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by

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Abstract

Let M be a compact Kähler manifold equipped with a Hamiltonian action of a compact Lie group G. In [Invent. Math. **67** (1982), no. 3, 515–538], Guillemin and Sternberg showed that there is a geometrically natural isomorphism between the G-invariant quantum Hilbert space over M and the quantum Hilbert space over the symplectic quotient $M/\!\!/G$. This map, though, is not in general unitary, even to leading order in $\hbar$.

In [Comm. Math. Phys. **275** (2007), no. 2, 401–422], Hall and the author showed that when the metaplectic correction is included, one does obtain a map which, while not in general unitary for any fixed $\hbar$, becomes unitary in the semiclassical limit $\hbar \to 0$. The unitarity of the classical Guillemin–Sternberg map and the metaplectically corrected analogue is measured by certain functions on the symplectic quotient $M/\!\!/G$. In this paper, we give precise expressions for these functions, and compute complete asymptotic expansions for them as $\hbar \to 0$.

Keywords: Geometric Quantization, Symplectic Reduction, Asymptotic Expansion, Laplace's Method

1 Introduction.

Let M be a compact Kähler manifold with Kähler form ω. Suppose there exists a Hermitian line bundle ℓ with connection with curvature $-i\omega$. For each positive integer k, the geometric quantization $\mathcal{H}_M^{(k)}$ of M is defined to be the space of holomorphic sections of $\ell^{\otimes k}$. In the context of geometric quantization, k is interpreted as the reciprocal of Planck's constant $\hbar$.

Suppose moreover that G is a compact Lie group, with Lie algebra $\mathfrak{g}$, which acts on M in a Hamiltonian fashion with moment map $\Phi : M \to \mathfrak{g}^*$. Under sufficient regularity assumptions, the symplectic quotient $M/\!\!/G$ is again a compact Kähler manifold; denote the resulting Kähler form by $\widehat{\omega}$. Assuming that the action of G lifts, the bundle ℓ descends to a line bundle $\hat{\ell} \to M/\!\!/G$, and the connection descends to one with curvature $-i\widehat{\omega}$.

The space $\mathcal{H}_{M/\!\!/G}^{(k)}$ of holomorphic sections of $\hat{\ell}^{\otimes k}$ is the result of reducing *before* quantizing. On the other hand, one may *first* quantize and *then* reduce, which amounts to considering the space $\left(\mathcal{H}_M^{(k)}\right)^G$ of G-invariant sections of $\ell^{\otimes k}$.

A classical result of Guillemin and Sternberg [GS82] is that there is a natural invertible linear map A_k from the "first quantize then reduce" space $\left(\mathcal{H}_M^{(k)}\right)^G$ to the "first reduce then quantize" space $\mathcal{H}_{M/\!\!/G}^{(k)}$. From the point of view of quantum mechanics, though, it is not only the vector space structure of the quantization that is important, but also the inner product.

It is known that in general, the Guillemin–Sternberg map A_k is *not* unitary, and is not even unitary to leading order as $k \to \infty$ [Flu98], [Cha06], [HK06] [Li08], [Pao05], [MM07], [MZ05], [MZ06]. In [HK06], the author and Brian Hall showed that when the so-called metaplectic correction is introduced, one obtains an analogue B_k of the Guillemin–Sternberg map which, though still not unitary in general for any fixed k, becomes unitary in the semiclassical limit $k \to \infty$ (this was later shown to be the case in a more general setting by Hui Li [Li08]).

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The unitarity, or lack thereof, of the map A_k (resp. B_k) is measured by a certain function I_k (resp. J_k) on the symplectic quotient $M/\!\!/G$, with unitarity achieved at least when I_k (resp. J_k) is identically 1. One of the main results of [HK06] is that

$$\lim_{k\to\infty} J_k = 1,$$

where the limit is uniform on $M/\!\!/G$. (There is an analogous computation for the limit of the I_k; see Section 1.1 below). We should also mention that in [MZ05] and [MZ06], Ma and Zhang, as well as Ma and Marinescu in [MM07], show the existence of an asymptotic series which is related to that for I_k, and also compute the first term.

Our main results are explicit expressions for I_k and J_k (Theorem 1.1) as well as complete asymptotic expansion of the functions I_k and J_k as $k \to \infty$ (Theorem 1.2). We state these results precisely in Section 1.1 below.

One may ask whether there are any cases in which B_k is asymptotically unitary to all orders, that is, in which $\lim_{k\to\infty} J_k = 1 + o(k^{-\infty})$. Although we do not prove it here, our results suggest that such "exact asymptotics" are not possible for compact M (see the remark following Lemma 3.3). Our results, as well as those of [HK06], do not seem to depend crucially on the compactness of M, and indeed the obstruction to "exact asymptotics" disappears when M is noncompact.

In the rest of this section, we describe our main results precisely and then finish by recalling from [HK06] the precise definition of modified Guillemin–Sternberg-type map B_k. In Section 2 we build on the results of [HK06] to give precise expressions for the densities I_k and J_k which make the asymptotic computations possible. In Section 3 we prove our main result, Theorem 1.2, by applying previous results of the author [Kir08] to the case at hand.

1.1 Main results.

Let $(M^{2n}, \omega, J, B := \omega(\cdot, J\cdot))$ be a compact Kähler manifold with symplectic form ω, complex structure J and metric B. Let $\ell \to M$ be a Hermitian line bundle over M with connection ∇ with curvature $-i\omega$. Suppose that a compact Lie group G of dimension d acts on M (preserving the Kähler structure) in a Hamiltonian fashion with moment map $\Phi : M \to \mathfrak{g}^*$, and suppose moreover that the induced infinitesimal action on ℓ (given by the quantization of the components of the moment map) exponentiates to a global action of G on ℓ. We denote the components of the moment map by $\phi_\xi : M \to \mathbb{R}$, for $\xi \in \mathfrak{g}$.

Suppose that 0 is a value and a regular value of Φ, and moreover that G acts freely on the zero set $\Phi^{-1}(0)$. In this case, the symplectic quotient $M/\!\!/G := \Phi^{-1}(0)/G$ is a compact smooth manifold which inherits a Kähler structure from that of M; denote the induced symplectic form on $M/\!\!/G$ by $\widehat{\omega}$. The line bundle ℓ descends to a Hermitian line bundle $\widehat{\ell} \to M/\!\!/G$, and the connection ∇, restricted to G-invariant sections, induces a connection on $\widehat{\ell}$. Throughout, $x_0 \in \Phi^{-1}(0)$ will denote a point in the zero-section, and $[x_0] := G \cdot x_0$ will denote the corresponding point in the symplectic quotient.

The infinitesimal action of G on M can be continued to an infinitesimal action of the complexified group[1] $G_\mathbb{C}$ by setting $X^{\sqrt{-1}\xi} := JX^\xi$. This action exponentiates to an action of $G_\mathbb{C}$ on M. The saturation $G_\mathbb{C} \cdot \Phi^{-1}(0)$ of the zero set by the group $G_\mathbb{C}$ is called the stable set M_s. It is an open submanifold of M, and the complement is of complex codimension at least one. The (free) action of $G_\mathbb{C}$ on M_s gives the stable set the structure of a principle $G_\mathbb{C}$-bundle $\pi_\mathbb{C} : M_s \to M/\!\!/G$. Indeed, the complex structure on the symplectic quotient can be understood via the Kähler isomorphism

$$\Phi^{-1}(0)/G = M/\!\!/G = M_s/G_\mathbb{C}.$$

Moreover, the action $\Lambda : \exp(\sqrt{-1}\mathfrak{g}) \times \Phi^{-1}(0) \to M_s$ gives the stable set the structure of a trivial vector bundle[2] over $\Phi^{-1}(0)$ with fiber $\mathfrak{g}$ (see [HK06], [Li08], or [Sja95] for details).

[1] For each compact Lie group G there exists a unique Lie group $G_\mathbb{C}$ such that G is a maximal compact subgroup which sits inside $G_\mathbb{C}$ as a totally real submanifold, and such that the Lie algebra of $G_\mathbb{C}$ is the complexification of $\mathfrak{g}$. The group $G_\mathbb{C}$ is called the complexification of $G_\mathbb{C}$. It is diffeomorphic to T^*G, and the multiplication map $G \times \exp(\sqrt{-1}\mathfrak{g}) \to G_\mathbb{C}$ is a diffeomorphism. See [Kna02, Sec VII.1] for details.

[2] This does *not* imply that M_s is a trivializable $G_\mathbb{C}$-bundle over $M/\!\!/G$; in general the zero set $\Phi^{-1}(0)$ is a non-trivial G-bundle over $M/\!\!/G$.

The geometric quantization $\mathcal{H}_M^{(k)}$ of M is the space of holomorphic sections of $\ell^{\otimes k}$, $k \in \mathbb{N}$. We make $\mathcal{H}_M^{(k)}$ into a Hilbert space by equipping it with the inner product

$$\langle s,t\rangle := (k/2\pi)^{n/2} \int_M (s,t)\frac{\omega^n}{n!},$$

where (s,t) denotes the pointwise Hermitian product in $\ell^{\otimes k}$.

Let $K : \bigwedge^n \left(T^{1,0}M\right)^*$ denote the canonical bundle of M. Suppose that K admits a square root[3], and denote a choice of square root by $\sqrt{K}$. Sections of $\sqrt{K}$ are called half-forms, and $\sqrt{K}$ is called a half-form bundle. A section of K is said to be holomorphic if in each local holomorphic coordinate chart, the coefficient of $dz^1 \wedge \cdots \wedge dz^n$ is a holomorphic function. Suppose that the action of G lifts to an action on $\sqrt{K}$ which is compatible with the action on K induced by pushforward.

There is a natural inner product on the space of sections of $\sqrt{K}$: if $\mu,\nu \in \Gamma\left(\sqrt{K}\right)$, then $\mu^2 \wedge \bar{\nu}^2 \in \bigwedge^{2n} T^{\mathbb{C}}M$ is a (complex) volume form. We can trivialize the bundle $\bigwedge^{2n} TM$ by the (global, nowhere vanishing) section $\omega^n/n!$, and hence there is a function (μ,ν)—the pointwise inner product of μ and ν—such that

$$\mu^2 \wedge \bar{\nu}^2 =: (\mu,\nu)^2\omega^n/n!. \tag{1.1}$$

The metaplectic correction, by definition, amounts to considering $\ell^{\otimes k} \otimes \sqrt{K}$; that is, the (half-form) corrected quantization $\widehat{\mathcal{H}}_M^{(k)}$ of M is the space of holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K}$. The pairing (1.1) is a special case of the BKS pairing in geometric quantization [Woo91, Sec 10.2]. It defines a Hermitian form on $\ell^{\otimes k} \otimes \sqrt{K}$ and hence an inner product on $\widehat{\mathcal{H}}_M^{(k)}$: for sections $t_1,t_2 \in \Gamma(\ell^{\otimes k} \otimes \sqrt{K})$ which are locally represented by $t_j(x) = s_j(x) \otimes \mu(x)$, we set

$$(t_1,t_2)(x) = (s_1(x),s_2(x))(\mu,\nu)(x).$$

Let $\left(\mathcal{H}_M^{(k)}\right)^G$ denote the space of G-invariant holomorphic sections of $\ell^{\otimes k}$, and similarly $\left(\widehat{\mathcal{H}}_M^{(k)}\right)^G$ the space of G-invariant holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K}$. The restriction of a G-invariant holomorphic section $s \in \left(\mathcal{H}_M^{(k)}\right)^G$ to $\Phi^{-1}(0)$ descends to a section of $\widehat{\ell}$ which we denote by $A_k s$. In [GS82], Guillemin and Sternberg show that $A_k s$ is holomorphic, and moreover that A_k is an isomorphism of vector spaces.

In [HK06], the author and Brian Hall showed that A_k is generically not unitary, and does not even become approximately unitary as $k \to \infty$. In Section 1.3 below, we will recall from [HK06] a similar map $B_k : \left(\widehat{\mathcal{H}}_M^{(k)}\right)^G \to \widehat{\mathcal{H}}_{M/\!\!/G}^{(k)}$, for k sufficiently large, relating the quantum spaces in the presence of the metaplectic correction. To define the map B_k requires more than just "restrict and descend" because two half-forms on M pair to give an $(n,0)$-form on M. But two half-forms on the quotient should pair to give an $(n-d,0)$-form, so a mechanism to reduce the degree is needed. The map B_k turns out to be essentially a square root of the map "restrict to $\Phi^{-1}(0)$, contract with the vectors in the directions of the infinitesimal G-action, and descend the G-invariant result to the quotient". The map B_k is also in general not unitary, but it *does* become approximately unitary as $k \to \infty$.

To measure the unitarity of the maps A_k and B_k, the author and Brian Hall showed in [HK06] that there exist functions $I_k \in C^\infty(M/\!\!/G)$, and for k sufficiently large functions $J_k \in C^\infty(M/\!\!/G)$, such that

$$\begin{aligned} \int_M |s|^2\,\frac{\omega^n}{n!} &= \int_{M/\!\!/G} |A_k s|^2\, I_k \frac{\widehat{\omega}^{n-d}}{(n-d)!}, \text{ for every } s \in \left(\mathcal{H}_M^{(k)}\right)^G, \text{and} \\ \int_M |r|^2\,\frac{\omega^n}{n!} &= \int_{M/\!\!/G} |B_k r|^2\, J_k \frac{\widehat{\omega}^{n-d}}{(n-d)!}, \text{ for every } r \in \left(\widehat{\mathcal{H}}_M^{(k)}\right)^G. \end{aligned} \tag{1.2}$$

[3] A square root of K is a line bundle, denoted by $\sqrt{K}$, such that $\sqrt{K} \otimes \sqrt{K} = K$. Such a line bundle exists if the second Stiefel–Whitney class of M vanishes. If a square root of K exists, then there are (up to equivalence) two nonisomorphic square roots.

Clearly, A_k (resp. B_k) is unitary if $I(k)$ (resp. $J(k)$) is identically 1. The main result of [HK06] is that for each $x_0 \in \Phi^{-1}(0)$,

$$\lim_{k\to\infty} I_k([x_0]) = 2^{-d/2} vol(G \cdot x_0), \text{and}$$
$$\lim_{k\to\infty} J_k([x_0]) = 1,$$

where both limits are uniform. This means in particular, that in the presence of the metaplectic correction, quantization commutes unitarily with symplectic reduction in the semiclassical limit. Moreover, in the uncorrected case, if $vol(G \cdot x_0)$ is not constant, then A_k does not converge to (a constant multiple) of a unitary map. Indeed, this fact, as well as results implying or equivalent to the uncorrected limit (1.2), have been previously studied. To the best of our knowledge, the first case was the thesis of Flude [Flu98], followed in various other forms by [Cha06], [MZ05], [MZ06], [MM07], and [Pao05] (we refer the reader to the discussion in [HK06] for more details), and most recently in greater generality in [Li08].

We will find expressions for the densities I_k and J_k in terms of the geometric data, and compute complete asymptotic expansions for both densities as $k \to \infty$.

To state our results precisely, fix an Ad-invariant inner product on $\mathfrak{g}$ and, with respect to it, and orthonormal basis $\{\xi_j\}_{j=1}^d$ such that the corresponding Haar measure $dvol_G$ on G is normalized to $\int_G dvol_G = 1$. Introduce polar coordinates $\xi = (\rho, \Omega)$ on $\mathfrak{g}$, where

$$\rho := \sqrt{(\xi^1)^2 + \cdots (\xi^d)^2}$$

and $\Omega \in S^{d-1}$ is a point in the unit sphere; in particular, $\xi = \rho\Omega$. The Lie algebra $\mathfrak{g}$ acts on M infinitesimally, and we denote the vector field giving the action of $\xi \in \mathfrak{g}$ by $X^\xi \in \Gamma(TM)$.

For a function $f \in C^1(M)$, we define its gradient as the image of df under the isomorphism between T^*M and TM given by the Kähler metric $B = \omega(\cdot, J\cdot)$; that is, $df = B(\operatorname{grad} f, \cdot)$. The divergence of a vector field $X \in \Gamma(TM)$ is defined by $\operatorname{div} X := \mathcal{L}_X \omega^n/\omega^n$, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, since for a Kähler manifold, the Liouville form $\omega^n/n!$ corresponds to the Riemannian volume. These are related to the Laplacian by

$$\Delta f = \operatorname{div} \operatorname{grad} f. \tag{1.3}$$

Our first main result is the following.

Theorem 1.1 *The densities I_k and J_k may be expressed as*

$$I_k([x_0]) = \left(\frac{k}{2\pi}\right)^{d/2} vol(G \cdot x_0)^2\, \mathfrak{j}_1(k, x_0), \ \ and \tag{1.4}$$

$$J_k([x_0]) = \left(\frac{k}{\pi}\right)^{d/2} vol(G \cdot x_0) \mathfrak{j}_{1/2}(k, x_0) \tag{1.5}$$

where

$$\mathfrak{j}_a(k, x_0) := \int_{\mathfrak{g}} \exp\left\{\int_0^1 -2k\phi_\xi(e^{it\xi}x_0) + a\,\Delta\phi_\xi(e^{it\xi}x_0)dt\right\} d^d\xi. \tag{1.6}$$

Moreover, we will find that $\mathfrak{j}_a(k, x_0)$ (and hence I_k and J_k) admits an entire asymptotic expansion

$$\mathfrak{j}_a(k, x_0) \sim k^{-d/2} \sum_{j=0}^{\infty} \zeta_{2j}^{(a)}(x_0) k^{-j}$$

as $k \to \infty$, where the coefficients are given explicitly in Theorem 1.2 below. The results of [HK06] may be interpreted as the statement that $\zeta_0^{(1)} = \zeta_0^{(1/2)} = \pi^{d/2} vol(G \cdot x_0)^{-1}$.

Our second main result is a computation of the coefficients ζ_j. The coefficients can be expressed, and computed, more efficiently in terms of certain combinatorial quantities which we will introduce in Section 3. We state here our results in a direct form, where the geometric content can be clearly seen. The concise version appears as Theorem 3.1 in Section 3.

Theorem 1.2 *For* $\mathrm{j}_a(k)$ *as defined in (1.6),*

$$\mathrm{j}_a(k,x_0) = k^{-d/2}\sum_{j=0}^{\infty}\zeta_{2j}^{(a)}(x_0)k^{-j} + o(k^{-\infty}), \tag{1.7}$$

where the coefficients are given by

$$\begin{aligned}
\zeta_j^{(a)}(x_0) = \tfrac{1}{2}\,\Gamma\left(\tfrac{d+j}{2}\right)\int_{S^{d-1}}\Bigg[&\left|X^\Omega\right|^{-(d+j)}\sum_{m=0}^{j}\frac{a^{j-m}}{(j-m)!}\\
&\times\sum_{l=1}^{j-m}\sum_{\mathcal{P}_{j,l}(\vec{n})}c(j-m;\vec{n})(\Delta\phi_\Omega)^{n_1}\left(\left(JX^\Omega\right)\Delta\phi_\Omega\right)^{n_2}\cdots\left(\left(JX^\Omega\right)^l\Delta\phi_\Omega\right)^{n_l}\\
&\times\sum_{r=1}^{m}\binom{-\frac{d+j}{2}}{r}\left|X^\Omega\right|^{-2r}\sum_{\mathcal{Q}_{m,r}(\vec{n})}\frac{2^r}{(n_1+2)!\cdots(n_r+2)!}(JX^\Omega)^{n_1+1}\phi_\Omega\cdots(JX^\Omega)^{n_r+1}\phi_\Omega\Bigg]_{x_0}d\Omega
\end{aligned} \tag{1.8}$$

where

$$\begin{aligned}
\binom{\alpha}{r} &:= \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!},\\
c(j;n_1,n_2,\dots,n_j) &:= \frac{j!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(j!)^{n_j}n_j!},
\end{aligned}$$

the sums in the second and third lines of (1.8) are taken over the sets

$$\begin{aligned}
\mathcal{P}_{j,l}(\vec{n}) &= \{(n_1,\dots,n_l)\in\mathbb{Z}^l_{\geq 0} : n_1+\cdots+n_l = j-l+1 \text{ and } n_1+2n_2+\cdots+l\,n_l = j\}, \text{ and}\\
\mathcal{Q}_{m,r}(\vec{n}) &= \{(n_1,\dots,n_r)\in\mathbb{Z}^r_{\geq 1} : n_1+\cdots+n_r = m\},
\end{aligned}$$

and empty sums are understood to be 1*.*

For example, the first two terms are

$$\begin{aligned}
\zeta_0^{(a)} &= \tfrac{1}{2}\,\Gamma\left(\tfrac{d}{2}\right)\int_{S^{d-1}}\left|X^\Omega_{x_0}\right|^{-d}d\Omega, \text{ and}\\
\zeta_2^{(a)} &= \tfrac{d}{4}\,\Gamma\left(\tfrac{d}{2}\right)\int_{S^{d-1}}\left|X^\Omega_{x_0}\right|^{-(d+2)}\Bigg[a(JX^\Omega_{x_0}\Delta\phi_\Omega(x_0) + a\,(\Delta\phi_\Omega(x_0))^2)\\
&\quad -\tfrac{d+2}{2}\,\frac{\frac{a}{3}\Delta\phi_\Omega(x_0)(JX^\Omega_{x_0})^2\phi_\Omega(x_0)+\frac{1}{12}(JX^\Omega_{x_0})^3\phi_\Omega(x_0)}{\left|X^\Omega_{x_0}\right|^2} + \binom{-\frac{d+2}{2}}{2}\frac{((JX^\Omega_{x_0})^2\phi_\Omega(x_0))^2}{9\left|X^\Omega_{x_0}\right|^4}\Bigg]d\Omega.
\end{aligned}$$

Remarks.

1. The function
$$JX^\Omega\phi_\Omega(x_0) = \omega(X^\Omega, JX^\Omega) = \left|X^\Omega_{x_0}\right|^2$$
is strictly positive (since we assume G acts freely on the zero set). We write it as $|X^\Omega|^2$ when we want to emphasize this positivity.
2. By the general theory of [Kir08], the first term can be expressed in terms of the determinant H of the Hessian of $2\int_0^1\phi_\xi(e^{it\xi}x_0)dt$ at $\xi=0$. This determinant was computed in [HK06, Lemma 3.1, Thm 4.1] to be $H = 2^d vol(G\cdot x_0)^2$, from which it follows by [Kir08, Prop. 1] and Lemma 3.3 that
$$\zeta_0^{(a)} = \tfrac{1}{2}\,\Gamma\left(\tfrac{d}{2}\right)\int_{S^{d-1}}\left|X^\Omega_{x_0}\right|^{-d}d\Omega = \frac{\Gamma(\frac{d}{2})}{2}\frac{2^{d/2+1}\pi^{d/2}}{\Gamma(\frac{d}{2})\sqrt{H}} = \pi^{d/2}vol(G\cdot x_0)^{-1}.$$

$\Diamond$

We conclude this section by recalling the definition from [HK06] of the half-form corrected Guillemin–Sternberg type map B_k discussed in the introduction. For a p-form α, denote the (left) contraction with vector fields $X_1, \ldots, X_r$ by $\mathfrak{i}\left(\bigwedge_{j=1}^r X_j\right)\alpha := \alpha(X_1, X_2, \ldots, X_r, \cdot, \ldots, \cdot)$.

Theorem 1.3 *[HK06, Thm 3.1] There exists a linear map $B : \Gamma(M, \sqrt{K})^G \to \Gamma(M/\!\!/G, \sqrt{\widehat{K}})$, unique up to an overall sign, with the property that*

$$\pi_{\mathbb{C}}^* \left[(B\nu)^2\right] = \left[\mathfrak{i}\left(\bigwedge\nolimits_j X^{\xi_j}\right)(\nu^2)\right]\Big|_{M_s}.$$

For any open set U in $M/\!\!/G$, if ν is holomorphic in a neighborhood V of $\pi_{\mathbb{C}}^{-1}(U)$, then $B\nu$ is holomorphic on U.

For each k, there is a linear map $B_k : \Gamma(M, \ell^{\otimes k} \otimes \sqrt{K})^G \to \Gamma(M/\!\!/G, \hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}})$, unique up to an overall sign, with the property that

$$B_k(s \otimes \nu) = A_k(s) \otimes B(\nu)$$

for all $s \in \Gamma(\ell^{\otimes k})$ and $\nu \in \Gamma(\sqrt{K})$. This map takes holomorphic sections of $\ell^{\otimes k} \otimes \sqrt{K}\Big|_V$ to holomorphic sections of $\hat{\ell}^{\otimes k} \otimes \sqrt{\widehat{K}}\Big|_U$.

2 The densities I_k and J_k.

In this section, we will build on the results of [HK06] to find the expressions (1.4) and (1.5) for the densities I_k and J_k (resp.). Much of the groundwork was already done in [HK06], but we need more precise results to get the full asymptotic expansion.

In [HK06], it was shown that

$$I_k([x_0]) = vol(G \cdot x_0)(k/2\pi)^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) \exp\left\{-2k \int_0^1 \phi_\xi(e^{it\xi}x_0)dt\right\} d^d\xi \tag{2.1}$$

and that

$$J_k([x_0]) = (k/\pi)^{d/2} \int_{\mathfrak{g}} \tau(\xi, x_0) \exp\left\{-\int_0^1 \left(2k\phi_\xi(e^{it\xi}x_0) + \frac{\mathcal{L}_{JX^\xi}\omega^n}{2\omega^n}(e^{it\xi}x_0)\right) dt\right\} d^d\xi, \tag{2.2}$$

where τ is the Jacobian of the diffeomorphism $\Lambda : \mathfrak{g} \times \Phi^{-1}(0) \to M_s$ given by $\Lambda(\xi, x_0) := e^{i\xi}x_0$. It was also shown that as $k \to \infty$, the contribution to I_k (resp. J_k) coming from the complement of a ball of finite radius is exponentially small, so it is enough to consider the integrals restricted to the unit ball $B := \{\xi \in \mathfrak{g} : |\xi| \leq 1\}$.

The main result of this section is the following computation of the Jacobian τ in terms of the geometric data.

Theorem 2.1 *The Jacobian of the map $\Lambda : (\xi, x_0) \in \mathfrak{g} \times \Phi^{-1}(0) \to e^{i\xi}x_0 \in M_s$ is given by*

$$\tau(\xi, x_0) = vol(G \cdot x_0) \exp\left\{\int_0^1 \Delta\phi_\xi(e^{it\xi}x_0)dt\right\}.$$

Using Theorem 2.1 to simplify the densities (2.1) and (2.2) above yields Theorem 1.1. The proof of Theorem 2.1 depends on two technical lemmas (Lemmas 2.2 and 2.3) which we defer to the end of this section.

Proof. For each $\xi \in \mathfrak{g}$, the $G_{\mathbb{C}}$-action yields a map $e^{-i\xi} : M_s \to M_s$ given by $x \mapsto e^{-i\xi}x$. Let $\mu \in \Gamma\left(\bigwedge^{2n} T^*M\right)$ be the volume form defined at each point by

$$\mu_{e^{i\xi}x_0} := \left((e^{-i\xi})^* \frac{\omega^n}{n!}\right)_{e^{i\xi}x_0}.$$

Since μ is nonvanishing, we can use it to trivialize $\bigwedge^{2n} T^*M$; in particular, there is a function $\delta \in C^\infty(M)$ such that

$$\frac{\omega^n}{n!} = \delta\mu. \tag{2.3}$$

Differentiating in the direction of JX^ξ and dividing by $\omega^n/n!$, we obtain (using Lemma 2.2)

$$\frac{\mathcal{L}_{JX^\xi}\omega^n}{\omega^n} = \frac{JX^\xi(\delta)}{\delta} = JX^\xi \log\delta. \tag{2.4}$$

Fix a point $\xi = \rho\Omega$ and define a path $\gamma_\Omega(t) := e^{it\Omega}x_0$ for $t \in [0,\rho]$. Then $\dot\gamma_\Omega(t) = JX^\Omega_{e^{it\Omega}x_0}$. Hence, $JX^\Omega \log\delta = d\log\delta(\dot\gamma_\Omega)$. Integrating $d\log\delta$ along the path $\gamma_\Omega(t)$ therefore yields

$$\delta(e^{i\rho\Omega}x_0) = \exp\left\{\int_0^\rho \frac{\mathcal{L}_{JX^\Omega}\omega^n}{\omega^n}\right\}\delta(x_0). \tag{2.5}$$

Now, by definition $\mu_{x_0} = \omega^n_{x_0}/n!$ for $x_0 \in \Phi^{-1}(0)$ which implies $\delta(x_0) = 1$ for $x_0 \in \Phi^{-1}(0)$. Combining Lemma 2.3 with (2.3) and (2.5) yields

$$\Lambda^*\omega^n/n! = vol(G\cdot x_0)\exp\left\{\int_0^\rho \frac{\mathcal{L}_{JX^\Omega}\omega^n}{\omega^n}\right\} d^d\xi \wedge dvol_{\Phi^{-1}(0)}.$$

Finally, to complete the proof, observe that by (1.3), $\mathcal{L}_{JX^\xi}\omega^n/\omega^n = \operatorname{div}\operatorname{grad}\phi_\Omega = \Delta\phi_\Omega$. ■

We now prove the lemmas that are used in the proof of Theorem 2.1. Consider the volume form μ on M_s given by

$$\mu_{e^{i\xi}x_0} := (e^{-i\xi})^*\frac{\omega^n}{n!}.$$

Lemma 2.2 *For each $\eta \in \mathfrak{g}$, we have $\mathcal{L}_{JX^\eta}\mu = 0$.*

Proof. Let $X_1, \cdots, X_{2n}$ be vector fields on M_s in a neighborhood of $e^{i\xi}x_0$. Then

$$\begin{aligned}(\mathcal{L}_{JX^\eta}\mu)_{e^{i\xi}x_0}(X_1,\ldots,X_{2n}) &= \lim_{s\to 0}\frac{1}{s}\left[(e^{is\eta})^*\mu_{e^{i\xi}x_0} - \mu_{e^{i\xi}x_0}\right](X_1,\ldots,X_{2n})\\ &= \lim_{s\to 0}\frac{1}{s}\left[\mu_{e^{is\eta+i\xi}x_0}(e^{is\eta}_*X_1,\ldots,e^{is\eta}_*X_{2n}) - \mu_{e^{i\xi}x_0}(X_1,\ldots,X_{2n})\right]\\ &= \lim_{s\to 0}\frac{1}{sn!}\Big[\omega^n_{x_0}(e_*^{-is\eta-i\xi+is\eta}X_1,\ldots,e_*^{-is\eta-i\xi+is\eta}X_{2n})\\ &\qquad\qquad - \omega^n_{x_0}(e_*^{-i\xi}X_1,\ldots,e_*^{-i\xi}X_{2n})\Big] = 0.\end{aligned}$$

■

Lemma 2.3 $(\Lambda^*\mu)_{(\xi,x_0)} = vol(G\cdot x_0)d^d\xi \wedge dvol_{\Phi^{-1}(0)}$.

Proof. Since both $\Lambda^*\mu$ and $d^d\xi \wedge dvol_{\Phi^{-1}(0)}$ are top dimensional and the latter is nowhere vanishing, there exists a function $h(\xi,x_0)$ such that $(\Lambda^*\mu)_{(\xi,x_0)} = h(\xi,x_0)d^d\xi \wedge dvol_{\Phi^{-1}(0)}$.

We will show that h is independent of ξ, from which we will conclude that $h(x_0) = vol(G\cdot x_0)$ by restricting to the zero set, where it is known [HK06, eqn (4.7) and Lemma 5.4] that

$$(\Lambda^*\mu)_{(0,x_0)} = (\Lambda^*\omega^n/n!)_{(0,x_0)} = vol(G\cdot x_0)d^d\xi \wedge dvol_{\Phi^{-1}(0)}.$$

To show that h is independent of ξ, let $\eta \in \mathfrak{g}$. Then

$$\Lambda_*(\eta,0)_{e^{i\xi}x_0} = \frac{d}{ds}\bigg|_{s=0}\Lambda(\xi+s\eta,x_0) = \frac{d}{ds}\bigg|_{s=0} e^{i(\xi+s\eta)}x_0 = JX^\eta_{e^{i\xi}x_0}.$$

By Lemma 2.2, for each $\eta \in \mathfrak{g}$,

$$\left(\mathcal{L}_{(\eta,0)}\Lambda^*\mu\right)_{(\xi,x_0)} = (\Lambda^*\mathcal{L}_{JX^\eta}\mu)_{(\xi,x_0)} = 0,$$

which implies $\frac{\partial}{\partial\xi_j}h(\xi,x_0) = 0$ for each basis vector ξ_j; that is, $h = h(x_0)$. ■

3 The expansion.

In this section, we first introduce some combinatorial objects to simplify the statement of Theorem 1.2. We then recall results of the author [Kir08] which can, after some computations, be used to arrive at Theorem 1.2. Finally, we will carry out these computations, thus arriving at our proof of Theorem 1.2.

To state Theorem 1.2 in a more useful form, we recall here some combinatorial objects related to Bell polynomials; we refer the interested reader to [Com74, Ch. 3] for more details. The **partial Bell polynomials** $\mathcal{B}_{j,l} = \mathcal{B}_{j,l}(x_1, x_2, \ldots, x_l)$, combinatorial functions on the set $\{x_1, \ldots, x_l\}$ which can be defined in terms of a formal double series expansion, are given explicitly by

$$\mathcal{B}_{j,l}(x_1, \ldots, x_l) = \sum_{\mathcal{P}_j(\vec{n})} c(j; \vec{n})\, x_1^{n_1} x_2^{n_2} \cdots x_l^{n_l} \tag{3.1}$$

where

$$c(j, \vec{n}) := c(j; n_1, n_2, \ldots, n_j) := \frac{j!}{(1!)^{n_1} n_1! (2!)^{n_2} n_2! \cdots (l!)^{n_l} n_l!}$$

and the sum is taken over the set $\mathcal{P}_{j,l}(\vec{n})$ consisting of all (ordered) l-tuples of nonnegative integers $\vec{n} := (n_1, n_2, \ldots, n_l)$ such that $n_1 + \cdots + n_l = j - l + 1$ and $n_1 + 2n_2 + \cdots l\, n_l = j$, that is,

$$\mathcal{P}_{j,l}(\vec{n}) = \left\{ (n_1, \ldots, n_l) \in \mathbb{Z}^l_{\geq 0} : \begin{array}{c} n_1 + n_2 + \cdots + n_l = j - l + 1 \\ n_1 + 2n_2 + \cdots + l\, n_l = j \end{array} \right\}. \tag{3.2}$$

The partial Bell polynomials are classical combinatorial objects and are known to satisfy many recursion (and other) identities. We will find useful the combinations

$$B_j(x_1, \cdots, x_j) := \sum_{l=1}^{j} \mathcal{B}_{j,l}(x_1, \ldots, x_l), \tag{3.3}$$

which are known as the **complete exponential Bell polynomials.**[4]

Related, though much simpler, are the polynomials $\mathcal{C}_{m,r} = \mathcal{C}_{m,r}(x_1, x_2, \cdots)$ defined by

$$(x_1 t + x_2 t^2 + x_3 t^3 + \cdots)^r = \sum_{m=r}^{\infty} \mathcal{C}_{m,r} t^m.$$

These polynomials can be computed recursively via the relation

$$\mathcal{C}_{m,r} = \sum_{j=r-1}^{m-1} x_{m-j} \mathcal{C}_{j,r-1}$$

with initial data $\mathcal{C}_{m,1}(x_1, x_2, \ldots) = x_m$. Alternatively, $\mathcal{C}_{m,r}$ is the sum of all ordered products of r elements of the set $\{x_1, x_2, \ldots\}$ such that the subscripts add to m:

$$\mathcal{C}_{m,r}(x_1, \ldots, x_m) = \sum_{\mathcal{Q}_{m,r}(\vec{n})} x_{n_1} x_{n_2} \ldots x_{n_r}, \tag{3.4}$$

where

$$\mathcal{Q}_{m,r}(\vec{n}) = \{(n_1, \cdots, n_r) \in \mathbb{Z}^r_{>0} : n_1 + n_2 + \cdots + n_r = m\}.$$

To state our main theorem more concisely, let $f, g, h \in C^\infty(\mathfrak{g} \times \Phi^{-1}(0))$ be

$$\begin{aligned} f(\rho, \Omega, x_0) &:= 2 \int_0^\rho \phi_\Omega(e^{it\Omega x_0}) dt, \\ h(\rho, \Omega, x_0) &:= \int_0^\rho \Delta\phi_\Omega(e^{it\Omega} x_0) dt, \text{ and} \\ g(\rho, \Omega, x_0) &:= \exp\{a h(\rho, \Omega, x_0)\} \end{aligned} \tag{3.5}$$

[4] Note that the sum starts at 1. This is in contrast to some other definitions in the literature; generally, sums starting at zero are called more simply the complete Bell polynomials.

The expression of Theorem 1.2 we give below is in terms of the Bell polynomials introduced above and the radial derivatives of f, g and h (which are computed below in Lemma 3.3). It will turn out that the leading order behavior of f is quadratic, so we define

$$f_j(\Omega, x_0) := \frac{1}{(j+2)!}\frac{\partial^{j+2}}{\partial\rho^{j+2}} f(\rho,\Omega,x_0)\bigg|_{\rho=0}.$$

For g we define the usual Taylor coefficients

$$g_j(\Omega, x_0) := \frac{1}{j!}\frac{\partial^j}{\partial\rho^j g(\rho,\Omega,x_0)}\bigg|_{\rho=0}$$

and similarly for h (we will drop the Ω and x_0 dependence to ease notation).

The following is a more concise version of our main Theorem 1.2.

Theorem 3.1 *For $\mathrm{j}_a(k)$ as defined in (1.6),*

$$\mathrm{j}_a(k, x_0) = k^{-d/2}\sum_{j=0}^{\infty}\zeta_{2j}^{(a)}(x_0)k^{-j} + o(k^{-\infty}),$$

where the coefficients are given by

$$\begin{aligned}
\zeta_j^{(a)} &= \tfrac{1}{2}\Gamma\left(\tfrac{d+j}{2}\right)\int_{S^{d-1}}\left[f_0^{-(d+j)}(\Omega)\sum_{m=0}^{j} g_{j-m}\sum_{r=1}^{j}\binom{-\frac{d+j}{2}}{r}\frac{\mathcal{C}_{m,r}(f_1,\dots,f_m)}{f_0(\Omega)^r}\right]d\Omega \qquad (3.6)\\
&= \tfrac{1}{2}\Gamma\left(\tfrac{d+j}{2}\right)\int_{S^{d-1}}\left[f_0^{-(d+j)}\sum_{m=0}^{j}\frac{a^{j-m}}{(j-m)!}B_{j-m}(h_1, 2!h_2,\cdots,m!h_m)\sum_{r=1}^{m}\binom{-\frac{d+j}{2}}{r}\frac{\mathcal{C}_{m,r}(f_1,\dots,f_m)}{{f_0}^r}\right]d\Omega,
\end{aligned}$$

where

$$\begin{aligned}
p!h_p &= (JX^{\Omega}_{x_0})^{p-1}\Delta\phi_\Omega(x_0),\\
f_p &= \frac{2}{(p+2)!}(JX^{\Omega}_{x_0})^{p+1}\phi_\Omega(x_0) = \frac{2}{(p+2)!}(JX^{\Omega}_{x_0})^{p}\left|X^{\Omega}_{x_0}\right|^2,
\end{aligned}$$

the polynomial $\mathcal{C}_{m,r}(f_1,\dots,f_m)$ (defined in (3.4)) is the sum of all ordered products of r elements of the set $\{f_1, f_2,\dots,f_m\}$ such that the subscripts add to m, the polynomial B_{j-m} is the complete Bell polynomial defined by (3.1) and (3.4), and empty sums are understood to be 1.

Remark.

To obtain naive "exact asymptotics", that is, term-by-term cancelation of the tail of the series (1.7), we see from (3.6) that it is necessary that $f = 2\int_0^\rho \phi_\Omega(e^{i\rho\Omega t}x_0)dt$ be quadratic in ρ. Otherwise the set $\{f_1, f_2,\dots,f_N\}$ is nontrivial for all $N > 1$ which yields nontrivial $\mathcal{C}_{m,r}(f_1,\dots,f_m)$ terms at all orders. For compact M, it is not possible that f be quadratic, since if it were, then twice differentiating implies $JX^\Omega\phi_\Omega(e^{i\rho\Omega}x_0) = \left|X^\Omega_{e^{i\rho\Omega}x_0}\right|^2 = const$. But as $\rho\to\infty$, the path $e^{i\rho\Omega}x_0$ approaches a point x_∞ which is fixed by e^Ω (see [Ler05]) so that we must rather have $\left|X^\Omega_{e^{i\rho\Omega}x_0}\right|^2 \to 0$ as $\rho\to\infty$. ◊

Using the linearity of the moment map and $\xi = \rho\Omega$, we have

$$\int_0^1 \phi_\xi(e^{it\xi}x_0)dt = \int_0^\rho \phi_\Omega(e^{it\Omega}x_0)dt$$

and

$$\int_0^1 \Delta\phi_\xi(e^{it\xi}x_0)dt = \int_0^\rho \Delta\phi_\Omega(e^{it\Omega}x_0)dt.$$

Therefore, in terms of the functions f and g defined in (3.5), the density j_a may be written

$$\mathrm{j}_a = \int_{\mathfrak{g}} e^{-kf} g \, d^d\xi = \int_{\mathfrak{g}} e^{-kf} e^{ah} \, d^d\xi,$$

which is, for each fixed $x_0 \in \Phi^{-1}(0)$, a Laplace type integral. It follows from [HK06] that it is enough to consider the integral over the unit ball $B := \{\xi \in \mathfrak{g} : |\xi| \leq 1\}$.

For completeness, we quote the result from [Kir08] which we need to obtain Theorem 1.2. Let $\{\xi^1, \ldots, \xi^d\}$ be coordinates on $\mathbb{R}^d$. Denote by $S^{d-1} = \{|\xi| = 1\} \subset \mathbb{R}^d$ the unit sphere and introduce polar coordinates $\rho := \sqrt{(\xi^1)^2 + \cdots (\xi^d)^2}$ and $\Omega = \xi/|\xi| \in S^{d-1}$.

Suppose that R is a region in $\mathbb{R}^d$ containing 0 as an interior point, and let f and g be measurable functions on R. Suppose f attains its unique minimum of 0 at 0. Assume moreover there exists $N > 0$ and

- $N+1$ continuous functions $f_j(\Omega),\ j = 0, \ldots, N$ with $f_0 > 0$ such that for some $\nu > 0$

$$f(\rho, \Omega) = \rho^\nu \sum_{j=0}^{N} f_j(\Omega)\rho^j + o(\rho^{N+\nu}) \text{ as } \rho \to 0, \text{ and} \tag{3.7}$$

- $N+1$ functions $g_j(\Omega),\ j = 0, \ldots, N$ such that for some $\lambda > 0$

$$g(\rho, \Omega) = \rho^{\lambda-d} \sum_{j=0}^{N} g_j(\Omega)\rho^j + o(\rho^{N+\lambda-d}) \text{ as } \rho \to 0. \tag{3.8}$$

Theorem 3.2 *[Kir08] With the hypotheses above, there exists an asymptotic expansion*

$$\int_B e^{-kf} g \, d^d x = \sum_{j=0}^{N} \zeta_j k^{-(\lambda+j)/\nu} + o(k^{-(N+\lambda)/\nu}) \tag{3.9}$$

where the coefficients are given by

$$\zeta_j = \tfrac{1}{\nu}\Gamma\left(\tfrac{j+\lambda}{\nu}\right) \int_{S^{d-1}} \left[f_0^{-(j+\lambda)/\nu} \sum_{m=0}^{j} g_{j-m} \sum_{r=1}^{m} \binom{-\frac{j+\lambda}{\nu}}{r} f_m^{(r)} f_0^{-r} \right] d\Omega,$$

where $f_m^{(r)} = \mathcal{C}_{m,r}(f_1, \ldots, f_m)$ is the sum[5] *of all ordered products of r elements of $\{f_1, f_2, \ldots, f_m\}$ such that the subscripts add to m, $\binom{\alpha}{r} := \alpha(\alpha-1)\cdots(\alpha-r+1)/r!$, and empty sums are understood to be* 1.

To apply Theorem 3.2, we need asymptotic expansions of f and g near 0. We will use their Taylor series:

Lemma 3.3 *For f and g as defined in (3.5),*

$$f = \rho^2 \sum_{j=0}^{\infty} \frac{2\rho^j}{(j+2)!} \left(JX^\Omega\right)^{j+1} \phi_\Omega + o(\rho^\infty), \ \rho \to 0$$

and

$$\begin{aligned} g &= \sum_{j=0}^{\infty} \frac{\rho^j a^j}{j!} B_j(\Delta\phi_\Omega(x_0), JX^\Omega_{x_0}\Delta\phi_\Omega(x_0), \ldots, (JX^\Omega_{x_0})^{j-1}\Delta\phi_\Omega(x_0)) + o(\rho^\infty), \ \rho \to 0 \\ &= \sum_{j=0}^{\infty} \rho^j \left[\frac{a^j}{j!} \sum_{l=0}^{j} \sum_{\mathcal{P}_{j,l}(\vec{n})} c(j, \vec{n}) \prod_{p=1}^{l} \left(\left(JX^\Omega_{x_0}\right)^{p-1} \Delta\phi_\Omega(x_0) \right)^{n_p} \right] + o(\rho^\infty), \ \rho \to 0, \end{aligned}$$

where B_j is the complete exponential Bell polynomial defined in (3.3) and $\mathcal{P}_{j,l}(\vec{n})$ is defined in (3.2).

[5]For example, $f_6^{(3)} = 6f_1f_2f_3 + 3f_1^2f_4 + f_2^3$,

Proof. The fundamental theorem of calculus yields

$$\partial_\rho^n f(e^{i\rho\Omega}x_0) = 2\partial_\rho^{n-1}\phi_\Omega(e^{i\rho\Omega}x_0) = 2(JX^\Omega)^{n-1}\phi_\Omega(e^{i\rho\Omega}x_0).$$

Moreover, $f(x_0) = \partial_\rho f(x_0) = 0$, so that

$$f = \sum_{j=2}^{\infty} \frac{2\rho^j}{j!}\left[(JX^\Omega)^{j-1}\phi_\Omega\right]_{x_0}$$

as desired.

To compute the Taylor series for g, we first recall that the Taylor series for $\exp(h(\rho))$ near $\rho = 0$ can be expressed using Faà di Bruno's formula as [Com74, Sec. 3.4]

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{\rho^j}{j!}\exp(h(0))B_j(h'(0),\dots,h^{(j)}(0)) \\ &= \sum_{j=0}^{\infty} \frac{\rho^j}{j!}\left[\exp(h(0))\sum_{l=1}^{j}\sum_{\mathcal{P}_{j,l}(\vec{n})} c(j;\vec{n})\prod_{p=1}^{l}\left(h^{(p)}(0)\right)^{n_p}\right], \end{aligned} \tag{3.10}$$

where B_j is the complete exponential Bell polynomial (3.3), and $\mathcal{P}_{j,l}(\vec{n})$ is defined in (3.2). Taking $h(\rho) = a\int_0^\rho \Delta\phi_\Omega(e^{i\Omega t}x_0)dt$ in (3.10) and using $\partial_\rho g(e^{i\rho\Omega}x_0) = JX^\Omega g(e^{i\rho\Omega}x_0)$, $\partial_\rho^{(l)} h = (JX^\Omega)^{l-1}\Delta\phi_\Omega$, and $g(x_0) = \exp(h(0)) = 1$ completes the proof. ■

We are now ready to prove our main Theorems 1.2 and 3.1.

Proof of Theorems 1.2 and 3.1. Take $\nu = 2$ and $\lambda = d$. From Lemma 3.3, we see that

$$f_j(\Omega, x_0) = \frac{2}{(j+2)!}(JX^\Omega_{x_0})^j \left|X^\Omega_{x_0}\right|^2 \tag{3.11}$$

and

$$\begin{aligned} g_j(\Omega, x_0) &= \frac{a^j}{j!}B_j(\Delta\phi_\Omega(x_0), JX^\Omega_{x_0}\Delta\phi_\Omega(x_0),\dots,(JX^\Omega_{x_0})^j\Delta\phi_\Omega(x_0)) \\ &= \frac{a^j}{j!}\sum_{l=0}^{j}\sum_{\mathcal{P}_{j,l}(\vec{n})} c(j,\vec{n})\prod_{p=1}^{l}\left(\left(JX^\Omega_{x_0}\right)^{p-1}\Delta\phi_\Omega(x_0)\right)^{n_p}. \end{aligned} \tag{3.12}$$

Plugging these into Theorem 3.2 applied to $\mathfrak{j}_a(k,x_0) \sim \int_B e^{-kf} g\, d^d\xi$ yields

$$\mathfrak{j}_a(k,x_0) = \sum_{j=0}^{\infty}\zeta_j k^{-(j+d)/2} + o(k^{-\infty})$$

where

$$\zeta_j = \tfrac{1}{2}\Gamma\left(\tfrac{j+d}{2}\right)\int_{S^{d-1}}\left[\left|X^\Omega_0\right|^{-(d+j)}\sum_{m=0}^{j}\frac{a^{j-m}B_{j-m}}{(j-m)!}\sum_{r=1}^{m}\binom{-\frac{j+d}{2}}{r}\left|X^\Omega_{x_0}\right|^{-2r} f_m^{(r)}\right]d\Omega \tag{3.13}$$

in which

$$\begin{aligned} f_m^{(r)} &= \mathcal{C}_{m,r}\left(\tfrac{2}{3!}(JX^\Omega_{x_0})^2\phi_\Omega, \tfrac{2}{4!}(JX^\Omega_{x_0})^3\phi_\Omega,\dots,\tfrac{2}{(m+2)!}(JX^\Omega_{x_0})^{m+1}\phi_\Omega\right) \\ &= \sum_{\mathcal{Q}_{m,r}(\vec{n})}\frac{2^r}{(n_1+2)!\cdots(n_r+2)!}(JX^\Omega_{x_0})^{n_1+1}\phi_\Omega\cdots(JX^\Omega)^{n_r+1}\phi_\Omega \end{aligned} \tag{3.14}$$

is the sum of all ordered products of r terms of the set $\{f_1(\Omega,x_0), f_2(\Omega,x_0),\dots\}$ whose subscripts add to m and

$$\begin{aligned} B_{j-m} &= B_{j-m}(\Delta\phi_\Omega(x_0), JX^\Omega_{x_0}\Delta\phi_\Omega(x_0),\dots,(JX^\Omega_{x_0})^{j-m-1}\Delta\phi_\Omega(x_0)) \\ &= \sum_{l=0}^{j-m}\sum_{\mathcal{P}_{j-m,l}(\vec{n})} c(j-m,\vec{n})\prod_{p=1}^{l}\left(\left(JX^\Omega_{x_0}\right)^{p-1}\Delta\phi_\Omega(x_0)\right)^{n_p}. \end{aligned} \tag{3.15}$$

Aside from simply substituting $f_m^{(r)}$ and B_{j-m}, we can make one significant simplification:

Lemma 3.4 *$\zeta_j = 0$ for j odd.*

Proof. The linearity of the infinitesimal action of $G_{\mathbb{C}}$ on M implies $X^{-\Omega} = -X^{\Omega}$ and $JX^{-\Omega} = -JX^{\Omega}$. Related is the linearity of the moment map in the component index: $\phi_{-\Omega} = -\phi_{\Omega}$. These facts together imply that $f_j(-\Omega, x_0) = (-1)^j f_j(\Omega, x_0)$. Since the sum of the subscripts of the terms appearing in $f_m^{(r)}$ is m, we have $f_m^{(r)}(-\Omega, x_0) = (-1)^m f_m^{(r)}(\Omega, x_0)$. Finally, we conclude that

$$g_{j-m}(-\Omega, x_0) f_m^{(r)}(-\Omega, x_0) = (-1)^j g_{j-m}(\Omega, x_0) f_m^{(r)}(\Omega, x_0)$$

which implies that for j odd, the integrand appearing in ζ_j is antisymmetric with respect to $\Omega \mapsto -\Omega$ so that the integral is 0 for j odd. ■

Making the substitutions of (3.14) and (3.15) into (3.13) and replacing j by $2j$ (Lemma 3.4) yields Theorems 1.2 and 3.1. ■

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