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Abstract

We study the fully entangled fraction (FEF) of arbitrary mixed states. New upper bounds of FEF are derived. These upper bounds make complements on the estimation of the value of FEF. For weakly mixed quantum states, an upper bound is shown to be very tight to the exact value of FEF.

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Quantum entanglement plays crucial roles in quantum information processing such as quantum computation [1, 2], quantum teleportation [3, 4], dense coding [5], quantum cryptographic schemes [6], entanglement swapping [7] and remote states preparation (RSP) [8–10]. For instance in terms of a classical communication channel and a quantum resource (a nonlocal entangled state like an EPR-pair of particles), the teleportation protocol gives ways to transmit an unknown quantum state from a sender to a receiver that are spatially separated. When the sender and receiver share a maximally entangled pure state, the state can be perfectly teleported. However when the shared entangled state is an arbitrary mixed state ρ , then the optimal fidelity of teleportation is given by [4, 11],

$$f(\rho) = \frac{d\mathcal{F}(\rho)}{d+1} + \frac{1}{d+1}, \quad (1)$$

which solely depends on the fully entangled fraction (FEF) $\mathcal{F}(\rho)$ of ρ .

In fact the quantity FEF plays essential roles in many other quantum information processing such as dense coding, entanglement swapping and quantum cryptography (Bell inequalities). Thus it is very important to compute the FEF of general quantum states. Unfortunately, precise formula of FEF has been only obtained for two qubits systems [12]. For high dimensional systems, it becomes quite difficult to derive an analytic formula for FEF. In [13] we have derived an upper bound of FEF to give an estimation of the value

of FEF. In this paper, we derive more tight upper bounds for FEF. These bounds make complements on the estimation of FEF.

Let \mathcal{H} be a d -dimensional complex vector space with computational basis $|i\rangle$, $i = 1, \dots, d$. The fully entangled fraction of a density matrix $\rho \in \mathcal{H} \otimes \mathcal{H}$ is defined by

$$\mathcal{F}(\rho) = \max_{\phi \in \epsilon} \langle \phi | \rho | \phi \rangle, \quad (2)$$

where ϵ denotes the set of $d \times d$ -dimensional maximally entangled pure states. (2) can be also alternatively expressed as

$$\mathcal{F}(\rho) = \max_U \langle \psi_+ | (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle, \quad (3)$$

where the maximization is taken over all unitary transformations U , $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ is the maximally entangled state and I is the corresponding identity matrix.

Let h and g be $d \times d$ matrices such that $h|j\rangle = |(j+1) \bmod d\rangle$, $g|j\rangle = \omega^j|j\rangle$, with $\omega = \exp\{\frac{-2i\pi}{d}\}$. We can introduce d^2 linear-independent $d \times d$ -matrices $U_{st} = h^t g^s$, which satisfy

$$U_{st} U_{s't'} = \omega^{st' - ts'} U_{s't'} U_{st}, \quad \text{Tr}(U_{st}) = d \delta_{s0} \delta_{t0}. \quad (4)$$

$\{U_{st}\}$ also satisfy the condition for *bases of unitary operators* in the sense of [14], i.e.

$$\begin{cases} \text{tr}(U_{st} U_{s't'}^\dagger) = d \delta_{tt'} \delta_{ss'}, \\ U_{st} U_{st}^\dagger = I. \end{cases} \quad (5)$$

$\{U_{st}\}$ form a complete basis of $d \times d$ -matrices, namely, for any $d \times d$ matrix W , W can be expressed as

$$W = \frac{1}{d} \sum_{s,t} \text{tr}(U_{st}^\dagger W) U_{st}. \quad (6)$$

From $\{U_{st}\}$, we can introduce the generalized Bell-states,

$$|\Phi_{st}\rangle = (I \otimes U_{st}^*) |\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{i,j} (U_{st})_{ij}^* |ij\rangle, \quad \text{and} \quad |\Phi_{00}\rangle = |\psi_+\rangle, \quad (7)$$

$|\Phi_{st}\rangle$ are all maximally entangled states and form a complete orthogonal normalized basis of $\mathcal{H} \otimes \mathcal{H}$.

Theorem 1: For any quantum state $\rho \in \mathcal{H} \otimes \mathcal{H}$, the fully entangled fraction defined in (2) and (3) fulfills the following inequality:

$$\mathcal{F}(\rho) \leq \max_j \{\lambda_j\}, \quad (8)$$

where λ_j s are the eigenvalues of the real part of the matrix $M = \begin{pmatrix} T & iT \\ -iT & T \end{pmatrix}$, T is a $d^2 \times d^2$ matrix with entries $T_{n,m} = \langle \Phi_n | \rho | \Phi_m \rangle$ and Φ_j is the maximally entangled basis states defined in (7).

Proof: From (6), any $d \times d$ unitary matrix U can be represented as $U = \sum_{k=1}^{d^2} z_k U_k$, where $z_k = \frac{1}{d} \text{Tr}(U_k^\dagger U)$, U_k are the unitary matrices defined in (4). Define

$$x_l = \begin{cases} \text{Re}[z_l], 1 \leq l \leq d^2; \\ \text{Im}[z_l], d^2 < l \leq 2d^2 \end{cases} \quad \text{and} \quad U'_l = \begin{cases} U_l, 1 \leq l \leq d^2; \\ i * U_l, d^2 < l \leq 2d^2. \end{cases} \quad (9)$$

Then the unitary matrix U can be rewritten as $U = \sum_{k=1}^{2d^2} x_k U'_k$. The necessary unitary condition of U , $\text{Tr}(UU^\dagger) = d$, requires that $\sum_k x_k^2 = 1$. Set

$$F(\rho) \equiv \langle \psi_+ | (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle = \sum_{m,n=1}^{2d^2} x_m x_n M_{mn}, \quad (10)$$

where M_{mn} is the entry of the matrix M defined in theorem. From the hermiticity of ρ it is easily verified that

$$M_{mn}^* = M_{nm}. \quad (11)$$

To maximize $F(\rho)$ under constraints we get the following equation

$$\frac{\partial}{\partial x_k} \{ F(\rho) + \lambda (\sum_l x_l^2 - 1) \} = 0. \quad (12)$$

Taking into account (11) we obtain an eigenvalue equation,

$$\sum_{n=1}^{2d^2} \text{Re}[M_{k,n}] x_n = -\lambda x_k. \quad (13)$$

Therefore

$$\mathcal{F}(\rho) = \max_U F(\rho) \leq \max_j \{\eta_j\}, \quad (14)$$

where $\eta_j = -\lambda_j$ is the corresponding eigenvalues of the real part of matrix M . \square

The upper bound derived in [13] says that for any $\rho \in \mathcal{H} \otimes \mathcal{H}$, the fully entangled fraction $\mathcal{F}(\rho)$ satisfies

$$\mathcal{F}(\rho) \leq \frac{1}{d^2} + 4 \|N^T(\rho) N(P_+)\|_{KF}, \quad (15)$$

where $N(\rho)$ denotes the correlation matrix with entries $n_{ij}(\rho)$ given in the expression of ρ

$$\rho = \frac{1}{d^2} I \otimes I + \frac{1}{d} \sum_{i=1}^{d^2-1} r_i(\rho) \lambda_i \otimes I + \frac{1}{d} \sum_{j=1}^{d^2-1} s_j(\rho) I \otimes \lambda_j + \sum_{i,j=1}^{d^2-1} n_{ij}(\rho) \lambda_i \otimes \lambda_j, \quad (16)$$

$\lambda_i, i = 1, \dots, d^2 - 1$, are the generators of the $SU(d)$ algebra with $Tr\{\lambda_i\lambda_j\} = 2\delta_{ij}$, $r_i(\rho) = \frac{1}{2}Tr\{\rho\lambda_i(1) \otimes I\}$, $s_j(\rho) = \frac{1}{2}Tr\{\rho I \otimes \lambda_j(2)\}$, $n_{ij}(\rho) = \frac{1}{4}Tr\{\rho\lambda_i(1) \otimes \lambda_j(2)\}$, P_+ stands for the projection operator to $|\psi_+\rangle$, $N(P_+)$ is similarly defined to $N(\rho)$, N^T stands for the transpose of N , $\|N\|_{KF} = Tr\sqrt{NN^\dagger}$ is the Ky Fan norm of N . This upper bound was used to improve the distillation protocol proposed in [15]. Here we show that the upper bound in (8) is different from that in (15) by an example.

Example 1: We consider the bound entangled state [16]

$$\rho(a) = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix}. \quad (17)$$

From Fig. 1 we see that for $0 \leq a < 0.572$, the upper bound in (8) is larger than that in (15). But for $0.572 < a < 1$ the upper bound in (8) is always lower than that in (15), i.e. the upper bound (8) is tighter than (15) in this case.

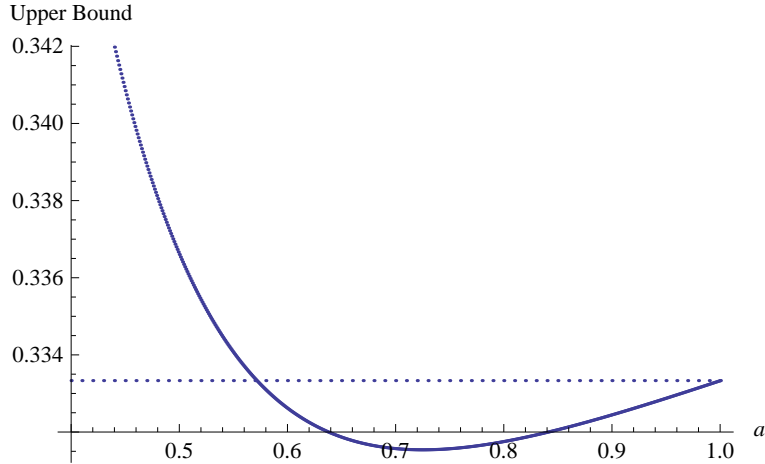


FIG. 1: Upper bound of $\mathcal{F}(\rho)$ from (8) (solid line) and upper bound from (15) (dashed line).

By using the operator norm, we have further

Theorem 2: For any $\rho \in \mathcal{H} \otimes \mathcal{H}$, the fully entangled fraction $\mathcal{F}(\rho)$ satisfies

$$\mathcal{F}(\rho) \leq \max_i \{\lambda_i\} \quad (18)$$

where λ_i s are the eigenvalues of ρ .

Proof: For any quantum state ρ and unitary U , we have

$$\begin{aligned} \langle \psi_+ | (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle &\leq \| \langle \psi_+ | \cdot \| \| (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle \| \\ &\leq \| (I \otimes U^\dagger) \rho (I \otimes U) \| \| | \psi_+ \rangle \|^2 = \| \rho \| \| | \psi_+ \rangle \|^2 = \| \rho \|, \end{aligned} \quad (19)$$

where $\| \rho \|$ stands for the operator norm, $\| \rho \| = \sup(\| \rho |x\rangle \| : \| |x\rangle \| = 1)$, $\| |x\rangle \| = \sqrt{\langle x|x \rangle}$. We have used the Cauchy-Schwarz inequality to obtain the first inequality. The second inequality is due to the basic property of operator norm. The followed equality follows from the fact that unitary transformation does not change the operator norm.

From [17] $\| \rho \|$ is an eigenvalue of ρ , actually, it is the maximal eigenvalue of ρ , i.e. $\| \rho \| = \max_i \{\lambda_i\}$ where λ_i s are the eigenvalues of ρ , which ends the proof. \square

This bound can give rise to further

Corollary: Let $|\psi\rangle = \sum_{ij} a_{ij} |ij\rangle$ with $\| |\psi\rangle \| = 1$ be the normalized eigenvector of ρ with respect to the maximal eigenvalue λ_{max} . If the matrix A with elements $A_{ij} = \sqrt{d} a_{ij}$ are unitary, the upper bound derived in (18) becomes the exact value of FEF.

Proof: A simple computation shows that

$$\begin{aligned} \mathcal{F}(\rho) &\leq \lambda_{max} = \langle \psi | \rho | \psi \rangle = \frac{1}{d} \sum_{ij,kl} \sqrt{d} a_{ij}^* \langle ij | \rho \sqrt{d} a_{kl} | kl \rangle \\ &= \sum_{i,k} \frac{1}{d} \langle ii | I \otimes A^\dagger \rho I \otimes A | kk \rangle = \langle \psi_+ | I \otimes A^\dagger \rho I \otimes A | \psi_+ \rangle \leq \mathcal{F}(\rho). \end{aligned}$$

Thus we have $\mathcal{F}(\rho) = \lambda_{max}$. \square

According to the corollary, we can find out when the upper bound derived in theorem 2 becomes the exact value of FEF.

Example 2: Consider the $3 \otimes 3$ state [18]

$$\rho = \frac{2}{7} | \psi_+ \rangle \langle \psi_+ | + \frac{\alpha}{7} \sigma_+ + \frac{5-\alpha}{7} \sigma_-, \quad (20)$$

where $\sigma_+ = \frac{1}{3}(|01\rangle\langle 01| + |12\rangle\langle 12| + |20\rangle\langle 20|)$, $\sigma_- = \frac{1}{3}(|10\rangle\langle 10| + |21\rangle\langle 21| + |02\rangle\langle 02|)$. ρ is entangled when $3 < \alpha \leq 5$. The maximal eigenvalue of ρ is $\frac{2}{7}$, with the corresponding normalized eigenvector $\{ \frac{1}{\sqrt{3}}, 0, 0, 0, \frac{1}{\sqrt{3}}, 0, 0, 0, \frac{1}{\sqrt{3}} \}$. The matrix A related to this eigenvector is just the 3×3 identity matrix which is obviously unitary. Thus we have for the state (20), $\mathcal{F}(\rho) = \frac{2}{7}$.

The following upper bound of FEF gives a very tight estimation of FEF for weakly mixed quantum states.

Theorem 3: For any bipartite quantum state $\rho \in \mathcal{H} \otimes \mathcal{H}$, the following inequality holds:

$$\mathcal{F}(\rho) \leq \frac{1}{d} (\text{Tr} \sqrt{\rho_A})^2, \quad (21)$$

where ρ_A is the reduced matrix of ρ .

Proof: Note that the FEF for pure state $|\psi\rangle$ is given by [15]

$$\mathcal{F}(|\psi\rangle) = \frac{1}{d} (\text{Tr} \sqrt{\rho_A^{|\psi\rangle}})^2, \quad (22)$$

where $\rho_A^{|\psi\rangle}$ is the reduced matrix of $|\psi\rangle\langle\psi|$.

For mixed state $\rho = \sum_i p_i \rho^i$, we have

$$\begin{aligned} \mathcal{F}(\rho) &= \max_U \langle \psi_+ | (I \otimes U^\dagger) \rho (I \otimes U) | \psi_+ \rangle \\ &\leq \sum_i p_i \max_U \langle \psi_+ | (I \otimes U^\dagger) \rho^i (I \otimes U) | \psi_+ \rangle \\ &= \frac{1}{d} \sum_i p_i (\text{Tr} \sqrt{\rho_A^i})^2 = \frac{1}{d} \sum_i (\text{Tr} \sqrt{p_i \rho_A^i})^2. \end{aligned} \quad (23)$$

Let λ_{ij} be the real and nonnegative eigenvalues of the matrix $p_i \rho_A^i$. Recall that for any function $F = \sum_i (\sum_j x_{ij}^2)^{\frac{1}{2}}$ subjected to the constraints $z_j = \sum_i x_{ij}$ with x_{ij} being real and nonnegative, the inequality $\sum_j z_j^2 \leq F^2$ holds. It follows that

$$\mathcal{F}(\rho) \leq \frac{1}{d} \sum_i \left(\sum_j \sqrt{\lambda_{ij}} \right)^2 \leq \frac{1}{d} \left(\sum_j \sqrt{\sum_i \lambda_{ij}} \right)^2 = \frac{1}{d} (\text{Tr} \sqrt{\rho_A})^2, \quad (24)$$

which ends the proof. \square

We now give an example to show that when the quantum state is weakly mixed, theorem 3 will be a very good estimation for the FEF.

Example 3: Consider the following $3 \otimes 3$ mixed state: $\rho = \frac{1-p}{9} I_9 + p |\psi\rangle\langle\psi|$, where $|\psi\rangle = \sqrt{\frac{x^2+2}{3}} \{x, 0, 0, 0, 1, 0, 0, 0, 1\}$ is a pure state with one parameter x . To show the effectiveness of (21), we compare it with the single fraction of entanglement $F_s = \langle \psi_+ | \rho | \psi_+ \rangle$. As seen from the Fig. 2, for weakly mixed states (with larger parameter p), the bound provides excellent estimation of the FEF.

We have studied the fully entangled fraction that has tight relations with many quantum information processing. New upper bounds for FEF have been derived. They make complements on estimation of the value of FEF. The conditions for the bounds to be exact or to be more tight have been analyzed. These bounds provide a better estimation of FEF and can be use in related information processing, e.g. to detect the entanglement of the non-local source used in quantum teleportation.

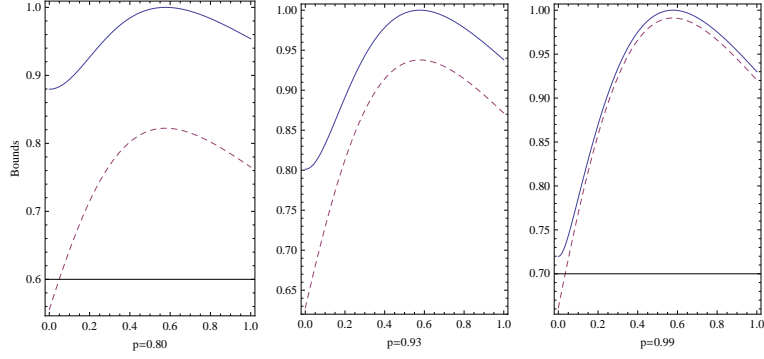


FIG. 2: Upper bound of $\mathcal{F}(\rho)$ from (21) (solid line) and single fraction of entanglement F_s

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