 2.4 for some of those) are not covered by the result of [CC].

Let us point out that the definition introduced in [CC] looks slightly different from ours. Indeed Capogna and Citti follow the definition of viscosity solutions for degenerate parabolic equations used by Evans and Spruck in [ES], i.e.

Definition 3.3. *Let $\Gamma_0 = \{x \in \mathbb{R}^n \mid u_0(x) = 0\}$ hypersurface in \mathbb{R}^n . We say that $\Gamma(t) = \{x \in \mathbb{R}^n \mid u(t, x) = 0\}$ is a generalized evolution by horizontal mean curvature flow if u satisfies the initial condition $u(0, x) = u_0(x)$ and*

- (1) *for any $\varphi \in \mathcal{C}^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local minimum at (t_0, x_0) , then*

$$\begin{cases} \varphi_t - \Delta_0 \varphi + \Delta_{0, \infty} \varphi \geq 0, & \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) \neq 0 \\ \varphi_t - \Delta_0 \varphi + \left\langle (\mathcal{X}^2 \varphi)^*, \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle \geq 0, \text{ for some } |\eta| \leq 1, \text{ at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) = 0 \end{cases}$$

(2) for any $\varphi \in \mathbb{C}^2(\mathbb{R}^n \times (0, +\infty))$ such that $u - \varphi$ has a local maximum at (t_0, x_0) , then

$$\left\{ \begin{array}{l} \varphi_t - \Delta_0 \varphi + \Delta_{0,\infty} \varphi \leq 0, \quad \text{at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) \neq 0 \\ \varphi_t - \Delta_0 \varphi + \left\langle (\mathcal{X}^2 \varphi)^* \frac{\eta}{|\eta|}, \frac{\eta}{|\eta|} \right\rangle \leq 0, \text{ for some } |\eta| \leq 1, \text{ at } (t_0, x_0), \text{ if } \mathcal{X}\varphi(t_0, x_0) = 0 \end{array} \right.$$

This gives a more general notion of weak solution. Obviously, solutions in the sense of Definition 3.2 are solutions in the sense of Definition 3.3, just choose for η an eigenvector corresponding to λ_{\min} or λ_{\max} . Hence comparison principles for Definition 3.3 imply comparison for the evolution introduced in Definition 3.2.

It is known that in the euclidean case the two definitions are equivalent (see [Gi]), while this equivalence is not clear in the sub-Riemannian case. We would like to remark that the results proved for general nonlinear degenerate parabolic equations in [Gi] (like equivalence of the definitions, comparison principles, existence, etc.) rely on techniques which are not applicable in our case. The main difference between the usual degenerate parabolic equations and the level set equation for the evolution by horizontal mean curvature flow is that equation (16) is discontinuous at the points $(p, x) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\sigma(x)p = 0$, which is a space-variable-dependent set which has *non-zero dimension* in p .

To conclude this section we are going to have a closer look at the case of the Heisenberg group. Since $(\mathcal{X}^2 u)^* = \sigma(x)(D^2 u)\sigma^t(x)$ is a symmetric 2×2 matrix, there are exactly two eigenvalues. As the trace is the sum of the eigenvalues, we can rewrite Definition 3.2 at singular points as

$$\varphi_t - \Delta_0 \varphi + \lambda_{\max}((\mathcal{X}^2 \varphi)^*) = \varphi_t - \lambda_{\min}((\mathcal{X}^2 \varphi)^*) \geq 0$$

and

$$\varphi_t - \Delta_0 \varphi + \lambda_{\min}((\mathcal{X}^2 \varphi)^*) = \varphi_t - \lambda_{\max}((\mathcal{X}^2 \varphi)^*) \leq 0$$

In the particular case of rotational surfaces around the z -axis in the Heisenberg group, it is easy to see that the level set equation is continuous up to the set of characteristic points. Let $\Gamma_0 = \{(x, y, z) \in \mathbb{H}^1 \mid |z| = f(r)\}$, i.e. $u(t, (x, y, z)) = |z| - f(t, r)$ with $r = \sqrt{x^2 + y^2}$. The level set equation at the non characteristic points with $0 < r \ll 1$, is equal to

$$f_t = \frac{4(f'(r))^3 + r^3 f''(r)}{4r(f'(r))^2 + r^3} \quad (20)$$

We already remarked that the characteristic points correspond to $r = 0$ with vanishing Neuman condition $f'(0) = 0$. So by the first-order Taylor expansion for the function $f'(r)$ in 0 (that is $f'(r) = f'(0) + f''(0)r + o(r) = f''(0)r$), we can deduce

$$\lim_{r \rightarrow 0} \frac{f'(r)}{r} = f''(0)$$

Hence, at the characteristic points, the matrix $(\mathcal{X}^2 u)^*(r = 0)$ has an eigenvalue equal to $f''(0)$ with multiplicity 2. So in such a case the level set equation is continuous and we obtain $f_t - f''(0) = 0$, whenever $r = 0$. In particular, there is a rich class of examples for which the velocity in the characteristic point is non-zero.

4. CONTROLLED DIFFUSION PROCESSES.

Let us first recall some elementary facts from stochastic analysis for continuous semi-martingales which can be found in any standard textbook such as e.g. [KS]. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a Filtration $\{\mathcal{F}_t\}_{t \geq 0}$ let $x_i(t)$ be continuous and adapted (i.e. $\xi(t)$ is \mathcal{F}_t -measurable), and let $B(t)$ be a Brownian motion adapted to the filtration. Then the Itô integral $\xi dB(t)$ is defined as the following limit (as the step size of the partition decreases) in $L^2(\Omega)$:

$$\int_0^t \xi(s) dB(s) \stackrel{L^2}{:=} \lim_{N \rightarrow +\infty} \sum_{i=1}^N \xi(t_i) (B(t_{i+1}) - B(t_i)).$$

Note that this holds actually in a far more general setting: The convergence holds in the space of continuous square integrable martingales, the deterministic partition may be replaced by one constructed via stopping times, the integrand ξ need not be continuous, but merely previsible, and the Brownian motion as integrator can be replaced by any square-integrable continuous (semi-) martingale $\eta(t)$. In latter, more general case, we write $(\xi d\eta)(t)$ for the Itô-integral.

The Stratonovich integral $\xi \circ d\eta$ is defined as

$$\int_0^t \xi(s) \circ d\eta(s) \stackrel{L^2}{:=} \lim_{N \rightarrow +\infty} \sum_{i=1}^N \frac{\xi(t_i) + \xi(t_{i+1})}{2} (\eta(t_{i+1}) - \eta(t_i)),$$

both integrals are related by the formula

$$\xi \circ d\eta = \xi d\eta + \frac{1}{2} d\langle \xi, \eta \rangle$$

where $\langle \xi, \eta \rangle$ denotes the quadratic covariation of the processes ξ and η which is defined as

$$\int_0^t d\langle \xi, \eta \rangle(s) \stackrel{L^2}{=} \lim_{N \rightarrow +\infty} \sum_{i=1}^N (\xi(t_{i+1}) - \xi(t_i)) (\eta(t_{i+1}) - \eta(t_i))$$

The chain rule looks classical if we use the Stratonovich integral. In fact, for any smooth f , the process $f(\xi(t))$ satisfies

$$d[f(\xi(t))] = f'(\xi(t)) \circ d\xi,$$

which can be re-written as

$$d[f(\xi(t))] = f'(\xi(t)) d\xi + \frac{1}{2} f''(\xi(t)) d\langle \xi, \xi \rangle.$$

Note that, whenever $\xi = B$ is a Brownian motion, we have $d\langle \xi, \xi \rangle = d\langle B, B \rangle = dt$ and the formula above is the well known Itô formula. This establishes the basic connection between second order PDE and stochastic processes which yields an extension of the classical method of characteristics to the case of second order equations.

We would like to point out that we will use the Stratonovich calculus for defining our controlled stochastic processes since, because the chain rule is the classical one, it does not depend on the chosen parametrization and so it is intrinsic in Riemannian and sub-Riemannian geometries (see e.g. [H]). Nevertheless the Itô calculus will be very useful for proofs and computations (see Sec. 5).

4.1. The stochastic control problem.

It is well known that viscosity solutions of certain second-order equations are closely related to the value function of stochastic control problems, see e.g. [FS]. The relation between solutions of degenerate equations like in Definition 3.2 and stochastic control problems is more complicated. Nevertheless, Soner and Touzi (in [ST2, ST3]) and, using another approach, Buckdahn, Cardaliaguet and Quincampoix (in [BCQ]) derived a stochastic representation for a set evolving by mean curvature flow (in the euclidean case). The following stochastic optimal control problem (24) has very much in common with its Euclidean predecessors [ST2, ST3, BCQ], where now we have to replace the Itô by a Stratonovich SDE which reflects the fact that we do not work in an euclidean space.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space and B is a m -dimensional Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We define the set of admissible controls by

$$\mathcal{V} = \{(\nu(s))_{s \geq 0} \text{ predictable} \mid \nu(s) \in S_m, \nu \geq 0, I_m - \nu^2 \geq 0, \text{Tr}(I_m - \nu^2) = 1\}.$$

Under suitable assumptions, each $\nu(s)$ determines a (unique) control path $\xi^{t,x,\nu(\cdot)}$ as a solution to the SDE

$$\begin{cases} d\xi^{t,x,\nu(\cdot)}(s) = \sqrt{2}\sigma^t(\xi^{t,x,\nu(\cdot)}(s)) \circ dB^\nu(s), & s \in (t, T] \\ dB^\nu(s) = \nu(s)dB(s), & s \in (t, T] \\ \xi^{t,x,\nu(\cdot)}(t) = x \end{cases} \quad (21)$$

where $\circ dB^\nu$ denotes the integral w.r.t to B^ν in the sense of Stratonovich. Using the relation $\xi \circ d\eta = \xi d\eta + \frac{1}{2}\langle \xi, \eta \rangle$ between the Stratonovich and the Itô formulation, we get the following equivalent Itô formulation for SDE (21)

$$\begin{cases} d\xi^{t,x,\nu(\cdot)}(s) = \sqrt{2}\sigma^t(\xi^{t,x,\nu(\cdot)}(s))\nu(s)dB(s) \\ \quad + \sum_{i,j=1}^m (\nu^2(s))_{ij} \nabla_{X_i} X_j(\xi^{t,x,\nu(\cdot)}(s))ds, & s \in (t, T] \\ \xi^{t,x,\nu(\cdot)}(t) = x \end{cases} \quad (22)$$

where $\nabla_{X_i} X_j = DX_j \cdot X_i$ is the (Euclidean) derivative of the vector field X_j in direction X_i . A straightforward application of Itô's formula gives for smooth bounded $u : \mathbb{R}^n \rightarrow \mathbb{R}$ that

$$\begin{aligned} du(\xi^{t,x,\nu}(s)) &= \sqrt{2} \sum_{i=1}^m X_i(u)(\xi^{t,x,\nu}(s))\nu(s)dB(s) \\ &\quad + \sum_{i,j=1}^m (\nu(s))_{ij}^2 \left[\sum_{k,l=1}^n u_{kl} X_i^k X_j^l + \sum_{k=1}^n u_k \nabla_{X_i} X_j^k \right] (\xi^{t,x,\nu}(s)) ds, \end{aligned}$$

where we used the notation $X_i = (X_i^1, \dots, X_i^n) \in \mathbb{R}^n$, $u_x = \frac{\partial u}{\partial x_k}$ and $u_{kl} = \frac{\partial^2 u}{\partial x_k \partial x_l}$, so that the previous identity can be written as

$$du(\xi^{t,x,\nu}(s)) = \sqrt{2} \sum_{i=1}^m X_i(u)(\xi^{t,x,\nu}(s))\nu(s)dB(s) + \text{tr} [(\nu(s))^2 (\mathcal{X}^2 u)] (\xi^{t,x,\nu}(s)) ds$$

From now on, we assume that the matrix $\sigma(x)$ as well as the drift

$$\mu(x) := \sum_{i,j=1}^m \nabla_{X_i} X_j(x)$$

are Lipschitz in x . Under the Lipschitz condition, classical results for stochastic ODEs give that for any fixed control ν , (21) has a unique strong (see e.g. [YZ], Chapter 1, Corollary 6.1). Recall that the main difference between the notions of strong and weak solutions is that, in the first case, the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and the Brownian motion B are fixed while a weak solutions mean that there exists a process on *some* filtered probability space equipped with an adapted Brownian motion which satisfies the equation, for more details see Definitions 6.2 and 6.5, [YZ]. This difference becomes very important for the stochastic control problem, i.e. when considering

$$\inf_{\nu} \mathbb{E}[f(\xi^{t,x,\nu}(T))], \quad (23)$$

where usually f is a suitably regular terminal cost function and $\xi^{t,x,\nu}(\cdot)$ are solutions of a controlled Itô SDE as e.g. (22). It is clear that the properties of the previous minimum problem depend on the set where we take the infimum. Hence we define the set \mathcal{A} of all the weak-admissible controlled pairs ([YZ], Definition 4.2) which are, roughly speaking, 6-tuple $\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, B(\cdot), \nu(\cdot))$ such that $(\xi^{t,x,\nu}(\cdot), (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}))$ is a weak solution of the controlled SDE (21), w.r.t. the control ν and the Brownian motion B in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$. Under certain structural assumptions for the control set and assuming sufficient regularity of the coefficients, the existence of an optimal control for a large class of problems as in (23) is known, (see for example Theorem 5.3 in [YZ]). For these results it is crucial to use the weak formulation.

Following [ST2, ST3, BCQ], for a given bounded and uniformly continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, we define the value function associated to the stochastic control problems (21), as

$$V(t, x) = \inf_{\nu \in \mathcal{A}} \operatorname{ess\,sup}_{\omega \in \Omega} g(\xi^{t,x,\nu(\cdot)}(T)(\omega)) \quad (24)$$

where the set \mathcal{A} is the set of the weak-admissible controlled pairs, defined above. From now to on we will often omit the dependency on ω .

In the Euclidean case (i.e. $\sigma(x) = Id$) the value function (24) is the solution of the level set equation for the evolution by mean curvature flow (backward) in the viscosity sense (cf. [BCQ], Theorem 1.1),

$$\begin{cases} -V_t = \Delta V - \Delta_{\infty} V, & x \in \mathbb{R}^n, t \in [0, T) \\ V(T, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (25)$$

Our goal is to show that this result is still true in the general sub-Riemannian case. We conclude this section which was devoted to stochastic control problems with a remark.

Remark 4.1 (Non-Lipschitz coefficients). *If the coefficients of the matrix $\sigma(x)$ and of the drift part $\mu(x)$ are smooth but not globally Lipschitz, the solutions of the SDE could explode in finite time. There are results on non-explosion for some classes of non-Lipschitz coefficients, but we will not investigate this issue further,*

but instead assume global in time existence of solutions for the controlled SDEs. In many important examples, e.g. in the Heisenberg group, the Lipschitz condition is satisfied and so the non-explosion follows.

4.2. Properties of the value function.

In this section we study the main properties of the value function as defined by (24).

Lemma 4.1 (Comparison Principle). *Let g_1, g_2 be bounded and uniformly continuous functions with $g_1 \leq g_2$ on $[0, T] \times \mathbb{R}^n$, and let V_i , $i \in \{1, 2\}$, be defined as in (24) with g_i as terminal cost. Then*

$$V_1(x, t) \leq V_2(x, t), \quad \text{on } [0, T] \times \mathbb{R}^n$$

The proof is obvious and therefore omitted.

Lemma 4.2 (Value function is geometric). *Let g be bounded and uniformly continuous, and let V_g be defined as in (24) with g as terminal cost. Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and strictly increasing. Then*

$$\varphi(V_g(t, x)) = V_{\varphi(g)}(t, x)$$

Proof. As φ is increasing and continuous, $\varphi(\inf A) = \inf \varphi(A)$ for any bounded set $A \subseteq \mathbb{R}$. Hence, for any measurable function $f : \Omega \rightarrow \mathbb{R}$, it is trivial to note:

$$\varphi(\text{ess sup } f) = \text{ess sup } (\varphi(f))$$

and so we can easily conclude the proof. \square

Remark 4.2. *Lemmas 4.1 and 4.2 allow us to conclude, reasoning as in [Gi], that the sublevel set $\{V(t, x) \leq 0\}$ depends only on the set $\{g(x) \leq 0\}$, and not on the specific form of g . Therefore we could introduce a new weak notion for the evolution by horizontal mean curvature by defining the zero-level set of the value function $V(t, x)$ as generalized solution at time t , which is well-defined. Moreover, in Theorem 5.1, we will show that $V(t, x)$ solves (in the viscosity sense) the level set equation for the evolution by horizontal mean curvature flow. Unfortunately, without comparison principles and the resulting uniqueness for solutions of the PDE, we cannot say that this new definition is consistent with the classical evolution for regular hypersurfaces.*

Lemma 4.3 (Boundedness). *Assume that g is bounded and that there exists a weak solution of the controlled SDE (21) for at least one control $\nu(s)$, then the value function $V(t, x)$ defined in (24) is bounded.*

Proof. The property follows immediately once we know that the infimum is taken over a non-empty set. \square

In order to investigate the continuity of the value function, we have to restrict our attention to the case of Carnot groups.

Lemma 4.4 (Continuity in space). *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group, and suppose $g : \mathbb{G} \rightarrow \mathbb{R}$ is bounded and uniformly continuous on the one-point-compactification of \mathbb{G} , i.e. it is uniformly continuous on \mathbb{G} and there exists $\lim_{|x| \rightarrow \infty} g(x)$. Then $V(t, x)$ defined in (24) is continuous in space.*

Proof. Denote by $L_a(\cdot)$ the left translation in the Carnot group by the element $a \in \mathbb{G}$. As \mathbb{G} is a Lie group, we may assume that the vector fields X_1, \dots, X_m are left-invariant, i.e.

$$X_i(a \cdot x) = X_i(L_a(x)) = DL_a X_i(x), \quad (26)$$

for $i = 1, \dots, m$, where DL_a is the derivative of the left translation (see e.g. [W] for more details on Lie groups). Let $\xi^{t,x,\nu(\cdot)}$ be a constrained codimension one horizontal Brownian motion, then, by the chain rule for Stratonovich integrals, it holds

$$\begin{aligned} L_a \circ (\sqrt{2}\sigma^t(\xi^{t,x,\nu(\cdot)}(s)) \circ dB^\nu(s)) \\ = \sqrt{2}(DL_a(\sigma^t(\xi^{t,x,\nu(\cdot)}(s)) \circ dB^\nu(s)) = \sqrt{2}\sigma^t(L_a(\xi^{t,x,\nu(\cdot)}(s))) \circ dB^\nu(s) \end{aligned} \quad (27)$$

where we have used (26) for the last equality. Hence the left translation of a codimension 1 horizontal Brownian motion yields another one. Now fix a point x , $\epsilon > 0$ and choose a control ν_x such that

$$V(t, x) + \epsilon \geq \text{ess sup } g(\xi^{t,x,\nu_x(\cdot)}(T)).$$

Let $a = y \cdot x^{-1}$. By (27), the path $\eta^{t,y,\nu_x(\cdot)}$ starting at the time t in y , is equal to $L_a(\xi^{t,x,\nu_x(\cdot)})$. (Note that the control ν_x is the same for both points x and y .) Therefore

$$\begin{aligned} V(t, y) &\leq \text{ess sup } g(\eta^{t,y,\nu_x(\cdot)}(T)) = \text{ess sup } g(L_a(\xi^{t,x,\nu_x(\cdot)}(T))) \\ &= \text{ess sup } (g(\xi^{t,x,\nu_x(\cdot)}(T)) + (g(L_a(\xi^{t,x,\nu_x(\cdot)}(T))) - g(\xi^{t,x,\nu_x(\cdot)}(T)))) \\ &\leq V(t, x) + \epsilon + \text{ess sup } |g(L_a(\xi^{t,x,\nu_x(\cdot)}(T))) - g(\xi^{t,x,\nu_x(\cdot)}(T))| \end{aligned}$$

Choose a large number $R > 0$ then

$$\begin{aligned} \text{ess sup}_{\omega \in \Omega} \left| g(\xi^{t,x,\nu_x(\cdot)}(T)(\omega)) - g(L_a(\xi^{t,x,\nu_x(\cdot)}(T)(\omega))) \right| \\ \leq \sup_{\{z \in \mathbb{R}^n: |z| < R\}} |g(z) - g(a \cdot z)| + \sup_{\{z \in \mathbb{R}^n: |z| \geq R\}} |g(z) - g(a \cdot z)| =: A + B \end{aligned}$$

where we set $z = \xi^{t,x,\nu_x(\cdot)}(T)(\omega)$ and so $a \cdot z = L_a(\xi^{t,x,\nu_x(\cdot)}(T)(\omega))$.

Note that $|a \cdot x| \rightarrow \infty$ if $|x| \rightarrow \infty$. Therefore we can use the continuity of g at ∞ to find a sufficiently large R such that $B < \epsilon$. As, by continuity of the group operation, $|a \cdot x - x| \rightarrow 0$ (uniformly on compact sets) as $|a| = |y \cdot x^{-1}| \rightarrow 0$, we can use the uniform continuity of g to find $\delta > 0$ such that $V(t, y) \leq V(t, x) + 3\epsilon$ for $|x - y| < \delta$. Reversing the role of x and y yields the continuity. \square

5. EXISTENCE OF A GENERALIZED EVOLUTION BY HORIZONTAL MEAN CURVATURE FLOW IN SUB-RIEMANNIAN MANIFOLDS.

Using the value function for the stochastic control problem introduced in the previous section as representation for the viscosity solution of equation (16), we get an existence result for the generalized evolution by horizontal mean curvature flow as given in Definition 3.2. By classical results (see e.g. [FS], [T]), it is known how to find the equation solved by value functions of the form $\inf_{\nu \in \mathcal{A}} \mathbb{E}[g(\xi^{t,x,\nu(\cdot)}(T))]$. Unfortunately, the value function $V(t, x)$ defined in (24) looks different, because of the essential supremum instead of the expectation. Hence the idea (already used in [BCQ]) is to approximate formula (24) with functions that look like the infimum of an expectation and then to pass to the limit, essentially using the fact that the L_p -norm of a fixed nonnegative function converges to the essential supremum as

$p \rightarrow \infty$.

Since we are only able to show that the value functions defined in (24) are lower semicontinuous, we need to use the viscosity theory for discontinuous functions. Next we will recall the definition. For more details on this theory, we refer to [Ba].

Definition 5.1. *A locally bounded function $u : \mathbb{R}^n \times [0, T] \rightarrow \mathbb{R}$ is a discontinuous viscosity solution of equation (25), if $u^*(t, x)$ is a viscosity subsolution and $u_*(t, x)$ is a viscosity supersolution of the same equation, where u^* and u_* are respectively the upper and lower semicontinuous envelope of u , i.e.*

$$u^*(t, x) := \inf\{v(t, x) \mid v \text{ cont. and } v \geq u\} = \limsup_{r \rightarrow 0^+} \{u(s, y) \mid |y - x| \leq r, |t - s| \leq r\},$$

$$u_*(t, x) := \sup\{v(t, x) \mid v \text{ cont. and } v \leq u\} = \liminf_{r \rightarrow 0^+} \{u(s, y) \mid |y - x| \leq r, |t - s| \leq r\}.$$

The main result of this paper is the following existence theorem.

Theorem 5.1. *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded and Hölder continuous, $T > 0$ and $\sigma(x) = [X_1(x), \dots, X_m(x)]^t$ a $m \times n$ -Hörmander matrix with $m \leq n$ and smooth coefficients. Assuming that $\sigma(x)$ and $\sum_{i=1}^m \nabla_{X_i} X_j(x)$ are Lipschitz (in order to have non-explosion for the solution of the SDE), then the value function $V(t, x)$ defined by (24) is a bounded lower semicontinuous viscosity solution of the level set equation for the evolution by horizontal mean curvature flow (25), with terminal condition $V(T, x) = g(x)$.*

Remark 5.1 (Examples). *We recall that the Lipschitz requirement of the theorem is always satisfied if the vector fields X_1, \dots, X_m and their derivatives are smooth with at most linear growth at infinity. Hence the previous theorem covers many sub-Riemannian geometries, as, for example, the Heisenberg group, the Grušin plane and the roto-translation geometry, introduced in Examples 2.1, 2.2 and, 2.3.*

Next we give a proof that works rigorously just under additional, strong assumptions on the regularity. In fact, for passing to the limit, we will need a strong convergence of the approximating functions together with all the derivatives involved. In general those assumptions are not satisfied and indeed $V(t, x)$ is just a viscosity solution and not a classic solution. In the Appendix, we will present a viscosity proof that follows the one used in the euclidean case by Buckdahn, Cardaliaguet and Quincampoix in [BCQ]. Nevertheless, the following proof gives a clear idea of why the optimal control, realizing the infimum in (24), is given by the projection on the tangent space of the level set of the solution at that point.

Before we start with the heuristic arguments, let us introduce a regularization which will be used both for the heuristics and for a rigorous viscosity proof. As the essential supremum is the limit of L^p -norms, we consider, following [BCQ], for any $1 < p < +\infty$, the following value functions:

$$V_p(t, x) = \inf_{\nu \in \mathcal{A}} \left(\mathbb{E}[g^p(\xi^{t,x,\nu(\cdot)}(T))] \right)^{1/p}. \quad (28)$$

Note that we can replace g by $ag + b$ for real numbers a and b and therefore, as g is bounded, assume $C \geq g(x) \geq 1$, (for any $C > 0$)

The idea is to derive the PDE solved by the value functions (28) and then to show that V is their limit as $p \rightarrow +\infty$ and solves a limit-equation which is exactly the level set equation for the evolution by horizontal mean curvature flow, (16). In fact, we have

Lemma 5.1. *Under the assumptions of Theorem 5.1, we have*

$$V(t, x) = \lim_{p \rightarrow +\infty} V_p(t, x) \quad \text{for all } (t, x) \in [0, T] \times \mathbb{R}^N \text{ (pointwise convergence)} \quad (29)$$

with $V(t, x)$ as in (24)

As the L^p norms are increasing and bounded by the essential supremum, for each fixed control, it is clear that $V(t, x) \geq \lim V_p(t, x)$. In order to show equality, we can argue as in [BCQ], noting that in our case the controlled SDE has a drift part, but it depends on the control only through ν^2 , and our control set is, as the one in [BCQ], convex in ν^2 . Hence similar arguments, based on [YZ], apply. Let us now explain heuristically how to find the equation solved by V .

Heuristic proof. We first look at

$$U_p(t, x) = V_p^p(t, x) = \inf_{\nu \in \mathcal{V}} \mathbb{E} [g^p(\xi^{t, \nu, x}(T))] \quad (30)$$

It is known (see e.g. [FS], [T]) that $U_p(t, x)$ is a viscosity solution of

$$\begin{cases} -(U_p)_t + H(x, DU_p, D^2U_p) = 0, & x \in \mathbb{R}^n, t \in [0, T) \\ U_p(T, x) = g^p(x), & x \in \mathbb{R}^n \end{cases} \quad (31)$$

where

$$H(x, p, S) = \sup_{\nu \in \mathcal{A}} \left[-\frac{1}{2} \text{Tr} [F(x, \nu) F^t(x, \nu) S] + \langle \mu(x, \nu), p \rangle \right] \quad (32)$$

whenever $\xi^{t, \nu, x}(T)$ is a solution of the stochastic ODE

$$\begin{cases} d\xi^{t, x, \nu(\cdot)} = \sqrt{2} F(\xi^{t, x, \nu(\cdot)}(s), \nu) dB(s) + \mu(\xi^{t, x, \nu(\cdot)}(s), \nu) ds, & s \in (t, T) \\ \xi^{t, x, \nu(\cdot)}(t) = x \end{cases}$$

In order to compute the structure of the Hamiltonian associated to (30), we have to consider the Itô formulation of the stochastic control problem, given in (22), that means $F(x, \nu) = \sqrt{2} \sigma^t(x) \nu(s)$ and $\mu(x, \nu) = \sum_{i, j=1}^m (\nu^2(s))_{i, j} \nabla_{X_i} X_j(x)$. Hence (32) becomes

$$H(x, p, S) = \sup_{\nu \in \mathcal{A}} \left[-\text{Tr}(\sigma(x) S \sigma^t(x) \nu^2(s)) + \sum_{i, j=1}^m (\nu^2(s))_{i, j} \langle \nabla_{X_i} X_j(x), p \rangle \right] \quad (33)$$

for any $x, p \in \mathbb{R}^n$ and any symmetric $n \times n$ matrix S .

Unlike for euclidean mean curvature flow, our Hamiltonian depends on the gradient and on space, not just on S , (In fact in the euclidean case $\nabla_{X_i} X_j = 0$ and $\sigma(x) = Id$). Assuming g locally Lipschitz (or locally Hölder) and $\sigma(x)$ smooth (hence locally Lipschitz), it is possible to show the continuity of U_p , exactly as in Propositions 1.3 and 1.4 in [T]. So U_p is a (continuous) viscosity solution of (31) with H given by (33).

Once we know the equation solved by $U_p(t, x) = V_p^p(t, x)$, we can easily find the corresponding PDE for V_p .

Since $C \geq g(x) \geq 1$, $V_p \geq 1 > 0$, too, and we can divide by pV_p^{p-1} . Assuming that all functions involved are smooth, a trivial calculation tells that V_p solves

$$\begin{cases} -(V_p)_t + H(x, DV_p, (p-1)V_p^{-1}DV_p(DV_p)^t + D^2V_p) = 0, & x \in \mathbb{R}^n, t \in [0, T) \\ V_p(T, x) = g(x), & x \in \mathbb{R}^n \end{cases} \quad (34)$$

Whenever V_p is just continuous, we can show that it solves equation (34) in the viscosity sense, by applying the previous calculation to the (smooth) test functions. The continuity for V_p follows from the continuity for U_p .

The following computations conclude the proof, assuming V_p and V are so regular that

$$(V_p)_t \rightarrow V_t, \quad DV_p \rightarrow DV, \quad D^2V_p \rightarrow D^2V \quad \text{as } p \rightarrow +\infty$$

As we already remarked, those assumptions are in general not satisfied and so we will give in the Appendix a more rigorous viscosity proof.

However, the following computation explains why the supremum in (33) is attained whenever $\bar{\nu} = I_m - n_0 \otimes n_0$, where n_0 is the horizontal normal to the level set (at the non-characteristic points).

So, let us first assume $\mathcal{X}V(t, x) = \sigma(x)DV(t, x) \neq 0$ (implies $\mathcal{X}V_p(t, x) \neq 0$, at least, for large p). We can write explicitly the Hamiltonian in (34):

$$\begin{aligned} H(x, DV_p, (p-1)V_p^{-1}DV_p(DV_p)^t + D^2V_p) = \\ \sup_{\nu \in \mathcal{A}} \left[-(p-1)\text{Tr}[V_p^{-1}\nu\nu^t(\sigma(x)DV_p)(\sigma(x)DV_p)^t] + \text{Tr}[\nu\nu^t\sigma(x)D^2V_p\sigma^t(x)] \right. \\ \left. + \sum_{i,j=1}^m (\nu^2(s)_{i,j} \langle \nabla_{X_i} X_j(x), DV_p \rangle) \right] \quad (35) \end{aligned}$$

Recalling that $(\mathcal{X}^2V_p)^* = \sigma(x)D^2V_p\sigma^t(x) + A(x, DV_p)$ where $A(x, p)$ is defined by (7), we observe that

$$\text{Tr}(\mathcal{X}^2V_p)^* = \text{Tr}(\sigma(x)D^2V_p\sigma^t(x)) + \text{Tr}(A(x, DV_p))$$

The trace of the first order part is

$$\text{Tr}A(x, DV_p) = \sum_{i=1}^m \langle \nabla_{X_i} X_i(x), DV_p \rangle$$

so Hamiltonian (35) can be rewritten in ‘‘horizontal notation’’ as

$$\begin{aligned} H(x, DV_p, (p-1)V_p^{-1}DV_p(DV_p)^t + D^2V_p) = \\ \sup_{\nu \in \mathcal{A}} \left[-(p-1)V_p^{-1}\text{Tr}[\nu\nu^t(\mathcal{X}V_p)(\mathcal{X}V_p)^t] + \text{Tr}[\nu\nu^t(\mathcal{X}^2V_p)^*] \right] \quad (36) \end{aligned}$$

Note that

$$-(p-1)V_p^{-1}\text{Tr}[\nu\nu^t(\mathcal{X}V_p)(\mathcal{X}V_p)^t] = -(p-1)V_p^{-1}\text{Tr}[(\nu^t\mathcal{X}V_p)(\nu^t\mathcal{X}V_p)^t] \leq 0,$$

and so it goes to $-\infty$ as $p \rightarrow +\infty$. Hence, in order to attain the supremum, we need (at least for large p) that $\nu^t\mathcal{X}V_p = 0$. Since the horizontal gradient is in the direction of the horizontal normal, the optimal control $\bar{\nu}$ has to coincide with the projection on the tangent space, that means

$$\bar{\nu} = I_m - n_0 \otimes n_0 \quad (37)$$

To get the level set equation, we have to write Hamiltonian (35) in a bit different way. Let I_m be the $m \times m$ identity-matrix, we can replace ν^2 by $I_m - a \otimes a$ with $a \in \mathbb{R}^m$, then, for any $n \times n$ matrix S , it holds

$$\sup_{\nu \in \mathcal{A}} [-\operatorname{Tr}(\nu^2 S)] = \sup_{\nu \in \mathcal{A}} [-\operatorname{Tr}((I_m - a \otimes a)S)] = -\operatorname{Tr}[S] + \max_{|a|=1} \langle Sa, a \rangle$$

Using the optimal control (37) and recalling that $n_0 = \mathcal{X}V/|\mathcal{X}V|$ and $S = (\mathcal{X}^2V)^*$, we can conclude that the limit Hamiltonian, as $p \rightarrow \infty$, is

$$H(x, DV, D^2V) = -\operatorname{Tr}[(\mathcal{X}^2V)^*] + \left\langle (\mathcal{X}^2V)^* \frac{\mathcal{X}V}{|\mathcal{X}V|}, \frac{\mathcal{X}V}{|\mathcal{X}V|} \right\rangle = -\Delta_0 V + \Delta_{0,\infty} V$$

So the limit equation of problems (34), as $p \rightarrow +\infty$, is exactly the level set equation for the horizontal evolution by mean curvature, at non-characteristic points.

Next we would like to investigate what happens at the characteristic points, so we assume $\mathcal{X}V(t, x) = 0$. Passing to the limit in (36), the first order disappears whatever the control ν looks like. Then we get, for any control $\nu = I_m - a \otimes a$ and $S = (\mathcal{X}^2V)^*$,

$$\begin{aligned} 0 &= -V_t + \sup_{\nu \in \mathcal{A}} \left[-\operatorname{Tr}[\nu \nu^t (\mathcal{X}^2V)^*] \right] = -V_t - \operatorname{Tr}(\mathcal{X}^2V)^* + \max_{|a|=1} \operatorname{Tr}(aa^t (\mathcal{X}^2V)^*) \\ &= -V_t - \Delta_0 V + \lambda_{\max}(\mathcal{X}^2V)^* \end{aligned}$$

So we find, as expected, the upper semicontinuous regularization of the equation.

We like to conclude this proof with some remarks on the regularity of $V(t, x)$ and the initial condition.

First we can note that $V(t, x)$ is bounded since the datum g is. Moreover, since V_p is a non-decreasing sequence of continuous functions, we have

$$V(t, x) = \lim_{p \rightarrow +\infty} V_p(t, x) = \sup_{p > 1} V_p(t, x), \quad (38)$$

Hence $V(t, x)$ is, a priori, just lower semicontinuous. To conclude, we observe that $V(T, x) = \lim_{p \rightarrow +\infty} V_p(T, x) = g(x)$. \square

Example 5.1. *In the case of the Heisenberg group, all the hypersurfaces quoted in Examples 2.4 are covered by the existence result, given by Theorem 5.1. In fact, whenever the hypersurface is compact and C^1 (like in the Euclidean ball), we can choose as initial datum g , the signed Euclidean distance in a neighborhood, extended continuously by constants outside, so that the corresponding g is bounded and Lipschitz continuous. For the Carnot-Carathéodory ball, we can define $g(x) = d(x, 0) - 1$ locally around the surface and constant outside. The datum in this case is bounded and locally Hölder continuous (since it is Lipschitz w.r.t. the Carnot-Carathéodory metric d). The same holds for the Korányi ball.*

We would like to show that the functions defined in (24) are continuous in time and space and to get so the existence of a continuous viscosity solution of the level set equation for horizontal mean curvature flow.

Unfortunately, so far we are able to show directly the continuity of the value function (24) just w.r.t. x and for Carnot groups (Lemma 4.4).

Following the strategy introduced in [BCQ], it is possible to get the continuity of the solution via comparison principles for viscosity subsolutions and supersolutions. The problem with generalizing this idea to the sub-Riemannian setting is that

comparison principles are known just for special initial data in Carnot groups, hence so far we are able to get a full continuous existence result just for the class of initial data covered by Theorem 3.1 by [CC].

Proposition 5.1. *Let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ bounded and Hölder continuous, $T > 0$ and $\sigma(x)$ an $m \times n$ -Hörmander matrix like in Theorem 5.1. If comparison principles for (25) hold, then the value function $V(t, x)$ defined by (24) is the unique continuous viscosity solution of the level set equation (25), satisfying $V(T, x) = g(x)$.*

Let $V^\sharp(t, x)$ the half-relaxed upper-limit of $V_p(t, x)$, i.e.

$$V^\sharp(t, x) = \limsup_{\substack{(s, y) \rightarrow (t, x) \\ p \rightarrow +\infty}} V_p(s, y) \quad (39)$$

We remark that, in our case $V^\sharp(t, x) = V^*(t, x)$, (more details on this point, will be given in the Appendix). We can in this way prove the following lemma, which is the key-point in order to apply comparison principles and obtain the continuity of the solution.

Lemma 5.2. *For any $x \in \mathbb{R}^n$, $V^\sharp(T, x) \leq g(x)$.*

For a proof of the lemma, we refer to the Appendix.

Proof of Proposition 5.1. By Theorem 5.1 and Lemma 5.2, once we have comparison principles, it is trivial to conclude. In the viscosity proof of 5.1 (given in the Appendix), we will show that $V^*(t, x) = V^\sharp(t, x)$ is a viscosity subsolution while $V_*(t, x) = V(t, x)$ is a viscosity supersolution of equation (25). Since, by Lemma 5.2, $V^\sharp(T, x) \leq g(x)$ while $g(x) = V(T, x)$, comparison principles imply $V^\sharp(t, x) \leq V(t, x)$. Moreover $V^\sharp(t, x) \geq V(t, x)$ by definition. Hence $V^\sharp(t, x) = V(t, x)$, which means $V(t, x)$ upper semicontinuous. Since $V(t, x)$ is already lower semicontinuous as supremum of continuous functions, we conclude that $V(t, x)$ is continuous. \square

The previous conditional result becomes a full result in the cases where comparison principles are known, as in some particular hypersurfaces in Carnot groups ([CC]).

Corollary 5.1. *In Carnot groups and under the assumptions of Theorem 5.1 and Theorem 3.1, the value function $V(t, x)$ defined in (24) is the unique continuous viscosity solution of the level set equation (25), with $V(T, x) = g(x)$.*

6. APPENDIX.

In this appendix we give the proofs omitted in previous section.

Next proof uses ideas from [BCQ], applied to the horizontal case, to give a formal viscosity proof of the main theorem.

Proof of Theorem 5.1. We have to show that $V(t, x)$ solves the horizontal level set equation in the sense of Definition 5.1.

First we recall that, since $V(t, x)$ is lower semicontinuous in time and space, then $V_*(t, x) = V(t, x)$. Let $V^\sharp(t, x)$ the half-relaxed upper-limit of $V_p(t, x)$, defined in (39). We remark that $V^\sharp \geq V$ and V^\sharp upper semicontinuous. Since the upper semicontinuous envelope V^* is the smallest upper semicontinuous function above V , then $V^*(t, x) \leq V^\sharp(t, x)$. Moreover since $V_p(t, x)$ is non-decreasing, $V_p(t, x) \leq V(t, x)$, for any x, t and $p > 1$. Taking the limsup in x, t, p , we get also the reverse inequality

$V^\sharp(t, x) \leq V^*(t, x)$, hence $V^* = V^\sharp$. Therefore, to verify Definition 5.1, we have to show that $V(t, x)$ is a viscosity supersolution and $V^\sharp(t, x)$ is a viscosity subsolution.

First we show that $V(t, x)$ is a supersolution of (25). So let $\varphi \in C^1([0, T]; C^2(\mathbb{R}^n))$ such that $V - \varphi$ has a local maximum at (t, x) . Two different cases occur.

If $\mathcal{X}\varphi(t, x) \neq 0$, we have to verify that

$$-\varphi_t(t, x) - \Delta_0\varphi(t, x) + \Delta_{0, \infty}\varphi(t, x) \geq 0 \quad (40)$$

while, if $\mathcal{X}\varphi(t, x) = 0$, we need to check

$$-\varphi_t(t, x) - \Delta_0\varphi(t, x) + \lambda_{\max}((\mathcal{X}^2\varphi)^*(t, x)) \geq 0 \quad (41)$$

Note that, for any $p > 1$, there exists (t_p, x_p) such that $V_p - \varphi$ has a local minimum at (t_p, x_p) and $(t_p, x_p) \rightarrow (t, x)$. In fact, we can always assume that (t, x) is a strict minimum in some $B_R(t, x)$. Set $K = \overline{B_{\frac{R}{2}}(t, x)}$, the sequence of minimum points (t_p, x_p) converge to some $(\bar{t}, \bar{x}) \in K$. As V is the limit of the V_p and lower semicontinuous, therefore a standard argument yields that (\bar{x}, \bar{t}) is a minimum, hence it equals (x, t) . Since V_p is a solution of (34), then

$$-\varphi_t(t_p, x_p) + H(x_p, (p-1)V_p^{-1}D\varphi(D\varphi)^t + D^2\varphi)(t_p, x_p) \geq 0$$

In the case $\sigma(x)D\varphi(t, x) \neq 0$, we can write the Hamiltonian in a more explicit way. Set

$$\begin{aligned} S_1 &= (p-1)V_p^{-1}\mathcal{X}\varphi(t_p, x_p)(\mathcal{X}\varphi(t_p, x_p))^t \\ S_2 &= (\mathcal{X}^2\varphi)^*(t_p, x_p) \end{aligned}$$

$$\begin{aligned} H(x_p, S_1, S_2) &= -\text{Tr}(S_1 + S_2) + \lambda_{\max}(S_1 + S_2) = -\text{Tr}(S_1) - \text{Tr}(S_2) + \lambda_{\max}(S_1 + S_2) \\ &= -(p-1)V_p^{-1}(t_p, x_p)|\mathcal{X}\varphi(t_p, x_p)|^2 - \Delta_0\varphi(t_p, x_p) + \lambda_{\max}(S_1 + S_2) \end{aligned} \quad (42)$$

since the trace operator is linear and $\text{Tr}((\mathcal{X}\varphi(x_p))(\mathcal{X}\varphi)^t(x_p)) = |\mathcal{X}\varphi(x_p)|^2$.

Now we need the following result.

Lemma 6.1 ([BCQ], Lemma 1.2). *Let S a symmetric $m \times m$ -matrix such that the space of the eigenvectors associated to the maximum eigenvalue is of dimension one. Then, $S \rightarrow \lambda_{\max}(S)$ is C^1 in a neighborhood of S . Moreover, $D\lambda_{\max}(S)(H) = \langle Ha, a \rangle$, for any $a \in \mathbb{R}^m$ eigenvector associated to $\lambda_{\max}(S)$ and $|a| = 1$.*

We apply previous lemma to the matrix

$$S_p = V_p^{-1}(t, x)\mathcal{X}\varphi(t_p, x_p)(\mathcal{X}\varphi(t_p, x_p))^t$$

which clearly satisfies the assumptions of previous lemma.

Expanding the Hamiltonian (42) around S_p and then, passing to the limit as $p \rightarrow +\infty$, we get exactly (40).

The case $\mathcal{X}\varphi(t, x) = 0$ is much easier. We have just to use the subadditivity of the function $S \rightarrow \lambda_{\max}(S)$ and remark that, since V_p is supersolution

$$\begin{aligned} 0 &\leq -\varphi_t + H(x_p, DV_p, (p-1)V_p^{-1}D\varphi(D\varphi)^t + D^2\varphi) \\ &\leq -\varphi_t - (p-1)V_p^{-1}|\mathcal{X}\varphi|^2 - \text{Tr}((\mathcal{X}^2\varphi)^*) + \lambda_{\max}((p-1)V_p^{-1}\mathcal{X}\varphi(\mathcal{X}\varphi)^t + (\mathcal{X}^2\varphi)^*) \\ &\quad \leq -\varphi_t - (p-1)V_p^{-1}|\mathcal{X}\varphi|^2 - \Delta_0\varphi + (p-1)V_p^{-1}|\mathcal{X}\varphi|^2 + \lambda_{\max}((\mathcal{X}^2\varphi)^*) \end{aligned}$$

at the point (t_p, x_p) . So, passing to the limit as $p \rightarrow +\infty$, we find (41).

To verify the subsolution property for $V^* = V^\sharp$, let $\varphi \in C^1([0, T]; C^2(\mathbb{R}^n))$ such that $V^\sharp - \varphi$ has a maximum at (t_0, x_0) and we may assume that such a maximum is strict. Let (t_p, x_p) a sequence of maximum points of $V_p - \varphi$, we can find a subsequence converging to x . Hence, since V_p are solutions of (34), we have

$$0 \leq -\varphi_t + H(x, (p-1)\varphi^{-1}D\varphi(D\varphi)^t + D^2\varphi) \quad (43)$$

at (t_p, x_p) . We define, for any $z > 0$, $x, d \in \mathbb{R}^n$ and any $n \times n$ symmetric matrix S .

$$\begin{aligned} H_p(x, z, d, S) = & -\frac{(p-1)}{z}|\sigma(x)d + A(x, d)|^2 - \text{Tr}(\sigma^t(x)S\sigma(x) + A(x, d)) \\ & + \lambda_{\max}\left(\frac{(p-1)}{z}(\sigma(x)d)(\sigma(x)d)^t + \sigma^t(x)S\sigma(x) + A(x, d)\right) \end{aligned} \quad (44)$$

and

$$\overline{H}(x, d, S) = \begin{cases} -\text{Tr}(\sigma^t(x)S\sigma(x) + A(x, d)) + \\ \quad \left\langle (\sigma^t(x)S\sigma(x) + A(x, d)) \frac{\sigma(x)d}{|\sigma(x)d|}, \frac{\sigma(x)d}{|\sigma(x)d|} \right\rangle, & |\sigma(x)d| \neq 0 \\ -\text{Tr}(\sigma^t(x)S\sigma(x)) + \lambda_{\max}(\sigma^t(x)S\sigma(x)), & |\sigma(x)d| = 0 \end{cases} \quad (45)$$

Since $\lambda_{\max}(\sigma^t S \sigma + A) \geq \lambda_{\min}(\sigma^t S \sigma + A)$, it is clear that

$$\overline{H}_*(x, d, S) = H_*(x, d, S)$$

Moreover, as in [BCQ], we can observe that

$$H_p(x, z, d, S) \geq \overline{H}(x, d, S), \quad \text{for any } z$$

which is trivial for $|\sigma(x)d| = 0$ (by (44)) and for $|\sigma(x)d| \neq 0$, it follows taking $a = \frac{\sigma(x)d}{|\sigma(x)d|}$ in the definition of maximum eigenvalue as $\lambda_{\max}(\tilde{S}) = \max_{|a|=1} \langle \tilde{S}a, a \rangle$. Set $z = \varphi^{-1}(t_p, x_p) > 0$, $d = D\varphi(t_p, x_p)$, $S = D^2\varphi(t_p, x_p)$, by (43) taking the limsup as $p \rightarrow +\infty$, we can deduce that

$$0 \geq \varphi_t + H_*(x, D\varphi, D^2\varphi)$$

at (t, x) . That concludes the proof since it shows that the value function $V(t, x)$ satisfies Definition 5.1. \square

We conclude by giving a proof of Lemma 5.2, the key point for obtaining the continuity of the functions defined by the stochastic representation formula via the comparison principle.

Proof of Lemma 5.2. Assuming that it is not true, there exists a point x_0 such that

$$V^\sharp(T, x_0) \geq g(x_0) + \varepsilon$$

for $\varepsilon > 0$ sufficiently small. Then we use as test function

$$\varphi(t, x) = \alpha(T - t) + \beta|x - x_0|^2 + g(x_0) + \frac{\varepsilon}{2}$$

with $\alpha > -C\beta$, with C a constant depending just on the data of the problem and the point x_0 (in the Euclidean case $C = -2(n-1)$) and $\beta > 1$ sufficiently large. Now we can find a sequence $(t_x, x_k) \rightarrow (T, x_0)$ and $p_k \rightarrow +\infty$ as $k \rightarrow +\infty$ such that $V_{p_k} - \varphi$ has a positive local maximum at some point (s_k, y_k) , for any $k > 1$ (see [BCQ] for more details). To get the contradiction we need to use the fact that V_p is

a solution (so in particular a subsolution) of equation (34) and, by the subsolution condition, get

$$\alpha + C\beta \leq 0$$

which contradicts the choice $\alpha > -C\beta$.

Unfortunately in our case, unlike in the Euclidean case, the test function, inserted in the equation for V_p , does not give a constant number since the Hamiltonian depends on the space-variable. Nevertheless, we can observe that the functions V_p are bounded uniformly in p so, by the growth of $|x - x_0|$, the maximum points are such that $y_k \in \overline{B_R(x_0)} =: K$, with R independent of k .

Using this remark and the continuity of the terms which we get calculating H in φ and at the point (s_k, y_k) , we will be able to show the following lower bound

$$\alpha + C\beta \leq -\varphi_t + H((p-1)\varphi^{-1}D\varphi(D\varphi)^t + D^2\varphi) \leq 0$$

and so conclude.

First we need to remark that

$$\varphi_t(t, x) = -\alpha, \quad D\varphi(t, x) = 2\beta|x - x_0|, \quad D^2\varphi(t, x) = Id$$

Remarking that at the point (s_k, y_k) , we have

$$\begin{aligned} 0 \geq (p-1)\varphi^{-1}D\varphi(D\varphi)^t + D^2\varphi &\geq \alpha - \text{Tr}(\sigma(y_k)\sigma^t(y_x) + A(x, x - x_0)) \\ &\quad + \lambda_{\min}(\sigma(x)\sigma^t(x) + (A(x, x - x_0))) \end{aligned}$$

Recalling that there is a compact set K such that $y_k \in K$ for all k , we get by continuity

$$0 \geq (p-1)\varphi^{-1}D\varphi(D\varphi)^t + D^2\varphi \geq \alpha + 2C\beta$$

with

$$C = -\max_{x \in K} \text{Tr}(\sigma(x)\sigma^t(x)) - \max_{x \in K} A(x, x - x_0) + \min_{x \in K} \lambda_{\min}(\sigma(x)\sigma^t(x)) + \min_{x \in K} \lambda_{\min}(A(x, x - x_0))$$

With such an estimate, we are able to obtain the same contradiction as in the Euclidean case, choosing $\alpha > -C\beta$. \square

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