Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The Carnot-Caratheodory distance and the infinite Laplacian

by

Thomas Bieske, Federica Dragoni, and Juan J. Manfredi

Preprint no.: 70 2008

THE CARNOT-CARATHÉODORY DISTANCE AND THE INFINITE LAPLACIAN 6/03/08

THOMAS BIESKE, FEDERICA DRAGONI, JUAN MANFREDI

Abstract. In $\mathbb{R}^n$ equipped with the Euclidean metric, the distance from the origin is smooth and infinite harmonic everywhere except the origin. Using geodesics, we find a geometric characterization of when the distance from the origin in an arbitrary Carnot-Carathéodory space is viscosity infinite harmonic at a point outside the origin. We specifically show that at points in the Heisenberg group and Grushin plane where this condition fails, the distance from the origin is not a viscosity subsolution. We also show that at the origin, the distance function is not a viscosity supersolution.

1. Introduction

In the Euclidean setting, it is easy to show by direct computation that the distance from the origin is (smoothly) infinite harmonic away from the origin. Such a result can not be obtained in Carnot-Carathéodory spaces because the distance function is not necessarily smooth off the origin [12, 17, 18]. We therefore, examine the infinite Laplace equation from the viewpoint of viscosity solutions. We discover that the distance from the origin does not even satisfy the infinite Laplace equation in the viscosity sense at all points. Using geodesics, we find a geometric characterization of when the distance from the origin is indeed a viscosity solution. The Heisenberg group and Grushin plane will be examined closer, showing that when this condition fails at a point, the distance need not be a viscosity solution. This differs from the eikonal equation, in which the distance from the origin is a viscosity solution to the eikonal equation at all points outside the origin [12]. At the origin, the distance function is a viscosity subsolution to the infinite Laplace equation, but not a viscosity supersolution.

We divide the paper up as follows: Section 2 discusses Carnot-Carathéodory spaces while Section 3 concerns viscosity infinite harmonic functions and their key properties. Section 4 is the main section, presenting the infinite Laplace material discussed above.

Date: June 03, 2008.

2000 *Mathematics Subject Classification.* Primary: 53C17,22E25,35H20,53C22.

Key words and phrases. Carnot-Carathéodory spaces, infinite Laplacian, viscosity solutions.

2. Carnot-Carathéodory spaces

In this section, we will briefly discuss the spaces under consideration. We first note that a general Carnot-Carathéodory space is a manifold of topological dimension n. The tangent space is generated by linearly independent vector fields $X_1, X_2, \ldots, X_m$, with $m \leq n$, that satisfy Hörmander's condition. That is, the vector fields and their Lie brackets span the tangent space at each point. By Chow's Theorem (See, for example, [3].) any two points can be joined by a curve whose tangent vector lies in $\operatorname{span}\{X_i\}_{i=1}^m$. The natural distance between points x and y, denoted $d(x,y)$, is the infimum of lengths of such curves joining x and y. Thus, Carnot-Carathéodory spaces are length spaces.

Calculus on Carnot-Carathéordory spaces is defined using the given vector fields $X_1, X_2, \ldots, X_m$. The horizontal gradient of a smooth function u is given by

$$\mathfrak{X}u = (X_1 u, X_2 u, \ldots, X_m u)$$

and the symmetrized second order horizontal derivative matrix $(\mathfrak{X}^2 u)^\star$ has entries

$$(\mathfrak{X}^2 u)^\star_{ij} = \frac{1}{2}\big(X_i u X_j u + X_j u X_i u\big)$$

for $i, j = 1, 2, \ldots, m$. Using these derivatives, the main operator we consider is the infinite Laplace operator, defined by

$$\Delta_\infty u = \langle (\mathfrak{X}^2 u)^\star \, \mathfrak{X}u, \mathfrak{X}u \rangle = \sum_{i,j=1}^m X_i u X_j u X_i X_j u.$$

We concern ourselves with three main types of Carnot-Carathéodory spaces, namely Carnot groups, Grushin-type spaces and Riemannian manifolds. Carnot groups, denoted $\mathcal{G}$, are Carnot-Carathéodory spaces with a non-abelian algebraic group law. The tangent space is a stratified nilpotent Lie algebra, denoted $\mathfrak{g}$, with decomposition

$$\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_w$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_1, V_j] = V_{1+j}$. The exponential map $\exp : \mathfrak{g} \to \mathcal{G}$ can be taken to be the identity map. The exponential map is used to define natural dilations δ_r for $r > 0$ on $\mathcal{G}$ via the dilations on $\mathfrak{g}$, also denoted δ_r, given by $\delta_r(V_i) = r^i V_i$.

The (first) Heisenberg group $\mathbb{H}$ is a Carnot group that will deserve our close attention. The Heisenberg algebra h can be identified with $\mathbb{R}^3$ in coordinates (x_1, x_2, x_3) spanned by a basis consisting of vector fields X_1, X_2, and X_3 given by

$$X_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \ X_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3}, \ \text{and} \ X_3 = \frac{\partial}{\partial x_3}.$$

By taking Lie brackets, it is easy to see that $w=2$ and $\delta_r(x_1,x_2,x_3)=(rx_1,rx_2,r^2x_3)$. The Heisenberg group also has a smooth gauge bi-Lipschitz equivalent to the Carnot-Carathéodory distance function. For a point $x=(x_1,x_2,x_3)$, it is given by

$$g(x,0)=\left((x_1^2+x_2^2)^2+16x_3^2\right)^{\frac{1}{4}}.$$

For further details concerning Carnot groups and the Heisenberg group, the interested reader is directed to [3], [4], [14], and the references therein.

The second class of spaces under consideration, Grushin-type spaces, lack an algebraic group structure. Their tangent space is constructed by considering $\mathbb{R}^n$ with coordinates $(x_1,x_2,\ldots,x_n)$ and the vector fields

$$X_i=\rho_i(x_1,x_2,\ldots,x_{i-1})\frac{\partial}{\partial x_i}$$

for $i=2,3,\ldots,n$ where ρ_i is a (possibly constant, but not identically zero) polynomial. We decree that $\rho_1\equiv 1$ so that

$$X_1=\frac{\partial}{\partial x_1}.$$

Points in the Grushin-type space are also denoted $(x_1,x_2,\ldots,x_n)$. Global dilations do not, in general, exist. A special Grushin-type space under consideration is the Grushin plane, denoted $\mathbb{G}$, which has $n=2$ and $\rho_2=x_1$. For further results on Grushin-type spaces, the interested reader is directed to [3], [5], [6] and the references therein.

Note that when $m=n$ and the vectors vanish nowhere, we are in the Riemannian case. See [2] and [7] for further discussion.

3. Viscosity infinite harmonic functions and comparison with cones

As discussed above, our main equation under consideration is the infinite Laplace equation given by

$$-\Delta_\infty u=0.$$

We now define appropriate weak solutions to this equation. Namely,

Definition 1. An infinite harmonic function u is a continuous function that is a viscosity solution to the infinite Laplace equation. That is, u satisfies the following:

- IL1 For any point x_0 and function ϕ with $X_iX_j\phi$ continuous for all i,j such that $u(x_0)=\phi(x_0)$ and $u(x)\leq\phi(x)$ near x_0, we have $-\Delta_\infty\phi(x_0)\leq 0$ (u is a viscosity infinite harmonic subsolution).
- IL2 For any point x_0 and function ψ with $X_iX_j\psi$ continuous for all i,j such that $u(x_0)=\psi(x_0)$ and $u(x)\geq\psi(x)$ near x_0, we have $-\Delta_\infty\psi(x_0)\geq 0$ (u is a viscosity infinite harmonic supersolution).

It is an open problem if infinite harmonic functions are uniquely determined by their boundary values in general Carnot-Carathéodory spaces. The uniqueness theorems in Carnot groups, Grushin-type spaces and Riemannian manifolds motivated our focus on these spaces. Jensen in $\mathbb{R}^n$ [15], Bieske, in the Heisenberg group [4], Grushin-type spaces [6] and Riemannian manifolds [7] and Wang in Carnot groups [20] proved the following theorem.

Theorem A. *Let Ω be a bounded domain and let $\theta\colon \partial\Omega \to \mathbb{R}$ be a continuous function. Then the Dirichlet problem*

$$\begin{cases} -\Delta_\infty u = 0 & \text{on } \Omega \\ u = \theta & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

has a unique viscosity solution u.

It is well-known that in Euclidean space, when $\theta(y) = a + b\, d(x, y)$ for some fixed x and $a, b \in \mathbb{R}$, the viscosity solution to Equation (3.1) is $u(y) = \theta(y)$ on all of $\overline{\Omega}$ and such solutions are cones [10]. It is natural to ask if this is still the case in Carnot-Carathéodory spaces. There are two kinds of cones when using the Carnot-Carathéodory distance function. The first kind are called *infinite harmonic cones* and are defined using viscosity solutions of the infinite Laplacian. That is,

Definition 2. Let $a, b \in \mathbb{R}$. Given a point x and an open set U, we define the function $D : \partial(U \setminus \{x\}) \to \mathbb{R}$ by

$$D(y) = a + b\, d(x, y).$$

The *infinite harmonic cone* based on (U, x) is the unique viscosity infinite harmonic function $\omega^{a,b}_{U,x}$ in $U \setminus \{x\}$ such that

$$\omega^{a,b}_{U,p} = D \quad \text{on} \quad \partial(U \setminus \{x\}).$$

The second kind are called *metric cones* and are defined by extending the function D in the definition above to all of $\overline{U}$. As discussed above, in Euclidean space these two definitions coincide. However, we will show below that in Carnot groups, Grushin-type spaces and Riemannian manifolds, these definitions produce different cones.

The first use of these cones is to define the *comparison with cones* property. Namely,

Definition 3. Let U be an open set, and let $u : U \to \mathbb{R}$. Then u *enjoys comparison with infinite harmonic cones from above in U* if for every open $V \subset U$ and $a, b \in \mathbb{R}$ for which

$$u(y) \le \omega^{a,b}_{U,x}(y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \le \omega^{a,b}_{U,x}(y)$$

in V. A similar definition holds for when the function u *enjoys comparison with infinite harmonic cones from below in U*. The function u *enjoys comparison with* *infinite harmonic cones in* U exactly when it enjoys comparison with infinite harmonic cones from above and below.

The function u *enjoys comparison with metric cones from above in* U if for every open $V \subset U$ and $a, b \in \mathbb{R}$ with $b \geq 0$ for which

$$u(y) \leq D^+(y) \stackrel{\text{def}}{=} a + b\, d(x,y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \leq D^+(y)$$

in V. The function u *enjoys comparison with metric cones from below in* U if for every open $V \subset U$ and $a, b \in \mathbb{R}$ with $b \geq 0$ for which

$$u(y) \geq D^-(y) \stackrel{\text{def}}{=} a - b\, d(x,y)$$

holds on $\partial(V \setminus \{x\})$, then we have

$$u(y) \geq D^-(y)$$

in V. The function u *enjoys comparison with metric cones in* U exactly when it enjoys comparison with metric cones from above and below.

The uniqueness of infinite harmonic functions, proved via a comparison principle produces the following lemma.

Lemma 3.1. *(*[6, 7]*) A viscosity infinite harmonic supersolution in* U *enjoys comparison with infinite harmonic cones from below in* U*. Similarly, a viscosity infinite harmonic subsolution enjoys comparison with infinite harmonic cones from above in* U *and an infinite harmonic function enjoys comparison with infinite harmonic cones in* U*.*

Before discussing comparison with metric cones, we recall the definition of a Lipschitz function on a metric space.

Definition 4. Let (X, d) be a metric space and let Y be a proper subset of X. An L-Lipschitz function $F : Y \to \mathbb{R}$ is a function with

$$\operatorname{Lip}(F, Y) \stackrel{\text{def}}{=} \sup_{\substack{x,y \in Y \\ x \neq y}} \frac{|F(x) - F(y)|}{d(x,y)} \leq L < \infty. \tag{3.2}$$

Using Lipschitz functions, we can define the important concept of an absolute minimizing Lipschitz extension (AMLE) on a metric space.

Definition 5. Let (X, d) be a metric space and Y a proper subset of X. Given a Lipschitz function $F : Y \to \mathbb{R}$, we say that $u : X \to \mathbb{R}$ is an AMLE of F on X exactly when

(i) $\operatorname{Lip}(u, X) = \operatorname{Lip}(F, Y)$.

(ii) for any open set $U \subset\subset X$, we have
$$\mathrm{Lip}(u, U) = \mathrm{Lip}(u, \partial U).$$

Champion and De Pascale [8] proved the following theorem in length spaces. Recall that a length space is a space, such as a Carnot-Carathéodory space, in which the distance between two points is the infimum of the lengths of paths connecting the points.

Theorem B. *Let (X, d) be a length space and Y proper open subset of X, then $u : \overline{Y} \to \mathbb{R}$ is an AMLE if and only if u satisfies comparison with metric cones.*

In addition, the tug-of-war approach in [19] produces the following theorem.

Theorem C. *Let (X, d) be a length space and Y a proper subset of X. For any given Lipschitz function $F : Y \to \mathbb{R}$, there exists a unique AMLE of F on X.*

Because both types of cones are used to characterize different mathematical concepts, it is natural to attempt to establish a relationship between metric cones and infinite harmonic cones. We first need to recall a result of Monti and Serra-Cassano [18].

Theorem D. *Given a point y in a Carnot group, Grushin-type space or Riemannian manifold, for almost every x, we have*
$$\|\mathfrak{X} d(x, y)\| \leq 1.$$

Using this result, we then have the following proposition.

Proposition 3.2. *([6, 7]) Given a pair (U, x) and $a, b \in \mathbb{R}$, the cone $\omega^{a,b}_{U,x}$ satisfies*
$$\begin{aligned}\omega^{a,b}_{U,x}(y) &\leq a + \mathrm{abs}(b)\, d(x, y)\\ \omega^{a,b}_{U,x}(y) &\geq a - \mathrm{abs}(b)\, d(x, y)\end{aligned}$$
for $y \in U$. Here, $\mathrm{abs}(\cdot)$ denotes absolute value.

We then have the following lemma.

Lemma 3.3. *A function that enjoys comparison with infinite harmonic cones from above enjoys comparison with metric cones from above. A function that enjoys comparison with infinite harmonic cones from below enjoys comparison with metric cones from below. Thus, infinite harmonic functions enjoy comparison with metric cones.*

Proof. Let U be an open set and $V \subset\subset U$. Let x be a point and let $a, b \in \mathbb{R}$ with $b \geq 0$ define a metric cone $D^+(y) = a + b\, d(x, y)$ such that $D^+ \geq u$ on $\partial(V \setminus \{x\})$. Let $\omega^{a,b}_{V,x}$ be the infinite harmonic cone in V equal to $D^+(y)$ on $\partial(V \setminus \{x\})$. Then $\omega^{a,b}_{V,x} \geq u$ on $\partial(V \setminus \{x\})$. Since u enjoys comparison with infinite harmonic cones from above, we have
$$u(y) \leq \omega^{a,b}_{V,x}(y) \text{ in } V.$$

The result follows from the theorem. The second statement is proved in a similar manner and omitted. The third follows from the first two. □

Combining the results in this section, we have the following corollary.

Corollary 3.4. *Let X be a Carnot group, Grushin-type space or Riemannian manifold, let Y be a bounded domain in X and let $F : \partial Y \to \mathbb{R}$ be a Lipschitz function. The function u is the unique AMLE of F into Y if and only if u is infinite harmonic in Y and $u = F$ on ∂Y.*

Proof. Let u be the unique AMLE. If v is the unique infinite harmonic function on Y with boundary data F, v enjoys comparison with infinite harmonic cones and thus comparison with metric cones. By uniqueness of AMLE's, v must equal u. □

4. The distance function and the infinite Laplacian

In the previous section, we showed that the infinite harmonic functions enjoy comparison with metric cones. The interesting question is whether the distance function itself is infinite harmonic. As mentioned above, this is the case in Euclidean space [10]. The answer to this question depends on the geometry of the space.

We begin with two geometric definitions concerning points in a domain U.

Definition 6. Let U be a bounded domain, and x an arbitrary point.

1 A point $y \in U$ is **geodesically near with respect to the point** x if
$$y \in \Lambda = \{ \bigcup_{z \in \partial(U \setminus \{x\})} \gamma : \gamma \text{ is a geodesic between } x \text{ and } z \}.$$
2 A point $y \in U$ that is not geodesically near is **geodesically far with respect to the point x**. That is, $y \notin \Lambda$.
3 A point $y \in U$ is **boundary near with respect to the point x** if there exists $z \in \partial U$ so that
$$d(x, y) < d(x, z).$$
4 A point $y \in U$ that is not boundary near is **boundary far with respect to the point x**. That is, for all $z \in \partial U$, we have
$$d(x, y) \geq d(x, z).$$

We drop the phrase "with respect to x" in these definitions when the point x is understood.

We first note that because geodesics need not be unique, the set Λ actually includes *all* geodesics between points x and z. Points that are geodesically near with respect to x lie on *some* geodesic from x to the boundary point z. Additionally, it is clear that geodesically near implies boundary near, or equivalently, boundary far implies geodesically far. We next note through the following examples that, unlike the Euclidean case, interior points need not be geodesically near.

Example 7. Boundary far points. Constant boundary data with $b > 0$. Consider the Riemann sphere with the spherical metric so that the geodesics are arcs of great circles. Let the domain U be the southern hemisphere and fix the point x as the north pole. Then, on the boundary of U, $a + b\, d(x, y)$ equals a fixed constant D for all $a, b \in \mathbb{R}$. Having constant boundary data, the corresponding infinite harmonic cone is the constant D. Clearly, we have $D < a + b\, d(x, y)$ in U and $D = a + b\, d(x, y)$ on ∂U. We note that the interior points are both boundary far and geodesically far.

Example 8. Boundary near does not imply geodesically near. Let x be the origin in the Heisenberg group $\mathbb{H}$ and consider the ball of radius R centered at the origin, denoted $B_R(0)$. All points in $B_R(0)$ are boundary near. However, not all points are geodesically near. In particular, the points in A_R are geodesically far. See the figure below.

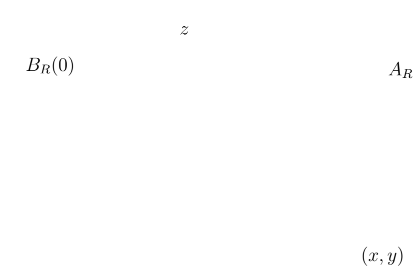

FIGURE 1. Heisenberg ball: set of points geodesically far from the origin.

Example 9. Boundary near does not imply geodesically near. Let x be the origin in the Grushin plane $\mathbb{G}$ and consider the ball of radius R centered at the origin, denoted $B_R(0)$. All points in $B_R(0)$ are boundary near. However, not all points are geodesically near. In particular, the points in A_R are geodesically far. This corresponds to the two-dimensional version of Figure 1 above.

We next fix $a, b \in \mathbb{R}$ with $b \geq 0$, a bounded domain U and a point x. We will write ω_D for the infinite harmonic cone in U with boundary data $D(y) \stackrel{\text{def}}{=} a + b\, d(x, y)$ on ∂U. We begin by considering cones with constant boundary data. In the case when $b = 0$, we have $\omega_D(y) = D(y) = a$ for all points y in $\overline{U}$. In the case when $b > 0$, we have the following proposition motivated by Example 7.

Proposition 4.1. *Let $D(y)$ be defined as above. If $D(y)$ is constant with $b > 0$ then $x \notin \overline{U}$.*

Proof. Suppose $x \in \overline{U}$. Then $x \in \partial(U \setminus \{x\})$ and $D(x) = a$. Thus, $D(y) = a$ for all $y \in \partial(U \setminus \{x\})$. Choose $x \neq z \in \partial(U \setminus \{x\})$. Then

$$a = D(z) = a + b\, d(x, z)$$

and since $b > 0$ we arrive at a contradiction. □

Because the boundary data is constant, the uniqueness of the infinite harmonic cones produces the constant infinite harmonic cone ω_D. We have the following theorem.

Theorem 4.2. *Let U be a bounded domain and $a, b \in \mathbb{R}$ with $b > 0$. Define $D(y) = a + b\, d(x, y)$ as above. Suppose $D(z) = K$ for $z \in \partial(U \setminus \{x\}) = \partial U$ for some constant K. Let ω_D be the (constant) infinite harmonic cone with boundary data K. Then the point $y \in U$ is boundary far with respect to x exactly when $\omega_D(y) < D(y)$.*

Proof. Suppose that y is boundary far with respect to x. Because y is an interior point to $U \setminus \{x\}$, there is an $r > 0$ so that the ball $B(y, r) \subset\subset (U \setminus \{x\})$. Let γ be a geodesic from x to y. Then, there is a point $\hat{x} \in (B(y, r) \setminus \{y\}) \cap \gamma$ with the property

$$d(x, y) = d(x, \hat{x}) + d(\hat{x}, y).$$

Using this property, we see that $D(y) > D(\hat{x})$. Suppose that $\omega_D(y) = D(y)$. We would then have

$$D(y) = \omega_D(y) = K = \omega_D(\hat{x}) \leq D(\hat{x}) < D(y).$$

We note that the penultimate inequality is a consequence of Proposition 3.2 and therefore conclude that $\omega_D(y) < D(y)$.

Suppose next that $\omega_D(y) < D(y)$. Then by Proposition 3.2, we have

$$K = \omega_D(y) < D(y).$$

That is, for any $z \in \partial(U \setminus \{x\})$,

$$a + b\, d(x, z) < a + b\, d(x, y).$$

Because $b > 0$, we conclude that y is boundary far with respect to x. □

The case of non-constant cones is more involved. We have the following partial result that parallels the constant case.

Theorem 4.3. *Let U, x, a, b be as in Theorem 4.2. Suppose that $D(z)$ is non-constant on $\partial(U \setminus \{x\})$ and let ω_D be the (non-constant) infinite harmonic cone with boundary data $D(z)$. Then we have the implications*

$$\begin{aligned} &y \textit{ is boundary far with respect to } x \Rightarrow \\ &\quad \omega_D(y) < D(y) \Rightarrow y \textit{ is geodesically far with respect to } x. \end{aligned}$$

Proof. We first observe that as a non-constant (continuous) infinite harmonic function on a compact set, we have that ω_D achieves its maximum on $\overline{U}$. By the strong maximum principle, which follows from the Harnack inequality [16], this maximum can occur only on the boundary.

Now assume that y is boundary far. Suppose $\omega_D(y) = D(y)$. Because y is boundary far and $b > 0$, for all $z \in \partial(U \setminus \{x\})$ we have $D(y) \geq D(z)$. That is,

$$\omega_D(y) \geq \omega_D(z)$$

for all $z \in \partial(U \setminus \{x\})$. This contradicts the fact that the maximum of ω_D occurs only on the boundary. We conclude that $\omega_D(y) < D(y)$.

The contrapositive of the second assertation is an observation in Section 1.4 of [1]. □

Ideally, we would like to prove both converse implications of the above theorem. This, however, is not possible, since if both converse statements are true, we would have proved that all geodesically far points are boundary far, which is not necessarily the case as Example 8 and Example 9 show. We conclude that the converse statements are not necessarily both true. We have the following lemma that addresses the first reverse implication.

Lemma 4.4. *Let $U, x, a, b, d(y)$ and $\omega_D(y)$ be as in Theorem 4.3. Additionally, suppose U has points that are boundary far with respect to x and points that are boundary near with respect to x. Then there exists a point $y \in U$ that is boundary near with $\omega_D(y) < D(y)$. Thus, $\omega_D(y) < D(y)$ does not necessarily imply that y is boundary far.*

Proof. Suppose that $\omega_D(y) < D(y)$ implies y is boundary far. Then the logically equivalent implication that y is boundary near implies $\omega_D(y) = D(y)$ would be true. We will show, however, that the latter implication is false.

By the continuity of the distance function, we may construct a sequence $\{y_n\}_{n\in\mathbb{N}}$ of points in U that are boundary near with respect to x and converge to the point $y \in U$ that is boundary far with respect to x. By our assumption, we have $\omega_D(y_n) = D(y_n)$. By continuity of the cone functions, this implies $\omega_D(y) = D(y)$. However, x is boundary far, and so Theorem 4.3, which showed that $\omega_D(y) < D(y)$, is contradicted. □

In order to examine the second reverse implication, we need to further explore when points are geodesically near, for at those points, we have the two cones are equal. We must, therefore, focus on the geodesics themselves. We recall the following definition.

Definition 10. Given a geodesic space (X, d), let $\gamma : [0, 1] \to X$ be a minimizing geodesic from $x \in X$ to $y \in X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Then γ is *extendable* if there is some $\varepsilon > 0$ so that the curve $\hat{\gamma} : [0, 1 + \varepsilon] \to X$ is a minimizing geodesic from x to $\hat{\gamma}(1 + \varepsilon)$ and $\hat{\gamma}|_{[0,1]} = \gamma$.

We note some important examples and non-examples of extendable geodesics.

Example 11. Let (X, d) be the Riemann sphere with the spherical metric. Geodesics from the north pole to the south pole are not extendable. Geodesics from the north pole to any other point are extendable. (Cf. Example 7.)

Example 12. Let (X, d) be the Heisenberg group $\mathbb{H}$. Then geodesics from the origin to all points off the x_3-axis are extendable, while geodesics terminating on the x_3-axis are not extendable [3].

Example 13. Let (X, d) be the Grushin plane $\mathbb{G}$. Geodesics from the origin ending at the x_2-axis are not extendable, while those ending off the x_2-axis are extendable [3].

We now relate points at which a geodesic is extendable to geodesically near points.

Proposition 4.5. *Fix a point x in a Carnot-Carathéodory space. Let y be an arbitrary point. Then there exists a bounded domain U with $y \in U$ so that y is geodesically near with respect to x if and only if there exists some geodesic γ from x to y that is extendable.*

Proof. Fix the point x. First, let U be a bounded domain so that y is geodesically near with respect to x. By definition, there is a geodesic from x to a point $z \in \partial(U \setminus \{x\})$ that meets y. The restriction is also a geodesic from x to y and is extendable to z.

Next, let γ be an extendable geodesic from x to y. Let $B(y, r)$ be the open ball centered at y with radius $r << 1$ so that there exists a point $z \in \hat{\gamma} \cap \partial B(y, r)$. Let U be a bounded domain containing y and having $z \in \partial(U \setminus \{x\})$. Then y lies on a geodesic from x to z and is therefore geodesically near with respect to x. $\square$

Theorem 4.3 leads to the following corollary.

Corollary 4.6. *Let x be a point in a Carnot-Carathéodory space where infinite harmonic functions are unique, i.e., Theorem A holds. Then the metric cones are infinite harmonic at points y where a geodesic from x to y is extendable. In particular, if x is the origin of the Heisenberg group $\mathbb{H}$ then the metric cones are infinite harmonic everywhere except possibly the x_3-axis and if x is the origin of the Grushin plane $\mathbb{G}$ then the metric cones are infinite harmonic everywhere except possibly the x_2-axis.*

Proof. Let y be a point where a geodesic from x to y is extendable. By Proposition 4.5, there is a domain U so that $y \in U$ is geodesically near with respect to x. By Theorem 4.3, $\omega_D(y) = D(y)$. $\square$

Our next goal is to remove the word "possibly" from the above two specific examples. We have the following theorem.

Theorem 4.7. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0, 0, x_3^0)$ with $x_3^0 \neq 0$ or $x_0 \in \mathbb{G}$ a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. Then there is a function ϕ with $X_i X_j \phi$ continuous for all i, j such that $\phi(x_0) = d(0, x_0)$ and $d(0, x) < \phi(x)$ near x_0 but*

$-\Delta_\infty \phi(x_0) > 0$. *Thus, the distance from the origin is not a viscosity subsolution to the infinite Laplace equation at these points.*

Proof. We begin with the Grushin plane $\mathbb{G}$ and recall the Grushin vector fields are $X_1 = \frac{\partial}{\partial x_1}$ and $X_2 = x_1 \frac{\partial}{\partial x_2}$. Let $x_0 = (0, x_2^0)$ with $x_2^0 \neq 0$ and let $x = (x_1, x_2)$ be near x_0. Note that the vector X_2 is the zero vector at x_0. Let $\phi : \mathbb{G} \to \mathbb{R}$ be the function

$$\phi(x) = \phi(x_1, x_2) = \sqrt{\pi}(x_1^4 + 4x_2^2)^{\frac{1}{4}} + \frac{1}{2}x_1 - 2x_1^2 + (x_2 - x_2^0)^4.$$

Then ϕ is smooth near x_0 with $X_1\phi(x_0) = \frac{1}{2}$ and $X_1X_1\phi(x_0) = -4$. We therefore have

$$-\Delta_\infty \phi(x_0) = -\langle (D^2\phi(x_0))^\star \mathfrak{X}\phi(x_0), \mathfrak{X}\phi(x_0)\rangle = -X_1X_1\phi(x_0)(X_1\phi(x_0))^2 = 1 > 0.$$

Thus, if $\phi(x) > d(0, x)$ near x_0, then $d(0, x)$ is not a viscosity subsolution at x_0 and is therefore not infinite harmonic at x_0. (Condition IL1 would not hold.)

Recalling that abs$(\cdot)$ is the absolute value, we note that via the geodesic formulas in [3], $\phi(x_0) = \sqrt{2\pi \operatorname{abs}(x_2^0)} = d(0, x_0)$ and so we only need to show that $d(0, x) < \phi(x)$ near x_0. If x is of the form $(0, x_2)$, then

$$d(0, x) = \sqrt{2\pi \operatorname{abs}(x_2)} \leq \sqrt{2\pi \operatorname{abs}(x_2)} + (x_2 - x_2^0)^4 = \phi(x)$$

with equality occurring only when $x = x_0$. At other points, we can see via a computer algebraic program that $\phi(x) - d(0, x) > 0$ in a neighborhood of x_0.

Similarly, in the Heisenberg group, we let $x_0 = (0, 0, x_3^0)$ with $x_3^0 \neq 0$ and let $\phi : \mathbb{H} \to \mathbb{R}$ be the function

$$\phi(x) = \phi(x_1, x_2, x_3) = \sqrt{\pi}((x_1^2 + x_2^2)^2 + 16x_3^2)^{\frac{1}{4}} + \frac{1}{2}(x_1 + x_2) - 2(x_1^2 + x_2^2) + (x_3 - x_3^0)^4.$$

Then ϕ is smooth near x_0 with $X_1\phi(x_0) = X_2\phi(x_0) = \frac{1}{2}$ and

$$(D^2\phi(x_0))^\star = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$$

so that, as above,

$$-\Delta_\infty \phi(x_0) = -\langle (D^2\phi(x_0))^\star \mathfrak{X}\phi(x_0), \mathfrak{X}\phi(x_0)\rangle = \big(2 \times \frac{1}{2}\big) + \big(2 \times \frac{1}{2}\big) = 2 > 0.$$

We again note that using [3], $\phi(x_0) = 2\sqrt{\pi \operatorname{abs}(x_3^0)} = d(0, x_0)$ and so we only need to show that $d(0, x) < \phi(x)$ near x_0. For points x of the form $(0, 0, x_3)$, we have

$$d(0, x) = 2\sqrt{\pi \operatorname{abs}(x_3)} \leq 2\sqrt{\pi \operatorname{abs}(x_3)} + (x_3 - x_3^0)^4 = \phi(x)$$

with equality only when $x_3 = x_3^0$. At other points, we proceed as in the Grushin plane case. $\square$

Having shown that the distance function is not a viscosity subsolution at these points, it is natural to ask if it is a viscosity supersolution there. We answer in the affirmative with the following theorem and corollary.

Theorem 4.8. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0,0,x_3^0)$ with $x_3^0 \neq 0$. Let $\overline{x} = (x_1, x_2)$. For real numbers η_1, η_2, η_3 and a 2×2 symmetric matrix X, consider the following inequalities based on the Taylor series* [4]*:*

$$d(x,0) \geq d(x_0,0) + x_1\eta_1 + x_2\eta_2 + o(d(x,x_0)) \text{ as } x \to x_0. \tag{4.1}$$

$$\begin{aligned} d(x,0) \leq\ & d(x_0,0) + x_1\eta_1 + x_2\eta_2 + (x_3 - x_3^0)\eta_3 \\ & + \frac{1}{2}\langle X\overline{x}, \overline{x}\rangle + o(d^2(x,x_0)) \text{ as } x \to x_0. \end{aligned} \tag{4.2}$$

If η_1, η_2, η_3 and X satisfy these inequalities, then $\eta_1 = \eta_2 = 0$.

Similarly, let $x_0 \in \mathbb{G}$ be a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. For real numbers ν_1, ν_2, ν_3, consider the following inequalities based on the Taylor series [5]*:*

$$d(x,0) \geq d(x_0,0) + x_1\nu_1 + o(d(x,x_0)) \text{ as } x \to x_0. \tag{4.3}$$

$$\begin{aligned} d(x,0) \geq\ & d(x_0,0) + x_1\nu_1 + 2(x_2 - x_2^0)\nu_2 \\ & + \frac{1}{2}(x_1)^2\nu_3 + o(d^2(x,x_0)) \text{ as } x \to x_0. \end{aligned} \tag{4.4}$$

If ν_1, ν_2 and ν_3 satisfy these inequalities, then $\nu_1 = 0$.

Proof. We shall prove only the Heisenberg case, the Grushin case is similar and omitted. If Equation (4.1) holds, it will hold for the points $x = (x_1, 0, x_3^0)$ as they approach x_0. Using the fact that $x \in B(0, d(x_0, 0))$ [3], we then have

$$0 \geq d(x,0) - d(x_0,0) \geq x_1\eta_1 + o(|x_1|) \text{ as } x \to x_0.$$

Dividing by $|x_1|$, we obtain

$$0 \geq \operatorname{sgn}(x_1)\eta_1 + o(1).$$

If η_1 is strictly negative, then choosing $x_1 < 0$ produces a contradiction and if η_1 is strictly positive, then choosing $x_1 > 0$ also produces a contradiction. We conclude that $\eta_1 = 0$. Similarly, $\eta_2 = 0$. If η_1, η_2, η_3 and X satisfy Equation (4.2) then η_1 and η_2 satisfy Equation (4.1). □

The following corollary gives the desired result.

Corollary 4.9. *Let $x_0 \in \mathbb{H}$ be a point of the form $(0,0,x_3^0)$ with $x_3^0 \neq 0$ or $x_0 \in \mathbb{G}$ a point of the form $(0, x_2^0)$ with $x_2^0 \neq 0$. Then the distance from the origin is a viscosity supersolution to the infinite Laplace equation at these points.*

Proof. We only prove the Heisenberg case, the Grushin case is similar and omitted. If Condition IL2 is vacuous, we are done. If a function ψ satisfies the hypotheses of Condition IL2, then by setting $X_1\psi(x_0) = \eta_1, X_2\psi(x_0) = \eta_2, X_3\psi(x_0) = \eta_3$ and

$(\mathfrak{X}^2\psi(x_0))^\star = X$, we have a solution to Equation (4.2)[4]. By the Theorem, $X_1\psi(x_0) = X_2\psi(x_0) = 0$ and so
$$-\Delta_\infty\psi(x_0) = 0 \geq 0.$$
Condition IL2 holds and thus the distance is a viscosity supersolution. □

We now consider the distance function at the origin. We recall from Section 2 that a Carnot-Carathéodory space is defined as an n-dimensional manifold whose tangent space is generated by m vectors. We also recall that for a vector v, $\overline{v}$ is the projection of v onto the space V_1. We then begin with the following theorem.

Theorem 4.10. *Given a Carnot-Carathéodory space, let* $\dim(V_2) = m_2$ *and let the point* x *have coordinates* $(x_1, x_2, \dots, x_n)$*. Recall that* $\overline{x} = (x_1, x_2, \dots, x_m)$*. For real numbers* $\eta_1, \eta_2, \dots, \eta_{m+m_2}$ *and a* $m \times m$ *symmetric matrix* X*, consider the following inequalities based on the Taylor series* [13]*:*

$$d(x,0) \leq \sum_{i=1}^{m} x_i\eta_i + o(d(x,0)) \text{ as } x \to 0. \tag{4.5}$$

$$d(x,0) \leq \sum_{i=1}^{m+m_2} x_i\eta_i + \frac{1}{2}\langle X\overline{x},\overline{x}\rangle + o(d^2(x,0)) \text{ as } x \to 0. \tag{4.6}$$

Then these inequalities hold for no choice of $\eta_1, \eta_2, \dots, \eta_{m+m_2}$ *or* X*.*

Proof. Suppose Equation (4.5) held for some $\eta_1, \eta_2, \dots, \eta_m$ and for all points x near the origin. In particular, it would hold for $x = (x_1, 0, \dots, 0)$, so that Equation (4.5) becomes
$$|x_1| \leq x_1\eta_1 + o(|x_1|).$$
Dividing by $|x_1|$ we have
$$1 \leq (\operatorname{sgn} x_1)\eta_1 + o(1).$$
For $x_1 > 0$ and letting $x_1 \to 0^+$, we see that $1 \leq \eta_1$. For $x_1 < 0$ and letting $x_1 \to 0^-$, we see that $1 \leq -\eta_1$. We then have
$$1 \leq \eta_1 \leq -1$$
and conclude no such η_1 can exist. If there are values $\eta_1, \eta_2, \dots, \eta_{m+m_2}$ and X that satisfy Equation (4.6) then $\eta_1, \eta_2, \dots, \eta_m$ satisfy Equation (4.5). □

The inability to satisfy these equations produces the following corollary.

Corollary 4.11. *In any Carnot-Carathéodory space, the distance from the origin is a viscosity subsolution to the infinite Laplace equation at the origin.*

Proof. Let ϕ be a function meeting the requirements of Definition 1. Then, $X_1\phi(0)$, $X_2\phi(0), \dots X_m\phi(0)$ and $(D^2\phi(0))^\star$ would satisfy Equation (4.6) [4, 5]. Thus, Condition IL1 is vacuous. □

We now will show that the distance from the origin need not be a supersolution at the origin.

Theorem 4.12. *In the Heisenberg group $\mathbb{H}$ and Grushin plane $\mathbb{G}$, the distance from the origin is not a vicosity supersolution to the infinite Laplace equation at the origin.*

Proof. Consider the function $h : \mathbb{H} \to \mathbb{R}$ given by

$$h(x) = h(x_1, x_2, x_3) = \frac{1}{2}(x_1 + x_2) + 2(x_1^2 + x_2^2) + x_3^4$$

and the function $w : \mathbb{G} \to \mathbb{R}$ given by

$$w(x) = w(x_1, x_2) = \frac{1}{2}x_1 + 2x_1^2 + x_2^4.$$

We consider first the Grushin case. First, we have $w(0) = 0 = d(0, 0)$ and we can compute $X_1w(0) = \frac{1}{2}$ and $X_1X_1w(0) = 4$. Thus, as in Theorem 4.7, we have $-\Delta_\infty w(0) = -1 < 0$. We only need to show that $d(x, 0) \geq w(x)$ near the origin. Any point x of the form $(0, x_2)$, we have $d(0, x) = \sqrt{2\pi \operatorname{abs}(x_2)}$ and $w(x) = x_2^4$. Thus for small x_2, we have $d(x, 0) > w(x)$. For other points, a graph of $w(x)$ versus $d(0, x)$ shows that $d(0, x) > w(x)$ with equality only at the origin.

The Heisenberg case is similar. We have $h(0) = 0 = d(0, 0)$ and

$$-\Delta_\infty h(0) = -\big(2 \times \frac{1}{2}\big) + \big(2 \times \frac{1}{2}\big) = -2 < 0.$$

We note that when a point x is of the form $(0, 0, x_3)$, we have $d(0, x) = \sqrt{4\pi \operatorname{abs}(x_3)}$ while $h(x) = x_3^4$ and so for x_3 near 0, we have $h(x) < d(0, x)$. As in the Grushin case, it is easy to see that $h(x) < w(x)$ near the origin. □

In summary, the Carnot-Carathéodory distance is infinite harmonic in the Heisenberg group and Grushin plane only at points where the geodesic is extendable. This fact suggests that the operator Δ_∞ is a better link to the metric of the space than the classical Laplacian Δ, especially in non-commutative geometries such as Carnot-Carathéodory spaces. At points away from the origin where the geodesics are not extendable, the distance function is a viscosity supersolution, but not a viscosity subsolution. At the origin, the opposite is true; the distance function is a viscosity subsolution, but not a viscosity supersolution. This situation can be better visualized in the Heisenberg group $\mathbb{H}$ and Grushin plane $\mathbb{G}$ through the following pictures.

FIGURE 2. Carnot-Carathéodory distance from the origin in $\mathbb{H}$.

FIGURE 3. Carnot-Carathéodory distance from the origin in $\mathbb{H}$.

FIGURE 4. Carnot-Carathéodory distance from the origin in $\mathbb{G}$.

FIGURE 5. Carnot-Carathéodory distance from the origin in $\mathbb{G}$.

References

[1] Aronsson, Gunnar.; Crandall, Michael.; Juutinen, Petri. A Tour of the Theory of Absolutely Minimizing Functions. Bull. AMS. **2004**, *41* (4), 439–505.

[2] Beatrous, Frank; Bieske, Thomas; Manfredi, Juan. *The maximum principle for vector fields,* The p-harmonic equation and recent advances in analysis. Contemp. Math. 370, Am. Math. Soc., Providence, RI, 2005, 1–9.

[3] Bellaïche, André. *The Tangent Space in Sub-Riemannian Geometry.* In *Sub-Riemannian Geometry*; Bellaïche, André., Risler, Jean-Jacques., Eds.; Progress in Mathematics; Birkhäuser: Basel, Switzerland. 1996; Vol. 144, 1–78.

[4] Bieske, Thomas. *On $\infty-$harmonic functions on the Heisenberg group* Comm. in PDE., **2002**, *27*, (3& 4) 727–761.

[5] Bieske, Thomas. *Lipschitz Extensions on generalized Grushin spaces,* Michigan Math J. **2005**, *53* (1), 3–31.

[6] Bieske, Thomas. *Properties of Infinite Harmonic Functions on Grushin-type Spaces* To appear in Rocky Mountain Journal.

[7] Bieske, Thomas. *Properties of Infinite Harmonic Functions in Riemannian Vector Fields* Preprint. Copy available at http://www.math.usf.edu/~tbieske

[8] Champion, T.; De Pascale, L. *A principle of comparison with distance functions for absolute minimizers,* J. of Convex Anal. **2007**, *14* (3), 515-541.

[9] Crandall, Michael. *Viscosity Solutions: A Primer*; Lecture Notes in Mathematics 1660; Springer-Verlag: Berlin, 1997.

[10] Crandall, M.G.; Evans, L.C.; Gariepy, R.F. *Optimal Lipschitz extensions and the infinity Laplace,* Calc. Var. P.D.E. **2001** *13* (2), 123–139.

[11] Crandall, Michael.; Ishii, Hitoshi.; Lions, Pierre-Louis. User's Guide to Viscosity Solutions of Second Order Partial Differential Equations. Bull. of Amer. Math. Soc. **1992**, *27* (1), 1–67.

[12] Dragoni, F. *Metric Hopf-Lax formula with semicontinuous data,* Disc and Cont Dynam Sys **2007**, *17* (4), 713–729.

[13] Folland, G.B.; Stein, Elias M. *Hardy Spaces on Homogeneous Groups*; Princeton University Press: Princeton, NJ. 1982.

[14] Heinonen, Juha. Calculus on Carnot Groups. In *Fall School in Analysis Report No. 68*, Fall School in Analysis, Jyväskylä, 1994; Univ. Jyväskylä: Jyväskylä, Finland. 1995; 1–31.

[15] Jensen, Robert. Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient. Arch. Ration. Mech. Anal. **1993**, *123*, 51–74.

[16] Lu, Guozhen.; Manfredi, Juan.; Stroffolini, Bianca. Convex Functions on the Heisenberg Group. Calc. Var. PDE **2004**, *19* (1), 1–22.

[17] Monti, Roberto. Some Properties of Carnot-Carathéodory Balls in the Heisenberg Group. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 11 **2000**, (3), 155–167.

[18] Monti, Roberto. Serra Cassano, Francesco. Surface measures in Carnot-Carathéodory spaces. Calc. Var. PDE. **2001**, *13*, 339–376.

[19] Peres, Y.; Schramm, O.; S. Sheeld, S.; Wilson, D.B. *Tug-of-war and the infinity Laplacian,* Rend. Sem. Mat. Univ. Politec. Torino 1989, Special issue on topics in nonlinear PDEs: **1991**, 15–68.

[20] Wang, Changyou. *The Aronsson equation for absolute minimizers of L^∞-functionals associated with vector fields satisfying Hörmander's condition.* Trans. Amer. Math. Soc. **2007**, *359*, 91–113.