Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Non-homogeneous Navier-Stokes systems with order-parameter dependent stresses

by

Helmut Abels, and Yutaka Terasawa

Preprint no.: 19 2009

Non-homogeneous Navier-Stokes systems with order-parameter dependent stresses

Helmut Abels[*], Yutaka Terasawa[†]

March 30, 2009

Abstract

We consider the Navier-Stokes system with variable density and variable viscosity coupled to a transport equation for an order parameter c. Moreover, an extra stress depending on c and ∇c, which describes surface tension like effects, is included in the Navier-Stokes system. Such a system arises e.g. for certain models for granular flows and as a diffuse interface model for a two-phase flow of viscous incompressible fluids. The so-called density-dependent Navier-Stokes system is also a special case of our system. We prove short-time existence of strong solution in L^q-Sobolev spaces with $q > d$. We consider the case of a bounded domain and an asymptotically flat layer with combination of a Dirichlet boundary condition and a free surface boundary condition. The result is based on a maximal regularity result for the linearized system.

Key words: Navier Stokes equations, free boundary value problems, maximal regularity, diffuse interface models, granular flows, non-stationary Stokes system
AMS-Classification: 76D05, 35Q30, 35R35, 76T99, 76D27, 76D45

1 Introduction and Main Results

We consider the following Navier-Stokes system with variable density and variable viscosity coupled to a transport equation for an order parameter c.

$$\varrho(c)(\partial_t v + v\cdot\nabla v) - \operatorname{div}(2\nu(c)Dv) + \nabla\tilde{q} = -\operatorname{div} F(c,\nabla c) \quad \text{for } x\in\Omega(t), \tag{1.1}$$

$$\operatorname{div} v = 0 \quad \text{for } x\in\Omega(t), \tag{1.2}$$

$$\partial_t c + v\cdot\nabla c = 0 \quad \text{for } x\in\Omega(t), \tag{1.3}$$

$$v|_{\Gamma_1} = 0 \quad \text{for } x\in\Gamma_1, \tag{1.4}$$

$$n\cdot T(c,v,\tilde{q})|_{\Gamma_{2,t}} = n\cdot F(c,\nabla c) \quad \text{for } x\in\Gamma_{2,t}, \tag{1.5}$$

$$v|_{t=0} = v_0 \quad \text{for } x\in\Omega_0 \tag{1.6}$$

[*]Max Planck Institute for Mathematics in Science, Inselstr. 22, 04103 Leipzig, Germany, e-mail: abels@mis.mpg.de

[†]Mathematical Institute, Tôhoku University, 980-8758 Sendai, Japan

for $t \in (0,T)$ and $\Omega(0) = \Omega_0$. Here v is the velocity of the fluid, p is the pressure, n is the exterior normal, and

$$T(c, v, \tilde{q}) = 2\nu(c)Dv - \tilde{q}I$$

is the usual stress tensor for Newtonian, incompressible fluids in the case of a variable viscosity $\nu(c) > 0$, where $Dv = \frac{1}{2}(\nabla v + \nabla v^T)$. The density $\varrho > 0$ depends explicitly and sufficiently smooth on the order parameter c. Moreover, $F(c, \nabla c)$ is an extra stress tensor describing surface tension like forces related to variations of the order parameter. Finally, we assume that $\partial\Omega(t) = \Gamma_1 \cup \Gamma_{2,t}$ for all $t \in [0,T]$, where $\Gamma_1, \Gamma_{2,t}$ are disjoint, closed (possibly empty), and sufficiently smooth surfaces. $\Gamma_{2,t}$ and Γ_1 describe the free boundary of the domain $\Omega(t)$ and the fixed part of the boundary, respectively. The motion of the free boundary is determined by the usual kinematic relation

$$V_{\Gamma_2(t)} = n \cdot v|_{\Gamma_2(t)}, \tag{1.7}$$

where $V_{\Gamma_2(t)}$ is the normal velocity of $\Gamma_2(t)$. Surface tension effects at the free boundary $\Gamma_2(t)$ (in relation to the exterior gas or vacuum) and exterior forces are neglected.

In applications F can be e.g.

$$F(c, \nabla c) = a(c)|\nabla c|^\alpha \nabla c \otimes \nabla c$$

for some $\alpha \geq 0$ and $a \in C^2(\mathbb{R})$. Then $F(c, \nabla c)$ describes capillary stresses related to a free energy of the form

$$E(c) = \int_\Omega \frac{a(c)|\nabla c|^{2+\alpha}}{2+\alpha}\, dx.$$

(Of course more general version of a free energy depending on c and ∇c and a corresponding extra stress can be treated too.) The latter form includes the case that the free energy density depends on $\nabla\varrho$ instead of ∇c since $\nabla\varrho = \varrho'(c)\nabla c$. In particular, if $\alpha = 0$ and c describes the concentration of two partly mixing incompressible viscous fluids, we recover a well-known diffuse interface model in the case when diffusion effects are neglected, cf. Gurtin et al. [10]. Note that in this case the system arises from the systems studied e.g. in [1, 2] if one chooses the mobility coefficient $m = 0$. Moreover, if $c = \varrho$ describes the density of the fluid and $\alpha = 0$, the system describes a continuum model for the motion of granular material as e.g. sand or powder, cf. Málek and Rajagopak [12]. The latter system was studied by Nakano and Tani [13] before, where short time existence of strong solutions in anisotropic L^2-Sobolev spaces in the case of Dirichlet boundary condition and a bounded domain was proved.

The purpose of the paper is to obtain local existence of strong solutions for the system above in anisotropic L^q-Sobolev space for a general F in bounded domains and asymptotically flat layers. The result is proved by transformation to Lagrangian coordinates, where both the evolution of the free boundary $\Gamma_{2,t}$ and the transport equation for c can be solved explicitly for given velocity v.

More precisely, let $X(\xi,t)$, $t>0$, be the trajectory of the mass particle, i.e., $X(\xi,t)$ solves

$$\partial_t X(\xi,t) = v(X(\xi,t),t), \qquad X(\xi,0)=\xi, \qquad \text{for } t\in(0,T), \xi\in\Omega_0.$$

Then (1.7) implies $X(\Gamma_{2,0},t)=\Gamma_{2,t}$ for $t\in(0,T)$, and (1.3) implies

$$c(X(\xi,t),t) = c_0(\xi) \quad \text{for all } t\in(0,T), \xi\in\Omega$$

since

$$\partial_t c(X(\xi,t),t) = (\partial_t c)(X(\xi,t),t) + (v\cdot\nabla c)(X(\xi,t),t) = 0.$$

Moreover, let $u(\xi,t)=v(X(\xi,t),t)$, $p(\xi,t)=\tilde{q}(X(\xi,t),t)$ be the velocity and the pressure of the fluid in Lagrangian coordinates. Then

$$X(\xi,t) = X_u(\xi,t) := \xi + \int_0^t u(\xi,\tau)d\tau$$

and the system (1.1)-(1.6) is transformed to

$$\begin{aligned}
\varrho(c_0)\partial_t u - \operatorname{div}_u(2\nu(c_0)D_u u) + \nabla_u p &= -\operatorname{div}_u F(c_0,\nabla_u c_0), && \text{in } Q_T, && (1.8)\\
\operatorname{div}_u u &= 0, && \text{in } Q_T, && (1.9)\\
u|_{\Gamma_1} &= 0, && \text{on } \Gamma_1\times(0,T), && (1.10)\\
n_u\cdot T_u(c_0,u,p)|_{\Gamma_2} &= n_u\cdot F(c_0,\nabla_u c_0) && \text{on } \Gamma_2\times(0,T), && (1.11)\\
u|_{t=0} &= u_0 && \text{in } \Omega_0 && (1.12)
\end{aligned}$$

where $\Omega\equiv\Omega_0=\Omega(0)$, $Q=\Omega\times(0,T)$, $\Gamma_2=\Gamma_{2,0}$, and

$$X_u(t,\xi) = \xi + \int_0^t u(\xi,\tau)d\tau. \tag{1.13}$$

Here

$$\nabla_u = A(u)\nabla, \qquad \operatorname{div}_u v = \nabla_u\cdot v = \operatorname{Tr}(A(u)\nabla v),$$

$$T_u(u,p) = 2\nu(c_0)D_u u - pI, \quad D_u v = \frac{1}{2}\left(\nabla_u v + (\nabla_u v)^T\right), \quad n_u(\xi,t) = \frac{A(u)n_\xi}{|A(u)n_\xi|},$$

where $A(u)=(D_\xi X_u)^{-T}(\xi,t)$ and n_ξ denotes the exterior normal at $\xi\in\partial\Omega$.

The main result of the paper is the following:

THEOREM 1.1 *Let $d\geq 2$, $d<q<\infty$, $q\neq 3$, and let Ω be a bounded domain with $W_q^{2-\frac{1}{q}}$-boundary or let $\Omega=\Omega_\gamma$ be an asymptotically flat layer with $W_q^{2-\frac{1}{q}}$-boundary, cf. Assumption 1.2 below. Moreover, we assume that $\nu,\varrho\colon\mathbb{R}\to\mathbb{R}$ are C^1-functions, $F\colon\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ is a C^2-function with $F(s,0)=0$ for all $s\in\mathbb{R}$, $c_0\in W_q^2(\Omega)$ and that $\nu(c_0)$ and $\varrho(c_0)$ are bounded from above and below by some positive constant. Then for every $u_0\in W_q^{2-\frac{2}{q}}(\Omega)^d$ with $\operatorname{div} u_0=0$, $u_0|_{\Gamma_1}=0$, and $(n2\nu(c_0)\cdot Du_0)_\tau|_{\Gamma_2}=0$ if $q>3$, there is some $T>0$ such that (1.8)-(1.13) have a unique solution $(u,p)\in W_q^{2,1}(Q_T)^d\times W_q^{1,0}(Q_T)$ with $p|_{\Gamma_2}\in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,T))$.*

Here and in the following we denote by f_τ and f_n the tangential and normal components, respectively, of a vector field f.

The advantage of the L^q-theory in comparison with the usual result based on L^2-Sobolev spaces is that less smoothness of the data is needed. We note that we also reduced the assumption on the smoothness of $\partial\Omega$ in comparison to the earlier work A. [3] in the case of an asymptotically flat layer with $\partial\Omega \in C^{1,1} \cap W_q^{2-\frac{1}{q}}$, constant viscosity and density, and $F \equiv 0$, and in comparison with Beale [6], where the corresponding L^2-theory was treated. Moreover, it improves and extends the result on the density-dependent Navier-Stokes equation in a bounded domain by Danchin [8], where $\partial\Omega \in C^{2+\varepsilon}$ and Dirichlet boundary conditions are assumed.

The proof is done with the aid of the Banach contraction-mapping principle using the unique solvability of the linearized system, i.e., the non-stationary Stokes system with variable viscosity:

$$\partial_t v - \operatorname{div}(2\nu(x)Dv) + \nabla p = f \qquad \text{in } \Omega \times (0,T), \tag{1.14}$$
$$\operatorname{div} v = g \qquad \text{in } \Omega \times (0,T), \tag{1.15}$$
$$v|_{\Gamma_1} = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{1.16}$$
$$n \cdot T(v,p)|_{\Gamma_2} = a \qquad \text{on } \Gamma_2 \times (0,T), \tag{1.17}$$
$$v|_{t=0} = v_0 \qquad \text{on } \Omega \tag{1.18}$$

where $v\colon \Omega \times (0,T) \to \mathbb{R}^d$ is the velocity of the fluid, $p\colon \Omega \times (0,T) \to \mathbb{R}$ is the pressure,

$$T(v,p) = 2\nu(x)Dv - pI,$$

$\nu\colon \Omega \to (0,\infty)$ is a variable given viscosity coefficient (independent of t), and $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is a suitable domain with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consisting of two closed, disjoint (possibly empty) components Γ_j, $j = 1,2$. More precisely, we assume the following:

Assumption 1.2 *Let Ω be a bounded domain with $W_r^{2-\frac{1}{r}}$-boundary for some $d < r \leq \infty$ or let $\Omega = \Omega_\gamma$ be an asymptotically flat layer with $W_r^{2-\frac{1}{r}}$-boundary, i.e.,*

$$\Omega_\gamma = \left\{x \in \mathbb{R}^d : a + \gamma_-(x') < x_d < b + \gamma_+(x')\right\},$$

where $x = (x', x_d)$, $a < b$, and $\gamma_\pm \in W_r^{2-\frac{1}{r}}(\mathbb{R}^{d-1})$ such that $\gamma_+(x') - \gamma_-(x') + b - a \geq \kappa > 0$ for all $x' \in \mathbb{R}^{d-1}$, $\lim_{|x'|\to\infty} \gamma_\pm(x') = 0$, and $\lim_{|x'|\to\infty} \nabla\gamma_\pm(x') = 0$ if $r = \infty$.

Moreover, let $\partial\Omega = \Gamma_1 \cup \Gamma_2$ consist of two closed, disjoint (possibly empty) components Γ_j, $j = 1,2$. In the case of an asymptotically flat layer, we will assume that $\Gamma_1 = \{(x', \gamma_-(x') : x' \in \mathbb{R}^{d-1}\}$ and $\Gamma_2 = \{(x', \gamma_+(x') : x' \in \mathbb{R}^{d-1}\}$.

The maximal regularity result we prove and apply is the following.

THEOREM 1.3 *Let $0 < T < \infty$, let Ω be as in Assumption 1.2, and let $\frac{3}{2} < q < \infty$ with $\max(q,q') \leq r$, $q \neq 3$. Moreover, assume that $\nu \in W^1_{r_1}(\Omega)$ for some $d < r_1 \leq$*

∞ *such that* $\max(q,q') \le r_1$ *and* $\nu(x) \ge \nu_0 > 0$. *Then for every* $f \in L^q(Q_T)^d$, $g \in W^{1,0}_q(Q_T)$ *with* $\partial_t g \in L^q(0,T;W^{-1}_{q,\Gamma_2}(\Omega))$, $g|_{\Gamma_2} \in W_q^{1-\frac1q,\frac12(1-\frac1q)}(\Gamma_2\times(0,T))$, $a \in W_q^{1-\frac1q,\frac12(1-\frac1q)}(\Gamma_2\times(0,T))^d$, *and* $v_0 \in W_q^{2-\frac2q}(\Omega)^d$ *satisfying the compatibility condition*

$$\operatorname{div} v_0 = g|_{t=0} \text{ in } W^{-1}_{q,\Gamma_2}(\Omega),\ \ v_0|_{\Gamma_1}=0,\ \ (n\cdot 2\nu Dv_0)_\tau|_{\Gamma_2} = a_\tau|_{t=0} \text{ if } q>3.$$

there is a unique solution $(v,p) \in W^{2,1}_q(Q_T)^d \times W^{1,0}_q(Q_T)$ *of (1.14)-(1.18). Moreover,*

$$\begin{aligned}&\|v\|_{W^{2,1}_q} + \|\nabla p\|_{L^q} + \|p|_{\Gamma_2}\|_{W_q^{1-\frac1q,\frac12(1-\frac1q)}} \\ &\le\ C\left(\|(f,\nabla g)\|_{L^q} + \|\partial_t g\|_{-1,0,q} + \|(g|_{\Gamma_2},a)\|_{W_q^{1-\frac1q,\frac12(1-\frac1q)}} + \|v_0\|_{W_q^{2-\frac2q}(\Omega)}\right),\end{aligned}\tag{1.19}$$

where $\|.\|_{-1,0,q} := \|.\|_{L^q(0,T;W^{-1}_{q,\Gamma_2})}$. *The constant* C *can be chosen independently of* $T\in(0,T_0]$ *for any fixed* $0<T_0<\infty$.

Precise definitions of the function spaces are given in Section 2 below.

The proof of the latter theorem is based on a recent result on the existence of a bounded H_∞-calculus for an associated (reduced) Stokes operator by the authors, cf. [5].

We note that first results on general non-stationary Stokes systems, including the case of variable viscosity, were obtained by Solonnikov [14, 15] in L^q-Sobolev spaces and weighted Hölder spaces in the case of a bounded domain with pure Dirichlet boundary conditions and $g=0$. Moreover, Bothe and Prüß [7] obtained unique solvability of general non-stationary Stokes systems in L^q-Sobolev spaces for the case of bounded and exterior domains with Dirichlet, Neumann, and Navier boundary conditions. Finally, we note that Ladyženskaja and Solonnikov [11] and later Danchin [8] obtained results for a similar non-stationary Stokes system with variable density instead of variable viscosity.

The structure of the article is as follows: In Section 2, we prepare some preliminary results. In Section 3, the unique solvability of the linear system is established, which is Theorem 1.3. Finally, in Section 4, the results of the linear theory are applied and Theorem 1.1 is proved.

2 Preliminaries and Notation

For $s\in\mathbb{R}$ we denote by $[s]$ the largest integer $\le s$ and set $\{s\} := s-[s] \in [0,1)$.

If $M\subseteq\mathbb{R}^d$ is measurable, $L^q(M)$, $1\le q\le\infty$ denotes the usual Lebesgue-space and $\|.\|_q$ its norm. Moreover, $L^q(M;X)$ denotes its vector-valued variant, where X is a Banach space. If $f\in L^q(M)$, $g\in L^{q'}(M)$, where $\frac1q+\frac1{q'}=1$, then

$$(f,g)_M := \int_M f(x)\overline{g(x)}dx.$$

If X is a Banach space and X' is its dual, then

$$\langle f, g\rangle \equiv \langle f, g\rangle_{X',X} = f(g), \qquad f \in X', g \in X,$$

denotes the duality product. Moreover, $C^\theta([0,T];X)$, $\theta \in (0,1)$, $T > 0$, denotes the space of Hölder continuous functions $f\colon [0,T] \to X$. Furthermore, $C^{k,1}(\overline{\Omega})$ denotes the space of k-times differentiable functions $f\colon \overline{\Omega} \to \mathbb{R}$ with uniformly Lipschitz continuous k-th derivatives, where $k \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^d$ is a domain.

Let $\Omega \subset \mathbb{R}^d$ be a domain. In the following $W^s_q(\Omega)$, $s \geq 0$, $1 \leq q < \infty$, denotes the usual Sobolev-Slobodeckij space normed by

$$\|u\|^q_{s,q} = \sum_{|\alpha|\leq s} \|D^\alpha u\|^q_q \qquad \text{if } s \in \mathbb{N}_0,$$

$$\|u\|^q_{s,q} = \sum_{|\alpha|\leq [s]} \|D^\alpha u\|^q_q + \sum_{|\alpha|=[s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^q}{|x-y|^{n+q\{s\}}}\, dx\, dy \quad \text{if } s \notin \mathbb{N}_0.$$

Moreover, $W^s_q(\Omega;X)$ denotes its vector-valued variant, where X is a Banach space. Finally, $W^s_q(\partial\Omega)$ is defined in the same way as above with the Lebesgue measure replaced by the surface measure.

We note that Assumption 1.2 on the domain Ω implies the Assumption 1 in [5], see A. [4] for the details. In particular, since Ω has a C^1-boundary due to $r > d$, the usual Sobolev embedding theorem holds for $W^1_q(\Omega)$. Hence $W^1_q(\Omega) \hookrightarrow L^\infty(\Omega)$ for all $d < q < \infty$ and we have the following fundamental lemma:

Lemma 2.1 *Let $1 < q < \infty$ and $d < p \leq \infty$ such that $q \leq p$ and let Ω be a domain as in the Assumption 1.2. Then $\pi(f,g)(x) := f(x)g(x)$ defines a continuous, bilinear mapping $\pi\colon W^1_q(\Omega) \times W^1_p(\Omega) \to W^1_q(\Omega)$.*

The anisotropic Sobolev-Slobodeckij space is defined as

$$W^{2s,s}_q(Q_T) = L^q(0,T;W^{2s}_q(\Omega)) \cap W^s_q(0,T;L^q(\Omega)), \ s \geq 0$$

normed by

$$\|u\|^q_{2s,s,q} = \|u\|^q_{L^q(0,T;W^{2s}_q(\Omega))} + \|u\|^q_{W^s_q(0,T;L^q(\Omega))}.$$

Moreover, we define $W^{m,0}_q(Q_T) = L^q(0,T;W^m_q(\Omega))$, $m \in \mathbb{N}$, and denote by $\|.\|_{m,0,q}$ the corresponding norm.

The following lemma will be used to reduce to zero boundary and initial values.

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, $d < r \leq \infty$ be as in Assumption 1.2, let $\frac{3}{2} < q < \infty$, $q \neq 3$, and let $0 < T \leq \infty$. Moreover, let $\nu \in W^1_{r_1}(\Omega)$ with $\nu(x) \geq \nu_0 > 0$ for some $d < r_1 \leq \infty$ such that $r_1 \geq q$. Then:*

1. *For every $u_0 \in W^{2-\frac{2}{q}}_q(\Omega)$ with $u_0|_{\Gamma_1} = 0$ there is some $u \in W^{2,1}_q(Q_T)$ with $u|_{t=0} = u_0$, $u|_{\Gamma_1\times(0,T)} = 0$. Moreover, there is some $C > 0$ independent of $T \in (0,\infty]$ such that*

$$\|u\|_{W^{2,1}_q(Q_T)} \leq C\|u_0\|_{W^{2-\frac{2}{q}}_q(\Omega)}.$$

2. *For every* $a \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))^d$ *with* $a|_{t=0} = 0$ *if* $q > 3$ *there is some* $A \in W_q^{2,1}(Q_T)^d$ *with* $A|_{t=0} = 0$, $A|_{\Gamma_1} = 0$, *and*

$$(n \cdot 2\nu DA)_\tau|_{\Gamma_2} = a_\tau, \qquad \operatorname{div} A|_{\Gamma_2} = a_n.$$

Moreover,

$$\|A\|_{W_q^{2,1}(Q_T)} \leq C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$$

where C *can be chosen independently of* $T \in (0,\infty]$.

Proof: With the aid of the coordinate transformations due to [5, Proposition 1] and a suitable partition of unity the first statement can be easily reduced to the case of a half-space $\mathbb{R}^d_+$, which is well-known, cf. e.g. Grubb [9, Appendix].

In order to prove 2., let $A \in W_q^{2,1}(Q_T)^d$ with $A|_{t=0} = A|_{\partial\Omega} = 0$, and $\partial_n A|_{\Gamma_2} = \nu^{-1}a$ such that $\|A\|_{2,1,q} \leq C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$. – As before, the existence of A can be reduced to the corresponding statement in $\mathbb{R}^d_+$. – Then

$$\begin{aligned}(n \cdot 2\nu DA)_\tau|_{\Gamma_2} &= (\nu\nabla_\tau A_n + \nu\partial_n A_\tau)|_{\Gamma_2} = 0 + a_\tau,\\ \operatorname{div} A|_{\Gamma_2} &= (\operatorname{div}_\tau A_\tau + \partial_n A_n)|_{\Gamma_2} = 0 + a_n.\end{aligned}$$

The constant C can be chosen independently of T since we can extend a to $\tilde{a} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,\infty))^d$ such that $\|\tilde{a}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q} \leq C\|a\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}$, where C does not depend on T, and restrict the corresponding $\widetilde{A} \in W_q^{2,1}(\Omega \times (0,\infty))^d$ to $(0,T)$ afterwards. The latter extension to $(0,\infty)$ can be done by first extending a in an even way around $t = T$ to a function defined on $(0,2T)$ and then extending by zero, which yields an $\tilde{a} \in W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,\infty))^d$ since $\tilde{a}|_{t=2T} = a|_{t=0} = 0$ if $q > 3$. ■

For the following we denote

$$\begin{aligned}W^1_{q,\Gamma_2}(\Omega) &= \begin{cases}\{u \in W^1_q(\Omega) : u|_{\Gamma_2} = 0\} & \text{if } \Gamma_2 \neq \emptyset,\\ \{u \in W^1_q(\Omega) : \int_\Omega u\,dx = 0\} & \text{if } \Gamma_2 = \emptyset,\end{cases}\\ W^{-1}_{q,\Gamma_2}(\Omega) &= \begin{cases}\left(W^1_{q',\Gamma_2}(\Omega)\right)'. & \text{if } \Gamma_2 \neq \emptyset,\\ \{f \in W^{-1}_{q,0}(\Omega) := (W^1_{q'}(\Omega))' : \langle f, 1\rangle = 0\} & \text{if } \Gamma_2 = \emptyset.\end{cases}\end{aligned}$$

Here $W^1_{q,\Gamma_2}(\Omega)$ is equipped with the norm $\|\nabla \cdot\|_{L^q(\Omega)}$. Moreover, we note that Ω is a bounded domain if $\Gamma_2 = \emptyset$.

Lemma 2.3 *Let* Ω *be as in Assumption 1.2 and let* $1 < q < \infty$. *Then for every* $F \in W^{-1}_{q,\Gamma_2}(\Omega)$ *and* $a \in W_q^{1-\frac{1}{q}}(\Gamma_2)$ *there is a unique* $p \in W^1_{q,\Gamma_2}(\Omega)$ *such that*

$$(\nabla p, \nabla\varphi)_\Omega = \langle F, \varphi\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q,\Gamma_2}} \quad \text{for all } \varphi \in W^1_{q',\Gamma_2}(\Omega), \tag{2.1}$$

$$p|_{\Gamma_2} = a \quad \text{on } \Gamma_2. \tag{2.2}$$

Moreover, there is some constant C_q independent of F such that

$$\|\nabla p\|_{L^q(\Omega)} \leq C_q \left(\|F\|_{W^{-1}_{q,\Gamma_2}(\Omega)} + \|\nabla A\|_{L^q(\Omega)} \right)$$

for every $A \in W^1_q(\Omega)$ with $A|_{\Gamma_2} = a$.

We refer to [5, Lemma 2] and [4, Corollary A.2] for the proof.

We will frequently use the following lemma:

Lemma 2.4 *Let Ω be as in Assumption 1.2 and let $q > d$. Then for any $F \in C^1(\overline{U})$, where $U \subset \mathbb{R}^m$, $m \geq 1$, is open, and any $R > 0$ there is some constant $C = C(R,F) > 0$ such that*

$$\begin{aligned} \|F(u)\|_{W^1_q(\Omega)} &\leq C \\ \|F(u) - F(v)\|_{L^\infty(\Omega)} &\leq C\|u - v\|_{W^1_q(\Omega)} \end{aligned}$$

for all $u, v \in W^1_q(\Omega)$ with $\|(u,v)\|_{W^1_q} \leq R$ and $u(x), v(x) \in \overline{U}$ for all $x \in \Omega$. If even $F \in C^2(\overline{U})$, then

$$\|F(u) - F(v)\|_{W^1_q(\Omega)} \leq C\|u - v\|_{W^1_q(\Omega)}$$

for all $u, v \in W^1_q(\Omega)$ with $\|(u,v)\|_{W^1_q} \leq R$ and $u(x), v(x) \in \overline{U}$ for all $x \in \Omega$.

Proof: The proof follows easily from the Sobolev embedding $W^1_q(\Omega) \hookrightarrow L^\infty(\Omega)$, the chain rule, and the representation

$$F(u) - F(v) = \int_0^1 DF(tu + (1-t)v)\, dt \cdot (u - v).$$

■

3 Non-Stationary Stokes Equations

As in the case of the generalized Stokes resolvent equations, cf. [5], (1.14)-(1.18) can (at least formally) be reduced to the *non-stationary reduced Stokes equations*

$$\partial_t v - \operatorname{div}(\nu \nabla v) + \nabla P_\nu v - \nabla \nu^T \nabla v^T = f_r \quad \text{in } Q_T, \tag{3.1}$$

$$v|_{\Gamma_1} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.2}$$

$$T_1' u = a_r \quad \text{on } \Gamma_2 \times (0,T), \tag{3.3}$$

$$v|_{t=0} = v_0 \quad \text{in } \Omega, \tag{3.4}$$

where $T_1' v$ is defined by

$$(T_1' v)_\tau = (n \cdot 2\nu Dv)_\tau|_{\Gamma_2}, \qquad (T_1' v)_n = \nu \operatorname{div} v|_{\Gamma_2}. \tag{3.5}$$

For given $\nu \in W^1_q(\Omega)$ with $\nu(x) \geq \nu_0 > 0$ the reduced Stokes operator A_q on $L^q(\Omega)^d$ is defined as

$$\begin{aligned} A_q v &= -\operatorname{div}(\nu \nabla v) + \nabla P v - \nabla \nu^T \nabla v^T \qquad (3.6)\\ \mathcal{D}(A_q) &= \left\{ v \in W^2_q(\Omega)^d : v|_{\Gamma_1} = 0, T_1' v|_{\Gamma_2} = 0 \right\}, \end{aligned}$$

Moreover, $Pv \equiv p_1 \in W^1_q(\Omega)$ with $p_1|_{\Gamma_2} \in W_q^{1-\frac{1}{q}}(\Gamma_2)$ if $\Gamma_2 \neq \emptyset$ and $\int_\Omega p_1\, dx = 0$ if $\Gamma_2 = \emptyset$ is defined as the solution of

$$\begin{aligned} (\nabla p_1, \nabla \varphi)_\Omega &= (\nu(\Delta - \nabla \operatorname{div}) v, \nabla \varphi)_\Omega + (Dv, 2\nabla \nu \otimes \nabla \varphi)_\Omega, \qquad (3.7)\\ p_1|_{\Gamma_2} &= 2\nu \partial_n v_n \qquad (3.8) \end{aligned}$$

for all $\varphi \in W^1_{q',\Gamma_2}(\Omega)$. Note that the right-hand-side of (3.7) defines a bounded linear functional on $W^1_{q',\Gamma_2}(\Omega)$. The existence of a solution follows from Lemma 2.3. Hence $P\colon W^2_q(\Omega)^d \to W^1_q(\Omega)$ is a bounded linear operator. The following result follows from [5, Theorems 1,2, and 3].

Theorem 3.1 *Let $1 < p, q < \infty$, $0 < T < \infty$, and let Ω be as in Assumption 1.2 and assume that $\max(q, q') \leq r$. Moreover, assume that $\nu \in W^1_{r_1}(\Omega)$ for some $d < r_1 \leq \infty$ such that $\max(q, q') \leq r_1$ and $\nu(x) \geq \nu_0 > 0$. Then for every $f \in L^p(0, T; L^q(\Omega)^d)$ there is a unique solution $v \in W^1_p(0, T; L^q(\Omega)^d) \cap L^p(0, T; \mathcal{D}(A_q))$ of*

$$\begin{aligned} v'(t) + A_q v(t) &= f(t), \qquad 0 < t < T,\\ v(0) &= 0 \end{aligned}$$

Moreover, there is some constant $C > 0$ independent of f such that

$$\|v'\|_{L^p(0,T;L^q)} + \|A_q v\|_{L^p(0,T;L^q)} \leq C \|f\|_{L^p(0,T;L^q)}.$$

Remark 3.2 Obviously, the constant C above can be chosen uniformly in $0 < T \leq T_0$ for any $0 < T_0 < \infty$.

From the latter theorem and Lemma 2.2, we deduce:

Theorem 3.3 *Let $0 < T < \infty$, let Ω, r be as in Assumption 1.2, let $\frac{3}{2} < q < \infty$ with $\max(q, q') \leq r$, $q \neq 3$, and let $\nu \in W^1_{r_1}(\Omega)$ for some $d < r_1 \leq \infty$ such that $\max(q, q') \leq r_1$ and $\nu(x) \geq \nu_0 > 0$. Moreover, let $(f_r, a_r, v_0) \in L^q(Q_T)^d \times W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))^d \times W_q^{2-\frac{2}{q}}(\Omega)^d$ satisfy the compatibility conditions*

1. *$\operatorname{div} v_0 = g|_{t=0}$ in $W^{-1}_{q,\Gamma_2}(\Omega)$, $v_0|_{\Gamma_1} = 0$,*

2. *$(n \cdot 2\nu D v_0)_\tau|_{\Gamma_2} = a_\tau|_{t=0}$ if $q > 3$.*

Then there is a unique solution $v \in W_q^{2,1}(Q_T)^d$ *of (3.1)-(3.4), which satisfies*

$$\|v\|_{W_q^{2,1}(Q_T)} \leq C\left(\|f_r\|_{L^q(Q_T)} + \|a_r\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2\times(0,T))} + \|v_0\|_{W_q^{2-\frac{2}{q}}(\Omega)}\right)$$

for some $C > 0$ *independent of* f_r, a_r, v_0*. The constant can be chosen uniformly in* $0 < T \leq T_0$ *for any* $0 < T_0 < \infty$*.*

Proof: The theorem follows immediately from Theorem 3.1 if $a_r = v_0 = 0$. The general case can be easily reduced to the latter case by first subtracting a suitable extension of u_0 and then a suitable extension of a_r, cf. Lemma 2.2. ■

Now we are able to prove Theorem 1.3. A proof in a more general case can be found in [4]. For a similar proof in the case of constant viscosity and an asymptotically flat layer with mixed boundary conditions we refer to [3].

Proof of Theorem 1.3: For almost every $t \in (0,T)$ let $p_2(.,t) \in W_q^1(\Omega)$ with $p_2|_{\Gamma_2} \in W_q^{1-\frac{1}{q}}(\Gamma_2)$ if $\Gamma_2 \neq \emptyset$ and $\int_\Omega p_2\,dx = 0$ else be the solution of

$$(\nabla p_2(.,t), \nabla\varphi) = (f(t) + \nu\nabla g(t), \nabla\varphi)_\Omega + \langle \partial_t g(t), \varphi\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} \tag{3.9}$$

for all $\varphi \in W^1_{q',\Gamma_2}(\Omega)$ and $p_2|_{\Gamma_2} = -a_n$, cf. Lemma 2.3. Now we define $f_r = f - \nabla p_2 + \nu\nabla g$. Then

$$\|f_r\|_q \quad\leq\quad C\left(\|(f,\nabla g)\|_q + \|\partial_t g\|_{L^q(0,T;W^{-1}_{q,\Gamma_2})} + \|a_n\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}\right)$$

with C independent of T. Moreover, let $(a_r)_\tau = a_\tau$ and $(a_r)_n = g|_{\Gamma_2}$.

Now let $v \in W_q^{2,1}(Q_T)^d$ be the solution of the reduced Stokes equations with right-hand side (f_r, a_r^+). Then (v,p) with $\nabla p = \nabla P v + \nabla p_2$ solves (1.14) and (1.16)-(1.18) by construction. Hence it only remains to prove that $\operatorname{div} v = g$.

First of all, because of (3.9),

$$-(f_r(t), \nabla\varphi)_\Omega = \langle \partial_t g(t), \varphi\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}}, +(\nu\nabla g(t), \nabla\varphi)_\Omega \tag{3.10}$$

for all $\varphi \in W^1_{q,\Gamma_2}(\Omega)$ and almost every $t \in (0,T)$. On the other hand, if $v \in W_q^{2,1}(\Omega)^d$ solves (3.1)-(3.4), then

$$-(f_r, \nabla\varphi)_\Omega = \langle \partial_t \operatorname{div} v, \varphi\rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} + (\nu\nabla \operatorname{div} v, \nabla\varphi)_\Omega \tag{3.11}$$

for all $\varphi \in W^1_{q,\Gamma_2}(\Omega)$ because of

$$\begin{aligned}&(\operatorname{div}(\nu\nabla v), \nabla\varphi)_\Omega - (\nabla P v, \nabla\varphi)_\Omega + (\nabla\nu^T\nabla v^T, \nabla\varphi)_\Omega \\ =\;& (\nu\Delta v, \nabla\varphi)_\Omega - (\nabla P v, \nabla\varphi)_\Omega + (Dv, 2\nabla\nu\otimes\nabla\varphi)_\Omega = (\nu\nabla \operatorname{div} v, \nabla\varphi)_\Omega\end{aligned} \tag{3.12}$$

for all $\varphi \in W^1_{q,\Gamma_2}(\Omega)$ and almost every $t \in (0,T)$ due to (3.7). Moreover, since $\operatorname{div} v - g \in W^1_{q,\Gamma_2}(\Omega)$, Proposition 3.4 below implies $\operatorname{div} v = g$. ■

Proposition 3.4 *Let Ω, r be as in Assumption 1.2, $1 < q < \infty$ with $\max(q,q') \leq r$, $q \neq 3$, and let $\nu \in W^1_{r_1}(\Omega)$ for some $d < r_1 \leq \infty$ such that $\max(q,q') \leq r_1$ and $\nu(x) \geq \nu_0 > 0$. Moreover, let $u \in L^q(0,T;W^1_{q,\Gamma_2}(\Omega))$, $0 < T < \infty$, be such that $\partial_t u \in L^q(0,T;W^{-1}_{q,\Gamma_2}(\Omega))$, $u|_{t=0} = 0$ in $W^{-1}_{q,\Gamma_2}(\Omega)$, and*

$$\int_0^T \langle \partial_t u, \varphi \rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} + (\nu \nabla u, \nabla \varphi)_{Q_T} = 0 \tag{3.13}$$

for all $\varphi \in L^{q'}(0,T;W^1_{q',\Gamma_2}(\Omega))$. Then $u = 0$.

Proof: Let $\psi \in L^{q'}(0,T;W^1_{q',\Gamma_2}(\Omega))$ be arbitrary and let $v \in W^{2,1}_{q'}(Q_T)^d$ be a solution of the reduced Stokes equations (3.1)-(3.4) with right-hand side $f_r = \nabla \psi$, $a = 0$, and $v_0 = 0$. Then by (3.11)

$$-(\nabla \psi, \nabla \varphi)_{Q_T} = \int_0^T \langle \partial_t \operatorname{div} v, \varphi \rangle_{W^{-1}_{q',\Gamma_2}, W^1_{q,\Gamma_2}} \, dt + (\nu \nabla \operatorname{div} v, \nabla \varphi)_{Q_T}$$

for all $\varphi \in L^q(0,T;W^1_{q,\Gamma_2}(\Omega))$. Now, choosing $\varphi(x,t) = u(x,T-t)$, we obtain

$$\begin{aligned}
&-(\nabla u(T-.), \nabla \psi)_{Q_T} \\
&= \int_0^T \langle \partial_t \operatorname{div} v(t), u(T-t) \rangle_{W^{-1}_{q',\Gamma_2}, W^1_{q,\Gamma_2}} \, dt + (\nu \nabla \operatorname{div} v, \nabla u(T-.))_{Q_T} \\
&= \int_0^T \langle (\partial_t u)(T-t), \operatorname{div} v(t) \rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} \, dt + (\nu \nabla u(T-.), \nabla \operatorname{div} v)_{Q_T} = 0
\end{aligned}$$

due to (3.13). Here we have used

$$\int_0^T \langle \partial_t v, w \rangle_{W^{-1}_{q,\Gamma_2}, W^1_{q',\Gamma_2}} \, dt = \langle v(t), w(t) \rangle_{W_q^{1-\frac{2}{q}}, W_{q'}^{1-\frac{2}{q'}}} \Big|_{t=0}^{T} - \int_0^T \langle v, \partial_t w \rangle_{W^1_{q,\Gamma_2}, W^{-1}_{q',\Gamma_2}} \, dt$$

for all $v \in L^q(0,T;W^1_{q,\Gamma_2}) \cap W^1_q(0,T;W^{-1}_{q,\Gamma_2})$, $w \in L^{q'}(0,T;W^1_{q',\Gamma_2}) \cap W^1_{q'}(0,T;W^{-1}_{q',\Gamma_2})$, where we note that $L^s(0,T;W^1_s) \cap W^1_s(0,T;W^{-1}_s) \hookrightarrow BUC([0,T];W_s^{1-\frac{2}{s}})$ for all $1 < s < \infty$.

Since $\psi \in L^{q'}(0,T;W^1_{q',\Gamma_2}(\Omega))$ was arbitrary, we conclude $\nabla u(t) = 0$ for almost every $t \in (0,T)$ due to Lemma 2.3. Hence $\partial_t u = 0$ due to (3.13) and therefore $u = 0$ since $u|_{t=0} = 0$. ■

4 Short-Time Existence for the Non-Linear System

Recall that our coupled Navier-Stokes system of interest, (1.1)-(1.6), reads in Lagrangian coordinates as follows:

$$\begin{aligned}
\varrho(c_0)\partial_t u - \operatorname{div}_u(2\nu(c_0)D_u u) + \nabla_u p &= -\operatorname{div}_u F(c_0, \nabla_u c_0) && \text{in } Q_T,\\
\operatorname{div}_u u &= 0 && \text{in } Q_T,\\
u|_{\Gamma_1} &= 0 && \text{on } \Gamma_1 \times (0,T),\\
n_u \cdot T_u(c,u,p)|_{\Gamma_2} &= n_u \cdot F(c_0, \nabla_u c_0) && \text{on } \Gamma_2 \times (0,T),\\
u|_{t=0} &= u_0 && \text{in } \Omega
\end{aligned}$$

where $\Gamma_2 = \Gamma_{2,0}, \Omega = \Omega(0)$ and

$$X_u(t,\xi) \;=\; \xi + \int_0^t u(\xi,\tau)d\tau, \qquad T_u(u,p) = 2\nu(c_u)D_u u - pI.$$

We will solve the system locally in time by a linearization technique. The linearization of the problem (around 0) is the following system:

$$\begin{aligned}
\partial_t w - \operatorname{div}(2\nu(c_0)\varrho_0^{-1}Dw) + \nabla p &= f && \text{in } Q_T,\\
\operatorname{div} w &= g, && \text{in } Q_T,\\
w|_{\Gamma_1} &= 0, && \text{on } \Gamma_1 \times (0,T),\\
n \cdot T(w,p)|_{\Gamma_2} &= a, && \text{on } \Gamma_2 \times (0,T),\\
w|_{t=0} &= w_0 && \text{in } \Omega,
\end{aligned}$$

where we have set $w := \varrho(c_0)v$, $w_0 := \varrho(c_0)v_0$, $\varrho_0 = \varrho(c_0)$, and

$$T(w,p) = 2\nu(c_0)\varrho(c_0)^{-1}Dw - pI.$$

For this system, we can apply Theorem 1.3 to obtain $w \in W_q^{2,1}(Q_T)$. From this, we can obtain the estimates of $v = \varrho(c_0)^{-1}w \in W_q^{2,1}(Q_T)$ since $c_0 \in W_q^2(\Omega)$.

We can formulate the initial-boundary value problem as an abstract fixed point equation:

$$Lv \;=\; G(v) + h \quad \Leftrightarrow v = L^{-1}G(v) + L^{-1}h, \tag{4.1}$$

where $v = (u,p)^T \in X_T$, and $h = (0,0,0,\varrho(c_0)u_0)^T$

$$\begin{aligned}
X_T \;:=\;& \big\{(u,p) : u \in W_q^{2,1}(Q_T)^d, u|_{\Gamma_1} = 0, \operatorname{div} u|_{t=0} = 0, p \in W_q^{1,0}(Q_T),\\
& p|_{\Gamma_2} \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\Gamma_2 \times (0,T)), (n \cdot Su|_{t=0})_\tau|_{\Gamma_2} = 0 \text{ if } q > 3\big\},\\
Y_T \;:=\;& \big\{(f,g,a,u_0) : f \in L^q(Q_T)^d, g \in W_q^{1,0}(Q_T), \partial_t g \in L^q\left(0,T; W_{q,\Gamma_2}^{-1}(\Omega)\right),\\
& g|_{t=0} = 0, a \in W_q^{1-\frac{1}{q},\frac{1}{2}-\frac{1}{2q}}(\Gamma_2), a_\tau|_{t=0} = 0, u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^d, \operatorname{div} u_0 = 0,\\
& u_0|_{\Gamma_1} = 0, (n \cdot Su_0)_\tau|_{\Gamma_2} = 0 \text{ if } q > 3\big\},
\end{aligned}$$

and

$$
\begin{aligned}
L(u,p) &:= \begin{pmatrix} \partial_t(\varrho_0 u) - \operatorname{div}(2\nu(c_0)\varrho_0^{-1} D(\varrho(c_0)u)) + \nabla p \\ \operatorname{div}(\varrho_0 u) \\ n \cdot T(\varrho_0 u, p)|_{\Gamma_2} \\ \varrho_0 u_0 \end{pmatrix} \\
G(u,p) &:= (g_1(u,p), g_2(u), g_3(u,p), 0)^T .
\end{aligned}
$$

with

$$
\begin{aligned}
g_1(u,p) &:= \operatorname{div}_u(2\nu(c_0)D_u u) - \operatorname{div}(2\nu(c_0)Du) - \operatorname{div}(\nu(c_0)\varrho_0^{-1}(\nabla\varrho_0 \otimes u + u \otimes \nabla\varrho_0)) \\
&\quad + \nabla p - \nabla_u p - (\operatorname{div}_u - \operatorname{div})F(c_0, \nabla_u c_0), \\
g_2(u) &:= \begin{cases} \nabla\varrho(c_0)\cdot u + \varrho(c_0)(\operatorname{div} u - \operatorname{div}_u u) & \text{if } \Gamma_2 \neq \emptyset, \\ \nabla\varrho(c_0)\cdot u + \varrho(c_0)(\operatorname{div} u - \operatorname{div}_u u) + \frac{1}{|\Omega|}\int_\Omega \varrho_0 \operatorname{div}_u u\, dx & \text{if } \Gamma_2 = \emptyset, \end{cases} \\
g_3(u,p) &:= n \cdot T_0(u,p)|_{\Gamma_2} - n_u \cdot T_u(u,p)|_{\Gamma_2} + n \cdot \big(\nu(c_0)\varrho_0^{-1}(\nabla\varrho_0 \otimes u + u \otimes \nabla\varrho_0)\big)\big|_{\Gamma_2} \\
&\quad + n_u \cdot F(c_0, \nabla_u c_0)|_{\Gamma_2} .
\end{aligned}
$$

Because of Theorem 1.3 and $\varrho(c_0) \in W^2_q(\Omega)$, L^{-1} exists and $\|L^{-1}\|_{\mathcal{L}(Y_T,X_T)} \leq C(T_0)$ for any $0 < T \leq T_0$ and fixed $0 < T_0 < \infty$.

Note that, if $\Gamma_2 = \emptyset$, then Ω is necessarily a bounded domain due to our assumptions. The modification in this case implies the necessary compatibility condition $\int_\Omega g_2(w)\, dx = 0$. Moreover, we note that

$$
g_2(u) \;=\; \nabla\varrho_0 \cdot u + \varrho_0 \operatorname{Tr}((I - A(u))\nabla u) \tag{4.2}
$$

if $\Gamma_2 \neq \emptyset$.

We mention that $X_T = X'_T \times \left\{ p \in W^{1,0}(Q_T) : p|_{\Gamma_2} \in W_q^{\frac{1}{q'},\frac{1}{2q'}}(\Gamma_2 \times (0,T)) \right\}$, where X'_T shall be normed by

$$
\|u\|_{X'_T} = \|u\|_{W^{2,1}_q(Q_T)} + \|u\|_{W^1_q(0,T;W^{-1}_{q,\Gamma_2})} + \|u|_{t=0}\|_{W_q^{2-\frac{2}{q}}(\Omega)} .
$$

This implies that

$$
\|u\|_{BUC([0,T];W_q^{2-\frac{2}{q}}(\Omega))} \leq C\|u\|_{X'_T} \tag{4.3}
$$

uniformly in $0 < T \leq 1$.

The proof of Theorem 1.1 relies on the following result.

THEOREM 4.1 *Let $d \geq 2$, $q > d, q \neq 3$ $\kappa > 0$ and let Ω be as in Assumption 1.2 with $r = q$, and let $G\colon X_T \to Y_T, T > 0$, be defined as above. Then for every $R > 0$ there is some $T_0 > 0$ such that*

$$
\|G(u) - G(v)\|_{Y_T} \leq \kappa \|u - v\|_{X_T}
$$

for all $u, v \in \overline{B_R(0)} \subset X_T$ and $0 < T \leq T_0$.

The main task now is to prove the latter theorem.

To this end we will use the following lemma.

Lemma 4.2 *Let $d \geq 2$, $q > d$, and $R > 0$. Moreover, let $F(u) = X_u$ and $Z = W^2_q(\Omega)^d$ or $F(u) = A(u)$ and $Z = W^1_q(\Omega)^{d\times d}$. Then there is some $T_0 = T_0(R) > 0$ and a constant $C > 0$ such that for all $0 < T \leq T_0$*

$$\sup_{0\leq t\leq T} \|F(u) - F(v)\|_Z \leq CT^{\frac{1}{q'}}\|u - v\|_{2,1,q}, \tag{4.4}$$

$$\sup_{0\leq t\leq T} \left(\int_0^t \frac{\|\Delta_h(F(u) - F(v))(.,t)\|^q_Z}{h^{1+\frac{q}{2q'}}}\,dh\right)^{\frac{1}{q}} \leq CT^{\frac{1}{2q'}}\|u - v\|_{2,1,q}, \tag{4.5}$$

$$\left(\int_0^T \int_0^t \frac{\|\Delta_h(F(u) - F(v))(.,t)\|^q_Z}{h^{1+\frac{q}{2q'}}}\,dh\,dt\right)^{\frac{1}{q}} \leq CT^{\frac{1}{q'}+\frac{1}{2q'}}\|u - v\|_{2,1,q} \tag{4.6}$$

for all $u, v \in W^{2,1}_q(\Omega_T)^n$ with $\|u\|_{2,1,q}, \|v\|_{2,1,q} \leq R$, where $\Delta_h f(t) = f(t) - f(t - h)$.

The proof in the present situation is identical with the proof in [3, Lemma 4.1].

Remark 4.3 We note (4.4) implies that for every $R > 0$ there is some $0 < T_0 \leq 1$ such that

$$\|X_u - \mathrm{id}\,\|_{L^\infty(0,T;C^{0,1}(\overline{\Omega}))} + \|\nabla X_u - I\|_{L^\infty(Q_T)} \leq \frac{1}{2}$$

for all $0 < T \leq T_0$, $\|u\|_{W^{2,1}_q} \leq R$. In particular, this implies that $A(u) = (D_\xi X_u)^{-T}$ is well-defined and $\|A(u)\|_{L^\infty(Q_T)} \leq 2$. Moreover, $X_u\colon \overline{\Omega} \to \overline{\Omega}$ is a C^1-diffeomorphism under the latter conditions.

Lemma 4.4 *Let $R > 0$, let $T_0 = T_0(R)$ be as above, and let $0 < T \leq T_0$. Then $g_1\colon X_T \to L^q(Q_T)^d$ is a bounded mapping such that*

$$\|g_1(v_1) - g_1(v_2)\|_{L^q(Q_T)} \leq C(R)T^{\frac{1}{q'}}\|v_1 - v_2\|_{X'_T} \tag{4.7}$$

uniformly in $0 < T \leq T_0$ and $v_1, v_2 \in X_T$ with $\|(v_1, v_2)\|_{X'_T} \leq R$.

Proof: First of all,

$$\begin{aligned}&\mathrm{div}_u(2\nu(c_0)D_u u) - \mathrm{div}(2\nu(c_0)Du)\\ &\quad= L_1(x, \widetilde{X}_u, \nabla\widetilde{X}_u; \nabla^2 u, \nabla p) + \mathrm{Tr}(A(u)\nu(c)\nabla A(u)2Du),\end{aligned}$$

and

$$-\,\mathrm{div}_u F(c_0, \nabla_u c_0) = L_2(x, \nabla\widetilde{X}_u; \nabla c_0, \nabla^2 c_0, \nabla^2\widetilde{X}_u)$$

where L_j, $j = 1, 2$, are uniformly bounded w.r.t x, continuously differentiable w.r.t. the second (and third) variable, and linear w.r.t. the variable after ";". Moreover, $\widetilde{X}_u := \int_0^t u(x,\tau)d\tau = X_u(x,t) - x$. Hence we can decompose $g_1(u,p)$ as

$$\begin{aligned}g_1(u,p) &= L_1(x, \nabla\widetilde{X}_u; \nabla u, \nabla^2 u, \nabla p) - \mathrm{div}(\nu(c_0)\varrho_0^{-1}(\nabla\varrho_0 \otimes u + u \otimes \nabla\varrho_0))\\ &\quad+ L_2(x, \nabla\widetilde{X}_u; \nabla c_0, \nabla^2 c_0, \nabla^2\widetilde{X}_u) + \mathrm{Tr}(A(u)\nu(c)\nabla A(u)2Du).\end{aligned} \tag{4.8}$$

Using the structural properties of L_1, we have the following estimates

$$\begin{aligned}
&\|L_1(x,\nabla\widetilde{X}_{u_1};\nabla u,\nabla^2 u,\nabla p)-L_1(x,\nabla\widetilde{X}_{u_2};\nabla u,\nabla^2 u,\nabla p)\|_{L^q(Q_T)}\\
&\leq\ C\|\nabla X_{u_1}-\nabla X_{u_2}\|_{L^\infty(Q_T)}\left(\|u\|_{W^{2,1}_q(Q_T)}+\|\nabla p\|_{L^q(Q_T)}\right)\\
&\leq\ CT^{\frac{1}{q'}}\|u_1-u_2\|_{W^{2,1}_q}\left(\|u\|_{W^{2,1}_q}+\|\nabla p\|_{L^q(Q_T)}\right)
\end{aligned}\tag{4.9}$$

due to (4.4). Since $L_1(x,0;\nabla u,\nabla^2 u,\nabla p)=0$, (4.9) implies

$$\begin{aligned}
&\|L_1(x,\nabla\widetilde{X}_u;\nabla v,\nabla^2 v,\nabla p)\|_{L^q(Q_T)}\\
&\leq\ CT^{\frac{1}{q'}}\|u\|_{W^{2,1}_q(Q_T)}\left(\|v\|_{W^{2,1}_q(Q_T)}+\|\nabla p\|_{L^q(Q_T)}\right).
\end{aligned}$$

Altogether these estimates yield (4.7) for $g_1(v)$ replaced by $L_1(x,\nabla\widetilde{X}_u;\nabla u,\nabla^2 u,\nabla p)$ with $v=(u,p)$.

For L_2 we have similarly

$$\begin{aligned}
&\|L_2(x,\nabla\widetilde{X}_u;\nabla c_0,\nabla^2 c_0,\nabla^2\widetilde{X}_u)-L_2(x,\nabla\widetilde{X}_v;\nabla c_0,\nabla^2 c_0,\nabla^2\widetilde{X}_v)\|_{L^q(Q_T)}\\
&\leq\ \|L_2(x,\nabla X_u;0,0,\nabla^2(\widetilde{X}_{u-v}))\|_{L^q(Q_T)}\\
&\quad+\|L_2(x,\nabla X_u;\nabla c_0,\nabla^2 c_0,\nabla^2\widetilde{X}_v)-L_2(x,\nabla\widetilde{X}_v;\nabla c_0,\nabla^2 c_0,\nabla^2\widetilde{X}_v)\|_{L^q(Q_T)}\\
&\leq\ C_R\|\nabla^2\widetilde{X}_{u-v}\|_{L^q(Q_T)}+C_R\|\nabla\widetilde{X}_{u-v}\|_{L^\infty(0,T;W^1_q)}\cdot\|(\nabla c_0,\nabla^2 c_0,\nabla^2\widetilde{X}_v)\|_{L^q(Q_T)}\\
&\leq\ C_RT^{\frac{1}{q'}}\|u-v\|_{W^{2,1}_q(Q_T)}
\end{aligned}$$

for all $u,v\in W^{2,1}_q(Q_T)$ with $\|(u,v)\|_{W^{2,1}_q}\leq R$ due to (4.4) again.

To estimate the last term in (4.8), we use the fact that $A(u)\nu(c)\in L^\infty(Q_T)$, $\widetilde{A}(u)\in L^\infty(Q_T)$ where $\widetilde{A}(u)$ is defined by $\nabla A(u)=\widetilde{A}(u)\nabla^2 X_u$ and the following estimate:

$$\|\nabla^2 X_u\cdot Du\|_{L^q(Q_T)}\leq\|\nabla^2 X_u\|_{L^\infty(0,T;L^q)}\|\nabla u\|_{L^q(0,T;L^\infty)}\leq CT^{\frac{1}{q'}}\|u\|_{W^{2,1}_q(Q_T)}$$

Finally, using Lemma 2.1, it is easy to obtain

$$\|\nabla\varrho_0\otimes u-\nabla\varrho_0\otimes v\|_{L^\infty(0,T;W^1_q)}\ \leq\ C\|u-v\|_{L^\infty(0,T;W^1_q)}\leq C\|u-v\|_{X'_T}.$$

due to (4.3). This yields easily the estimate (4.7) for this part due to $\|f\|_{L^q(0,T;X)}\leq T^{\frac{1}{q'}}\|f\|_{L^\infty(0,T;X)}$. ■

Lemma 4.5 *Let $R>0$ and let $T_0=T_0(R)$ be as above. To this end we use $g_2\colon X'_T\to Y_{2,T}:=L^q(0,T;W^1_q(\Omega))\cap W^1_q(0,T;W^{-1}_{q,\Gamma_2}(\Omega))$ is a bounded mapping such that*

$$\|g_2(v_1)-g_2(v_2)\|_{Y_{2,T}}\leq C(R)T^{\frac{1}{q'}}\|v_1-v_2\|_{X'_T}$$

uniformly in $0<T\leq T_0$ and $v_1,v_2\in X_T$ with $\|(v_1,v_2)\|_{X'_T}\leq R$.

Proof: First of all, the estimate

$$\|g_2(u_1) - g_2(u_2)\|_{L^q(0,T;W^1_q)} \leq CT^{\frac{1}{q'}}\|u_1 - u_2\|_{X'_T}$$

can be obtained similarly as before. – It is almost identical with the corresponding estimates in [3, Proof of Lemma 4.3], where we note that $\nabla\varrho(c_0)\cdot u$ is linear w.r.t u, of lower order, and can simply be estimated as

$$\|\nabla\varrho(c_0)\cdot u\|_{L^q(0,T;W^1_q)} \leq CT^{\frac{1}{q'}}\|u\|_{L^\infty(0,T;W^1_q)} \leq C'T^{\frac{1}{q'}}\|u\|_{X'_T}.$$

due to (4.3). Hence it only remains to estimate the $W^1_q(0,T;W^{-1}_{q,\Gamma_2})$-norm. Then

$$(g_2(u),\varphi)_\Omega \;=\; -(u,\operatorname{div}((I - A(u)^T)\varrho_0\varphi))_\Omega + (\nabla\varrho_0\cdot u,\varphi)_\Omega$$

for all $\varphi \in W^1_{q,\Gamma_2}(\Omega)$. Therefore we obtain for all $\varphi \in W^1_{q',\Gamma_2}(\Omega)$ with $\|\varphi\|_{W^1_{q',\Gamma_2}} = 1$

$$\begin{aligned}\frac{d}{dt}(g_2(u(t)),\varphi)_\Omega &= -(\partial_t u,\operatorname{div}((I - A(u)^T)\varrho_0\varphi))_\Omega - (\nabla u,(\partial_t A(u)^T)\varrho_0\varphi)_\Omega\\ &\quad +(\nabla\varrho_0\cdot\partial_t u,\varphi)_\Omega,\end{aligned}$$

where

$$\begin{aligned}\|A(u_1) - A(u_2)\|_{L^\infty(0,T;W^1_q)} &\leq CT^{\frac{1}{q'}}\|u_1 - u_2\|_{L^q(0,T;W^2_q)},\\ \|\partial_t A(u_1) - \partial_t A(u_2)\|_{L^q(0,T;W^1_q)} &\leq C\|\nabla u_1 - \nabla u_2\|_{L^q(0,T;W^1_q(\Omega))}\end{aligned}$$

due to (4.4) and $\partial_t A(u_1) = DF(\nabla X_{u_j})\nabla u_j$; here $A(u_j) = F(\nabla X_{u_j})$. Hence

$$\begin{aligned}&\left|(\partial_t u(t),\operatorname{div}((A(u_1(t)) - A(u_2(t))^T)\varphi))_\Omega\right|\\ &\leq C\|\partial_t u(t)\|_{L^q(\Omega)}\|\|A(u_1)^T - A(u_2)^T\|_{L^\infty(0,T;W^1_q)}\\ &\leq CT^{\frac{1}{q'}}\|\partial_t u(t)\|_{L^q(\Omega)}\|u_1 - u_2\|_{X'_T}\end{aligned}$$

and

$$\begin{aligned}&\left|(\nabla u(t),(\partial_t(A(u_1(t)) - A(u_2(t)))^T\varphi))_\Omega\right|\\ &\leq CT^{\frac{1}{q'}}\|u\|_{L^\infty(0,T;W^1_q(\Omega))}\|\|\nabla u_1(t) - \nabla u_2(t)\|_{L^q(\Omega)}\\ &\leq CT^{\frac{1}{q'}}\|u\|_{X'_T}\|u_1(t) - u_2(t)\|_{W^1_q(\Omega)}\end{aligned}$$

for all $\varphi \in W^1_{q',\Gamma_2}(\Omega)$ with $\|\varphi\|_{W^1_{q',\Gamma_2}} \leq 1$. From these estimates one easily derives

$$\|\partial_t g_2(u_1) - \partial_t g_2(u_2)\|_{L^q(0,T;W^{-1}_{q,\Gamma_2})} \leq C(R)T^{\frac{1}{q'}}\|u_1 - u_2\|_{X'_T}$$

for all $u_1, u_2 \in X'_T$ with $\|(u_1,u_2)\|_{X'_T} \leq R$. In particular, this implies $g_2(u) = g_2(u) - g_2(0) \in W^1_q(0,T;W^{-1}_{q,\Gamma_2})$.

■

Lemma 4.6 *Let* $R > 0$ *and let* $T_0 = T_0(R) > 0$ *be as above. Then* $g_3\colon X_T \to W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}(\Gamma_2 \times (0,T))$ *is a bounded mapping such that*

$$\|g_3(v_1) - g_3(v_2)\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}} \le C(R)T^{\frac{1}{2q'}}\|v_1 - v_2\|_{X_T}$$

uniformly in $0 < T \le T_0$ *and* $v_1, v_2 \in X_T$ *with* $\|(v_1,v_2)\|_{X_T} \le R$.

Proof: Let $v_j = (u_j, p_j) \in X_T$ with $\|(v_1,v_2)\|_{X_T} \le R$. We estimate the terms

$$\nu(c_0)(n_{u_1} \cdot D_{u_1} - n \cdot D)u_1 - \nu(c_0)(n_{u_2} \cdot D_{u_2} - n \cdot D)u_2 \tag{4.10}$$
$$(n_{u_1} - n)p_1 - (n_{u_2} - n)p_2 \tag{4.11}$$
$$n_{u_1} \cdot F(c_0, \nabla_{u_1} c_0) - n_{u_2} \cdot F(c_0, \nabla_{u_2} c_0) \tag{4.12}$$
$$n \cdot \nu(c_0)\varrho_0^{-1}\left((\nabla\varrho_0 \otimes u_1 + u_1 \otimes \nabla\varrho_0) - (\nabla\varrho_0 \otimes u_2 + u_2 \otimes \nabla\varrho_0)\right) \tag{4.13}$$

on $\Gamma_2 \times (0,T)$ separately. First of all, in [3, Proof of Lemma 4.3] it was shown that

$$\|n_u \cdot D_u w - n_v \cdot D_v w\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}} \le C(R)T^{\frac{1}{2q'}}\|u - v\|_{W_q^{2,1}(Q_T)}\|w\|_{W_q^{2,1}(Q_T)} \tag{4.14}$$

and

$$\begin{aligned}&\|n_u p - n_v p\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}}\\ \le\ & C(R)T^{\frac{1}{2q'}}\|u - v\|_{W_q^{2,1}(Q_T)}\left(\|\nabla p\|_{L^q(Q_T)} + \|p\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}}\right)\end{aligned}$$

for all $(u,p), (v,p) \in X_T$ with norm bounded by R and all $0 < T \le T_0$ for some suitable $T_0 > 0$ and $w \in W_q^{2,1}(Q_T)$. – Note that in [3] the estimates are shown for an asymptotically flat layer with $C^{1,1}$-boundary. But the proof in the present situation is almost identically using only additionally $n \in W_r^{1-\frac{1}{r}}(\Gamma_2)$ and $\|nb\|_{W_q^{1-\frac{1}{q}}(\Gamma_2)} \le C\|n\|_{W_r^{1-\frac{1}{r}}(\Gamma_2)}\|b\|_{W_q^{1-\frac{1}{q}}(\Gamma_2)}$ for all $1 < q \le r$. – Using $n = n_0$, $D = D_0$, as well as the simple general relation

$$(f(x_1) - f(0))x_1 - (f(x_2) - f(0))x_2 = (f(x_1) - f(x_2))x_2 + (f(x_1) - f(0))(x_1 - x_2),$$

these estimates imply the estimates of $(n_{u_1} - n)p_1 - (n_{u_2} - n)p_2$ and

$$\|(n_{u_1} \cdot D_{u_1} - n \cdot D)u_1 - (n_{u_2} \cdot D_{u_2} - n \cdot D)u_2\|_{W_q^{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q})}} \le C(R)T^{\frac{1}{2q'}}\|u_1 - u_2\|_{X'_T}$$

for $v_j = (u_j, p_j)$ as in the statement of the lemma. This implies the same estimate with $\nu(c_0)$ as prefactor since $\nu(c_0) \in W_q^{1-\frac{1}{q}}(\Gamma_2)$. Hence we obtained the estimate of the term (4.10). Furthermore, the term (4.12) can be estimated in the similarly as in [3, Proof of Lemma 4.3].

Finally, to estimate (4.13) we use that

$$u \in BUC([0,T]; W_q^{2-\frac{2}{q}}(\Omega)) \cap C^{1-\frac{1}{q}}([0,T]; L^q(\Omega)) \hookrightarrow C^{1-\frac{s}{2}-\frac{1}{q}}([0,T]; W_q^s(\Omega))$$

for every $0 \leq s \leq 2 - \frac{2}{q}$ and $u \in W_q^{2,1}(Q_T)$. Therefore

$$\|u|_{\Gamma_2}\|_{C^\alpha([0,T];L^q(\partial\Omega))} \leq C_\alpha \|u\|_{X_T'}$$

for every $0 \leq \alpha < \frac{1}{2} - \frac{3}{2q}$ due to $\|u|_{\Gamma_2}\|_{W_q^{s-\frac{1}{q}}(\partial\Omega)} \leq C_s \|u\|_{W_q^s(\Omega)}$ for all $\frac{1}{q} < s \leq 1$. Hence

$$\|u|_{\Gamma_2}\|_{W_q^{\frac{1}{q'}, \frac{1}{2q'}}} \leq CT^{\frac{1}{q'}} \left(\|u\|_{C^{1-\frac{3}{2q}-\varepsilon}([0,T];L^q(\Gamma_2))} + \|u\|_{L^\infty(0,T;W_q^1)} \right) \leq CT^{\frac{1}{q'}} \|u\|_{X_T'},$$

for some $0 < \varepsilon < \frac{1}{2} - \frac{1}{q}$, which implies $\frac{1}{2q'} < 1 - \frac{3}{2q}$. Since $c_0, \nabla c_0 \in W_q^{1-\frac{1}{q}}(\partial\Omega)$, this yields the estimate of the last term. ■

Proof of Theorem 1.1: We apply the Banach fixed point theorem to the set $\overline{B_R(0)}$, where $R > 0$ is chosen so large that $\|L^{-1}h\|_{X_{\widetilde{T}}} \leq \frac{R}{2}$ for some $\widetilde{T} > 0$. Then, because of Theorem 4.1, there is some $0 < T \leq \widetilde{T}$ such that $L^{-1}G : \overline{B_R(0)}|_{X_T} \to \overline{B_R(0)}|_{X_T}$ is a contraction with Lipschitz constant $\kappa = \frac{1}{2}$. Now let $F(v) := L^{-1}G(v) + L^{-1}h,\ v \in X_T$. Then $\|F(v)\|_{X_T} \leq R$ for $v \in \overline{B_R(0)}|_{X_T}$ since $\|L^{-1}h\|_{X_T} \leq \|L^{-1}h\|_{X_T} \leq \frac{R}{2}$ and $\|L^{-1}G(v) - L^{-1}G(0)\|_{X_T} \leq \frac{R}{2}$. Hence the Banach fixed point theorem implies the existence of a unique fixed point $Lv = G(v)$, which is a solution of (1.8)-(1.13) if $\Gamma_2 \neq \emptyset$. Finally, if $\Gamma_2 = \emptyset$, then a fixed point $w = \varrho_0 u$ of $Lw = G(w)$ only satisfies

$$\operatorname{div}(\varrho(c_0)u) = g_2(u) \quad \Leftrightarrow \quad \varrho_0 \operatorname{div}_u u = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \operatorname{div}_u u \, dx =: K(t).$$

Therefore it remains to prove that $K(t) \equiv 0$, which follows from the next lemma. ■

Lemma 4.7 *If $L(u,p) = G(u,p)$ and $\Gamma_2 = \emptyset$, then $K(t) = 0$ for every $t \in [0,T]$, where $K(t)$ is defined as above.*

Proof: First of all, since $u|_{\partial\Omega} = 0$ and $\|I - X\|_{L^\infty(0,T;C^{0,1}(\overline{\Omega}))} \leq \frac{1}{2}$, $X(t) \colon \overline{\Omega} \to \overline{\Omega}$ is a C^1-diffeomorphism with $X|_{\partial\Omega} = \mathrm{id}_{\partial\Omega}$ for every $0 \leq t \leq T$. Hence $\Omega(t) := X(t,\Omega) = \Omega$ for all $t \in [0,T]$ and therefore $|\Omega(t)| = |\Omega_0|$ for all $0 \leq t \leq T$. Moreover, if $v(x,t)$ is defined by

$$v(X_u(\xi,t),t) = u(\xi,t) \qquad \text{for all } \xi \in \Omega_0, t \in [0,T],$$

then $\partial_t X(\xi,t) = v(X(\xi,t),t)$ and $X(\xi,0) = \xi$ and therefore

$$\begin{aligned} 0 &= \frac{d}{dt}|\Omega(t)| = \int_{\Omega(t)} \operatorname{div} v(x,t)\, dx = \int_{\Omega_0} \operatorname{div}_u u(x,t) \det DX(x,t)\, dx \\ &= K(t) \int_{\Omega_0} \varrho_0(x)^{-1} \det DX(x,t)\, dx. \end{aligned}$$

Since the latter integral is positive, we conclude $K(t) = 0$ for all $t \in [0,T]$. This implies $\operatorname{div}_u u = 0$. ∎

References

[1] H. Abels. Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities. *Preprint, Max Planck Institute for Mathematics in the Sciences, No. 5/2008, to appear in Comm. Math. Phys.*

[2] H. Abels. On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities. *Arch. Rat. Mech. Anal., DOI 10.1007/s00205-008-0160-2*.

[3] H. Abels. The initial value problem for the Navier-Stokes equations with a free surface in L^q-Sobolev spaces. *Adv. Diff. Eq.*, Vol. 10, No. 1:45–64, 2005.

[4] H. Abels. Nonstationary stokes system with variable viscosity in bounded and unbounded domains. *Preprint, Max Planck Institute for Mathematics in the Sciences, No. 12/2009*, 2009.

[5] H. Abels and Y. Terasawa. On Stokes Operators with variable viscosity in bounded and unbounded domains. *Math. Ann.,* DOI: 10.1007/s00208-008-0311-7, 344(2):381–429, 2009.

[6] J. T. Beale. *The initial value problem for the Navier-Stokes equations with a free surface*. Comm. Pure Appl. Math., Vol. 34, 359-392, 1981.

[7] D. Bothe and J. Prüss. L_P-theory for a class of non-Newtonian fluids. *SIAM J. Math. Anal.*, 39(2):379–421 (electronic), 2007.

[8] R. Danchin. Density-dependent incompressible fluids in bounded domains. *J. Math. Fluid Mech.*, 8(3):333–381, 2006.

[9] G. Grubb and V. A. Solonnikov. *Solution of parabolic pseudo-differential initial-boundary value problems.* J. Diff. Eq. 87, 256-304, 1990.

[10] M. E. Gurtin, D. Polignone, and J. Viñals. Two-phase binary fluids and immiscible fluids described by an order parameter. *Math. Models Methods Appl. Sci.*, 6(6):815–831, 1996.

[11] O. A. Ladyženskaja and V. A. Solonnikov. The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 52:52–109, 218–219, 1975. Boundary value problems of mathematical physics, and related questions of the theory of functions, 8.

[12] J. Málek and K. R. Rajagopal. On the modeling of inhomogeneous incompressible fluid-like bodies. *Mechanics of Materials*, 38:233–242, 2006.

[13] N. Nakano and A. Tani. An initial-boundary value problem for motion of incompressible inhomogeneous fluid-like bodies. *Preprint*.

[14] V. A. Solonnikov. L_p-estimates for solutions to the initial boundary-value problem for the generalized Stokes system in a bounded domain. *J. Math. Sci. (New York)*, 105(5):2448–2484, 2001. Function theory and partial differential equations.

[15] V. A. Solonnikov. Estimates of the solution of model evolution generalized Stokes problem in weighted Hölder spaces. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 336(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 37):211–238, 277, 2006.