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THE SEIDEL MORPHISM OF CARTESIAN PRODUCTS

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Abstract. We prove that the Seidel morphism of $(M \times M', \omega \oplus \omega')$ is naturally related to the Seidel morphisms of (M, ω) and (M', ω'), when these manifolds are monotone. We deduce a condition for loops of Hamiltonian diffeomorphisms of the product to be homotopically non trivial. This result was inspired by and extends results obtained by Pedroza [P].

All the symplectic manifolds we consider in this note are closed. A symplectic manifold (M, ω) is strongly semi-positive if (at least) one of the following conditions holds (c_1 denotes the first Chern class $c_1(TM, \omega)$):

(a) there exists $\lambda \geq 0$, such that for all $A \in \pi_2(M)$, $\omega(A) = \lambda c_1(A)$,
(b) c_1 vanishes on $\pi_2(M)$,
(c) the minimal Chern number N ($c_1(\pi_2(M)) = N\mathbb{Z}$) satisfies $N \geq n - 1$.

Under this assumption, Seidel introduced [S] a group morphism:

$$q_M \colon \tilde{\pi}_1(\mathrm{Ham}(M, \omega)) \longrightarrow QH_*(M, \omega)^\times,$$

where $QH_*(M, \omega)^\times$ denotes the group of invertible elements of $QH_*(M, \omega)$, the quantum homology of (M, ω). We recall that the identity of the group $QH_*(M, \omega)^\times$ is the fundamental class of M, which is denoted $[M]$.

As usual, $\mathrm{Ham}(M, \omega)$ denotes the group of Hamiltonian diffeomorphisms of (M, ω) and $\tilde{\pi}_1(\mathrm{Ham}(M, \omega))$ is a covering of $\pi_1(\mathrm{Ham}(M, \omega))$ which will be defined below. The inclusions of $\mathrm{Ham}(M, \omega)$ and $\mathrm{Ham}(M', \omega')$ in $\mathrm{Ham}(M \times M', \omega \oplus \omega')$ induce a map between the respective fundamental groups: $([g], [g']) \mapsto [g, g']$, where $[g, g']$ stands for $[(g, g')]$, the homotopy class of the loop (g, g'). The extension of this map to the coverings $\tilde{\pi}_1$ is straightforward. We denote it by

$$i \colon \tilde{\pi}_1(\mathrm{Ham}(M, \omega)) \times \tilde{\pi}_1(\mathrm{Ham}(M', \omega')) \longrightarrow \tilde{\pi}_1(\mathrm{Ham}(M \times M', \omega \oplus \omega')).$$

We also denote by

$$\kappa_Q \colon QH_*(M, \omega) \otimes QH_*(M', \omega') \longrightarrow QH_*(M \times M', \omega \oplus \omega')$$

the inclusion given by Künneth formula and the compatibility of the Novikov rings with the cartesian product (see §1 for definitions).

Let (M, ω) and (M', ω') be strongly semi-positive symplectic manifolds and let $\phi \in \tilde{\pi}_1(\mathrm{Ham}(M, \omega))$ and $\phi' \in \tilde{\pi}_1(\mathrm{Ham}(M', \omega'))$. When $(M \times M', \omega \oplus \omega')$ is strongly semi-positive (this is not necessarily the case, see discussion in Remark 8), one can, on one hand, compute the images of ϕ and ϕ' via the respective Seidel's morphisms and then see the result as an element in $QH_*(M \times M', \omega \oplus \omega')^\times$ via κ_Q. On the other hand, one can compute the image of $i(\phi, \phi')$, via the Seidel morphism of the product.

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In this short note, we prove that both computations coincide when the manifolds are monotone. A symplectic manifold is called monotone if it satisfies condition (a) above, with $\lambda > 0$. Notice that, if (M, ω) and (M', ω') are monotone with constants λ and λ', the product $(M \times M', \omega \oplus \frac{\lambda}{\lambda'}\omega')$ is monotone. In particular, if $\lambda = \lambda'$, $(M \times M', \omega \oplus \omega')$ is monotone.

Theorem 1. *Let (M, ω) and (M', ω') be closed monotone symplectic manifolds (with identical constants), $\phi \in \tilde{\pi}_1(\mathrm{Ham}(M, \omega))$ and $\phi' \in \tilde{\pi}_1(\mathrm{Ham}(M', \omega'))$, then*

$$q_{M \times M'}\big(i(\phi, \phi')\big) = \kappa_Q\big(q_M(\phi) \otimes q_{M'}(\phi')\big).$$

Remark 2. The monotonicity assumption ensures that there exists no non-constant pseudo-holomorphic sphere of first Chern number 0. This property is only used in the proof of Lemma 15, which states that a particular choice of almost complex structures is regular enough to compute Seidel's morphism. Thus, all the results of this note hold under the weaker assumption that both manifolds *and* their product are "strongly semi-positive, without non-constant pseudo-holomorphic spheres with first Chern number 0".

For strongly semi-positive manifolds admitting such spheres, the theorem is more difficult to prove but most probably holds, see Remark 17 (we do mean that it holds even without the use of virtual techniques, see Remark 4).

Even though the morphism q is interesting in itself, one usually looks for information concerning $\pi_1(\mathrm{Ham}(M, \omega))$ (rather than $\tilde{\pi}_1(\mathrm{Ham}(M, \omega))$). Now, the definition of quantum homology relies on a Novikov ring built from Γ, a quotient of $\pi_2(M)$. Moreover, Γ can be seen as a subgroup of the group of invertible elements of quantum homology via the map τ, defined by $\tau(\gamma) = [M] \otimes \gamma$ for all $\gamma \in \Gamma$. Seidel's morphism then induces a morphism $\bar{q}$ defined by the commutativity of the diagram:

$$\begin{tikzcd} \tilde{\pi}_1(\mathrm{Ham}(M, \omega)) \arrow[r] \arrow[d, "q"'] & \pi_1(\mathrm{Ham}(M, \omega)) \arrow[d, "\bar{q}"] \\ QH_*(M)^\times \arrow[r] & QH_*(M)^\times/\tau(\Gamma) \end{tikzcd} \tag{1}$$

The main consequence of Theorem 1 can be stated in terms of $\bar{q}$.

Corollary 3. *Let (M, ω), (M', ω') be as in the theorem. Let g and g' be loops of Hamiltonian diffeomorphisms of respectively (M, ω) and (M', ω'). If $[g, g']$ is trivial in $\pi_1(\mathrm{Ham}(M \times M', \omega \oplus \omega'))$, both loops g and g' are mapped to the identity via Seidel's morphism, namely, $\bar{q}_M([g]) = [M]$ and $\bar{q}_{M'}([g']) = [M']$.*

Remark 4. Notice that, in order to get (the same) results *directly on the fundamental groups*, one could also adopt the approach of Lalonde, McDuff and Polterovich [LMP]. Furthermore, following McDuff [M], one could address these questions by means of virtual techniques. This would probably provide a proof of the results contained in this note in great generality. We thank Shengda Hu for pointing this fact out to us.

From Corollary 3, it is easy to derive the following properties.

Corollary 5. *Let (M, ω) be monotone and let $[g] \in \pi_1(\mathrm{Ham}(M, \omega))$ such that $\bar{q}_M(g) \neq [M]$. Then for any monotone symplectic manifold (M', ω') (with the same monotonicity constant), the map*

$$\iota[g] \colon \pi_1(\mathrm{Ham}(M', \omega')) \longrightarrow \pi_1(\mathrm{Ham}(M \times M', \omega \oplus \omega'))$$

defined by $\iota[g]([g']) = [g, g']$*, is an inclusion. Moreover, if* $\bar{q}_M([g_1]) \neq \bar{q}_M([g_2])$*, then*

$$\iota[g_1](\pi_1(\mathrm{Ham}(M', \omega'))) \cap \iota[g_2](\pi_1(\mathrm{Ham}(M', \omega'))) = \emptyset.$$

Corollary 6. *Let* (M, ω) *be a monotone symplectic manifold. If Seidel's morphism* $\bar{q}$ *is injective, then the map*

$$\pi_1(\mathrm{Ham}(M, \omega)) \times \pi_1(\mathrm{Ham}(M, \omega)) \longrightarrow \pi_1(\mathrm{Ham}(M \times M, \omega \oplus \omega))$$

induced by the inclusion is injective.

We emphasize here the fact that the maps $\iota[g]$ appearing in Corollary 5 are maps between sets (and not group morphisms).

Example. *Let* ω_{st} *be the symplectic form on* $\mathbb{CP}^m$ *such that* $(\mathbb{CP}^m, \omega_{\mathrm{st}})$ *is monotone, with monotonicity constant* $1/(m+1)$*. Seidel* [S] *proved that there exists an element of degree* $m+1$ *in* $\pi_1(\mathrm{Ham}(\mathbb{CP}^m, \omega_{\mathrm{st}}))$[1]*. This explicit element comes from the action of* $\mathrm{U}(m+1)$ *on* $\mathbb{C}^{m+1}$*; we denote it by* α_m*.*

In order to obtain monotone products, we consider a multiple of the standard symplectic form, namely, we endow $\mathbb{CP}^m$ *with* $\omega_m = (m+1)\,\omega_{\mathrm{st}}$*. We also denote the element of order* $m+1$ *of* $\pi_1(\mathrm{Ham}(\mathbb{CP}^m, \omega_m))$ *by* α_m*.*

From Corollary 5 we deduce the following properties.

(1) *For* $1 \leq l \leq n$*, the inclusions* $\iota[\alpha_n^l]$*, given by* $\iota[\alpha_n^l]([g']) = [\alpha_n^l, g']$*, lead to* n *distinct copies of* $\pi_1(\mathrm{Ham}(\mathbb{CP}^{n'}, \omega_{n'}))$ *in* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n \times \mathbb{CP}^{n'}, \omega_n \oplus \omega_{n'}))$*.*

(1') *Similarly,* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n \times \mathbb{CP}^{n'}, \omega_n \oplus \omega_{n'}))$ *also contains* n' *distinct copies of* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n, \omega_n))$*, given by* $\iota'[\alpha_{n'}^{l'}]([g]) = [g, \alpha_{n'}^{l'}]$*, for* $1 \leq l' \leq n'$*.*

(2) *These injections intersect pairwise in a unique point, namely,* $\iota[\alpha_n^l]([g']) = \iota'[\alpha_{n'}^{l'}]([g])$ *if and only if* $g = \alpha_n^l$ *and* $g' = \alpha_{n'}^{l'}$*.*

(3) *Finally, the elements* $(\alpha_n, \mathrm{id}_{\mathbb{CP}^{n'}})$ *and* $(\mathrm{id}_{\mathbb{CP}^n}, \alpha_{n'})$ *are of respective orders* $n+1$ *and* $n'+1$ *(and* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n \times \mathbb{CP}^{n'}, \omega_n \oplus \omega_{n'}))$ *contains subgroups isomorphic to* $\mathbb{Z}_{n+1}$ *and* $\mathbb{Z}_{n'+1}$*).*

Notice that when $n = n'$ *the same statements hold with* ω_{st}*. In this particular case, Corollary 6 asserts that* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n, \omega_{\mathrm{st}})) \times \pi_1(\mathrm{Ham}(\mathbb{CP}^n, \omega_{\mathrm{st}}))$ *is included in* $\pi_1(\mathrm{Ham}(\mathbb{CP}^n \times \mathbb{CP}^n, \omega_{\mathrm{st}} \oplus \omega_{\mathrm{st}}))$

Remark 7. Another application of Theorem 1 comes from recent work of Hu and Lalonde [HL]. Indeed, they introduced a relative version (that is, defined with respect to a Lagrangian L) of Seidel's morphism and they proved that it is related to the Seidel morphism of the ambient manifold (W, Ω) via a map defined by Albers [A] (under suitable assumptions).

Let $(W, \Omega) = (M \times M, \omega \oplus (-\omega))$ and L be the graph of a Hamiltonian diffeomorphism of (M, ω). Combining Theorem 1 with the morphism introduced by Biran, Polterovich and Salamon [BPS] allows us to compare the relative Seidel morphism associated to L, not only to the absolute Seidel morphism associated to (W, Ω) but also to the one associated to (M, ω) (at least for "split" loops).

Remark 8. As mentioned above, being strongly semi-positive is not *a priori* compatible with the cartesian product. Nevertheless, as for monotone symplectic manifolds, the product of certain manifolds is automatically strongly semi-positive. Let, for instance, (M, ω) and (M', ω') both satisfy the (sub-)condition

[1] Actually, Seidel proved the existence of such an element for any complex Grassmannian $\mathrm{Gr}_k(\mathbb{C}^{m+1})$, $k \geq 1$

(a) with constants λ and λ'. If $\lambda = \lambda' = 0$, the product also satisfies this condition. (In that case, there is no non-constant pseudo-holomorphic sphere since such a sphere has positive symplectic area. Thus, Seidel's morphism is trivial for these manifolds and Theorem 1 is trivially satisfied.)

(b) then c_1 vanishes on $\pi_2(M)$, c_1' vanishes on $\pi_2(M')$. Thus the first Chern class of the tangent bundle of $M \times M'$ vanishes on $\pi_2(M \times M')$ and the product is strongly semi-positive.

(c) the minimal Chern numbers N and N' satisfy $N \geq n - 1$ and $N' \geq n' - 1$. The minimal Chern number of the product being the greatest common divisor of N and N', for the product to satisfy sub-condition (c), the gcd of N and N' has to satisfy $\gcd(N, N') \geq n + n' - 1$. Notice for example that if $N \geq 2n - 1$, then $(M \times M, \omega \oplus \omega)$ is strongly semi-positive.

There is also another remarkable particular case: when $\pi_2(M') = 0$, then of course if (M, ω) is monotone (respectively, satisfies (a), (b) or (c) with $N \geq n + n' - 1$), the product is monotone (respectively, strongly semi-positive). Thus Theorem 1 extends results obtained by Pedroza. Indeed, [P, Theorem 1.1] is given by our theorem, for $\pi_2(M') = 0$, $\phi' = \mathrm{id}$, and [P, Theorem 1.3] corresponds to the case where $M = M'$, $\pi_2(M) = 0$ and $\phi' = \phi$.

Concerning the latter result, we emphasize the fact that our result *does not require any type of asphericity condition* such as $\pi_2(M)$ trivial. This is important since, for $(M', \omega') = (M, \omega)$ with $\pi_2(M) = 0$, the involved Seidel morphisms are trivial (and the statement of Theorem 1 is trivially satisfied in this particular case).

Another noteworthy difference between this note and [P] is the approach to Seidel's morphism which we consider. Pedroza approaches the question via the point of view of Hamiltonian fibrations, we use the representation approach (in terms of automorphisms of Floer homology).

In the next section we recall the definitions of quantum homology (§1.1) and of Seidel's morphism (§1.2). This allows us to prove Corollary 3 from Theorem 1. Then we recall the construction of Floer homology and the representation viewpoint on Seidel's morphism (§1.3). Afterwards, we prove Theorem 1 (§2) up to a claim concerning the regularity of a particular choice of parameters (required to compute Seidel's morphism). Finally, we justify the claim (§3).

1. Making things precise

1.1. **The group Γ and the morphism κ_Q.** Following Seidel, we define Γ_M as the group of equivalence classes of elements in $\pi_2(M)$ under the equivalence relation $A \sim B$ if $\omega(A) = \omega(B)$ and $c_1(A) = c_1(B)$. Notice that the obvious map

$$\begin{aligned} \pi_\Gamma\colon\ \Gamma_M \times \Gamma_{M'} &\longrightarrow \Gamma_{M \times M'} \\ ([A], [A']) &\longmapsto [A, A'] \end{aligned}$$

is well-defined and surjective but in general not injective. Its kernel consists of pairs $([A], [A'])$ such that $\omega(A) + \omega'(A') = 0$ and $c_1(A) + c_1'(A') = 0$.

We recall that the (small) quantum homology is the Λ_M–module given as the tensor product $H_*(M, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Lambda_M$, where Λ_M is the Novikov ring defined as the group of formal sums $\sum m_\gamma \cdot \gamma$, with $\gamma \in \Gamma_M$, $m_\gamma \in \mathbb{Z}_2$ and satisfying the finiteness condition:

$$\forall C \in \mathbb{R},\ \#\{m_\gamma | \, m_\gamma \neq 0,\, \omega(\gamma) \leq C\} < \infty.$$

Since an element of the type $[M] \otimes \gamma$ is invertible in $QH_*(M,\omega)$, the formula $\tau(\gamma) = [M] \otimes \gamma$ defines a morphism $\tau\colon \Gamma_M \to QH_*(M,\omega)^\times$. Since this morphism is injective, $\Gamma_M \simeq \tau(\Gamma_M)$ can be seen as a subgroup of $QH_*(M,\omega)^\times$.

Now let $\kappa\colon H_*(M,\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H_*(M',\mathbb{Z}_2) \longrightarrow H_*(M \times M',\mathbb{Z}_2)$ be the inclusion given by Künneth formula (due to the fact that we use the field $\mathbb{Z}_2$ for coefficients, this is actually an isomorphism). We define κ_Q by the formula

$$\begin{aligned} \kappa_Q\colon QH_*(M,\omega) \otimes_{\mathbb{Z}_2} QH_*(M',\omega') &\longrightarrow QH_*(M \times M', \omega \oplus \omega') \\ \big((\alpha \otimes \gamma) \otimes (\alpha' \otimes \gamma')\big) &\longmapsto \kappa(\alpha \otimes \alpha') \otimes \pi_\Gamma(\gamma,\gamma') \end{aligned}$$

for simple tensors and extend it by linearity for general quantum elements. By definition of κ_Q and injectivity of κ, we deduce the following lemma.

Lemma 9. *Let $\kappa_Q(q \otimes q') = [M \times M'] \otimes \gamma_\times$ for some $\gamma_\times \in \Gamma_{M\times M'}$. There exist $\lambda \in \Lambda_M$ and $\lambda' \in \Lambda_{M'}$ such that $q = [M] \otimes \lambda$ and $q' = [M'] \otimes \lambda'$.*

1.2. **Seidel's morphism and the proof of Corollary 3.** Let (M,ω) be a monotone symplectic manifold. Following Seidel's notation, G denotes the set of smooth loops of Hamiltonian diffeomorphisms (based at the identity). Then $\pi_0(G) \simeq \pi_1(\mathrm{Ham}(M,\omega))$. Let $\mathcal{L}M$ be the set of free, smooth, contractible loops of M. $\widetilde{\mathcal{L}M}$ is the set of equivalence classes of pairs $(v,x) \in C^\infty(D^2,M) \times \mathcal{L}M$ such that v coincides with x on the boundary ∂D^2, under the equivalence relation

$$(v,x) \sim (v',x') \Leftrightarrow x = x' \text{ and } \omega(v\#\overline{v'}) = 0, c_1(v\#\overline{v'}) = 0.$$

Here $\overline{v'}$ is v' considered with the opposite orientation and $v\#\overline{v'}$ the sphere obtained by gluing the two discs along their common boundary. There is an action of G on $\mathcal{L}M$, given by $g \cdot x = [t \mapsto g_t(x(t))]$, which lifts to $\widetilde{\mathcal{L}M}$. We define $\widetilde{G}$ as the subset of $G \times \mathrm{Homeo}(\widetilde{\mathcal{L}M})$ consisting of pairs $(g,\tilde{g})$ such that $\tilde{g}$ is a lift of g (that is, $\tilde{g}(v,x) = (v', g\cdot x)$ for any $(v,x) \in \widetilde{\mathcal{L}M}$).

$\pi_0(\widetilde{G})$ is the covering of $\pi_1(\mathrm{Ham}(M,\omega))$ which was denoted $\tilde{\pi}_1(\mathrm{Ham}(M,\omega))$ above. Following Witten [W], Seidel introduced the morphism q by considering Hamiltonian fibre bundles over S^2, with fibre (M,ω). Roughly, since a fibre bundle over a disc is trivial, it is easy to see that such a fibre bundle over S^2 corresponds to the choice of a loop g of Hamiltonian diffeomorphisms (based at the identity). Seidel then derived invariants from the pseudo-holomorphic sections of these bundles by comparing them to some chosen equivalence class of sections (given by the choice of a lift of g).

Remark 10. Now that $\tilde{\pi}_1(\mathrm{Ham}(M,\omega))$ has been made precise, we will also use the following obvious notation $i([g,\tilde{g}],[g',\tilde{g}']) = [g,g';\tilde{g},\tilde{g}']$ to denote the morphism i defined above.

We can now deduce Corollary 3 from Theorem 1.

Proof of Corollary 3. Let g and g' be loops of Hamiltonian diffeomorphisms such that $[g,g']$ is trivial in $\pi_1(\mathrm{Ham}(M \times M', \omega \oplus \omega'))$. Then $\bar{q}([g,g'])$ is the identity of $QH_*(M \times M', \omega \oplus \omega')^\times / \tau(\Gamma_{M\times M'})$. By definition of $\bar{q}$ (that is, by commutativity of Diagram (1)), this amounts to the fact that for any lift $(g, g'; \tilde{g}, \tilde{g}')$ of (g, g'), $q([g, g'; \tilde{g}, \tilde{g}']) \in \tau(\Gamma_{M\times M'})$. We fix $\tilde{g}$ and $\tilde{g}'$ respective lifts of g and g'.

We know that there exists $\gamma_\times \in \Gamma_{M\times M'}$ such that $q([g, g'; \tilde{g}, \tilde{g}']) = [M\times M']\otimes\gamma_\times$. By Theorem 1, this gives that $\kappa_Q\big(q_M([g, \tilde{g}]) \otimes q_{M'}([g', \tilde{g}'])\big) = [M \times M'] \otimes \gamma_\times$.

By Lemma 9, this implies the existence of $\lambda \in \Lambda_M$ and $\lambda' \in \Lambda_{M'}$ such that $q_M([g, \tilde{g}]) = [M] \otimes \lambda$ and $q_{M'}([g', \tilde{g}']) = [M'] \otimes \lambda'$. Since,

$$[M] \otimes \lambda = [M] \otimes \sum m_\gamma \gamma = \sum m_\gamma([M] \otimes \gamma)$$

$q_M([g, \tilde{g}])$ is mapped to the identity element in $QH_*(M)^\times/\tau(\Gamma_M)$ (and similarly for $q_{M'}([g', \tilde{g}']))$; by definition of $\bar{q}$, so are $\bar{q}_M([g])$ and $\bar{q}_{M'}([g'])$. □

1.3. **An alternate description: Seidel's representation.** As noticed by Seidel, there is an alternate description of q as a representation of $\tilde{\pi}_1(\mathrm{Ham}(M, \omega))$ in terms of automorphisms of Floer homology.

We briefly recall the definition of the Floer homology of a closed (monotone) symplectic manifold (M, ω). Let H be a Hamiltonian function on M. The action functional is defined on $\widetilde{\mathcal{L}M}$ by the formula

$$\mathcal{A}_H([v, x]) = -\int v^*\omega + \int H_t(x(t))dt.$$

The critical points of $\mathcal{A}_H$, $\mathrm{Crit}(\mathcal{A}_H)$, are equivalence classes $[v, x]$ where x is a contractible periodic orbit of X_H, the Hamiltonian vector field generated by H. We recall that $\mathrm{Crit}(\mathcal{A}_H)$ is graded via the Conley–Zehnder index.

We now pick a regular almost complex structure on TM such that the pair (H, J) is regular (which means, in particular, that the critical points of $\mathcal{A}_H$ are non-degenerate, see §3 for precise definitions). A choice of such a regular pair is generic and for such a choice, one can define the Floer complex $(CF_*(H), \partial_{(H,J)})$ where $CF_k(H)$ is the group of formal sums $\sum_c m_c \cdot c$ where the c's are critical points of $\mathcal{A}_H$ of index k, $c \in \mathrm{Crit}_k(\mathcal{A}_H)$. The coefficients m_c are elements of $\mathbb{Z}_2$ and satisfy the finiteness condition

$$\#\{m_c |\, m_c \neq 0 \text{ and } \mathcal{A}_H(c) \geq C\} < \infty \tag{2}$$

for all real numbers C.

The differential $\partial_{(H,J)}\colon\ \mathrm{Crit}_k(\mathcal{A}_H) \to \mathrm{Crit}_{k-1}(\mathcal{A}_H)$ is defined by the formula

$$\partial_{(H,J)}(c) = \sum_{c'\in\mathrm{Crit}_{k-1}(\mathcal{A}_H)} \#_2[\mathcal{M}(c, c'; H, J)/\mathbb{R}] \cdot c'$$

where $\#_2\mathcal{M}(c, c'; H, J)$ is the cardinal (mod 2) of the set of curves $u\colon\ S^1 \times \mathbb{R} \to M$ satisfying

$$\partial_s u + J(u)(\partial_t u - X_H(u)) = 0 \tag{3}$$

and which admit a lift $\tilde{u}\colon\ \mathbb{R} \to \widetilde{\mathcal{L}M}$ with limits c and c'. Indeed, $\mathbb{R}$ acts on $\mathcal{M}(c, c'; H, J)$ by translation and the regularity condition satisfied by the pair (H, J) ensures that, for c and c' with index difference 1, $\mathcal{M}(c, c'; H, J)/\mathbb{R}$ is a compact 0–dimensional manifold.

Floer homology is the homology of this complex, and does not depend on the choice of the regular pair (H, J): $HF_*(M, \omega) = H_*(CF(H), \partial_{(H,J)})$. A proof of this well-known fact is given by the usual comparison morphism whose definition is recalled below.

Now, Seidel's morphism can be seen as a representation of $\tilde{\pi}_1(\mathrm{Ham}(M,\omega))$, namely, for any $[g,\tilde{g}] \in \tilde{\pi}_1(\mathrm{Ham}(M,\omega))$, one can associate an automorphism

$$\mathbf{S}[g,\tilde{g}]\colon\ HF_*(M,\omega) \longrightarrow HF_{*-2I(g,\tilde{g})}(M,\omega)$$

defined as the composition $\mathbf{S}[g,\tilde{g}] = H_*(\mathbf{c}(\mathbf{H},\mathbf{J}) \circ \mathbf{n}(g,\tilde{g}))$. The first morphism involved in this composition is the naturality morphism which is an identification of chain complexes

$$\mathbf{n}(g,\tilde{g})\colon\ CF_*(M,\omega;H^g,J^g) \longrightarrow CF_{*-2I(g,\tilde{g})}(M,\omega;H,J)$$

where the pair (H^g,J^g) is the pushforward of (H,J) by g, defined as

$$\begin{aligned} H^g(t,y) &= H(t,g_t(y)) - K_g(t,g_t(y)), \\ \text{and } J^g_t &= dg_t^{-1} J_t dg_t \end{aligned} \tag{4}$$

(K_g being a Hamiltonian generating the loop g). By straightforward computations, one can show that (H^g,J^g) is a regular pair if and only if (H,J) is regular.

The definition of the shift of indices $I(g,\tilde{g})$ is standard (it corresponds to the degree of a loop in $\mathrm{Sp}(2n,\mathbb{R})$ coming from a trivialization of TM over the cappings v's of the orbits x's – see the definition of $\widetilde{\mathcal{L}M}$). It is compatible with the cartesian product, in the following sense:

$$I(g,g';\tilde{g},\tilde{g}') = I(g,\tilde{g}) + I(g',\tilde{g}').$$

In view of this formula, the shift of indices will be implied in what follows.

The second morphism is the usual comparison morphism of Floer homology

$$\mathbf{c}(\mathbf{H},\mathbf{J})\colon\ CF_*(M,\omega;H,J) \longrightarrow CF_*(M,\omega;H^g,J^g).$$

It is defined by using $(\mathbf{H},\mathbf{J})$, any regular homotopy between (H,J) and (H^g,J^g). It induces an isomorphism in homology.

The correspondence between these two descriptions of Seidel's representation, namely, between $q[g,\tilde{g}]$ and $\mathbf{S}[g,\tilde{g}]$, goes via the Piunikhin–Salamon–Schwarz (PSS) morphism as well as the pair-of-pants product. These tools appeared in [PSS]. We recall that (under the monotonicity assumption) the PSS morphism is a canonical isomorphism

$$\mathrm{PSS}\colon\ QH_*(M,\omega) \longrightarrow HF_*(M,\omega)$$

between the quantum homology and the Floer homology of (M,ω), as modules over the Novikov ring. The pair-of-pants product is a product on Floer homology

$$\star_{\mathrm{pp}}\colon\ HF_*(M,\omega) \otimes HF_*(M,\omega) \longrightarrow HF_*(M,\omega)$$

defined on chain complexes by counting suitable moduli spaces of pair-of-pants. Given these tools, Seidel proved that

$$\mathbf{S}[g,\tilde{g}](b) = \mathrm{PSS}(q[g,\tilde{g}]) \star_{\mathrm{pp}} b \tag{5}$$

for all $b \in HF_*(M,\omega)$. This is the interpretation we use to prove Theorem 1.

2. Proof of the theorem

Let (M,ω) and (M',ω') be closed monotone symplectic manifolds (with the same monotonicity constant). Let (H,J) and (H',J') be respectively defined on (M,ω) and (M',ω'). We define on $M\times M'$ the Hamiltonian $H\oplus H'$ and the almost complex structure $J\oplus J'$ by the formulae

$$\begin{aligned}(H\oplus H')_t(x,x') &= H_t(x)+H_t(x')\\ (J\oplus J')_t(\xi,\xi') &= (J_t(\xi),J'_t(\xi'))\end{aligned}\tag{6}$$

for all $(x,x')\in M\times M'$ and all $(\xi,\xi')\in T_xM\otimes T_{x'}M'\simeq T_{(x,x')}(M\times M')$.

Remark 11. Notice that the pushforward, as defined by (4), of $(H\oplus H',J\oplus J')$ by any element of the form $(g,g')\in \mathrm{Ham}(M\times M',\omega\oplus\omega')$ is

$$\big((H\oplus H')^{g,g'},(J\oplus J')^{g,g'}\big)=(H^g\oplus H'^{g'},J^g\oplus J'^{g'}).$$

Even for regular pairs (H,J) and (H',J'), the pair $(H\oplus H',J\oplus J')$ is not *a priori* regular. As we shall see, the problem comes from the fact that the moduli spaces of simple spheres of the product is, in general, bigger than the product of the moduli spaces of simple spheres of each component. Thus, the complex structure $J\oplus J'$ is not automatically regular. In §3, we show that to go through the construction a weaker regularity condition is enough. This will give sense to the following claim.

Claim 1. *If (H,J) and (H',J') are regular pairs, the pair $(H\oplus H',J\oplus J')$ is regular enough. Moreover, if $(\mathbf{H},\mathbf{J})$ and $(\mathbf{H}',\mathbf{J}')$ are regular homotopies, then the homotopy $(\mathbf{H}\oplus\mathbf{H}',\mathbf{J}\oplus\mathbf{J}')$ is regular enough.*

We postpone the proof until the next section. As mentioned in Remark 2, the proof of the part concerning almost complex structures is the only place where we use (a property implied by) the monotonicity assumption.

Now, notice that

$$\mathrm{Crit}_k(\mathcal{A}_{H\oplus H'})=\bigcup_{l+l'=k}\mathrm{Crit}_l(\mathcal{A}_H)\times\mathrm{Crit}_{l'}(\mathcal{A}_{H'})$$

and that the action agrees with this decomposition, that is, for all $[v,x]\in\mathrm{Crit}_l(\mathcal{A}_H)$ and all $[v',x']\in\mathrm{Crit}_{l'}(\mathcal{A}_{H'})$:

$$\mathcal{A}_{H\oplus H'}([v,v';x,x'])=\mathcal{A}_H([v,x])+\mathcal{A}_{H'}([v',x']).$$

The finiteness condition (2) is such that

$$\bigoplus_{l+l'=k}CF_l(H)\otimes CF_{l'}(H')\simeq CF_k(H\oplus H').$$

This isomorphism induces a morphism in homology

$$\kappa\colon\ HF_*(M,\omega)\otimes HF_*(M',\omega')\longrightarrow HF_*(M\times M',\omega\oplus\omega').$$

Claim 2. *The following diagram commutes*

$$\begin{tikzcd}
HF_*(M,\omega)\otimes HF_*(M',\omega') \arrow[r, "\kappa"] \arrow[d, "\mathbf{S}[g,\tilde g]\otimes\mathbf{S}[g',\tilde g']"'] & HF_*(M\times M',\omega\oplus\omega') \arrow[d, "\mathbf{S}[g,g';\tilde g,\tilde g']"] \\
HF_*(M,\omega)\otimes HF_*(M',\omega') \arrow[r, "\kappa"] & HF_*(M\times M',\omega\oplus\omega')
\end{tikzcd}$$

Proof of Claim 2. Decomposing the automorphisms of Floer homology given by $[g,\tilde{g}]$, $[g',\tilde{g}']$ and $[g,g';\tilde{g},\tilde{g}']$ in terms of naturality and comparison morphisms, we want to prove that the following diagram commutes in homology

$$\begin{tikzcd} CF_*(H^g,J^g)\otimes CF_*(H'^{g'},J'^{g'}) \arrow[r] \arrow[d, "\mathbf{n}[g,\tilde{g}]\otimes\mathbf{n}[g',\tilde{g'}]"'] & CF_*(H^g\oplus H'^{g'},J^g\oplus J'^{g'}) \arrow[d, "\mathbf{n}[g,g';\tilde{g},\tilde{g}']"] \\ CF_*(H,J)\otimes CF_*(H',J') \arrow[r] \arrow[d, "\mathbf{c(H,J)}\otimes\mathbf{c(H',J')}"'] & CF_*(H\oplus H',J\oplus J') \arrow[d, "\mathbf{c(H\oplus H',J\oplus J')}"] \\ CF_*(H^g,J^g)\otimes CF_*(H'^{g'},J'^{g'}) \arrow[r] & CF_*(H^g\oplus H'^{g'},J^g\oplus J'^{g'}) \end{tikzcd}$$

where (H,J), (H',J'), etc are defined as above. Actually, the diagram even commutes at the chain level, with the choices we made (justified by Claim 1) and by Remark 11: the horizontal maps identify products of moduli spaces with moduli spaces of the product (for any type of moduli spaces involved by these morphisms). $\square$

By (5) which relates the two descriptions of Seidel's morphism, Claim 2 immediately amounts to the fact that for $b\in CF_*(M,\omega;H,J)$ and $b'\in CF_*(M',\omega';H',J')$,

(7)
$$\mathrm{PSS}(q[g,g';\tilde{g},\tilde{g}'])\star_{\mathrm{pp}}\kappa(b\otimes b')=\kappa\big((\mathrm{PSS}(q[g,\tilde{g}])\star_{\mathrm{pp}}b)\otimes(\mathrm{PSS}(q[g',\tilde{g'}])\star_{\mathrm{pp}}b')\big).$$

Claim 3. *The following diagram commutes*

$$\begin{tikzcd} QH_*(M,\omega)\otimes QH_*(M',\omega') \arrow[r, "\kappa_Q"] \arrow[d, "\mathrm{PSS}\otimes\mathrm{PSS}"'] & QH_*(M\times M',\omega\oplus\omega') \arrow[d, "\mathrm{PSS}"] \\ HF_*(M,\omega)\otimes HF_*(M',\omega') \arrow[r, "\kappa"] & HF_*(M\times M',\omega\oplus\omega') \end{tikzcd}$$

Proof of Claim 3. If the parameters (Hamiltonian functions, almost complex structures, Morse functions, metrics, etc) used to define the involved PSS morphisms are chosen as above, the products of moduli spaces are again identified with the moduli spaces of the product and the commutativity even holds at the chain level. $\square$

For the element $q[g,\tilde{g}]\otimes q[g',\tilde{g}']$, this commutativity amounts to

(8)
$$\kappa\big(\mathrm{PSS}(q[g,\tilde{g}])\otimes\mathrm{PSS}(q[g',\tilde{g}'])\big)=\mathrm{PSS}\big(\kappa_Q(q[g,\tilde{g}]\otimes q[g',\tilde{g'}])\big).$$

Since $[M\times M']=\kappa_Q([M]\otimes[M'])$, Claim 3 also leads to

$$\mathrm{PSS}([M\times M'])=\mathrm{PSS}\big(\kappa_Q([M]\otimes[M'])\big)=\kappa\big(\mathrm{PSS}([M])\otimes\mathrm{PSS}([M'])\big).$$

Thus, letting $b=\mathrm{PSS}([M])$ and $b'=\mathrm{PSS}([M'])$ in (7), one gets that

(9)
$$\mathrm{PSS}(q[g,g';\tilde{g},\tilde{g}'])=\kappa\big(\mathrm{PSS}(q[g,\tilde{g}])\otimes\mathrm{PSS}(q[g',\tilde{g'}])\big)$$

since the image via the PSS morphism of the fundamental class (the identity element of the group of invertible elements of quantum homology) acts trivially for the pair-of-pants product. Finally, (8) and (9) amount to

$$\mathrm{PSS}\big(q[g,g';\tilde{g},\tilde{g}']\big)=\mathrm{PSS}\big(\kappa_Q(q[g,\tilde{g}]\otimes q[g',\tilde{g'}])\big).$$

This completes the proof of the theorem, since the PSS morphism is an isomorphism.

3. Regularity of split pairs

In this section, we give precise definitions of regularity (for almost complex structures and pairs (H, J)), we define "regular enough pairs" and prove Claim 1. We consider the case of S^1–families of ω–compatible almost complex structures. This is sufficient to prove that Floer homology is well-defined. The case of 2–parameter families of almost complex structures (needed for instance for homotopies) works along the same lines.

Let $\mathcal{M}^s(J)$ denote the set of pairs $(t, w) \in S^1 \times C^\infty(S^2, M)$, where w is a J_t–holomorphic *simple* sphere in M. This set is the union over k of the subsets $\mathcal{M}^s_k(J)$ of pairs with spheres of first Chern number k. With S^1–families of almost complex structures, the linearization of the equation $\bar\partial_J = 0$ at (t, w) is given by

$$\begin{aligned} \hat{D}_J(t,w)\colon\ T_tS^1 \times C^\infty(w^*TM) &\longrightarrow \Omega^{0,1}(w^*(TM, J_t)) \\ \hat{D}_J(t,w)(\theta, W) &= D_{J_t}(w)W + \frac{1}{2}DJ(t)\theta \circ w \circ i \end{aligned} \tag{10}$$

where i is the complex structure of $S^2 \simeq \mathbb{CP}^1$ and $DJ(t)$ denotes the derivative of the S^1–family of almost complex structures at t.

Definition 12. *The S^1–family of almost complex structures J is* regular *if the linearized operator defined by (10) is onto for all $(t, w) \in \mathcal{M}^s(J)$.*

Now denote by $V_k(J) \subset S^1 \times M$ the set of pairs (t, x) for which there exists a non-constant, J_t–pseudo-holomorphic sphere w, with first Chern number $c_1(w) \leq k$ and such that $x \in \mathrm{im}(w)$.

Definition 13. *A pair (H, J) consisting of a family of almost complex structures J and a time-dependent Hamiltonian $H\colon\ S^1 \times M \to \mathbb{R}$ is* regular *if*

- **i.** *J is a regular S^1–family of almost complex structures,*
- **ii.** *the critical points of $\mathcal{A}_H$ are non-degenerate and for any orbit x of the Hamiltonian vector field and all $t \in S^1$, $(t, x(t)) \notin V_1(J)$,*
- **iii.** *the linearization of the operator (3) is onto for all $u \in \mathcal{M}(c, c'; H, J)$, and*
- **iv.** *if $\mathrm{ind}(u) \leq 2$ then for all t and s, $(t, u(s, t)) \notin V_0(J)$.*

It is well-known that for monotone symplectic manifolds the sets of regular almost complex structures and of regular pairs are dense.

3.1. Regular enough almost complex structures and pairs.

When J is regular, the following fundamental claims hold.

Claim 1. *For all integers k, the set $\mathcal{M}^s_k(J)$ is (either empty or) a smooth manifold of dimension $2n + 2k + 1$.*

Claim 2. *J is semi-positive, that is, for all $k < 0$, $\mathcal{M}^s_k(J)$ is empty.*

We can now define a weaker regularity condition for almost complex structures.

Definition 14. *An almost complex structure is* regular enough *if Claim 1 holds for $k = 0$, 1 and 2 and Claim 2 holds. A pair is* regular enough *if the almost complex structure is regular enough and if the pair satisfies conditions* ***ii-iv*** *of Definition 13.*

Roughly speaking, Claim 1, for $k = 0$ and 1 implies that $V_0(J)$ has codimension 4 and $V_1(J)$ has codimension 2 as subsets of $S^1 \times M$. Thus, for a regular enough almost complex structure, the choice of a Hamiltonian H such that the pair (H,J) satisfies conditions **ii** and **iv** of Definition 13 is generic.

Now, for such a pair, Floer homology is well-defined since bubbling is avoided. Indeed, condition **ii** forbids configurations of the type index–0 tube (that is, a "constant" tube which coincides with an orbit) with attached spheres whose first Chern numbers sum to 1. Condition **iv** forbids the appearance of configurations of the type index–2 tube with attached spheres whose first Chern numbers sum to 0.

3.2. **Split pairs are regular enough (proof of Claim 1 of Section 2).** Let (M,ω) and (M',ω') be closed monotone symplectic manifolds (with the same monotonicity constant). A split pair on $M\times M'$ is a pair of the form $(H\oplus H', J\oplus J')$, as defined by (6), where (H,J) and (H',J') are respectively defined on (M,ω) and (M',ω').

If J and J' are compatible with the respective symplectic forms, $J\oplus J'$ is obviously $(\omega\oplus\omega')$–compatible.

Lemma 15. *If J and J' are regular, then $J\oplus J'$ is regular enough.*

The main observation is the decomposition of $\mathcal{M}^s_k(J\oplus J')$ in terms of pseudo-holomorphic spheres in M and M'. More precisely, a simple sphere $w=(v,v')$ of $M\times M'$ is a pair of

- two simple spheres,
- a simple sphere and a constant sphere,
- a simple sphere and a multiply covered sphere,
- two multiply covered spheres (with relatively prime degrees).

Notice that a pair consisting of a constant sphere and a multiply covered one is a multiply covered sphere (of the product).

Remark 16. Since the linearized operator defined by (10) respects the splitting $w^*T(M\times M') = v^*TM\times v'^*TM'$, it is onto at (t,w) if and only if both its projections are onto (at (t,v) and (t,v')). Thus a split almost complex structure is not *a priori* regular, since we have no information about the operator related to J (for instance) at (t,v) for multiply covered v.

Proof of Lemma 15. We denote by $\mathcal{M}^{m,i}_l(J)$ the set of pairs (t,v) where v is an i–fold covered, J_t–pseudo-holomorphic sphere in M with first Chern number l. Notice that (by convention) $\mathcal{M}^{m,1}_l(J)=\mathcal{M}^s_l(J)$, but $\mathcal{M}^m_l(J)$ denotes the union of these sets over $i>1$ (that is, $\mathcal{M}^m_l(J)$ is the set of pairs whose sphere is *strictly* multiply covered). We get, for any integer k,

$$\mathcal{M}^s_k(J\oplus J') \simeq \bigcup_{l+l'=k} \mathcal{M}^s_l(J)\times_{S^1}\mathcal{M}^s_{l'}(J') \tag{11}$$

$$\cup\ \mathcal{M}^s_k(J)\times M' \,\cup\, M\times\mathcal{M}^s_k(J') \tag{12}$$

$$\cup \bigcup_{l+l'=k} \mathcal{M}^s_l(J)\times_{S^1}\mathcal{M}^m_{l'}(J') \,\cup\, \mathcal{M}^m_l(J)\times_{S^1}\mathcal{M}^s_{l'}(J') \tag{13}$$

$$\cup \bigcup_{\substack{l+l'=k,\\ i,j\geq 2\\ \gcd(i,j)=1}} \mathcal{M}^{m,i}_l(J)\times_{S^1}\mathcal{M}^{m,j}_{l'}(J') \tag{14}$$

(we recall that the fibered products $\mathcal{M}(J) \times_{S^1} \mathcal{M}(J')$ appearing above consist of the elements $\big((t,v),(t',v')\big)$ of $\mathcal{M}(J) \times \mathcal{M}(J')$ for which $t = t'$). The union of (11), (13) and (14) can be described as

$$\bigcup_{\substack{l+l'=k,\\ i,j\geq 1 \mid \gcd(i,j)=1}} \mathcal{M}_l^{m,i}(J) \times_{S^1} \mathcal{M}_{l'}^{m,j}(J').$$

However, (11) and (12) are really different from (13) and (14) and should be studied separately.

Since J and J' are regular, there is no pseudo-holomorphic sphere with negative first Chern number (Claim 2). Thus, from the decomposition above, we can already conclude that Claim 2 holds for $J \oplus J'$. Moreover, for a non-empty set appearing in the decomposition, we have l and l' non-negative and furthermore if $k = 0$, then $l = l' = 0$. Let now look at small values of k.

Since a non constant pseudo-holomorphic sphere has positive symplectic area, such a sphere in a monotone symplectic manifold cannot have a vanishing first Chern number. Hence $\mathcal{M}_0^s(J \oplus J') = \emptyset$. Moreover, the first Chern number of a non constant pseudo-holomorphic *(strictly) multiply covered* sphere is at least 2. Thus,

$$\mathcal{M}_1^s(J \oplus J') = \mathcal{M}_1^s(J) \times M' \, \cup \, M \times \mathcal{M}_1^s(J')$$

which is a smooth manifold of the expected dimension $2(n+n')+3$ (see Claim 1). Finally for $k = 2$, in the decomposition above we have either $l = 1$ and $l' = 1$ (and there is no multiply covered pseudo-holomorphic sphere), or $l = 0$ or $l' = 0$ (and then at least one sphere has to be constant). Since a pair consisting of a constant sphere and a multiply covered one is not simple, we can conclude that

$$\mathcal{M}_2^s(J \oplus J') = \mathcal{M}_1^s(J) \times_{S^1} \mathcal{M}_1^s(J') \, \cup \, \mathcal{M}_2^s(J) \times M' \, \cup \, M \times \mathcal{M}_2^s(J')$$

which is the union of three smooth manifolds, of the expected dimension, $2(n+n')+5$ (see Claim 1). This proves that Claim 1 holds when $k = 0$, 1 and 2. □

Remark 17. Lemma 15 is the only place where we *a priori* have to restrict our study to monotone symplectic manifolds. The main problem appearing in the general (strongly semi-positive) case, comes from the existence of non-constant pseudo-holomorphic spheres with vanishing first Chern number. When such spheres exist, the moduli spaces $M_k^s(J \oplus J')$ (for $k = 0$, 1 and 2) are more complicated.

However, the additional subsets are products of smooth manifolds (moduli spaces of constant or simple pseudo-holomorphic spheres for regular almost complex structures) and of moduli spaces of strictly multiply covered spheres: $\mathcal{M}_k^{m,i}(J)$ and $\mathcal{M}_k^{m,j}(J')$ with i, $j > 1$ and $k = 0$ or 2. Such moduli spaces are formed of finitely many copies of sets, in bijection with smooth manifolds of the expected dimension (or of codimension at leat 2 in the whole union).

Indeed, for a i–fold covered sphere to represent the homotopy class $[A] \in \pi_2(M)$, there has to be a primitive homotopy class $[B]$, with $[A] = i[B]$. Thus, for each homotopy class $[A]$, there are only finitely many integers i for which $[A]$ admits a representative which is i–fold covered. (Since we consider non-trivial pseudo-holomorphic spheres, they represent non-zero homotopy classes).

Now, for each of these integers, the set $\mathcal{M}_k^{m,i}(J)$ is in bijection with i disjoint copies of $\mathcal{M}^s_{\frac{k}{i}}(J)$ (by considering the underlying simple curve, the canonical degree–i map of $\mathbb{CP}^1$: $[z_1 : z_2] \mapsto [z_1^i : z_2^i]$ and the action of each i–th root of unity ξ: $[z_1 : z_2] \mapsto [\xi z_1 : z_2]$). By regularity of J, $\mathcal{M}^s_{\frac{k}{i}}(J)$ is of dimension: $\dim(M) + 2\frac{k}{i}$, that is, the expected dimension when $k = 0$ and the expected dimension minus (at least) two when $k > 0$.

Lemma 18. *If the pairs (H, J) and (H', J') are regular, then $(H \oplus H', J \oplus J')$ is regular enough.*

Proof. Lemma 15 proves that for such pairs $J \oplus J'$ is regular enough. We now prove conditions **ii-iv** of Definition 13.
[**ii**.] – By definition of the split Hamiltonian,

$$\mathrm{Crit}(\mathcal{A}_{H\oplus H'}) = \mathrm{Crit}(\mathcal{A}_H) \times \mathrm{Crit}(\mathcal{A}_{H'}).$$

Thus, the critical points of $\mathcal{A}_{H\oplus H'}$ are non-degenerate.

For such a critical point $[v, v'; x, x']$, and some $t \in S^1$, $(t; x(t), x'(t)) \in V_1(J \oplus J')$ amounts to the existence of a non-constant $(J \oplus J')_t$–pseudo-holomorphic sphere $w\colon S^2 \to M \times M'$, with first Chern number less or equal to 1, and passing through $(x(t), x'(t))$. We recall that there is no pseudo-holomorphic sphere (in M and M', for J and J') with negative first Chern number. Thus, w is the product of pseudo-holomorphic spheres in M and M', one of them (let say the component in M) being non-constant, with first Chern number 0 or 1. Thus, $(t, x(t))$ lies in $V_1(J)$ which contradicts the fact that (H, J) satisfies **ii**. Hence, such a w does not exist and $(H \oplus H', J \oplus J')$ satisfies **ii**.
[**iii**.] – The moduli spaces defining the differential of the Floer complex of the product split, that is,

$$\mathcal{M}\big((c, c'), (d, d'); H \oplus H', J \oplus J'\big) = \mathcal{M}(c, d; H, J) \times \mathcal{M}(c', d'; H', J')$$

and thus so does the operator $\bar\partial_{J\oplus J', H\oplus H'}$. Thus the linearized operator is onto if and only if both operators (in each component of the product) are onto; since (H, J) and (H', J') satisfy **iii**, so does $(H \oplus H', J \oplus J')$.
[**iv**.] – In view of the decomposition of the moduli spaces above, if the index of $(u, u') \in \mathcal{M}(c, d; H, J) \times \mathcal{M}(c', d'; H', J')$ is less or equal than 2, then $\mathrm{ind}(u) \le 2$ and $\mathrm{ind}(u') \le 2$. Then we can conclude, as in the proof of **ii**, that the existence of some t such that $(t; u(s, t), u'(s, t)) \in V_0(J \oplus J')$ implies that $(t, u(s, t)) \in V_0(J)$ (and/)or $(t, u'(s, t)) \in V_0(J')$. Thus, since (H, J) and (H', J') satisfy **iv**, so does $(H \oplus H', J \oplus J')$. (Notice that, in the monotone case, the set of non-constant pseudo-holomorphic spheres of first Chern number 0 in the product is empty and thus that condition **iv** is trivially satisfied. The previous justification does not use the assumption of monotonicity.) □

Lemma 18 proves Claim 1 of §2 for pairs consisting of time-dependent Hamiltonians and almost complex structures. The same arguments can be carried out for homotopies.

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