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für Mathematik
in den Naturwissenschaften
Leipzig

Nonstationary Stokes System with Variable Viscosity in Bounded and Unbounded Domains

(revised version: March 2009)

by

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Preprint no.: 12 2009

Nonstationary Stokes System with Variable Viscosity in Bounded and Unbounded Domains

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March 30, 2009

Dedicated to V.A. Solonnikov on the occasion of his 75th birthday

Abstract

We consider a generalization of the nonstationary Stokes system, where the constant viscosity is replaced by a general given positive function. Such a system arises in many situations as linearized system, when the viscosity of an incompressible, viscous fluid depends on some other quantities. We prove unique solvability of the nonstationary system with optimal regularity in L^q-Sobolev spaces, in particular for an exterior force $f \in L^q(Q_T)$. Moreover, we characterize the domains of fractional powers of some associated Stokes operators A_q and obtain a corresponding result for $f \in L^q(0,T;\mathcal{D}(A_q^\alpha))$. The result holds for a general class of domains including bounded domain, exterior domains, aperture domains, infinite cylinder and asymptotically flat layer with $W_r^{2-\frac{1}{r}}$-boundary for some $r > d$ with $r \geq \max(q,q')$.

Key words: Stokes equation, Stokes operator, unbounded domains, maximal regularity, domains of fractional powers
AMS-Classification: 35Q30, 76D07, 47F05

1 Introduction and Assumptions

We consider the following nonstationary Stokes-like system

$$\partial_t v - \operatorname{div}(2\nu(x,t)Dv) + \nabla p = f \qquad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} v = g \qquad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$v|_{\Gamma_1} = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{1.3}$$

$$n \cdot T_\nu(v,p)|_{\Gamma_2} = a \qquad \text{on } \Gamma_2 \times (0,T), \tag{1.4}$$

$$v|_{t=0} = v_0 \qquad \text{on } \Omega \tag{1.5}$$

where $v\colon \Omega \times (0,T) \to \mathbb{R}^d$ is the velocity of the fluid, $p\colon \Omega \times (0,T) \to \mathbb{R}$ is the pressure,

$$T_\nu(v,p) = 2\nu(x,t)Dv - pI$$

is the stress tensor, $Dv = \frac{1}{2}(\nabla v + \nabla v^T)$, $\nu\colon \Omega\times(0,T) \to (0,\infty)$ is a variable viscosity coefficient, and $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a suitable domain with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consisting of two closed, disjoint (possibly empty) components Γ_j, $j = 1,2$. Moreover, n denotes the exterior normal at $\partial\Omega$ and $f_\tau = f - (n\cdot f)n$ the tangential component of a vector field f. Finally, we denote $S(v) = 2\nu Dv$ and $Q_T = \Omega\times(0,T)$ for $T \in (0,\infty]$.

In the case that $\nu(x,t) = \nu_0 \in (0,\infty)$ is independent of (x,t) the latter system was extensively studied in many kinds of different domains relevant for mathematical fluid mechanics. But in many situations the viscosity ν of an incompressible fluid depends on some quantities as e.g. temperature or a concentration of a species. Moreover, we note that the case of variable density can reduced to case of variable viscosity up to a lower order term.

First results on general nonstationary Stokes systems, including the case of variable viscosity, were obtained by Solonnikov [26, 25] in L^q-Sobolev spaces and weighted Hölder spaces in the case of a bounded domain with pure Dirichlet boundary conditions and $g = 0$. Moreover, Bothe and Prüß [7] obtained unique solvability of general nonstationary Stokes systems in L^q-Sobolev spaces for the case of bounded and exterior domains with Dirichlet, Neumann, and Navier boundary conditions. Finally, we note that Ladyženskaja and Solonnikov [20] and later Danchin [9] obtained results for a similar nonstationary Stokes system with variable density instead of variable viscosity.

In [5] Terasawa and the author studied the corresponding Stokes resolvent system to (1.1)-(1.4) in a large class of unbounded domains. In the latter contribution it is shown that an associated reduced Stokes operator admits a bounded H^∞-calculus, which implies in particular that the reduced Stokes operator has maximal L^p-regularity for every $1 < p < \infty$. Based on this result, we will show unique solvability in L^q-Sobolev spaces for the system (1.1)-(1.4).

More precisely, the first main result is the following:

Theorem 1.1 *Let $0 < T < \infty$, $d < r_1, r_2 \leq \infty$, $1 < q < \infty$ such that $q, q' \leq \min(r_1, r_2)$ and $q \neq \frac{3}{2}, 3$, and let $\nu(x,t) = \nu_\infty + \nu'(x,t)$ with $\nu' \in BUC([0,T); W^1_{r_1}(\Omega))$, $\nu'|_{\Gamma_2} \in C^{\frac{1}{2}}([0,T); L^\infty(\Gamma_2))$ and $\nu(x,t) \geq \nu_0 > 0$. Moreover, assume that Ω is either a bounded domain, an exterior domain, a perturbed half-space, an aperture domain, an asymptotically flat layer, or an infinite cylinder with boundary of class $W^{2-\frac{1}{r_2}}_{r_2}$. Then for every $f \in L^q(Q_T)^d$, $g \in W^{1,0}_q(Q_T)$ with $\partial_t g \in L^q(0,T; \dot{W}^{-1}_{q,\Gamma_2}(\Omega))$, $g|_{\Gamma_2} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))$, $a \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))^d$, and $v_0 \in W_q^{2-\frac{2}{q}}(\Omega)^d$ satisfying the compatibility condition*

$$\operatorname{div} v_0 = g|_{t=0} \text{ in } \dot{W}^{-1}_{q,\Gamma_2}(\Omega),\ \ v_0|_{\Gamma_1} = 0 \text{ if } q > \frac{3}{2},\ \ (n\cdot 2\nu Dv_0)_\tau|_{\Gamma_2} = a_\tau|_{t=0} \text{ if } q > 3.$$

there is a unique solution $(v,p) \in W^{2,1}_q(Q_T)^d \times W^{1,0}_q(Q_T)$ *of (1.1)-(1.5). Moreover,*

$$\|v\|_{W^{2,1}_q} + \|\nabla p\|_{L^q} + \|p|_{\Gamma_2}\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}} \tag{1.6}$$
$$\leq C\left(\|(f,\nabla g)\|_{L^q} + \|\partial_t g\|_{-1,0,q} + \|(g|_{\Gamma_2},a)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}} + \|v_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right),$$

where $\|.\|_{-1,0,q} := \|.\|_{L^q(0,T;\dot{W}^{-1}_{q,\Gamma_2})}$. *The constant* C *can be chosen independently of* $T \in (0,T_0]$ *for any fixed* $0 < T_0 < \infty$.

Finally, if Ω *is a bounded domain and* $\Gamma_1 \neq \emptyset$, *then all statements hold true for* $0 < T \leq T_0 = \infty$.

For precise definitions of the domains and the function spaces we refer to Section 2 below. Theorem 1.1 will be a consequence of the corresponding result for a more general class of domain satisfying Assumption 2.1 below.

Finally, we note that in Section 5 below we will derive a more general statement for the case that $f \in L^q(0,T;\mathcal{D}(A_q^\alpha))$, $\alpha \in \mathbb{R}$ in the case of pure Dirichlet boundary conditions ($\Gamma_2 = \emptyset$), cf. Theorem 5.1 below. Here A_q is an associated Stokes operator and the domains of fractional powers are characterized in Section 4 below.

2 Preliminaries

We use the notation of [5]. We just recall that $f \in \dot{W}^1_q(\Omega)$ if $f \in L^q_{loc}(\overline{\Omega})$ and $\nabla f \in L^q(\Omega)$. Moreover,

$$W^1_{q,\Gamma_j}(\Omega) := \left\{f \in W^1_q(\Omega) : f|_{\Gamma_j} = 0\right\}, \qquad W^{-1}_{q,\Gamma_j}(\Omega) := \left(W^1_{q',\Gamma_j}(\Omega)\right)', j = 1,2,$$
$$\dot{W}^1_{q,\Gamma_2}(\Omega) := \left\{f \in \dot{W}^1_q(\Omega) : f|_{\Gamma_2} = 0\right\}, \qquad \dot{W}^{-1}_{q,\Gamma_2}(\Omega) := \left(\dot{W}^1_{q',\Gamma_2}(\Omega)\right)'$$

If $g \in L^q(\Omega)$, then we say $g \in \dot{W}^{-1}_{q,\Gamma_2}(\Omega)$ if there is some $R \in W^1_{q,\Gamma_1}(\Omega)^d$ such that $g = \operatorname{div} R$. In this case we define

$$\langle g_R, \varphi\rangle_{\dot{W}^{-1}_{q,\Gamma_2}(\Omega),\dot{W}^1_{q',\Gamma_2}(\Omega)} = -(R,\nabla\varphi)_\Omega \qquad \text{for all } \varphi \in \dot{W}^1_{q',\Gamma_2}(\Omega).$$

The element $g_R \in \dot{W}^{-1}_{q,\Gamma_2}(\Omega)$ is independent of the choice of $R \in W^1_{q,\Gamma_1}(\Omega)^d$ such that $g = \operatorname{div} R$ since

$$(R_1 - R_2, \nabla\varphi) = 0 \qquad \text{for all } \varphi \in \dot{W}^1_{q',\Gamma_2}(\Omega)$$

if $R_1, R_2 \in W^1_{q,\Gamma_1}(\Omega)^d$ and $\operatorname{div} R_1 = \operatorname{div} R_2$. The latter identity can be easily proved by approximating $R_1 - R_2$ by compactly supported functions in $W^1_{q,\Gamma_1}(\Omega)^d$. Moreover, we have $g = g_R$ in $\mathcal{D}'(\Omega)$ in the sense that

$$\langle g_R, \varphi\rangle_{\dot{W}^{-1}_{q,\Gamma_2}(\Omega),\dot{W}^1_{q',\Gamma_2}(\Omega)} = (g,\varphi)_\Omega \qquad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Therefore we will identify g_R with g in the following. Finally, we have

$$\|g\|_{\dot{W}^{-1}_{q,\Gamma_2}(\Omega)} \leq \inf_{R\in W^1_{q,\Gamma_1}(\Omega),\operatorname{div} R=g} \|R\|_{L^q(\Omega)}.$$

Moreover, we recall the general class of domains considered in [5].

Assumption 2.1 *Let $1 < q < \infty$, let $d < r_1, r_2 \leq \infty$ such that $q, q' \leq \min(r_1, r_2)$. Moreover, let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a domain and $\partial\Omega = \Gamma_1 \cup \Gamma_2$ with Γ_1, Γ_2 closed and disjoint satisfying the following conditions:*

(A1) *There is a finite covering of $\overline{\Omega}$ with relatively open sets U_j, $j = 1, \dots, m$, such that U_j coincides (after rotation) with a relatively open set of $\overline{\mathbb{R}^d_{\gamma_j}}$, where $\mathbb{R}^d_{\gamma_j} := \{(x', x_d) \in \mathbb{R}^d : x_d > \gamma_j(x')\}$, $\gamma_j \in W^{2-\frac{1}{r_2}}_{r_2}(\mathbb{R}^{d-1})$. Moreover, suppose that there are cut-off functions $\varphi_j, \psi_j \in C^\infty_b(\overline{\Omega})$, $j = 1, \dots, m$, such that φ_j, $j = 1, \dots, m$, is a partition of unity, $\psi_j \equiv 1$ on $\operatorname{supp} \varphi_j$, and $\operatorname{supp} \psi_j \subset U_j$, $j = 1, \dots, m$.*

(A2) *For every $f \in L^s(\Omega)^d$, $s = q, q'$, there is a unique decomposition $f = f_0 + \nabla p$ with $f_0 \in J_s(\Omega)$ and $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$ where*

$$\begin{aligned} J_s(\Omega) &:= \overline{\left\{f \in C^\infty_{(0)}(\Omega \cup \Gamma_2)^d : \operatorname{div} f = 0\right\}}^{L^s(\Omega)}, \\ \dot{W}^1_{s,\Gamma_2}(\Omega) &:= \left\{p \in \dot{W}^1_s(\Omega) : p|_{\Gamma_2} = 0\right\}. \end{aligned}$$

(A3) *For every $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$, $s = q, q'$, there is a decomposition $p = p_1 + p_2$ such that $p_1 \in W^1_s(\Omega)$ with $p_1|_{\Gamma_2} = 0$, $p_2 \in L^s_{\mathrm{loc}}(\overline{\Omega})$ with $\nabla p_2 \in W^1_s(\Omega)$ and $\|(p_1, \nabla p_2)\|_{W^1_s(\Omega)} \leq C\|\nabla p\|_s$.*

We refer to [5, Section 2] for some basic result for function spaces defined on domains Ω satisfying the assumptions above. We note that the standard Sobolev embedding theorem holds for domains as above. In particular, we have $W^1_r(\Omega) \hookrightarrow L^\infty(\Omega)$ and

$$\|fg\|_{W^1_q(\Omega)} \leq C_{q,r}\|f\|_{W^1_r(\Omega)}\|g\|_{W^1_q(\Omega)} \tag{2.1}$$

for all $1 \leq q \leq r$ and $r > d$.

Now we provide some examples of domains satisfying the assumptions above:

Definition 2.2 *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a domain and let $d < r \leq \infty$. Then*

1. *Ω is called an* exterior domain *with $W^{2-\frac{1}{r}}_r$-boundary, if $\mathbb{R}^d \setminus \Omega$ is compact and $\partial\Omega$ is locally the graph of a $W^{2-\frac{1}{r}}_r$-function in a suitable coordinate system.*

2. *Ω is called a* perturbed half space *with $W_r^{2-\frac{1}{r}}$-boundary, if $\Omega \cup B_R(0) = \mathbb{R}^d_+ \cup B_R(0)$ for some $R > 0$ and $\partial\Omega$ is locally the graph of a $W_r^{2-\frac{1}{r}}$-function in a suitable coordinate system.*

3. *Ω is called* aperture domain *with $W_r^{2-\frac{1}{r}}$-boundary if $\Omega \cup B_R(0) = \mathbb{R}^d_+ \cup \mathbb{R}^d_- \cup B_R(0)$ for some $R > 0$, where $\mathbb{R}^d_- = \{x \in \mathbb{R}^d : x_d < -c\}$ for some $c > 0$ and $\partial\Omega$ is locally the graph of a $W_r^{2-\frac{1}{r}}$-function in a suitable coordinate system.*

4. *Ω is called an* infinite cylinder *with $W_r^{2-\frac{1}{r}}$-boundary if $\Omega = \Omega' \times \mathbb{R}$, where $\Omega' \subset \mathbb{R}^{d-1}$ is a bounded domain with $W_r^{2-\frac{1}{r}}$-boundary.*

5. *Ω is called an* asymptotically flat layer *with $W_r^{2-\frac{1}{r}}$-boundary if*

$$\Omega = \left\{x \in \mathbb{R}^d : a + \gamma_-(x') < x_d < b + \gamma_+(x')\right\},$$

   where $x = (x', x_d)$, $a < b$, and $\gamma_\pm \in W_r^{2-\frac{1}{r}}(\mathbb{R}^{d-1})$ such that $\gamma_+(x') - \gamma_-(x') + b - a \geq \kappa > 0$ for all $x' \in \mathbb{R}^{d-1}$, $\lim_{|x'|\to\infty} \gamma_\pm(x') = 0$, and $\lim_{|x'|\to\infty} \nabla\gamma_\pm(x') = 0$ if $r = \infty$.

Obviously, all domains above satisfy the condition (A1). In the case of pure Dirichlet boundary conditions (A2) is known to be valid for all $1 < q < \infty$, cf. [4, 12, 13, 22, 11, 24], where $\partial\Omega \in C^1$ is only needed. In the Appendix we will show that (A2) is also valid for the domains above in the case that Γ_2 is compact.

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, an exterior domain, a perturbed half-space, an aperture domain, an asymptotically flat layer or an infinite cylinder with $W_{r_2}^{2-\frac{1}{r_2}}$-boundary. Moreover, assume that Γ_2 is compact unless Ω is an asymptotically flat layer. Moreover, we assume that $\Gamma_1 \neq \emptyset$ in the case of an asymptotically flat layer. Then assumptions (A1)-(A3) are valid.*

Proof: It is easy to see that (A1) is fulfilled for all kinds of domains with $W_{r_2}^{2-\frac{1}{r_2}}$-boundary mentioned above.

First let $\Gamma_2 = \emptyset$. Then the (A2) holds because of the standard L^q-Helmholtz decomposition for these kinds of domains, cf. [4, 12, 13, 22, 11, 24], where $\partial\Omega \in C^1$ is only needed. If Ω is an asymptotically flat layer and $\Gamma_2, \Gamma_1 \neq \emptyset$, then (A2) follows from [4, Corollary A.3]. The case $\Gamma_2 \neq \emptyset$ and Ω is not an asymptotically flat layer is proved in the Appendix, cf. Corollary A.2 below.

Finally, we come to the prove of (A3). First let $\Gamma_2 = \emptyset$. We note that (A3) is valid if $\Omega = \mathbb{R}^d$. In this case $p = p_1 + p_2$ with $p_2 = \mathcal{F}^{-1}[\varphi(\xi)\hat{f}(\xi)]$ for some $\varphi \in C_0^\infty(\mathbb{R}^d)$ with $\varphi \equiv 1$ on $B_1(0)$ satisfies the conditions in (A3), cf. [4, Remark 2.6.2]. Moreover, (A3) holds true if the following extension property is satisfied: For every $p \in \dot{W}^1_q(\Omega)$ there is an extension $\tilde{p} \in \dot{W}^1_q(\mathbb{R}^d)$ such that $\tilde{p}|_\Omega = p$ and $\|\nabla\tilde{p}\|_q \leq C\|\nabla p\|_q$. This is the case for every (ε, ∞)-domain, cf. [8], in particular, for exterior domains and aperture domains. This extension property does not hold for asymptotically flat layers, cf. [4, Section 2.4], and also not for infinite cylinders by similar arguments. Nevertheless (A3) is also valid in asymptotically flat layers due to [4, Lemma 2.4]. If $\Omega = \Omega' \times \mathbb{R}$ is an infinite cylinders, then for every $p \in \dot{W}^1_q(\Omega)$ we have

$$\tilde{p}_1 = \frac{1}{|\Omega'|}\int_{\Omega'} f(x', x_d)\, dx' \in \dot{W}^1_q(\mathbb{R}), \qquad \tilde{p}_2 = p - p_1 \in W^1_q(\Omega)$$

due to Poincaré's inequality in Ω'. Now $\tilde{p}_1 = \tilde{p}_3 + p_2$ for some $\tilde{p}_3, \partial_{x_d} p_2 \in W^1_q(\mathbb{R})$ as seen above in the case $\Omega = \mathbb{R}^d$. Hence $p_1 = \tilde{p}_2 + \tilde{p}_3$ and p_2 satisfy the conditions in (A3).

Finally, if Γ_2 is compact, then the construction for the case $\Gamma_2 = \emptyset$ can be easily modified to obtain $p_1|_{\Gamma_2} = p_2|_{\Gamma_2} = 0$. If $\Gamma_2 \neq \emptyset$ and Ω is an asymptotically flat layer, then (A3) is trivial since $\dot{W}^1_{q,\Gamma_2}(\Omega) = W^1_{q,\Gamma_2}(\Omega) := \{f \in W^1_q(\Omega) : f|_{\Gamma_2} = 0\}$. ■

As an immediate consequence of the existence of an L^q-Helmholtz decomposition due to (A2) we obtain:

Lemma 2.4 *Let Ω, q be as in Assumption 2.1. Then for every $F \in \dot{W}^{-1}_{q,\Gamma_2}(\Omega)$ and $a \in W^{1-\frac{1}{q}}_q(\Gamma_2)$ there is some $p \in \dot{W}^1_q(\Omega)$ such that*

$$(\nabla p, \nabla \varphi)_\Omega = \langle F, \varphi \rangle_{\dot{W}^{-1}_{q,\Gamma_2}, \dot{W}^1_{q,\Gamma_2}} \quad \text{for all } \varphi \in \dot{W}^1_{q',\Gamma_2}(\Omega), \tag{2.2}$$

$$p|_{\Gamma_2} = a \quad \text{on } \Gamma_2. \tag{2.3}$$

If $\Gamma_2 \neq \emptyset$, p is uniquely determined. If $\Gamma_2 = \emptyset$, then p is uniquely determined up to a constant. Moreover, there is some constant C_q independent of F, a such that

$$\|\nabla p\|_{L^q(\Omega)^d} \leq C_q \left(\|F\|_{\dot{W}^{-1}_{q,\Gamma_2}(\Omega)} + \|\nabla A\|_{L^q(\Omega)} \right).$$

We refer to [5, Lemma 2] for the proof.

Recall that the anisotropic Sobolev-Slobodeckij space is defined as

$$W^{2s,s}_q(M \times (0,T)) = L^q(0,T; W^{2s}_q(M)) \cap W^s_q(0,T; L^q(M))$$

for $s \geq 0$ normed by

$$\|u\|^q_{W^{2s,s}_q} = \|u\|^q_{L^q(0,T;W^{2s}_q(M))} + \|u\|^q_{W^s_q(0,T;L^q(M))},$$

where $M \in \{\Omega, \partial\Omega, \Gamma_1, \Gamma_2\}$. Moreover, we define $W^{m,0}_q(Q_T) = L^q(0,T; W^m_q(\Omega))$, $m \in \mathbb{N}$,

Using an extension operator $E\colon W^{1-\frac{1}{q}}_q(\Gamma_2) \to W^1_q(\Omega)$, cf. [5, Corollary 2], and (2.1), one easily gets

$$\|\nu a\|_{L^q(0,T;W^{1-\frac{1}{q}}_q(\Gamma_2))} \leq C_{q,r_1} \|\nu\|_{BUC([0,T);W^1_{r_1}(\Omega))} \|a\|_{L^q(0,T;W^{1-\frac{1}{q}}_q(\Gamma_2))}$$

for any $\nu \in BUC([0,T);W^1_{r_1}(\Omega))$, $a \in L^q(0,T;W_q^{1-\frac{1}{q}}(\Gamma_2))$, and $1 < q \leq r_1$, $r_1 > d$. Moreover, if $\nu|_{\Gamma_2} \in C^{\frac{1}{2}}([0,T);L^\infty(\Gamma_2))$, then

$$\|\nu a\|_{W_q^{\frac{1}{2q'}}(0,T;L^q(\Gamma_2))} \leq C_{q,r_1}\|\nu\|_{C^{\frac{1}{2}}([0,T);L^\infty(\Gamma_2))}\|a\|_{W_q^{\frac{1}{2q'}}(0,T;L^q(\Gamma_2))}$$

since $W_q^{\frac{1}{2q'}}(0,T;X)$ is normed by

$$\|a\|^q_{W_q^{\frac{1}{2q'}}(0,T;X)} = \|a\|^q_{L^q(0,T;X)} + \int_0^T\int_0^T \frac{\|a(s)-a(t)\|_X^q}{|s-t|^{1+\frac{q}{2q'}}}\,dt\,ds.$$

Altogether we obtain

$$\|\nu a\|_{W_q^{\frac{1}{q'},\frac{1}{2q'}}} \leq C_{q,r_1}\left(\|\nu\|_{BUC([0,T);W^1_{r_1}(\Omega))} + \|\nu\|_{C^{\frac{1}{2}-\varepsilon}([0,T);L^\infty(\Gamma_2))}\right)\|a\|_{W_q^{\frac{1}{q'},\frac{1}{2q'}}} \tag{2.4}$$

provided that $1 < q \leq r_1$, $r_1 > d$, and $\varepsilon > 0$ is sufficiently small.

Finally, we need some extension results for the traces spaces of $W^{2,1}_q(Q_T)$.

Lemma 2.5 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, $1 < q < \infty$ with $q \neq \frac{3}{2}, 3$, be as in Assumption 2.1, and let $0 < T \leq \infty$. Then*

1. *For every $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$ with $u_0|_{\Gamma_1} = 0$ if $q > \frac{3}{2}$ there is some $u \in W^{2,1}_q(Q_T)$ with $u|_{t=0} = u_0$, $u|_{\Gamma_1\times(0,T)} = 0$ if $q > \frac{3}{2}$. Moreover, there is some $C > 0$ independent of $T \in (0,\infty]$ such that*

$$\|u\|_{W^{2,1}_q(Q_T)} \leq C\|u_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}.$$

2. *For every $a \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,T))^d$ with $a|_{t=0} = 0$ if $q > 3$ there is some $A \in W^{2,1}_q(Q_T)^d$ with $A|_{t=0} = 0$, $A|_{\Gamma_1} = 0$, and*

$$(n\cdot 2\nu DA)_\tau|_{\Gamma_2} = a_\tau, \qquad \operatorname{div} A|_{\Gamma_2} = a_n.$$

   Moreover,

$$\|A\|_{W^{2,1}_q(Q_T)} \leq C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$$

   where C can be chosen independently of $T \in (0,\infty]$.

Proof: With the aid of the coordinate transformations due to [5, Proposition 1] and the partition of unity due to Assumption 2.1 the first statement is easily reduced to case of a half-space $\mathbb{R}^d_+$, which is well-known, cf. e.g Grubb [19, Appendix].

In order to prove 2., let $A \in W^{2,1}_q(Q_T)^d$ with $A|_{t=0}$, $A|_{\partial\Omega} = 0$, and $\partial_n A|_{\Gamma_2} = \nu^{-1}a$ such that $\|A\|_{2,1,q} \le C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$. – As before, the existence of A can be reduced to the corresponding statement in $\mathbb{R}^d_+$. – Then

$$\begin{aligned}(n\cdot 2\nu DA)_\tau|_{\Gamma_2} &= (\nu\nabla_\tau A_n + \nu\partial_n A_\tau)|_{\Gamma_2} = 0 + a_\tau,\\ \operatorname{div} A|_{\Gamma_2} &= (\operatorname{div}_\tau A_\tau + \partial_n A_n)|_{\Gamma_2} = 0 + a_n.\end{aligned}$$

The constant C can be chosen independently of T since we can extend a to $\tilde{a} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,\infty))^d$ such that $\|\tilde{a}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} \le C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$, where C does not depend on T, and restrict the corresponding $\widetilde{A} \in W^{2,1}_q(\Omega\times(0,\infty))^d$ to $(0,T)$ afterwards. The latter extension to $(0,\infty)$ can be done by first extending a in an even way around $t = T$ to a function defined on $(0,2T)$ and then extending by zero, which yields an $\tilde{a} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,\infty))^d$ since $\tilde{a}|_{t=2T} = a|_{t=0} = 0$ if $q > 3$.

■

3 Nonstationary Stokes Equations

As in the case of the generalized Stokes resolvent equations, cf. [5], (1.1)-(1.5) can (at least formally) be reduced to the *nonstationary reduced Stokes equations*

$$\begin{aligned}\partial_t v - \operatorname{div}(\nu\nabla v) + \nabla P_\nu v - \nabla\nu^T\nabla v^T &= f_r &&\text{in } Q_T, &(3.1)\\ v|_{\Gamma_1} &= 0 &&\text{on } \Gamma_1\times(0,T), &(3.2)\\ T_1' u &= a_r &&\text{on } \Gamma_2\times(0,T), &(3.3)\\ v|_{t=0} &= v_0 &&\text{in } \Omega, &(3.4)\end{aligned}$$

For given $\nu = \nu(t)$ the reduced Stokes operator $A_{q,\nu}$ on $L^q(\Omega)^d$ is defined as

$$\begin{aligned}A_{q,\nu} v &= -\operatorname{div}(\nu\nabla v) + \nabla P_\nu v - \nabla\nu^T\nabla v^T \qquad (3.5)\\ \mathcal{D}(A_{q,\nu}) &= \left\{v \in W^2_q(\Omega)^d : v|_{\Gamma_1} = 0, T'_{1,\nu}v|_{\Gamma_2} = 0\right\},\end{aligned}$$

where $T'_{1,\nu}v$ is defined by

$$(T'_{1,\nu}v)_\tau = (n\cdot 2\nu Dv)_\tau|_{\Gamma_2}, \qquad (T'_1 v)_n = \nu\operatorname{div} v|_{\Gamma_2}. \qquad (3.6)$$

Moreover, $P_\nu v \equiv p_1 \in \dot{W}^1_q(\Omega)$ with $p_1|_{\Gamma_2} \in W_q^{1-\frac{1}{q}}(\Gamma_2)$ is defined as the solution of

$$\begin{aligned}(\nabla p_1, \nabla\varphi)_\Omega &= (\nu(\Delta - \nabla\operatorname{div})v, \nabla\varphi)_\Omega + (Dv, 2\nabla\nu\otimes\nabla\varphi)_\Omega, &(3.7)\\ p_1|_{\Gamma_2} &= 2\nu\partial_n v_n &(3.8)\end{aligned}$$

for all $\varphi \in \dot{W}^1_{q',\Gamma_2}(\Omega) = \left\{\varphi \in \dot{W}^1_{q'}(\Omega) : \varphi|_{\Gamma_2} = 0\right\}$. Note that the right-hand-side of (3.7) defines a bounded linear functional on $\dot{W}^1_{q',\Gamma_2}(\Omega)$. The existence of a solution of

(3.7)-(3.8) that is unique (up to a constant if $\Gamma_2 = \emptyset$) follows from the existence of a unique Helmholtz decomposition, i.e., (A2), cf. Lemma 2.4. Hence

$$P_\nu \colon W^2_q(\Omega)^d \to \left\{ p \in \dot{W}^1_q(\Omega) : p|_{\Gamma_2} \in W_q^{1-\frac{1}{q}}(\Gamma_2) \right\}$$

is a bounded linear operator.

Finally, we note that the domain of $A_{q,\nu}$ depends on t unless $\nu(x,t)$ is independent of t or $\Gamma_2 = \emptyset$. In the case that ν is independent of t the following result follows from [5, Theorems 1,2, and 3].

Theorem 3.1 *Let $1 < p < \infty$, $0 < T < \infty$, and let Ω, q be as in Assumption 2.1. Then for every $f \in L^p(0,T; L^q(\Omega)^d)$ there is a unique solution $v \in W^1_p(0,T; L^q(\Omega)^d) \cap L^p(0,T; \mathcal{D}(A_q))$ of*

$$\begin{aligned} v'(t) + A_q v(t) &= f(t), \qquad 0 < t < T, \\ v(0) &= 0 \end{aligned}$$

Moreover,

$$\|v'\|_{L^p(0,T;L^q)} + \|A_q v\|_{L^p(0,T;L^q)} \leq C\|f\|_{L^p(0,T;L^q)}.$$

If Ω is a bounded domain and $\Gamma_1 \neq \emptyset$, then the statement is also true for $T = \infty$.

From the latter theorem and Lemma 2.5, we deduce:

Theorem 3.2 *Let $0 < T < \infty$ and let Ω, q, ν be as in Assumption 2.1. Moreover, let $(f_r, a_r, v_0) \in L^q(Q_T)^d \times W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))^d \times W_q^{2-\frac{2}{q}}(\Omega)^d$ satisfy the compatibility conditions*

1. *$v_0|_{\Gamma_1} = 0$ if $q > \frac{3}{2}$.*
2. *$(n \cdot 2\nu D v_0)_\tau|_{\Gamma_2} = a_\tau|_{t=0}$ if $q > 3$.*

Then there is a unique solution $v \in W^{2,1}_q(Q_T)^d$ of (3.1)-(3.4), which satisfies

$$\|v\|_{W^{2,1}_q(Q_T)} \leq C \left(\|f_r\|_{L^q(Q_T)} + \|a_r\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,T))} + \|v_0\|_{W_q^{2-\frac{2}{q}}(\Omega)} \right)$$

The constant C can be chosen independently of $T \in (0, T_0]$ for every $0 < T_0 < \infty$. If Ω is a bounded domain, an infinite cylinder or an asymptotically flat layer, $\Gamma_1 \neq \emptyset$, $\lim_{t\to\infty} \nu(x,t) = \nu_\infty$ in $W^1_q(\Omega)$, and $\lim_{T\to\infty} \|\nu - \nu_\infty\|_{C^{\frac{1}{2}}([T,\infty);L^\infty(\Gamma_2))} = 0$, then the statements hold true for $T = \infty$.

Proof: First assume that $\nu = \nu(x)$ is independent of $t \in (0,T)$. Then the theorem follows immediately from Theorem 3.1 if $a_r = u_0 = 0$. The general case $a_r, u_0 \neq 0$ can be easily reduced to the latter case by first subtracting a suitable extension of u_0 and then a suitable extension of a_r, cf. Lemma 2.5.

Next let $\nu=\nu(x,t)$ be time-dependent and fix some $t_0\in[0,T)$. Then by the first part the theorem holds if ν is replaced by $\nu_{t_0}(x):=\nu(x,t_0)$. Moreover,

$$\|\operatorname{div}(2\nu Dv)-\operatorname{div}(2\nu_{t_0}Dv)\|_{L^q(Q_T)} \leq C\|\nu-\nu_{t_0}\|_{BUC([0,T);W^1_{r_1}(\Omega))}\|v\|_{W^{2,1}_q(Q_T)}$$

due to (2.1) and

$$\begin{aligned}&\|n\cdot T_\nu(v,p)-n\cdot T_{\nu_{t_0}}(v,p)\|_{W_q^{1-\frac1q,\frac12(1-\frac1q)}}\\ &\leq C\left(\|\nu-\nu_{t_0}\|_{BUC([0,T);W^1_{r_1}(\Omega))}+\|\nu-\nu_{t_0}\|_{C^{\frac12-\varepsilon}([0,T);L^\infty(\Gamma_2))}\right)\\ &\quad\cdot\left(\|v\|_{W^{2,1}_q(Q_T)}+\|p|_{\Gamma_2}\|_{W_q^{1-\frac1q,\frac12(1-\frac1q)}}\right)\end{aligned}$$

due to (2.4) for $\varepsilon>0$ sufficiently small. Hence by a standard perturbation argument the theorem holds true provided that

$$\|\nu-\nu_{t_0}\|_{BUC([0,T);W^1_{r_1}(\Omega))}+\|\nu-\nu_{t_0}\|_{C^{\frac12-\varepsilon}([0,T);L^\infty(\Gamma_2))}\leq\delta_0$$

for some $\delta_0=\delta_0(t_0)>0$ and $\varepsilon>0$ as before. Choosing $t_0=0$, this implies that the theorem is true if T is replaced by $0<T'\leq T$ sufficiently small. Now let $0<T_m\leq T$ be the supremum of all $T'\in(0,T]$ such that the statement of the theorem is true if T is replaced by T'. Then $T_m=T$ since otherwise we can extend the solution operator by solving the system on $[T_m,T_m+\kappa)$ for some $\kappa>0$ such that

$$\|\nu-\nu_{T_m}\|_{BUC([T_m,T_m+\kappa);W^1_{r_1}(\Omega))}+\|\nu-\nu_{T_m}\|_{C^{\frac12-\varepsilon}([T_m,T_m+\kappa);L^\infty(\Gamma_2))}\leq\delta_0(T').$$

Therefore the statement of the theorem holds true for any $0<T<\infty$ with some $C=C(T)$. If Ω is a bounded domain, then the statement holds for $T=\infty$ since it holds for $[0,\infty)$ replaced by $[T',\infty)$ for some $T'>0$ sufficiently large due to $\lim_{t\to\infty}\nu(t)=\nu_\infty$ in $W^1_q(\Omega)$ and $\lim_{T\to\infty}\|\nu-\nu_\infty\|_{C^{\frac12}([T,\infty);L^\infty(\Gamma_2))}=0$. ■

Now we are able to proof Theorem 1.1. For a similar proof in the case of constant viscosity and an asymptotically flat layer with mixed boundary conditions we refer to [3].

Proof of Theorem 1.1: For almost every $t\in(0,T)$ let $p_2(.,t)\in\dot W^1_q(\Omega)$ with $p_2|_{\Gamma_2}\in W_q^{1-\frac1q}(\Gamma_2)$ be the solution of

$$(\nabla p_2(.,t),\nabla\varphi)=(f(t)+\nu\nabla g(t),\nabla\varphi)_\Omega+\langle\partial_t g(t),\varphi\rangle_{\dot W^{-1}_{q,\Gamma_2},\dot W^1_{q',\Gamma_2}} \tag{3.9}$$

for all $\varphi\in\dot W^1_{q',\Gamma_2}(\Omega)$ and $p_2|_{\Gamma_2}=-a_n$, cf. Lemma 2.4. Now we define $f_r=f-\nabla p_2+\nu\nabla g$. Then

$$\|f_r\|_q \leq C\left(\|(f,\nabla g)\|_q+\|\partial_t g\|_{L^q(0,T;\dot W^{-1}_{q,\Gamma_2})}+\|a_n\|_{1-\frac1q,\frac12(1-\frac1q),q}\right)$$

with C independent of T. Moreover, let $(a_r)_\tau = a_\tau$ and $(a_r)_n = g|_{\Gamma_2}$.

Now let $v \in W^{2,1}_q(Q_T)^d$ be the solution of the reduced Stokes equations with right-hand side (f_r, a_r^+). Then (v,p) with $\nabla p = \nabla P_\nu v + \nabla p_2$ solves (1.1) and (1.3)-(1.5) by construction. Hence it only remains to prove that $\operatorname{div} v = g$.

First of all, because of (3.9),

$$-(f_r(t), \nabla\varphi)_\Omega = \langle \partial_t g(t), \varphi\rangle_{\dot W^{-1}_{q,\Gamma_2}, \dot W^1_{q',\Gamma_2}}, + (\nu\nabla g(t), \nabla\varphi)_\Omega \tag{3.10}$$

for all $\varphi \in \dot W^1_{q,\Gamma_2}(\Omega)$ and almost every $t \in (0,T)$. On the other hand, since $v \in W^{2,1}_q(\Omega)^d$ solves (3.1)-(3.4),

$$-(f_r, \nabla\varphi)_\Omega = \langle \partial_t \operatorname{div} v, \varphi\rangle_{\dot W^{-1}_{q,\Gamma_2}, \dot W^1_{q',\Gamma_2}} + (\nu\nabla \operatorname{div} v, \nabla\varphi)_\Omega \tag{3.11}$$

for all $\varphi \in \dot W^1_{q,\Gamma_2}(\Omega)$ because of

$$\begin{aligned}&(\operatorname{div}(\nu\nabla v), \nabla\varphi)_\Omega - (\nabla P_\nu v, \nabla\varphi)_\Omega + (\nabla\nu^T\nabla v^T, \nabla\varphi)_\Omega \\ &\quad = \ (\nu\Delta v, \nabla\varphi)_\Omega - (\nabla P_\nu v, \nabla\varphi)_\Omega + (Dv, 2\nabla\nu\otimes\nabla\varphi)_\Omega = (\nu\nabla \operatorname{div} v, \nabla\varphi)_\Omega\end{aligned} \tag{3.12}$$

for all $\varphi \in \dot W^1_{q,\Gamma_2}(\Omega)$ and almost every $t \in (0,T)$ due to (3.7). Moreover, since $\operatorname{div} v - g \in \dot W^1_{q,\Gamma_2}(\Omega)$, Proposition 3.3 below implies $\operatorname{div} v = g$. ■

Proposition 3.3 *Let Ω, q be as in Assumption 2.1 and let $u \in L^q(0,T; W^1_{q,\Gamma_2}(\Omega))$, $0 < T < \infty$, be such that $\partial_t u \in L^q(0,T; W^{-1}_{q,\Gamma_2}(\Omega))$, $u|_{t=0} = 0$, and*

$$\int_0^T \langle \partial_t u, \varphi\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} + (\nu\nabla u, \nabla\varphi)_{Q_T} = 0 \tag{3.13}$$

for all $\varphi \in L^{q'}(0,T; W^1_{q',\Gamma_2}(\Omega))$. Then $u = 0$.

Proof: Let $\psi \in L^{q'}(0,T; \dot W^1_{q',\Gamma_2}(\Omega))$ be arbitrary and let $v \in W^{2,1}_{q'}(Q_T)^d$ be a solution of the reduced Stokes equations (3.1)-(3.4) with right-hand side $\tilde f = \nabla\psi$, $a = 0$, and $v_0 = 0$. Then by (3.11)

$$-(\nabla\psi, \nabla\varphi)_{Q_T} = \int_0^T \langle \partial_t \operatorname{div} v, \varphi\rangle_{W^{-1}_{q',\Gamma_2}, W^1_{q,\Gamma_2}}\, dt + (\nu\nabla \operatorname{div} v, \nabla\varphi)_{Q_T}$$

for all $\varphi \in L^q(0,T; W^1_{q,\Gamma_2}(\Omega))$. Now, choosing $\varphi(x,t) = u(x, T-t) \in L^q(0,T; W^1_{q,\Gamma_2}(\Omega))$, we obtain

$$\begin{aligned}&-(\nabla u(T-.), \nabla\psi)_{Q_T} \\ &\quad = \int_0^T \langle \partial_t \operatorname{div} v(t), u(T-t)\rangle_{W^{-1}_{q',\Gamma_2}, W^1_{q,\Gamma_2}}\, dt + (\nu\nabla \operatorname{div} v, \nabla u(T-.))_{Q_T} \\ &\quad = \int_0^T \langle (\partial_t u)(T-t), \operatorname{div} v(t)\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}}\, dt + (\nu\nabla u(T-.), \nabla \operatorname{div} v)_{Q_T} = 0\end{aligned}$$

due to (3.13). Here we have used

$$\int_0^T \langle \partial_t v, w\rangle_{W^{-1}_{q,\Gamma_2},W^1_{q',\Gamma_2}}\,dt = \langle v(t), w(t)\rangle_{W_q^{1-\frac{2}{q}},W_{q'}^{1-\frac{2}{q'}}}\Big|_{t=0}^{T} - \int_0^T \langle v, \partial_t w\rangle_{W^1_{q,\Gamma_2},W^{-1}_{q',\Gamma_2}}\,dt$$

for all $v \in L^q(0,T;W^1_{q,\Gamma_2}) \cap W^1_q(0,T;W^{-1}_{q,\Gamma_2})$, $w \in L^{q'}(0,T;W^1_{q',\Gamma_2}) \cap W^1_{q'}(0,T;W^{-1}_{q',\Gamma_2})$, where we note that $L^s(0,T;W^1_s) \cap W^1_s(0,T;W^{-1}_s) \hookrightarrow BUC([0,T];W_s^{1-\frac{2}{s}})$ for all $1 < s < \infty$.

Since $\psi \in L^{q'}(0,T;\dot{W}^1_{q',\Gamma_2}(\Omega))$ was arbitrary, we conclude $\nabla u(t) = 0$ for almost every $t \in (0,T)$ due to Lemma 2.4. Hence $\partial_t u = 0$ due to (3.13) and therefore $u = 0$ since $u|_{t=0} = 0$. ■

4 Domains of Fractional Powers for Stokes Operators

In the following let $\nu(x) = \nu_\infty + \nu'(x)$ with $\nu' \in W^1_{r_1}(\Omega)$ be independent of t and $\nu(x) \geq \nu_0 > 0$ for all $x \in \Omega$. For simplicity we denote $A_q = A_{q,\nu}$ from now on. As shown in [5] we have

$$\mathcal{D}((c + A_q)^\alpha) = (L^q(\Omega)^d, \mathcal{D}(A_q))_{[\alpha]},$$

where $(.,.)_{[\alpha]}$ denotes the complex interpolation functor. This is a consequence of the bounded imaginary powers of $c + A_q$, cf. [5, Theorem 1] and [17, Proposition 6.1]. Here again $c \in \mathbb{R}$ is such that $c + A_q$ is invertible and admits a bounded H^∞-calculus. This is the case for $c > 0$ sufficiently large and for $c = 0$ if Ω is a bounded domain with $W_{r_2}^{2-\frac{1}{r_2}}$-boundary and $\Gamma_1 \neq \emptyset$.

In the following we will restrict ourselves to the case of pure Dirichlet boundary conditions, i.e., $\Gamma_1 = \partial\Omega$, $\Gamma_2 = \emptyset$. Then

$$\mathcal{D}(A_q) = \left\{u \in W^2_q(\Omega)^d : u|_{\partial\Omega} = 0\right\} = \mathcal{D}(\Delta_q)^d,$$

where Δ_q denotes the Dirichlet realization of the Laplacian Δ on $L^q(\Omega)$, i.e., $\mathcal{D}(\Delta_q) = \left\{u \in W^2_q(\Omega) : u|_{\partial\Omega} = 0\right\}$. Moreover, we have:

Lemma 4.1 *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, satisfy (A1) for some $d < r_2 \leq \infty$ and let $1 < q < \infty$ with $q \leq r_2$. Then*

$$(L^q(\Omega), W^2_q(\Omega) \cap W^1_{q,0}(\Omega))_{[\alpha]} = \begin{cases} \left\{u \in H^{2\alpha}_q(\Omega) : u|_{\partial\Omega} = 0\right\} & \text{if } \frac{1}{q} < 2\alpha \leq 2, \\ H^{2\alpha}_q(\Omega) & \text{if } 0 \leq 2\alpha < \frac{1}{q}. \end{cases}$$

Here $H^{2\alpha}_q(\Omega)$ is the restriction of the Bessel potential space $H^{2\alpha}_q(\mathbb{R}^d)$ to Ω equipped with the quotient norm.

Proof: First of all, the statement can be localized as follows: Let φ_j, ψ_j, $j = 1, \dots, m$ be the cut-off functions due to (A1). Then the mapping

$$R\colon W^k_q(\Omega) \to \prod_{j=1}^m W^k_q(\mathbb{R}^d_{\gamma_j}) \qquad \text{with } Rv = (\varphi_j v)_{j=1}^m$$

is bounded for every $1 < q < \infty$ and $k = 0, 1, 2$. By complex interpolation $R\colon H^{2\alpha}_q(\Omega) \to \prod_{j=1}^m H^{2\alpha}_q(\mathbb{R}^d_{\gamma_j})$ for all $1 < q < \infty$ and $0 \leq \alpha \leq 1$ since $H^k_q(\Omega) = W^k_q(\Omega)$. Moreover, the mapping

$$Q\colon \prod_{j=1}^m H^{2\alpha}_q(\mathbb{R}^d_{\gamma_j}) \to H^{2\alpha}_q(\Omega) \qquad \text{with } Qw = \sum_{j=1}^m \psi_j w_j$$

is bounded for all $1 < q < \infty$ and $0 \leq \alpha \leq 1$ and $QRv = v$ for all $v \in H^{2\alpha}_q(\Omega)$. This shows that $H^{2\alpha}_q(\Omega)$ is a retract of $\prod_{j=1}^m H^{2\alpha}_q(\mathbb{R}^d_{\gamma_j})$. Therefore the statement can be reduced to the case of $\Omega = \mathbb{R}^d_\gamma$, $\gamma \in W^{2-\frac{1}{r_2}}_{r_2}(\mathbb{R}^{d-1})$. Using the coordinate transformation $F_\gamma\colon \mathbb{R}^d \to \mathbb{R}^d$ with $F_\gamma|_{\mathbb{R}^d_+}\colon \mathbb{R}^d_+ \to \mathbb{R}^d_\gamma$ due to [5, Proposition 1, Corollary 1], we have

$$F^*_\gamma\colon H^{2\alpha}_q(\mathbb{R}^d) \to H^{2\alpha}_q(\mathbb{R}^d), \qquad F^*_\gamma\colon H^{2\alpha}_q(\mathbb{R}^d_\gamma) \to H^{2\alpha}_q(\mathbb{R}^d_+)$$

for all $1 < q \leq r_2$, $0 \leq \alpha \leq 1$, where $(F^*_\gamma v)(x) = v(F_\gamma(x))$. Hence the statement for $\Omega = \mathbb{R}^d_\gamma$ follows from the case of a half-space, cf. [15]. ∎

Corollary 4.2 *Let $\Omega, \Gamma_1, \Gamma_2, q$ be as in Assumption 2.1 and assume that $\Gamma_2 = \emptyset$. Then*

$$\mathcal{D}((c + A_q)^\alpha) = \begin{cases} \left\{u \in H^{2\alpha}_q(\Omega)^d : u|_{\partial\Omega} = 0\right\} & \text{if } \frac{1}{q} < 2\alpha \leq 2 \\ H^{2\alpha}_q(\Omega)^d & \text{if } 0 \leq 2\alpha < \frac{1}{q}, \end{cases}$$

where $c = 0$ in the case of a bounded domain and $c > 0$ sufficiently large else.

Finally, we derive a corresponding result for a variant of the standard Stokes operator, namely

$$\begin{aligned} A_{q,\sigma} v &:= P_q A_q v = -P_q \operatorname{div}(\nu \nabla v) - P_q \nabla \nu^T \nabla v^T, \quad v \in \mathcal{D}(A_{q,\sigma}), \\ \mathcal{D}(A_{q,\sigma}) &:= \mathcal{D}(A_q) \cap J_q(\Omega) = \left\{v \in W^2_q(\Omega)^d : \operatorname{div} v = 0, v|_{\partial\Omega} = 0\right\}. \end{aligned}$$

In an important relation between the Stokes operator $A_{q,\sigma}$ and the reduced Stokes operator A_q is given by the following proposition, which is a variant of [2, Lemma 3.1]:

Lemma 4.3 *Let $\Omega \subseteq \mathbb{R}^d$, $n \geq 2$, $1 < q < \infty$, and $\delta \in (0, \pi)$ be as in Assumption 2.1. Moreover, assume that $(\lambda + A_{s,\sigma})^{-1}$ exists for some $\lambda \in \Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$ with $|\lambda| \geq R$ and $s = q, q'$. Then $(\lambda + A_{q,\sigma})^{-1}$ exists and*

$$A_q|_{J_q(\Omega)} = A_{q,\sigma}, \qquad (\lambda + A_q)^{-1}|_{J_q(\Omega)} = (\lambda + A_{q,\sigma})^{-1}. \tag{4.1}$$

Proof: The first statement can be seen as follows: If $v \in \mathcal{D}(A_q) \cap J_q(\Omega)$, then

$$(-\operatorname{div}(\nu \nabla v) + \nabla P v + \nabla \nu^T \nabla v^T, \nabla \varphi)_\Omega = (\nu \nabla \operatorname{div} v, \nabla \varphi)_\Omega = 0$$

for all $\varphi \in \dot{W}^1_{q'}(\Omega)$ because of (3.12). Hence $-\operatorname{div}(\nu \nabla v) + \nabla P_\nu v + \nabla \nu^T \nabla v^T \in J_q(\Omega)$ due to (A.3) below. Thus

$$A_q v = -\operatorname{div}(\nu \nabla v) + \nabla \nu^T \nabla v^T + \nabla P_\nu v = P_q(-\operatorname{div}(\nu \nabla v) + \nabla \nu^T \nabla v^T) = A_{q,\sigma} v$$

for all $v \in \mathcal{D}(A_q) \cap J_q(\Omega)$.

In order to prove the second relation let $v = (\lambda + A_q)^{-1} f$ with $f \in J_q(\Omega)$. Then multiplying $(\lambda + A_q)u = f$ by $\nabla \varphi$, $\varphi \in W^1_{q'}(\Omega)$ and using (3.12) we obtain

$$\lambda(\operatorname{div} v, \varphi)_\Omega + (\nu \nabla \operatorname{div} v, \nabla \varphi)_\Omega = 0 \qquad \text{for all } \varphi \in W^1_{q'}(\Omega).$$

Hence $\operatorname{div} v = 0$ because of Lemma 4.4 below if $\lambda \neq 0$. If $\lambda = 0$ and Ω is a bounded domain, we get $\operatorname{div} v = 0$ too by the unique solvability Lemma 2.4. Hence $v \in J_q(\Omega)$. Since by the first statement $\lambda + A_{q,\sigma} = (\lambda + A_q)|_{J_q(\Omega)}$ is injective, we finally conclude that $(\lambda + A_{q,\sigma})^{-1} f = u = (\lambda + A_q)^{-1} f$ for every $f \in J_q(\Omega)$. ■

Lemma 4.4 *Let $\Omega \subset \mathbb{R}^d$, $n \geq 2$, and $1 < q < \infty$ be as in Assumption 2.1. If $\lambda + A_{q'}$ is surjective for $\lambda \notin (-\infty, 0]$, then there is no non-trivial $g \in W^1_q(\Omega)$ solving*

$$\lambda(g, \varphi)_\Omega + (\nu \nabla g, \nabla \varphi)_\Omega \qquad \textit{for all } \varphi \in W^1_{q'}(\Omega). \tag{4.2}$$

Proof: Let $f \in L^{q'}(\Omega)^d$ be arbitrary and let $u \in \mathcal{D}(A_{q'})$ such that $(\lambda + A_{q'})u = f$. Then multiplying f with ∇g we observe that $\operatorname{div} u \in W^1_{q'}(\Omega)$ solves

$$-\lambda(\operatorname{div} u, g) - (\nu \nabla \operatorname{div} u, \nabla g) = (f, \nabla g) \qquad \text{for all } g \in W^1_q(\Omega)$$

due to (3.12). Hence, if $g \in W^1_q(\Omega)$ solves (4.2), then $(f, \nabla g) = 0$ for all $f \in L^{q'}(\Omega)^d$ and therefore $\nabla g = 0$. Because of (4.2) and $\lambda \neq 0$, we conclude $g = 0$. ■

THEOREM 4.5 *Let $\Omega \subseteq \mathbb{R}^d$, $n \geq 2$, $1 < q < \infty$, and $\delta \in (0, \pi)$ be as in Assumption 2.1 and let $A_{q,\sigma}$ be as above. Then there is some $R \geq 0$ such that $(\lambda + A_{q,\sigma})^{-1}$ exists for all $\lambda \in \Sigma_\delta$ with $|\lambda| \geq R$ and*

$$\|(\lambda + A_{q,\sigma})^{-1}\|_{\mathcal{D}(J_q(\Omega))} \leq \frac{C}{1 + |\lambda|} \qquad \textit{for all } \lambda \in \Sigma_\delta \setminus B_R(0).$$

If Ω is a bounded domain, then the statement even holds for $R = 0$ and $\lambda \in \Sigma_\delta \cup \{0\}$. Moreover, $R + A_{q,\sigma}$ (with $R = 0$ for bounded domains) possesses bounded imaginary powers w.r.t. $\Sigma_{\pi-\delta}$ and we have

$$\mathcal{D}(A^\alpha_{q,\sigma}) = \begin{cases} \left\{ u \in H^{2\alpha}_q(\Omega)^d : u|_{\partial\Omega} = 0 \right\} \cap J_q(\Omega) & \textit{if } \frac{1}{q} < 2\alpha \leq 2, \\ H^{2\alpha}_q(\Omega)^d \cap J_q(\Omega) & \textit{if } 0 \leq 2\alpha < \frac{1}{q}. \end{cases}$$

Proof: Except for the last statement the theorem is an immediate consequence of Lemma 4.3 and [5, Theorems 1, 2, and 3]. The last statement follows from Corollary 4.2 and

$$\mathcal{D}(A_{q,\sigma}^{\alpha}) = (L^q(\Omega)^d, \mathcal{D}(A_q))_{[\alpha]} \cap J_q(\Omega),$$

which we prove by a modification of the arguments of [16, Lemma 6]. First of all, since $A_{q,\sigma}$ possesses bounded imaginary powers, we have

$$\mathcal{D}(A_{q,\sigma}^{\alpha}) = (J_q(\Omega), \mathcal{D}(A_{q,\sigma}))_{[\alpha]}$$

due to [17, Proposition 6.1]. Moreover, since the space on the right-hand side is independent of the choice of ν, it is sufficient to consider the case $\nu \equiv 1$ in the following. We define a projection $\widetilde{P}_q \colon \mathcal{D}(A_q) \to \mathcal{D}(A_{q,\sigma})$ by

$$\widetilde{P}_q f = -(c + A_{q,\sigma})^{-1} P_q (c + A_q) f, \qquad f \in \mathcal{D}(A_q).$$

Because of $A_q|_{J_q(\Omega)} = A_{q,\sigma}$, we have $P_q f = f$ for all $f \in \mathcal{D}(A_{q,\sigma})$. Hence $\widetilde{P}_q \colon \mathcal{D}(A_q) \to \mathcal{D}(A_{q,\sigma})$ is a projection onto $\mathcal{D}(A_{q,\sigma})$. Moreover,

$$\begin{aligned} (\widetilde{P}_q f, g)_\Omega &= ((c + A_{q,\sigma})^{-1} P_q (c + A_q) f, g)_\Omega \\ &= ((c - \Delta) f, (c + A_{q',\sigma})^{-1} g)_\Omega = (f, (c - \Delta)(c + A_{q',\sigma})^{-1} g)_\Omega \end{aligned}$$

for all $f \in \mathcal{D}(A_q)$ and $g \in J_{q'}(\Omega)$ because of $(A_{q,\sigma} v, w)_\Omega = (v, A_{q',\sigma} w)_\Omega$ for all $v \in \mathcal{D}(A_{q,\sigma})$, $w \in \mathcal{D}(A_{q',\sigma})$. Hence

$$\left| (\widetilde{P}_q f, g)_\Omega \right| \leq C \|f\|_{L^q(\Omega)} \|g\|_{J_{q'}(\Omega)}$$

for all $f \in \mathcal{D}(A_q)$ and $g \in J_{q'}(\Omega)$. Hence $\widetilde{P}_q$ extends to a bounded projection from $L^q(\Omega)^d$ onto $J_q(\Omega)$ since $\mathcal{D}(A_{q,\sigma})$ is dense in $J_q(\Omega)$. With the aid of $\widetilde{P}_q$ and [27, Theorem 1.2.4] we conclude

$$(J_q(\Omega), \mathcal{D}(A_{q,\sigma}))_{[\alpha]} = \widetilde{P}_q (L^q(\Omega)^d, \mathcal{D}(A_q))_{[\alpha]} = (L^q(\Omega)^d, \mathcal{D}(A_q))_{[\alpha]} \cap J_q(\Omega).$$

This finishes the proof. ■

5 Nonstationary Stokes System in Fractional Sobolev Spaces

Let $c = 0$ if Ω is a bounded domain and let $c > 0$ be so large that $c + A_q$ is invertible and has bounded imaginary powers else. Because of Lemma 4.3, $c + A_{q,\sigma}$ is invertible and has bounded imaginary powers too. Therefore we denote by A either $c + A_q$ defined on $X = L_q(\Omega)^d$ or $c + A_{q,\sigma}$ defined on $X = J_q(\Omega)$. Moreover, let

$$X_\alpha = \mathcal{D}(A^\alpha) = \{x \in X : A^\alpha x \in X\}$$

if $\alpha \geq 0$ equipped with the norm $\|x\|_{X_\alpha} = \|A^\alpha x\|_X$ and let X_α be the completion of X with respect to $\|x\|_{X_\alpha} = \|A^\alpha x\|_X$ if $\alpha < 0$. Then by [6, Theorem 1.5.4, Proposition 1.5.5., Chapter V] $A_\alpha \colon \mathcal{D}(A_\alpha) \subset X_\alpha \to X_\alpha$ with $A_\alpha x = Ax$ for all $x \in \mathcal{D}(A_\alpha) := X_{1+\alpha}$ is an invertible operator with bounded imaginary powers for arbitrary $\alpha \in \mathbb{R}$. Hence by the result by Dore and Venni we obtain:

THEOREM 5.1 *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, $0 < T \leq \infty$, and let Ω, q be as in Assumption 2.1 and let A_α, $\alpha \in \mathbb{R}$, be as above. Then for every $f \in L^p(0,T;\mathcal{D}(A^\alpha))$ and $v_0 \in (X_\alpha, X_{1+\alpha})_{1-\frac{1}{p},p}$ there is a unique solution $v \in W^1_p(0,T;X_\alpha) \cap L^p(0,T;X_{1+\alpha})$ of*

$$\begin{aligned} v'(t) + Av(t) &= f(t), \qquad 0 < t < T, \\ v(0) &= v_0 \end{aligned}$$

Moreover, there is a constant C independent of f, v_0, T such that

$$\|v'\|_{L^p(0,T;X_\alpha)} + \|Av\|_{L^p(0,T;X_\alpha)} \leq C \left(\|f\|_{L^p(0,T;X_\alpha)} + \|v_0\|_{(X_\alpha,X_{1+\alpha})_{1-\frac{1}{p},p}} \right).$$

Proof: By Lion's trace method, cf. e.g. [21, Proposition 1.2.10], for every $v_0 \in (X_\alpha, X_{1+\alpha})_{1-\frac{1}{p},p}$ there is some $w \in W^1_p(0,\infty;X_\alpha) \cap L^p(0,\infty;X_{1+\alpha})$ such that $w|_{t=0} = v_0$ and the norm of w is bounded by a constant times the norm of v_0. Hence subtracting w from v we can reduce to the case $v_0 = 0$. The latter case now follows from Dore and Venni [10, Theorem 3.2] if $T < \infty$ and from Giga and Sohr [18, Theorem 2.1] if $T = \infty$. ■

Finally, we note that Corollary 4.2 can know be used to obtain a more explicit characterization of the condition $f \in L^p(0,T;\mathcal{D}(A^\alpha))$ and $v \in W^1_p(0,T;X_\alpha) \cap L^p(0,T;X_{1+\alpha})$.

A Helmholtz Decomposition for Mixed Boundary Conditions

In the following we will show that (A2) is also valid for bounded domains, exterior domains, perturbed half-spaces, and aperture domains with C^1-boundary provided that Γ_2 is compact. To this end we use:

Proposition A.1 *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and let $1 < q < \infty$. Then (A2) holds true if and only if there is a constant $C_q > 0$ such that for every $s = q, q'$ and $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$*

$$\|\nabla p\|_{L^s(\Omega)} \leq C_q \|F\|_{\dot{W}^{-1}_{s,\Gamma_2}}, \tag{A.1}$$

where $F \in \dot{W}^{-1}_{s,\Gamma_2}(\Omega)$ is defined by

$$(\nabla p, \nabla \varphi)_\Omega = \langle F, \varphi \rangle_{\dot{W}^{-1}_{s,\Gamma_2}, \dot{W}^1_{s,\Gamma_2}} \quad \text{for all } \varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega). \tag{A.2}$$

Moreover, if (A2) holds, then for $s = q, q'$ and $F \in \dot{W}^{-1}_{s,\Gamma_2}$ there is a unique solution $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$ (up to a constant if $\Gamma_2 = \emptyset$) such that (A.2) holds. Finally, if (A2) holds, then

$$J_s(\Omega) = \left\{ f \in L^s(\Omega)^d : (f, \nabla\varphi)_\Omega = 0 \textit{ for all } \varphi \in \dot{W}^1_{s,\Gamma_2}(\Omega) \right\} \tag{A.3}$$

Proof: First assume that (A2) holds. Identifying $\dot{W}^1_{s',\Gamma_2}(\Omega)$ with a closed subspace of $L^q(\Omega)^d$ via $\varphi \mapsto \nabla\varphi$, we can find for every $F \in \dot{W}^{-1}_{s,\Gamma_2}(\Omega)$ some $f \in L^s(\Omega)^d$ such that

$$\langle F, \varphi \rangle_{\dot{W}^{-1}_{s,\Gamma_2}, \dot{W}^1_{s,\Gamma_2}} = (f, \nabla\varphi)_\Omega \qquad \text{for all } \varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega)$$

and $\|f\|_{L^s(\Omega)} \leq \|F\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)}$ by the Hahn-Banach theorem. Now let $f = f_0 + \nabla p$ be the decomposition due to (A2). Then $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$ solves

$$(\nabla p, \nabla\varphi)_\Omega = (f, \nabla\varphi)_\Omega = \langle F, \varphi \rangle_{\dot{W}^{-1}_{s,\Gamma_2}, \dot{W}^1_{s,\Gamma_2}}$$

for all $\varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega)$ because of

$$(f_0, \nabla\varphi)_\Omega = 0 \qquad \text{for all } \varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega)$$

due to the density of $\left\{ f \in C^\infty_{(0)}(\Omega \cup \Gamma_2)^d : \operatorname{div} f = 0 \right\}$ in $J_s(\Omega)$ by definition.

The proof of the converse implication is a modification of the arguments in [24, Proof of Theorem 1.4], which we include for the convenience of the reader. First of all, if (A.1) holds, then $-\Delta_s \colon \dot{W}^1_{s,\Gamma_2}(\Omega) \to \dot{W}^{-1}_{s,\Gamma_2}(\Omega)$ with

$$\langle -\Delta_s p, \varphi \rangle_{\dot{W}^{-1}_{s,\Gamma_2}, \dot{W}^1_{s',\Gamma_2}} = (\nabla p, \nabla\varphi)_\Omega \qquad \text{for all } \varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega)$$

is a bounded linear operator with closed range and trivial kernel. Moreover, $(-\Delta_s)' = -\Delta_{s'}$. Therefore $\mathcal{R}(-\Delta_{s'}) = \dot{W}^{-1}_{s,\Gamma_2}(\Omega)$ by the closed range theorem. Hence we can define $P_s f = f - \nabla p$, where p is the unique solution of (A.2) for $\langle F, \varphi \rangle_{\dot{W}^{-1}_{q,\Gamma_2}, \dot{W}^1_{q',\Gamma_2}} = (f, \nabla\varphi)_\Omega$. Then $P_s \colon L^s(\Omega)^d \to L^s(\Omega)^d$ with

$$\begin{aligned} \mathcal{R}(P_s) &= \left\{ f \in L^s(\Omega)^d : (f, \nabla\varphi)_\Omega = 0 \text{ for all } \varphi \in \dot{W}^1_{s',\Gamma_2}(\Omega) \right\}, \\ \mathcal{N}(P_s) &= \left\{ \nabla p \in L^s(\Omega)^d : p \in \dot{W}^1_{s,\Gamma_2}(\Omega) \right\}. \end{aligned}$$

Moreover, P_s has closed range since $I - P_s$ has closed range and it is easy to see that $(P_s)' = P_{s'}$. Obviously, $J_s(\Omega) \subseteq \mathcal{R}(P_s)$ since $P_s f = f$ for all $f \in J_s(\Omega)$. For the converse inclusion it is enough to prove

$$(J_s(\Omega))^\perp \subseteq \mathcal{N}(P_{s'}),$$

where $Z^\perp = \{ f \in X' : \langle f, x \rangle = 0 \quad \text{for all } x \in X \}$. Then the closed range theorem implies $\mathcal{R}(P_s) = \mathcal{N}(P_{s'})^\perp \subseteq J_s(\Omega)$. Therefore let $f \in (J_s(\Omega))^\perp \subseteq L^{s'}(\Omega)^d$. Then due to [24, Theorem 1.1] there is some $p \in W^1_{s',loc}(\Omega)$ such that $f = \nabla p$ almost everywhere. Because of $\partial\Omega \in C^1$ and [23, Théorème, p.114], $p \in W^1_{s',loc}(\overline{\Omega})$ and therefore $f = \nabla p \in \mathcal{N}(P_{s'})$. In particular, this shows that (A.3) holds and finishes the proof. ■

Corollary A.2 *Let $1 < q < \infty$ and let $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, be a bounded domain, an exterior domain, a perturbed half-space, or an aperture domain with $W_r^{2-\frac{1}{r}}$-boundary for some $d < r \leq \infty$. Then (A2) holds for any choice of closed and disjoint $\Gamma_1, \Gamma_2 \subseteq \partial\Omega$ such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ provided that Γ_2 is a compact and locally a C^1-manifold.*

Proof: As noted in the proof of Lemma 2.3 in the case $\Gamma_2 = \emptyset$ the validity of (A2) is well-known. Therefore let Ω and Γ_1, Γ_2 with $\Gamma_2 \neq \emptyset$ be as in the assumptions. We prove (A2) with the aid of Proposition A.1. First of all, we note that by the Lemma of Lax-Milgram (A.2) has a unique solution $p \in \dot{W}^1_{2,\Gamma_2}(\Omega)$ for any $F \in \dot{W}^{-1}_{2,\Gamma_2}(\Omega)$. Moreover, $\dot{W}^{-1}_{2,\Gamma_2}(\Omega) \cap \dot{W}^{-1}_{r,\Gamma_2}(\Omega)$ is dense in $\dot{W}^{-1}_{r,\Gamma_2}(\Omega)$ for any $1 \leq r < \infty$, which can be easily using the representation $\langle F, \varphi\rangle = (f, \nabla\varphi)$, $f \in L^r(\Omega)^d$ from above. Hence it is enough to show that there is a constant C_q such that

$$\|\nabla p\|_{L^s(\Omega)} \leq C_q \|F\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)} \qquad \text{for all } F \in \dot{W}^{-1}_{s,\Gamma_2}(\Omega) \cap \dot{W}^{-1}_{2,\Gamma_2}(\Omega), s = q, q'.$$

To this end let $F \in \dot{W}^{-1}_{s,\Gamma_2}(\Omega) \cap \dot{W}^{-1}_{2,\Gamma_2}(\Omega)$ and let $p \in \dot{W}^1_{2,\Gamma_2}(\Omega)$ be the solution of (A.2) for $q = 2$. Moreover, and let $\psi \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\psi \subseteq \Gamma_{2,\varepsilon} = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \Gamma_2) < \varepsilon\}$ and $\psi \equiv 1$ on $\Gamma_{2,\varepsilon/2}$ and let $\Omega_b \subseteq \mathbb{R}^d$ be a bounded domain with C^1-boundary such that $\Omega_b \cap \Gamma_{2,\varepsilon} = \Omega \cap \Gamma_{2,\varepsilon}$ for some $\varepsilon > 0$ sufficiently small. Then $p_0 := (1 - \psi)p \in \dot{W}^1_s(\Omega)$ and $p_1 = \psi p \in W^1_{s,0}(\Omega_b)$. Moreover,

$$(\nabla p_0, \nabla\varphi)_\Omega = \langle F, \psi\varphi\rangle_{\dot{W}^{-1}_{s,\Gamma_2}, \dot{W}^1_{s',\Gamma_2}} + (2(\nabla\psi)p, \nabla\varphi)_\Omega + ((\Delta\psi)p, \varphi)_\Omega$$

for all $\varphi \in \dot{W}^1_{s'}(\Omega)$, where φ is chosen such that $\int_{\Gamma_{2,\varepsilon}\cap\Omega} \varphi\, dx = 0$. Hence

$$\|\nabla p_0\|_{L^s(\Omega)} \leq C \left(\|F\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)} + \|p\|_{L^s(\Omega\cap\Gamma_{2,\varepsilon})} \right)$$

because of (A2) in the case $\Gamma_1 = \partial\Omega$ and Proposition A.1. Similarly, one obtains

$$\|\nabla p_1\|_{L^s(\Omega_b)} \leq C \left(\|F\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)} + \|p\|_{L^s(\Omega\cap\Gamma_{2,\varepsilon})} \right)$$

by standard results for the Laplace equation with Dirichlet boundary conditions a bounded C^1-domains. Altogether, this implies

$$\|\nabla p\|_{L^s(\Omega)} \leq C \left(\|F\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)} + \|p\|_{L^s(\Omega\cap\Gamma_{2,\varepsilon})} \right).$$

Now we use a standard compactness argument to prove (A.1). Provided there is no $C_q > 0$ such that (A.1) holds, there is a sequence $p_j \in \dot{W}^1_{s,\Gamma_2}(\Omega)$ such that

$\|\nabla p_j\|_{L^q(\Omega)} = 1$ and $F_j \in \dot{W}^{-1}_{s,\Gamma_2}(\Omega)$ defined by (A.2) with u replaced by u_j satisfies $\|F_j\|_{\dot{W}^{-1}_{s,\Gamma_2}} \to_{j\to\infty} 0$. Hence there is some $p \in \dot{W}^1_{s,\Gamma_2}(\Omega)$ such that $p_j \rightharpoonup_{j\to\infty} p$ in $\dot{W}^1_{s,\Gamma_2}(\Omega)$ up to a subsequence. Therefore p solves (A.2) with $F \equiv 0$. Hence the same localization procedure as above shows $p \in \dot{W}^1_{2,\Gamma_2}(\Omega)$, where one uses that $P_q = P_2$ on $J_q(\Omega) \cap J_2(\Omega)$ in the case of $\Gamma_2 = \emptyset$, cf. [12, Lemma 5.6] and [13, Lemma 3.2] or [14, Theorem 5] for the case of an aperture domain. Therefore $\nabla p = p = 0$ since $\Gamma_2 \neq \emptyset$. Finally, since $p_j \to_{j\to\infty} p$ in $L^s(\Omega \cap \Gamma_{2,\varepsilon})$ because of the compact embedding $W^1_s(\Omega \cap \Gamma_{2,\varepsilon}) \hookrightarrow L^s(\Omega \cap \Gamma_{2,\varepsilon})$, we conclude

$$1 = \|\nabla p_j\|_{L^s(\Omega)} \leq C \left(\|F_j\|_{\dot{W}^{-1}_{s,\Gamma_2}(\Omega)} + \|p_j\|_{L^s(\Omega\cap\Gamma_{2,\varepsilon})} \right) \to_{j\to\infty} 0,$$

which is a contradiction. Hence (A.1) holds for some C_q and $s = q, q'$. Therefore (A2) holds due to Proposition A.1. ■

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