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Adapted complex structures and the geodesic flow

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Adapted complex structures and the geodesic flow

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Abstract

In this paper, we give a new construction of the *adapted complex structure* on a neighborhood of the zero section in the tangent bundle of a compact, real analytic Riemannian manifold. Motivated by the "complexifier" approach of T. Thiemann as well as certain formulas of V. Guillemin and M. Stenzel, we obtain the polarization associated to the adapted complex structure by applying the "imaginary-time geodesic flow" to the vertical polarization. Meanwhile, at the level of functions, we show that every holomorphic function is obtained from a function that is constant along the fibers by "composition with the imaginary-time geodesic flow." We give several equivalent interpretations of this composition, including a convergent power series in the vector field generating the geodesic flow.

Keywords: adapted complex structure, Grauert tube, geodesic flow, geometric quantization, Kähler structure, polarization
MSC (2000): 53D25, 32D15, 32Q15, 53D50, 81S10

1 Introduction

Let (M^n, g) be a compact, connected, real analytic Riemannian manifold. Let TM be the tangent bundle of M, permanently identified with the cotangent bundle by means of the metric. We denote points in TM as $z = (x, v)$, where $x \in M$ and $v \in T_xM$. Let T^RM denote the set of points $(x, v) \in TM$ with $g(v, v) < R^2$, i.e., the set of tangent vectors of length less than R. Let E be the "energy" function defined by $E(x, v) = \frac{1}{2}g(v, v)$. Let $\Phi_\sigma : TM \to TM$ then denote the time-σ geodesic flow, which is the Hamiltonian flow associated to the function E.

Working independently, L. Lempert and R. Szőke [LS91, Sző91] and V. Guillemin and M. Stenzel [GS91, GS92] introduced natural complex structures defined on T^RM, for some sufficiently small R. Lempert–Szőke characterized their complex structure in terms of geodesics—see Theorem 3.8. Meanwhile, Guillemin–Stenzel characterized their complex structure in terms of a Kähler potential and an involution—see Section 3.2. Both sets of authors show that there is a unique such complex structure on T^RM for some sufficiently small R. It turns out that the two characterizations are equivalent, and complex structures satisfying these characterizations are referred to as *adapted complex structures*. The adapted complex structure fits together with the canonical symplectic structure on $TM \cong T^*M$ to make TM into a Kähler manifold. Numerous authors have studied adapted complex structures from various points of view; see, for example, [Agu01, BH01, Sző01, Hal02b, Tot03].

There are certain special cases in which the adapted complex structure exists on all of TM, including the case in which M is a compact Lie group with a bi-invariant metric. In the compact group case, one obtains a nice quantization of TM by performing geometric quantization using the Kähler structure associated to the adapted complex structure. The resulting Hilbert space can be identified [Hal02a] with the generalized Segal–Bargmann space introduced in [Hal94, Hal97]. See also [FMMN05, FMMN06], [Hue06], and the discussion in Section 3.4.

In this paper, we build on the "complexifier" method of T. Thiemann [Thi96, Thi01, Thi06] to give a new construction of the adapted complex structure. Our main construction, though, is phrased in terms of involutive Lagrangian distributions.[1] If there is a Kähler structure on a symplectic manifold N, then the distribution $P = T^{(1,0)}N$, which determines the complex structure, is an involutive Lagrangian distribution. In the case $N = TM$, there is one obvious involutive Lagrangian distribution, called the *vertical distribution*, in which the distribution P_z at the point $z \in TM$ is the (complexification of) the vertical subspace.

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[1] In the language of geometric quantization, an involutive Lagrangian distribution is called a *polarization*.

Our main result is that the involutive Lagrangian distribution associated to the adapted complex structure can be obtained from the vertical distribution by the "time-i geodesic flow." The way we make sense of this is to first push forward the vertical distribution by Φ_σ, for real values of σ, obtaining a family $P_z(\sigma)$ of involutive Lagrangian distributions. Then we show that for all z in some $T^R M$, one can analytically continue the map $\sigma \to P_z(\sigma)$ to a holomorphic map of a disk of radius greater than 1 into the Grassmannian of Lagrangian subspaces of $T_z^{\mathbb{C}} TM$. This analytic continuation serves to define $P_z(i)$ on $T^R M$. We then verify that on some possibly smaller tube $T^{R'} M$, there is a complex structure for which $P_z(i)$ is, at each point, the $(1,0)$ subspace. We then show that this complex structure satisfies both the Lempert–Szőke conditions and the Guillemin–Stenzel conditions.

Once the construction of the adapted complex structure is complete, we relate the adapted complex structure to the imaginary-time geodesic flow at the level of functions. We show that holomorphic functions with respect to the adapted complex structure can be obtained from functions that are constant on the fibers by "composing with the time-i geodesic flow." What this means is the following. If a function ψ is constant along the leaves of the vertical distribution (the fibers of the tangent bundle), then we consider $\psi \circ \Phi_\sigma$. If ψ is real analytic, then for all z in some $T^R M$, the map $\sigma \to \psi(\Phi_\sigma(z))$ admits an analytic continuation to a disk of radius greater than 1. We show that the function $\psi(\Phi_i(z))$, interpreted in terms of the just-mentioned analytic continuation, is holomorphic on $T^R M$ and that every holomorphic function on $T^R M$ arises in this way. Equivalently, $\psi \circ \Phi_i$ can be constructed by a convergent power series in the vector field generating Φ_σ (this is essentially Thiemann's approach) or by analytically continuing the exponential map for M (this is essentially the approach of Guillemin and Stenzel).

Although the imaginary-time geodesic flow appears already in [GS92] and [Hal02b] and in [Thi96] at the level of functions, we believe that our approach sheds new light on the subject by giving a self-contained construction of the adapted complex structure using this flow. For example, this way of thinking provides a simple explanation for the formula expressing the adapted complex structure in terms of Jacobi fields (Section 3.3), and sheds light on the results of [FMMN05, FMMN06] (Section 3.4). We also hope that, as suggested by Thiemann, the use of other imaginary-time Hamiltonian flows will lead to the construction of new complex structures.

2 Main results

Denote by $V_z \subset T_z^{\mathbb{C}} TM$ the complexification of the vertical tangent space to TM at z. (That is, the vertical subspace at $(x, v) \in TM$ is the complexified tangent space to the fiber $T_x M$.) Our principal object of interest is the pushforward of the vertical distribution by the geodesic flow. For any real number σ, consider

$$P_z(\sigma) := (\Phi_\sigma)_* \left(V_{\Phi_{-\sigma}(z)}\right) \subset T_z^{\mathbb{C}} TM.$$

For each fixed $z \in TM$, we are going to analytically continue the map $\sigma \to P_z(\sigma)$ to a holomorphic map of a disk about the origin in $\mathbb{C}$ into the Grassmannian of n-dimensional complex subspaces of $T_z^{\mathbb{C}} TM$.

The first main result of this paper is the following theorem, a concatenation of Theorems 3.5, 3.6, and 3.7, which are proved in Section 3.

Theorem 2.1. *There exists $R > 0$ for which the following properties hold.*

1. *For each $z \in T^R M$, the family $\sigma \mapsto P_z(\sigma)$ can be analytically continued in σ to a ball of radius greater than 1 around the origin, thus giving a meaning to the expression $P_z(i)$.*

2. *For each $z \in T^R M$, $P_z(i)$ intersects its complex conjugate only at zero.*

3. *Let J be the unique almost complex structure on $T^R M$ such that the restriction of J to $P_z(i)$ is iI. Then J is integrable and fits together with the canonical symplectic form ω on TM so as to give a Kähler structure to $T^R M$. This means, in particular, that $P_z(i)$ is a Lagrangian subspace relative to ω.*

The family $P_z(i)$ of subspaces constitutes a positive involutive Lagrangian distribution, that is, a *Kähler distribution.*

Our next results show that the complex structure we construct is the same as the one constructed (independently and from different points of view) by Lempert–Szőke and Guillemin–Stenzel. See also Section 3.3, where we show how to compute the complex structure in terms of Jacobi fields, in a way that agrees with the corresponding calculations in [LS91].

Theorem 2.2. *The complex structure on $T^R M$ described in Theorem 2.1 is "adapted" to the metric on M in the sense of Lempert-Szőke.*

For the definition of "adapted," see the proof of Theorem 3.8 or Definition 4.1 in [LS91]. In light of the preceding theorem, we will henceforth refer to the complex structure in 2.1 as the adapted complex structure on $T^R M$. We should emphasize, though, that our proofs are independent of the work of Lempert–Szőke (and of that of Guillemin–Stenzel). The main interest in our results is the clear and unifying geometric picture that the adapted complex structure arises from the time-i complexified geodesic flow.

Theorem 2.3. *If R is as in Theorem 2.1, then the map $\phi(x,v) := (x,-v)$ is antiholomorphic with respect to the complex structure on $T^R M$ described in that theorem. Furthermore, the function $\kappa(x,v) := g(v,v)$ is a Kähler potential for the Kähler structure described in Point 3 of the theorem.*

This result shows that the complex structure agrees with that of Guillemin–Stenzel. (See the theorem on p. 568 of [GS91].)

Our next main result, from Section 4, describes holomorphic functions on $T^R M$ in terms of the imaginary-time geodesic flow. Informally, it says that composition of a vertically constant function with the time-i geodesic flow yields a holomorphic function.

Theorem 2.4. *Let R be as in Theorem 2.1 and suppose a function $f : M \to \mathbb{C}$ admits an entire analytic continuation $f_{\mathbb{C}}$ to $T^r M$ for some $r \leq R$. Then for each $z \in T^r M$, the map*

$$\sigma \mapsto (f \circ \pi \circ \Phi_\sigma)(z)$$

admits an analytic continuation in σ from $\mathbb{R}$ to $\{\sigma + i\tau \in \mathbb{C} : |\tau| < r/|v|\}$. Moreover, the function $f_{\mathbb{C}}$ is given by the value of the continuation at $\sigma = i$; this can be expressed informally as

$$f_{\mathbb{C}}(z) = (f \circ \pi \circ \Phi_i(z)). \tag{2.1}$$

Furthermore, $f_{\mathbb{C}}(z)$ can be expressed as an absolutely convergent series

$$f_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \frac{i^k}{k!} X_E^k (f \circ \pi)(z), \tag{2.2}$$

where X_E is the Hamiltonian vector field associated to E. In this expansion, $(X_E^k f)(x,v)$ is a homogeneous polynomial of degree k with respect to v for each fixed $x \in M$, so that (2.2) may be thought of as a "Taylor series in the fibers."

The expression (2.2) embodies the "complexifier" approach advocated by Thiemann [Thi96, Thi01], although Thiemann does not address the question of convergence. Recall that we are thinking of the Lagrangian distribution associated to the adapted complex structure as being obtained from the vertical distribution by means of the time-i geodesic flow. Theorem 2.4 simply expresses this idea at the level of functions. The function $\psi := f \circ \pi$ is constant along the leaves of the vertical distribution and the function $f_{\mathbb{C}}$ is "constant along the leaves of the complex distribution" associated to the adapted complex structure, meaning that the derivative of $f_{\mathbb{C}}$ is zero in the directions of the $(0,1)$ subspace. (It is customary, for example in geometric quantization, to take derivatives in the direction of the *complex conjugate* of the subspaces P_z of a Lagrangian distribution.) Theorem 2.4 says, informally, that $f_{\mathbb{C}} = \psi \circ \Phi_i$.

Note that $\pi(\Phi_\sigma(x,v)) = \exp_x(v)$, where $\exp_x$ is the geometric exponential map. Thus, (2.1) is formally equivalent to the assertion that $f_{\mathbb{C}}(x,v) = f(\exp_x(iv))$. This expression for $f_{\mathbb{C}}$ is already implicit in the work of Guillemin–Stenzel (Section 5 of [GS92]).

3 Time-i geodesic flow

In this section, we consider the imaginary-time geodesic flow as it acts on Lagrangian distributions. On the tangent bundle TM (which is permanently identified with the cotangent bundle), we have the vertical distribution, consisting of the vertical subspace of the tangent space to TM at each point. Since the geodesic flow is, for each time, a symplectomorphism, pushing forward the vertical distribution by the time-σ geodesic flow gives another Lagrangian distribution. Thus, at each point $z \in TM$, we have a family $P_z(\sigma)$ of Lagrangian subspaces of $T_z^{\mathbb{C}} TM$ parameterized by the real number σ. In this section we will show that for all z in a neighborhood of the zero section, we can analytically continue this family to $\sigma = i$.

Furthermore, we will see that the family of subspaces $P_z(i)$ constitutes an involutive complex Lagrangian distribution on $T^R M$, and is also the distribution of $(1,0)$ vectors for a complex structure on TM (that is, it intersects its complex conjugate only at 0). It is not hard to see (Theorem 3.8) that this complex structure is "adapted" in the sense of Lempert–Szőke. It can also be shown (Section 3.2) that this complex structure satisfies the conditions of Guillemin–Stenzel (the theorem on p. 568 of [GS91]).

3.1 Basic properties

For each point $z \in TM$, let $V_z \subset T_z^{\mathbb{C}} TM$ denote the *complexification* of the vertical subspace of the tangent space.

Definition 3.1. *For each fixed $\sigma \in \mathbb{R}$ and point $z \in TM$, define the subspace $P_z(\sigma) \subset T_z^{\mathbb{C}} TM$ by*

$$P_z(\sigma) = (\Phi_\sigma)_*(V_{\Phi_{-\sigma}(z)}). \tag{3.1}$$

We consider the Lagrangian Grassmannian $\mathcal{L}_z$, consisting of all n-dimensional Lagrangian subspaces of $T_z^{\mathbb{C}} TM$. This is a complex manifold. (It is a complex submanifold of the Grassmannian of all n-dimensional complex subspaces of $T_z^{\mathbb{C}} TM$.) We then wish to extend the map $\sigma \in \mathbb{R} \to P_z(\sigma) \in \mathcal{L}_z$ to a holomorphic map of some domain in $\mathbb{C}$ into $\mathcal{L}_z$ and, if possible, evaluate at $\sigma = i$. To avoid ambiguity in the value of $P_z(\sigma)$ at $\sigma = i$, we limit our attention to points z such that the map $\sigma \to P_z(\sigma)$ can be analytically continued to a disk of radius greater than 1 about the origin.

We now develop the relevant properties of the subspaces $P_z(i)$. Most of these properties can be established either "by analyticity" (i.e., they hold for real values of σ and then by analytic continuation for complex values of σ as well) or "by continuity" (i.e., we can verify by direct computation that they hold on the zero section and thus by continuity they hold also on a neighborhood of the zero section).

Theorem 3.2 (Behavior on the zero section)**.** *For a point $(x, 0)$ in the zero section, identify $T_{(x,0)}TM$ with $T_xM \oplus T_xM$ (tangent space to the zero section direct sum tangent space to the fiber). Then for all $\sigma \in \mathbb{R}$, $(\Phi_\sigma)_*$ evaluated at $(x, 0)$ is the linear map represented by the block matrix*

$$\begin{pmatrix} I & \sigma I \\ 0 & I \end{pmatrix}.$$

Proof. The composition property of the flow Φ_σ implies that the restriction of $(\Phi_\sigma)_*$ to the tangent space at a fixed point z is a one-parameter group of linear transformations of T_zTM. This one-parameter group is then the exponential of some linear transformation A. The value of A is computed by differentiating $(\Phi_\sigma)_*|_{T_zTM}$ with respect to σ and evaluating at $\sigma = 0$. Since Φ_σ is smooth as a map of $TM \times \mathbb{R}$ into TM, we can interchange the derivative with respect to σ with the derivative in the space variable that computes the differential $(\Phi_\sigma)_*$. Differentiating Φ_σ with respect to σ gives the vector field X_E. Thus, working in local coordinates, A will be the matrix of derivatives of the vector field X_E evaluated at z.

If $q_1, \ldots, q_n$, $p_1, \ldots, p_n$ are local coordinates of the usual sort on $TM \cong T^*M$, then

$$X_E(q,p) = g^{jk}(q,p)p_j \frac{\partial}{\partial q^k} - \frac{\partial g^{jk}}{\partial x^l} p_j p_k \frac{\partial}{\partial p_l}$$

(sum convention). Differentiating and evaluating at $(q, 0)$, we can see that A is the block matrix $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$ and so $\exp(\sigma A)$ is the matrix in the statement of the theorem. ∎

We now turn to an important scaling property of the subspaces $P_z(\sigma)$. This property reflects that the energy function E is homogeneous of degree 2 in each fiber.

Theorem 3.3 (Scaling)**.** *For $c \in \mathbb{R}$, let $N_c : TM \to TM$ denote the scaling map in the fibers: $N_c(x, v) = (x, cv)$. Then for all real values of σ, all nonzero values of c, and all $z \in TM$, we have*

$$P_{N_c z}(\sigma) = (N_c)_*(P_z(c\sigma)).$$

It follows that if the map $\sigma \to P_z(\sigma)$ has a holomorphic extension to a disk of radius R then the map $\sigma \to P_{N_c z}(\sigma)$ has a holomorphic extension to a disk of radius R/c.

Proof. We make use of the relation $\Phi_\sigma = N_c \Phi_{c\sigma} N_{1/c}$, which holds because the energy function E is homogeneous of degree 2 in each fiber.

To compute $P_{N_c z}(\sigma)$, we take the vertical tangent space V at the point $\Phi_{-\sigma}(N_c z)$ and push it forward by $(\Phi_\sigma)_*$. But

$$(\Phi_\sigma)_* = (N_c)_*(\Phi_{c\sigma})_*(N_{1/c})_*,$$

and $(N_{1/c})_*$ maps the vertical subspace at $\Phi_{-\sigma}(N_c z)$ to the vertical subspace V' at $N_{1/c}(\Phi_{-\sigma}(N_c z)) = \Phi_{-c\sigma}(z)$. Thus,

$$\begin{aligned} P_{N_c z}(\sigma) &= (N_c)_*(\Phi_{c\sigma})_*(V') \\ &= (N_c)_*(P_z(c\sigma)). \end{aligned}$$

■

Lemma 3.4 (Analyticity)**.** *Let $\mathcal{L}$ denote the (complex) Lagrangian Grassmannian bundle over TM (with fiber $\mathcal{L}_z$). The map $P : TM \times \mathbb{R} \to \mathcal{L}$ given by (3.1) is analytic.*

Proof. Choose a local analytic frame $v^1, v^2, \ldots, v^n$ for the vertical distribution V. over an open set $(\Phi_{-\sigma})(U) \subset TM$. Then $\{u^j(\sigma) := (\Phi_\sigma)_* v^j\}$ is a local frame for $P(\sigma)$ over the open set $U \subset TM$. The geodesic flow is the Hamiltonian flow of the analytic function $E(x, v) = \frac{1}{2} g(v, v)$ and TM is analytic, so by standard results for ordinary differential equations, the geodesic flow Φ_σ depends analytically on σ and on the initial conditions [Koh99, Prop 3.37]. (That is, Φ_σ is analytic as a map of $TM \times \mathbb{R}$ into TM.) It follows that the pushforward map $(\Phi_\sigma)_*$ is also analytic, whence the frames $\{u_z^j(\sigma)\}$ depend analytically on σ and z. The Lagrangian subspace spanned by these vectors then also depends analytically on z and σ. ■

For each z, the map $\sigma \to P_z(\sigma)$ has a holomorphic extension to *some* disk. It then follows from the above scaling result that for all sufficiently small c, the map $\sigma \to P_{N_c z}(\sigma)$ has a holomorphic extension to a disk of radius greater than 1. This observation is the key to the following existence result.

Theorem 3.5 (Existence)**.** *There exists $R > 0$ such that: (1) for each $z \in T^R M$ the map $\sigma \to P_z(\sigma) \in \mathcal{L}_z$ admits a holomorphic extension to a disk in $\mathbb{C}$ of radius greater than* 1, *and this radius is locally bounded away from 1.*

Proof. In an analytic local coordinate system on TM with origin at some z_0, we have the associated analytic local trivialization of the tangent bundle of TM. Then we may think of the map $(z, \sigma) \to P_z(\sigma)$ as a map of $(U \subset \mathbb{R}^{2n}) \times \mathbb{R}$ into the Grassmannian of n-dimensional complex submanifolds of $\mathbb{C}^{2n}$. By the lemma, this map is analytic. There exists, then, an open set V in $\mathbb{C}^{2n} \times \mathbb{C}$ containing $U \times \mathbb{R}$ to which this map has a holomorphic extension. The set V contains a set of the form $W_1 \times W_2$, where W_1 is a neighborhood of the origin in $\mathbb{C}^{2n}$ and W_2 is a disk of some radius $R > 0$ around the origin in $\mathbb{C}$. This shows that there exists $R > 0$ such that for all z in some neighborhood of z_0, the map $\sigma \to P_z(\sigma)$ has a holomorphic extension to a disk of radius R in $\mathbb{C}$. This amounts to saying that the maximum radius of such extensions is locally bounded away from zero.

Let S now denote the unit tangent bundle in TM. Since S is compact, the above argument shows that there is some minimum radius R such that for all $z \in S$, the map $\sigma \to P_z(\sigma)$ extends holomorphically to a disk of radius R. It then follows from Theorem 3.3 that for all $z \in T^R M$, the map $\sigma \to P_z(\sigma)$ extends holomorphically to a disk of radius greater than 1. Furthermore, this radius is bounded away from 1 as long as z stays away from the boundary of $T^R M$. ■

Theorem 3.6 (Integrability)**.** *Suppose that for some $R > 0$, the map $\sigma \to P_z(\sigma) \in \mathcal{L}_z$ has a holomorphic extension to a disk of radius greater than 1 for all $z \in T^R M$. Then $P(i)$ is an integrable distribution on $T^R M$.*

Proof. Let $\mathcal{K}_n$ denote the set of n-dimensional complex subspaces of $\mathbb{C}^{2n}$. The group $GL(2n, \mathbb{C})$ acts transitively on $\mathcal{K}_n$ and the stabilizer subgroup of a point in $\mathcal{K}_n$ is isomorphic to the group H of $2n \times 2n$ matrices for which the lower left $n \times n$ block is zero. Because H is a closed complex subgroup, $GL(2n, \mathbb{C})$ is a holomorphic fiber bundle over $\mathcal{K}_n$ with fiber H. We can pick a local holomorphic section in a neighborhood of any point. This means that for $P_0 \in \mathcal{K}_n$, there is a neighborhood U of P_0 and a map $g(\cdot)$ from U into $GL(2n, \mathbb{C})$ such that $V = g(V)(V_0)$ for all $V \in U$. We can then choose a basis $u_1, \ldots, u_n$ for V_0 and we get a basis $u_1(V), \ldots, u_n(V)$ for each $V \in U$ by setting $u_j(V) = g(V) u_j$. The vectors $u_j(V)$ depend holomorphically on V.

Now consider a point $z_0 \in TM$ and $\sigma_0 \in \mathbb{C}$ with $|\sigma| \leq 1$. We choose a local trivialization of TTM near z_0, so that $P_z(\sigma)$ can be thought of as an n-dimensional subspace of $\mathbb{C}^{2n}$. By Lemma 3.4, we can pick a basis $u_z^1(\sigma), \ldots, u_z^n(\sigma)$ for $P_z(\sigma)$ that depends smoothly on z (in a neighborhood of z_0) and holomorphically on σ (in a neighborhood of σ_0), because the subspace $P_z(\sigma)$ depends holomorphically on σ and the u_z^j's depend holomorphically on the subspace. It follows that $[u_z^j(\sigma), u_z^k(\sigma)]$ depends holomorphically on σ.

Now, for all real values of σ, $P(\sigma)$ is just the push-forward of the vertical distribution by a diffeomorphism of TM and therefore $P(\sigma)$ is integrable. The above argument with $\sigma_0 = 0$ shows that $[u_z^j(\sigma), u_z^k(\sigma)]$ is zero, and hence $P(\sigma)$ is involutive, in a neighborhood of the origin. If there were some $t \in [0,1]$ for which $P(it)$ is not involutive, we could let τ be the infimum of the set of all such t's. Because $P(\sigma)$ is involutive for σ in a neighborhood of the origin, we have $\tau > 0$. Applying the above argument with $\sigma_0 = i\tau$, we have that $[u_z^j(\sigma), u_z^k(\sigma)]$ is holomorphic in σ and zero for $\sigma = it$, for $\tau - \varepsilon < t < \tau$, hence zero in a neighborhood of $i\tau$. This is a contradiction, and we conclude that $P(it)$ is involutive for all $t \in [0,1]$. ■

Theorem 3.7 (Kähler structure). *In a neighborhood of the zero section, $P_z(i)$ intersects its complex conjugate only at zero. Thus there is some $R' > 0$ (possibly smaller than the radius in the existence theorem) such that for $z \in T^{R'}M$, the distribution $P_z(i)$ is the $(1,0)$-tangent space of a complex structure.*

Moreover, there exists some $R \in (0, R']$ such that for $z \in T^R M$, the distribution $P_z(i)$ is positive with respect to the canonical symplectic form, that is, $-i\omega(Z, \overline{Z}) > 0$ for every $Z \in P_z(i)$. Hence, $P_z(i)$ defines a Kähler structure on $T^R M$.

Proof. Choose a local analytic frame $\{X_j\}_{j=1}^n$ of $P(i)$ over an open set $U_\alpha \subset TM$ which intersects the zero section in the (nonempty, open in M) set $U_0 = M \cap U_\alpha$. On the zero section, we know that $P(i)$ looks like the Euclidean $(1,0)$-bundle; in particular, $X_j(p) - \overline{X}_j(p) \neq 0$ for every $p \in U_0$. Since the vector fields $X_j - \overline{X}_j$ are analytic (hence continuous), it follows that $X_j(p) - \overline{X}_j(p) \neq 0$ for all p in some open (in TM) neighborhood V with $U_0 \subset V \subset U_\alpha$.

Do this for each open set U_α intersecting the zero section, then take a maximal tube inside union of the Vs (we only need to consider a finite set of U_αs, and hence of Vs, since M is compact) to obtain $T^{R_0}M$.

To see that $P_z(i)$ is positive, we work in the same local trivializations. On the zero section, the distribution $P_{(x,0)}(i)$ is just that of the standard Euclidean complex structure on $\mathbb{R}^{2n}$, so that $-i\,\omega_{(x,0)}(X_j, \overline{X}_j) > 0$. But $-i\,\omega_z(X_j, \overline{X}_j)$ is a continuous (in fact analytic) function of z, so it must be positive in a neighborhood of the zero section. Again, by the compactness of M we obtain a neighborhood of the zero section on which $P_z(i)$ is positive. Combined with the fact that the distribution $P(i)$ is Lagrangian, this shows that the complex structure defined by the distribution $P(i)$ is ω-compatible, whence $T^R M$ is Kähler. ■

Theorem 3.8 (Adaptedness). *Let R be any positive number such that $P_z(i)$ exists (in the sense of Theorem 3.5) for all $z \in T^R M$ and is disjoint from its complex conjugate. Then the associated complex structure is "adapted" in the sense of Lempert–Szőke (Definition 4.1 in [LS91]).*

Remark 3.9. In Section 3.3, we show $P_z(i)$ can be computed in terms of Jacobi fields in a way that agrees with formulas in [LS91] for the adapted complex structure.

Proof. The main point is that the geodesic flow Φ_t leaves invariant the tangent bundle of a geodesic, as a surface inside TM, and the action of Φ_t on this surface is the same as the geodesic flow for the real line.

Consider a point $z = (x, v)$ in TM with $|v| = 1$ and let γ be the geodesic satisfying $\gamma(0) = x$ and $\dot\gamma(0) = v$. Let $S_R \subset \mathbb{C}$ denote the strip $\{(\sigma, \tau) |\ |\tau| < R\}$. Define a map $\Psi_z : S_R \to T^R M$ by $\Psi_z(\sigma, \tau) = (\gamma(\sigma), \tau\dot\gamma(\sigma))$. Then a complex structure on $T^R M$ is called "adapted" (cf. [LS91, Def. 4.1]) if for all $z = (x, v)$ with $|v| = 1$, Ψ_z is holomorphic as a map of $S_R \subset \mathbb{C}$ into $T^R M$.

Now, the geodesic flow on $T^R M$ leaves each strip inside $T^R M$ of the form $\Psi_z(S_R)$ invariant. In fact, for each $t \in \mathbb{R}$, the time-t geodesic flow on such a strip corresponds to the map $(\sigma, \tau) \to (\sigma + t\tau, \tau)$. (That is to say, $\Phi_t(\Psi_z(\sigma, \tau)) = \Psi_z(\sigma + t\tau, \tau)$.) Furthermore, for each σ, the curve $s \to \Psi_z(\sigma, \tau + s)$ is "vertical," i.e., lying in a single fiber. It follows that the vector

$$\frac{\partial}{\partial\tau}\Psi_z(\sigma + t\tau, \tau) = (\Psi_z)_*(\partial/\partial\tau) + t(\Psi_z)_*(\partial/\partial\sigma) \tag{3.2}$$

belongs to $P_u(t)$ for all real numbers t and each point u in the strip. (Every point in the strip is of the form $\Psi_z(\sigma + t\tau, \tau)$ for some σ and τ.)

Now, the right-hand side of (3.2) is clearly a holomorphic vector-valued function of t for $t \in \mathbb{C}$. The family $P_u(t)$ of subspaces also depends holomorphically on t for t in a ball of radius greater than 1 around the origin. It follows that the right-hand side of (3.2) is in $P_u(t)$ for all t in that ball, including $t = i$. This means that $(\Psi_z)_*(\partial/\partial\tau) + i(\Psi_z)_*(\partial/\partial\sigma)$ belongs to $P_u(i)$. Recalling that $P_u(i)$ is the $J = i$ subspace and $\overline{P_u(i)}$ is the $J = -i$ subspace, a little algebra shows that $J((\Psi_z)_*(\partial/\partial\sigma)) = (\Psi_z)_*(\partial/\partial\tau)$. This shows that Ψ_z is holomorphic. ■

3.2 Involution and Kähler potential

We next obtain a Kähler potential for the adapted complex structure and show that the map $(x, v) \longmapsto (x, -v)$ is antiholomorphic. These results will show that the complex structure we have constructed satisfies the conditions of the theorem on p. 568 of [GS91]. Scaling (or the homogeneity property of E) gets used in both cases.

Theorem 3.10 (Involution)**.** *Let R be as in Theorem 3.5. Then the map $(x, v) \longmapsto (x, -v)$ is antiholomorphic with respect to the adapted complex structure on $T^R M$.*

Proof. Because $P(\sigma) = \overline{P(\sigma)}$ for $\sigma \in \mathbb{R}$, it is easily seen that $P(\sigma - i\tau) = \overline{P(\sigma + i\tau)}$. By Theorem 3.3, we have $(N_{-1})_* P(\sigma) = P(-\sigma)$. On $T^R M$, analytically continuing and evaluating at $\sigma = i$ yields $(N_{-1})_* P(i) = P(-i) = \overline{P(i)}$. This shows that N_{-1} (i.e., the map $(x, v) \longmapsto (x, -v)$) is antiholomorphic. ∎

Theorem 3.11 (Kähler potential)**.** *The function $\kappa(x, v) := g(v, v) = 2E(x, v)$ is a Kähler potential for the adapted complex structure on $T^R M$. Specifically, $\operatorname{Im} \bar{\partial}\kappa = \Theta$, where Θ is the canonical 1-form on $TM \cong T^*M$.*

Proof. Let Θ be the canonical 1-form on TM, given by $\Theta_{(x,v)}(Z) = g(v, \pi_* Z)$. We begin by proving that for all $Z \in P_z(\sigma) \subset T_z^{\mathbb{C}} TM$ we have

$$\Theta(Z) = \sigma Z(E), \quad Z \in P(\sigma) \tag{3.3}$$

for all σ in any disk around the origin in $\mathbb{C}$ on which $P_z(\sigma)$ is defined holomorphically. This is the same as saying that the 1-forms Θ and σdE agree on vectors in $P(\sigma)$. We first establish this when σ in in $\mathbb{R}$. Suppose X is a vector field lying in the vertical tangent space at each point in TM, and let $X^\sigma = (\Phi_\sigma)_*(X)$. We wish to show that the function

$$u(z, \sigma) := \Theta(X^\sigma) - \sigma X^\sigma(E)$$

is identically zero for all σ. Let us differentiate with respect to σ. Since $dX^\sigma/d\sigma = [X_E, X^\sigma]$ and $X_E(E) = 0$, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial \sigma} &= \Theta([X_E, X^\sigma]) - X^\sigma(E) - \sigma X_E(X^\sigma(E)) \\
&= X_E(\Theta(X^\sigma)) - X^\sigma(\Theta(X_E)) - d\Theta(X_E, X^\sigma) - X^\sigma(E) - \sigma X_E(X^\sigma(E)) \\
&= X_E u,
\end{aligned}$$

where in the second line we use the identity $d\Theta(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$ and in the third line we use the computation $\Theta(X_E) = -2E$ along with the fact that $d\Theta(X_E, X^\sigma) = dE(X^\sigma) = X^\sigma(E)$ (definition of Hamiltonian vector field).

Now, the solution to the equation $\partial u/\partial\sigma = X_E u$ is just $u(z, \sigma) = f(\Phi_\sigma(z))$, where $f(z) = u(z, 0)$. In our case, $u(z, 0) = 0$, because Θ is zero on the vertical subspace, and so u is identically equal to zero. Since X^σ can take any value in $P(\sigma)$ (for appropriate choice of X), we conclude that (3.3) holds for all real σ. Using local holomorphic frames as in the proof of Theorem 3.6, we can easily establish (3.3) for all σ in any disk around the origin on which $P_z(\sigma)$ is defined holomorphically.

Returning now to the proof of the Kähler potential, we start with the two facts that $d = \partial + \bar{\partial}$ and ω is of type $(1, 1)$. Then

$$\begin{aligned}
\omega = -d\Theta &= -(\partial + \bar{\partial})(\Theta^{1,0} + \Theta^{0,1}) \\
&= -\partial\Theta^{0,1} + \bar{\partial}\Theta^{1,0}
\end{aligned} \tag{3.4}$$

since $\partial\Theta^{1,0} = \bar{\partial}\Theta^{0,1} = 0$.

If we now apply (3.3) with $\sigma = i$ and $\sigma = -i$ (for $z \in T^R M$), we obtain

$$\begin{aligned}
\Theta^{1,0}(X) &= \Theta(X^{1,0}) = iX^{1,0}(E) = i\partial E(X), \text{ and} \\
\Theta^{0,1}(X) &= \Theta(X^{0,1}) = -iX^{0,1}(E) = -i\bar{\partial} E(X).
\end{aligned}$$

Hence,

$$\Theta = \Theta^{1,0} + \Theta^{0,1} = i\partial E - i\bar{\partial} E = 2 \operatorname{Im} \bar{\partial} E = \operatorname{Im} \bar{\partial}\kappa.$$

∎

3.3 Computation of the adapted complex structure in terms of Jacobi fields

We now give a more-or-less explicit way of computing the adapted complex structure. The results of this subsection also give a more direct way of verifying that the complex structure we have defined coincides with the adapted complex structure of Lempert and Szőke.

Fix a point $z = (x, v)$ in TM. Choose a basis $\{v_j\}$ for T_xM and let ξ_j and η_j denote the horizontal and vertical lifts, respectively, of v_j to T_zTM. We now push these vectors forward by the geodesic flow, defining a family of tangent vectors along the curve $\Phi_\sigma(z)$:

$$\xi_j(\sigma) := (\Phi_\sigma)_*(\xi_j)$$
$$\eta_j(\sigma) := (\Phi_\sigma)_*(\eta_j).$$

Finally, we project these vectors down to M by defining

$$v_j(\sigma) := \pi_*(\xi_j(\sigma))$$
$$w_j(\sigma) := \pi_*(\eta_j(\sigma)).$$

The vector fields v_j and w_j, defined along the geodesic $\pi(\Phi_\sigma(x, v))$, are Jacobi fields. (We refer the reader to [Jos95, Chap. 4] for background on Jacobi fields.) Note that $\xi_j(0) = \xi_j$, $\eta_j(0) = \eta_j$, $v_j(0) = v_j$, and $w_j(0) = 0$.

For all sufficiently small σ, $\{v_j(\sigma)\}$ is a basis for the tangent space to M at the point $\pi(\Phi_\sigma(z))$ and $\{\xi_j(\sigma), \eta_j(\sigma)\}$ is a basis for the tangent space to TM at the point $\Phi_\sigma(z)$. We let $f_z(\sigma)$ denote the matrix expressing $w_j(\sigma)$ in terms of $v_j(\sigma)$:

$$w_j(\sigma) = \sum\nolimits_k f_z(\sigma)_j^k v_k(\sigma). \tag{3.5}$$

It then follows that

$$\pi_*\left[\eta_j(\sigma) - \sum\nolimits_k f_z(\sigma)_j^k \xi_k(\sigma)\right] = 0.$$

This says that the vectors $\eta_j(\sigma) - \sum f_v(\sigma)_j^k \xi_k(\sigma)$ are contained in the vertical subspace of $T_{\Phi_\sigma(z)}TM$, and by the independence of ξ, η, these vectors actually span the vertical subspace.

Note that from the way $\xi_j(\sigma)$ and $\eta_j(\sigma)$ are defined, we have $(\Phi_\sigma)_*\xi_j(-\sigma) = \xi_j$ and $(\Phi_\sigma)_*\eta_j(-\sigma) = \eta_j$. Thus the vectors

$$\eta_j - \sum\nolimits_k f_z(-\sigma)_j^k \xi_k \in T_zTM \tag{3.6}$$

span the push-forward (to T_zTM) of the vertical subspace of $T_{\Phi_{-\sigma}(z)}TM$.

It follows that if, for a fixed z, the matrix-valued function $f_z(\cdot)$ admits an analytic continuation to a disk of radius greater than 1, then the family of subspaces $P_z(\sigma)$ also admits an analytic continuation, defined by putting in a complex value for σ into (3.6) and taking the span of the resulting vectors. (Since the η's are independent of the ξ's, any collection of n vectors of the form (3.6) are linearly independent.) We may encapsulate the preceding observations in the following theorem.

Theorem 3.12. *For a given $z \in TM$, let $f(\cdot)$ be the matrix-valued function defined, for sufficiently small $\sigma \in \mathbb{R}$, by (3.5). If f admits a matrix-valued holomorphic extension to a disk of radius R, then so also do the subspaces $P_z(\cdot)$, by setting $P_z(\sigma)$ equal to the span of the vectors in (3.6).*

We can demonstrate directly, using the matrix f, that the complex structure arising from the time-i geodesic flow coincides with the adapted complex structure as defined by Lempert–Szőke [LS91].

Lempert and Szőke show that for $v \in T^RM$, the adapted complex tensor $J_v \in End(T_vTM)$ is specified by [LS91, Eq. (5.8)]

$$J_v\xi_j = \left((\operatorname{Im} f(i))^{-1}\right)_j^k \left(\eta_k - \left(\operatorname{Re} f^l{}_k(i)\right)\xi_l\right), \ j = 1, \ldots, n. \tag{3.7}$$

Since $\{\xi_j\}_{j=1}^n$ is linearly independent, the $(1,0)$-tangent space $T_v^{1,0}TM$ is spanned by $\{\xi_j^{1,0}\}_{j=1}^n$ where the projection onto the $(1,0)$-tangent space is given by

$$\xi^{1,0} := \frac{1}{2}(1 - iJ)\xi.$$

Writing $f(i) = f_1 + if_2$ (i.e., $f_1 = \operatorname{Re} f(i)$ and $f_2 = \operatorname{Im} f(i)$), it follows from (3.7) that

$$\xi_j^{1,0} = \frac{i}{2}\sum\nolimits_k \left[\left(\overline{f(i)}f_2^{-1}\right)_j^k \xi_k \ - \ \left(f_2^{-1}\right)_j^k \eta_k\right]. \tag{3.8}$$

The $(1,0)$-tangent space is therefore spanned by the vectors $\sum_k \left[(\overline{f(i)} f_2^{-1})^k{}_j \xi_k - (f_2^{-1})^k{}_j \eta_k \right]$.

A short computation using the fact that $-i\omega(\xi^{1,0}, \xi^{0,1}) > 0$ (Theorem 3.7) shows that $\operatorname{Im} f(i) = f_2$ is positive definite, and hence invertible. Thus, the $(1,0)$-tangent space of the adapted complex structure, as defined by Lempert–Szőke, is spanned by the vectors $\sum_k \overline{f(i)}^k{}_j \xi_k - \eta_j$; that is, it is $P(i)$.

3.4 Connections to geometric quantization

The arguments we have used in this section also show that for any $\tau > 0$, we can obtain a well-defined Kähler distribution on some $T^R M$ by considering the subspace $P_z(i\tau)$ at each point $z \in T^R M$. In light of our scaling result, Theorem 3.3, this is equivalent to taking the adapted complex structure and rescaling by a factor of τ in the fibers. (That is, $P_z(i\tau)$ is the same as $(N_{1/\tau})_* P_{N_\tau z}(i)$.) We obtain, then, a one-parameter family J_τ of complex structures indexed by the positive real number τ. If f is a holomorphic function on $T^R M$ with respect to the adapted complex structure, then the function $f_\tau(x,v) := f(x, \tau v)$ is holomorphic with respect to J_τ. Note that in the limit as τ approaches zero, f_τ becomes constant along the fibers of TM. This reflects the idea that the Lagrangian distribution associated to J_τ is obtained from the vertical distribution by the time-$i\tau$ geodesic flow, so that the J_τ-distribution converges to the vertical distribution as τ tends to zero.

Now, given *any* complex structure on (a neighborhood of the zero section in) TM, one can rescale by a constant in the fibers to obtain a one-parameter family of complex structures. Furthermore, the distribution associated to this family of complex structures will converge to the vertical distribution as τ tends to zero. What is interesting in the case we are considering is that the one-parameter family of complex structures obtained by scaling the adapted complex structure can also be obtained by *starting* with the vertical distribution and applying the imaginary-time geodesic flow.

The procedure of geometric quantization associates to each integrable Lagrangian distribution, or *polarization*, the Hilbert space of sections of a certain line bundle[2] which are covariantly constant along the distribution.

In [Hal02a], the first author has considered geometric quantization on $TM \cong T^*M$ in the case that M is a compact Lie group K with a bi-invariant metric. The paper [Hal02a] considers the pairing map between the vertically polarized Hilbert space and the Kähler-polarized Hilbert space (with half-forms) associated to the adapted complex structure. It turns out that this pairing map is unitary (up to a constant) and coincides (up to a constant) with the generalized Segal–Bargmann transform, which was introduced in [Hal94, Hal97] and defined in terms of the heat equation on K. This result is surprising because the procedures of geometric quantization apparently have nothing to do with the heat equation.

In [FMMN05, FMMN06], Florentino–Mattias–Mourão–Nunes looked at the results of [Hal02a] in terms of the one-parameter family of complex structures described above, namely those obtained from the adapted complex structure by scaling in the fibers. Using ideas similar to the ones in [ADW91], these authors consider a parallel transport in the Hilbert bundle associated to the family of complex structures. This means that the base of their bundle is the positive half-line and the fiber over a point τ is the geometric quantization Hilbert space associated to the complex structure J_τ. They compute this parallel transport and show that it is given in terms of the heat equation on K. This calculation goes a long ways toward clarifying the results of [Hal02a].

The present paper adds one more clarifying insight to the picture: the one-parameter family of complex structures that Florentino–Mattias–Mourão–Nunes are considering are all obtained from the vertical distribution by means of the imaginary-time geodesic flow. Since the "transport" at the classical level (the Lagrangian distributions) is given in terms of the imaginary-time geodesic flow, it is not surprising that the transport at the quantum level (the Hilbert spaces) is given in terms of the heat equation. (The quantization of the energy function $E(x,v) = \frac{1}{2} g(v,v)$, by whatever quantization procedure one prefers, generally comes out to be the Laplacian plus a multiple of the scalar curvature, where in the case of a compact group with a bi-invariant metric, the scalar curvature is a constant.)

Thiemann makes a similar point in [Thi96, Thi01]. He proposes that the transition from functions of the position variables to the holomorphic functions can be made by using the imaginary-time geodesic flow (as explained in Section 4). He then argues that the quantum counterpart of this transition (the Segal–Bargmann transform) should be achieved by the quantum counterpart of the imaginary time geodesic flow, namely, the heat semigroup. Actually, Thiemann proposes that *any* (sufficiently regular) Hamiltonian flow can be used in place of the geodesic

[2] In geometric quantization, one studies *prequantum* line bundles, that is, complex Hermitian line bundles with connection with curvature $-i\omega$.

flow, though very few examples have been considered so far. In a future paper, we hope to look at the Hamiltonian flow associated to a charged particle in a magnetic field.

Meanwhile, J. Huesbschmann [Hue06] has examined the results of [Hal02a] from the point of view of the Kirillov character formula. In Huebschmann's approach, the heat equation enters because the characters are eigenfunctions for the Laplacian.

4 Holomorphic functions

In the previous section, we looked at the looked at the action of the imaginary-time geodesic flow on the vertical distribution. In this section, we look at the action of the imaginary-time geodesic flow on functions. Associated to any distribution P, there is a naturally associated class of functions, namely, those functions f such that for every z and every $X \in \overline{P_z}$, $Xf = 0$. (It is customary to consider functions that are constant in the directions of $\overline{P}$ rather than P.) In the case of the vertical distribution $\overline{P} = P$ and the functions constant in the P-directions are just the functions that are constant along each fiber of the tangent bundle. In the case of the distribution $P(i)$, the functions with derivative zero in the directions of $\overline{P(i)}$ are the holomorphic functions with respect to the adapted complex structure. We would like to see that these two classes of functions are related by the time-i geodesic flow.

Suppose ψ is a function whose derivatives in the vertical directions are zero. The it is easily seen that the derivatives of $\psi \circ \Phi_\sigma$ in the directions of $P(-\sigma)$ are zero. If we formally set $\sigma = i$, we see that the derivatives of $\psi \circ \Phi_i$ should be zero in the direction of $P(-i) = \overline{P(i)}$. (Note that $P(\sigma)$ is real for real σ, so that $P(\sigma - i\tau) = \overline{P(\sigma + i\tau)}$.) The conclusion is that if ψ is constant along the leaves of the vertical distribution, then we expect $\psi \circ \Phi_i$ (suitably interpreted) to be holomorphic with respect to the adapted complex structure on $T^R M$. In this section we will fulfill that expectation and give three different but equivalent ways of interpreting the expression $\psi \circ \Phi_i$.

If ψ is constant along the vertical distribution, then $\psi = f \circ \pi$ for some function f on M. We wish to compose with Φ_σ and then "set $\sigma = i$" to obtain the function

$$f(\pi(\Phi_i(z))). \tag{4.1}$$

The first way to interpret (4.1) is to look at the map

$$\sigma \to f(\pi(\Phi_\sigma(z))) \tag{4.2}$$

for a fixed f and z. This function is well defined for all real σ, and we can attempt to analytically continue it to a ball of radius greater than 1 about the origin in the complex plane. If such an analytic continuation exists, the value at $\sigma = i$ can but understood as the value of (4.1).

The second way to interpret (4.3) is to think of the action of the geodesic flow on functions as (formally) the exponential of the Hamiltonian vector field X_E. This approach is the one proposed by Thiemann in [Thi96]. Since

$$\frac{d}{d\sigma} f \circ \pi \circ \Phi_\sigma = X_E(f \circ \pi \circ \Phi_\sigma),$$

we have, at least formally,

$$f \circ \pi \circ \Phi_\sigma = e^{\sigma X_E}(f \circ \pi),$$

where

$$\begin{aligned} e^{\sigma X_E}\phi &= \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} X_E^k \phi \\ &= \sum_{k=0}^{\infty} \frac{\sigma^k}{k!} \{E, \{E, \ldots \{E, \phi\} \ldots\}\}. \end{aligned}$$

This is Thiemann's point of view. (See also [HM02], where this method is worked in a very explicit way in the case that M is a sphere.)

The third way to interpret (4.1) is to observe that

$$\pi(\Phi_\sigma(x, v)) = \exp_x(v),$$

where $\exp_x : T_x M \to M$ is the geometric exponential map. Suppose we embed M in a real-analytic fashion as a totally real submanifold of some complex manifold X of complex dimension n, as in work of Bruhat–Whitney

[WB59] and Grauert [Gra58]. Then for each x, the exponential map can be extended to a holomorphic map of a ball around the origin in $T_x^{\mathbb{C}}M$ into X. If f is real-analytic on M, then it has a holomorphic extension $f_{\mathbb{C}}$ to a neighborhood of M in X. Thus, the analytic continuation in σ of (4.2) can be accomplished by setting $f(\pi(\Phi_\sigma(z)))$ equal to $f_{\mathbb{C}}(\exp_x(\sigma v))$ for σ in a small enough ball in $\mathbb{C}$ that $\sigma v \in T_x^{\mathbb{C}}M$ will lie in the domain of the analytic continuation of $\exp_x$. Putting $t = i$ gives

$$f(\pi(\Phi_i(z))) = f_{\mathbb{C}}(\exp_x(iv)), \tag{4.3}$$

where on the right-hand side of (4.3), $\exp_x(iv)$ refers to the analytically continued exponential map.

What (4.3) is really saying is that we should identify $T^R M$ with a neighborhood of M in X by means of the map $(x, v) \to \exp_x(iv)$. (Thus, the value of a holomorphic function on $T^R M$ at (x, v) is obtained by evaluating a holomorphic function on X at $\exp_x(iv)$.) This identification is implicit in Section 5 of [GS92] and was made explicit in a personal communication of Stenzel with the first author. The same identification is also used in the work of S. Halverscheid [Hal02b].

We now provide theorems showing that the holomorphic functions on $T^R M$ with respect to the adapted complex structure can indeed be obtained by means of the imaginary-time geodesic flow, using any one of the three interpretations discussed above.

Theorem 4.1. *Let R be as in Theorem 2.1 and suppose a real-analytic function $f : M \to \mathbb{C}$ admits an analytic continuation $f_{\mathbb{C}} : T^r M \to \mathbb{C}$ for some $0 < r \leq R$. Then for each $z \in T^r M$, the map*

$$\sigma \mapsto (f \circ \pi \circ \Phi_\sigma)(z)$$

admits an analytic continuation in σ from $\mathbb{R}$ to $\{\sigma + i\tau \in \mathbb{C} : |\tau| < r/ |v|\}$. Moreover, the function $f_{\mathbb{C}}$ is given by the value of the continuation at $\sigma = i$,

$$f_{\mathbb{C}}(z) = (f \circ \pi \circ \Phi_i(z)),$$

and can be expressed as an absolutely convergent power series

$$f_{\mathbb{C}}(z) = \sum_{k=0}^{\infty} \frac{i^k}{k!} \left(X_E^k(f \circ \pi)\right)(z). \tag{4.4}$$

Remark 4.2. The function $X_E^k(f \circ \pi)$ can be written $\{E, \ldots, \{E, f \circ \pi\} \ldots\}$ (k Poisson brackets). A computation then shows that, for each fixed $x \in M$, the function $v \to X_E^k(f \circ \pi)(x, v)$ is a homogeneous polynomial of degree k. This means that (4.4) may be thought of as the real-variable Taylor series of $f_{\mathbb{C}}$ in the fibers. (The analogous construction in the case of the real line would be the expansion of a holomorphic function $f(x + iy)$ as a power series in y for each fixed x.)

Remark 4.3. Building on work of L. Boutet de Monvel, Guillemin and Stenzel [GS92, Thm. 5.2] have shown that the following result holds for all sufficiently small R: A function f on M admits an analytic continuation to $T^R M$ that is smooth up to the boundary if and only if f is of the form $\exp(-RP)g$ for some smooth function g. Here $P = \sqrt{\Delta}$, where Δ is the (positive) Laplacian. This condition on f implies that the Taylor series expansion of $\exp(RP)$, when applied to f, converges (in, say, the sup norm). The convergence on $T^R M$ of the Taylor series expansion of $\exp(iX_E)$, when applied to $f \circ \pi$, is an analogous result in our approach to the subject.

Theorem 4.4. *Suppose M is real-analytically embedded into a complex manifold X as a totally real submanifold of maximal dimension. Then there exists $R > 0$ such that (1) for all $x \in M$, the geometric exponential map $\exp_x$ extends to a holomorphic map of a ball of radius R in $T_x^{\mathbb{C}}M$ into X, (2) the map $(x, v) \to \exp_x(iv)$ is a diffeomorphism of $T^R M$ into X, and (3) the pullback of the complex structure on X to $T^R M$ by this map is the adapted complex structure on $T^R M$.*

It follows that if $f : M \to \mathbb{C}$ has a holomorphic extension $f_{\mathbb{C}}$ to a neighborhood of M in X, the corresponding holomorphic extension of f to $T^R M$ with respect to the adapted complex structure is given by $(x, v) \to f_{\mathbb{C}}(\exp_x(iv))$.

Remark 4.5. If we take X to be $T^R M$ itself with the adapted complex structure, then the identification in the theorem becomes simply an identity: $\exp_x(iv) = (x, v)$.

Proof of Theorem 4.1. It follows from the notion of adaptedness that the map $\sigma \longmapsto \exp_x(\sigma v) = \pi(\Phi_\sigma(x, v))$ has an analytic continuation in σ to a strip for fixed (x, v). The continuation is given by $\sigma + i\tau \longmapsto N_\tau \Phi_\sigma(x, v)$. Thus $t \mapsto (f \circ \pi \circ \Phi_t)(z)$ has a continuation given by $\sigma + i\tau \longmapsto f_{\mathbb{C}}(N_\tau \Phi_\sigma(x, v))$.

Now, the restriction of E to each fiber is a homogeneous polynomial of degree 2, whereas the restriction to each fiber of $f \circ \pi$ is constant. It then follows by a simple inductive computation that the restriction to each fiber of $X_E^k(f \circ \pi)$ is a homogeneous polynomial of degree k. When restricted to a line through the origin in T_xM, (4.4) is just the Taylor series of the map $t \mapsto (f \circ \pi \circ \Phi_t)(z)$, which converges to the function in any disk lying within the strip. ■

Proof of Theorem 4.4. This result is already implicitly contained in Section 5 of [GS92], in the assertion on p. 638 that the tubes M_ε defined on p. 637 coincide with the ones defined by the level sets of the function $\rho(x, v) = \sqrt{g(v, v)}$. Nevertheless, it is illuminating to give a direct proof.

There exists a biholomorphism from some T^RM with the adapted complex structure to a neighborhood U of M in X. If we use this biholomorphism to identify U with T^RM, then it suffices to verify the identity in Remark 4.5. But if we analytically continue $\exp_x$, then the map $c \to \exp_x(cv)$ will be holomorphic for c belonging to a neighborhood of the origin in $\mathbb{C}$. So it suffices to analytically continue the map $\sigma \to \exp_x(\sigma v)$.

Now, since the complex structure on T^RM is "adapted" in the sense of Lempert–Szőke (Theorem 3.8), the map sending $\sigma + i\tau$ to $(\gamma(\sigma), \tau\dot{\gamma}(\sigma))$ is a holomorphic map of a strip in $\mathbb{C}$ into T^RM. If γ is the geodesic with initial value x and initial derivative v, then $\exp_x(\sigma v)$ is nothing but $\gamma(\sigma v)$, which we identify with $(\gamma(\sigma v), 0) \in TM$. Thus, the analytic continuation of the map $\sigma \to \exp_x(\sigma v)$ is the map $\sigma + i\tau \to (\gamma(\sigma), \tau\dot{\gamma}(\sigma))$. Evaluating at i gives $\exp_x(iv) = (\gamma(0), \dot{\gamma}(0)) = (x, v)$. ■

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