

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The Metric Geometry of the Manifold of
Riemannian Metrics over a Closed Manifold

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Preprint no.: 21

2009



THE METRIC GEOMETRY OF THE MANIFOLD OF RIEMANNIAN METRICS OVER A CLOSED MANIFOLD

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ABSTRACT. We prove that the L^2 Riemannian metric on the manifold of all smooth Riemannian metrics on a fixed closed, finite-dimensional manifold induces a metric space structure. As the L^2 metric is a weak Riemannian metric, this fact does not follow from general results. In addition, we prove several results on the exponential mapping and distance function of a weak Riemannian metric on a Hilbert/Fréchet manifold. The statements are analogous to, but weaker than, what is known in the case of a Riemannian metric on a finite-dimensional manifold or a strong Riemannian metric on a Hilbert manifold.

1. INTRODUCTION

This paper is the first in a pair studying the metric geometry of the Fréchet manifold \mathcal{M} of all C^∞ Riemannian metrics on a fixed closed, finite-dimensional, orientable manifold M . This manifold has a natural weak Riemannian metric called the L^2 metric, which will be defined in Section 2. The main result of the paper is the following.

Theorem. (\mathcal{M}, d) , where d is the distance function induced from the L^2 metric, is a metric space.

This is indeed a nontrivial theorem, as the fact that the L^2 metric is a weak (as opposed to strong) Riemannian metric implies that its induced distance function is *a priori* only a pseudometric. That is, some points may have distance zero from one another. The first authors we know to have recognized this are Michor and Mumford. In [14, 15], they found examples of weak Riemannian metrics on Fréchet manifolds of embeddings for which the distance between *any* two points is zero. They then constructed other weak Riemannian metrics that they proved induce metric space structures on the manifolds (i.e., have positive-definite distance functions).

The manifold of metrics and the L^2 metric have been of interest to mathematicians and physicists for some time. We first became interested in their study because of their applications to Teichmüller theory, developed by Fischer and Tromba [21]. If the base manifold M is a Riemann surface of genus greater than one, then the Teichmüller space of M is diffeomorphic to $\mathcal{M}_{-1}/\mathcal{D}_0$, where $\mathcal{M}_{-1} \subset \mathcal{M}$ is the submanifold of hyperbolic metrics (with constant scalar curvature -1) and \mathcal{D}_0 is the group of diffeomorphisms that are homotopic to the identity, acting by pull-back. The L^2 metric on \mathcal{M} descends to $\mathcal{M}_{-1}/\mathcal{D}_0$ and is isometric (up to a constant scalar factor) to the much-studied Weil-Petersson metric on Teichmüller space. The sequel [2] to this paper contains an application to the geometry of Teichmüller space with respect to a class of metrics we define on it that generalize the Weil-Petersson metric.

The first investigations of the geometry of the manifold of metrics, which essentially avoided the infinite-dimensional issues that arise, were undertaken by DeWitt as part of his Hamiltonian formulation of general relativity [5]. Soon thereafter, Ebin [6] used the L^2 metric to investigate the differential topology of \mathcal{M} , as well as its quotient by the diffeomorphism group.

Two decades later, the basic Riemannian geometry of the L^2 metric was independently investigated by Freed and Groisser [8] as well as Gil-Medrano and Michor [9]. (The latter paper even

Date: April 6, 2009.

This research was supported by the International Max Planck Research School *Mathematics in the Sciences* and the DFG Research Training Group *Analysis, Geometry and their Interaction with the Natural Sciences*.

considers such questions for open base manifolds M .) Many of the results of the two papers coincide (both compute the geodesics and curvature of the L^2 metric, for instance), and the crucial observation of both papers is that the computation of certain essential geometric quantities is *pointwise* in nature. To illustrate what this means, consider a geodesic on \mathcal{M} . This is a one-parameter family g_t of metrics on M , uniquely determined by an initial point g_0 and initial tangent vector g'_0 . That the geodesic equation is pointwise means that for each $x \in M$, the value of $g_t(x)$ depends *only* on $g_0(x)$ and $g'_0(x)$, and not on the values of g_0 or g'_0 at any other points of M .

As will be seen in the subsequent sections, the main challenge in studying the metric geometry of the L^2 metric is in moving from *pointwise* questions to *local* or even *global* questions.

The paper is organized as follows:

In Section 2, we present the necessary facts and notation needed for studying the manifold of Riemannian metrics. We also introduce two related Fréchet manifolds, the manifold of smooth volume forms and the manifold of Riemannian metrics inducing a given volume form, that are of independent interest and play a role in the study of \mathcal{M} . We also give the geodesic equations of \mathcal{M} and the two other manifolds just mentioned.

In Section 3, we review the definition of a Riemannian metric on a Hilbert or Fréchet manifold, paying special attention to the distinction between weak and strong metrics. We go into more detail here than most papers on the subject, since our goal is to bring the differences between the metric (space) properties of the two kinds of Riemannian metrics into sharp focus. Following the definitions, we prove some results on the exponential mapping and the induced distance function of a weak Riemannian manifold, pointing out where these results are weaker than in the case of a strong Riemannian metric. While the approach is analogous to the strong case, we could not find a rigorous treatment in the weak case and so included them here.

In Section 4, we prove the main theorem mentioned above. The results of Section 3 turn out to be inadequate in this situation, but we give a direct proof for the L^2 metric.

Acknowledgements. The results of this paper formed a portion of my Ph.D. thesis ([3], where the reader may find the facts here presented in significantly greater detail), written at the Max Planck Institute for Mathematics in the Sciences in Leipzig and submitted to the University of Leipzig. I would like to thank my advisor Jürgen Jost for his many years of patient assistance. I am also grateful to Guy Buss, Christoph Sachse, and Nadine Große for many fruitful discussions and their careful proofreading.

2. THE MANIFOLD OF RIEMANNIAN METRICS

2.1. The L^2 Metric. For the entirety of the paper, let M denote a fixed closed, orientable, n -dimensional C^∞ manifold.

The basic facts about the manifold of Riemannian metrics given in this section can be found in [3, §2.5]

We denote by S^2T^*M the vector bundle of symmetric $(0, 2)$ tensors over M , and by \mathcal{S} the Fréchet space of C^∞ sections of S^2T^*M . The space \mathcal{M} of Riemannian metrics on M is an open subset of \mathcal{S} , and hence it is trivially a Fréchet manifold, with tangent space at each point canonically identified with \mathcal{S} . (For a detailed treatment of Fréchet manifolds, see, for example, [10]. For a more thorough treatment of the differential topology and geometry of \mathcal{M} , see [6].)

\mathcal{M} carries a natural Riemannian metric (\cdot, \cdot) , called the L^2 metric, induced by integration from the natural scalar product on S^2T^*M . Given any $g \in \mathcal{M}$ and $h, k \in \mathcal{S} \cong T_g\mathcal{M}$, we define

$$(h, k)_g := \int_M \text{tr}_g(hk) d\mu_g.$$

Here, $\text{tr}_g(hk)$ is given in local coordinates by $\text{tr}(g^{-1}hg^{-1}k) = g^{ij}h_{il}g^{lm}k_{jm}$, and μ_g denotes the volume form induced by g .

Throughout the paper, we use the notation d for the distance function induced from (\cdot, \cdot) by taking the infimum of the lengths of paths between two given points.

The basic Riemannian geometry of $(\mathcal{M}, (\cdot, \cdot))$ is relatively well understood. For example, it is known that the sectional curvature of \mathcal{M} is nonpositive (cf. [8, Cor. 1.17]), and the geodesics of \mathcal{M} are known explicitly (cf. Section 2.3).

We can and will consider related structures restricted to a point $x \in M$. Let $\mathcal{S}_x := S^2T_x^*M$ denote the vector space of symmetric $(0, 2)$ -tensors at x , and let $\mathcal{M}_x \subset \mathcal{S}_x$ denote the open subset of tensors inducing a positive definite scalar product on T_xM . Then \mathcal{M}_x is an open submanifold of \mathcal{S}_x , and its tangent space at each point is canonically identified with \mathcal{S}_x . For each $g \in \mathcal{M}_x$, we define a scalar product $\langle \cdot, \cdot \rangle_g$ on $T_g\mathcal{M}_x \cong \mathcal{S}_x$ by setting, for all $h, k \in \mathcal{S}_x$,

$$\langle h, k \rangle_g := \text{tr}_g(hk).$$

Then $\langle \cdot, \cdot \rangle$ defines a Riemannian metric on the finite-dimensional manifold \mathcal{M}_x .

2.2. A Product Manifold Structure for \mathcal{M} . \mathcal{M} can be written globally as a product manifold, with the factors given by the manifold of metrics inducing a given volume form and the manifold of volume forms on M . We sketch the details of this here. All facts that are given can be found (with proofs) either in [8, §1] or [3, §2.5.3].

The set of smooth volume forms on M , denoted by \mathcal{V} , is a Fréchet manifold. In fact, it is an open subset of $\Omega^n(M)$, the Fréchet space of highest-order differential forms on M (i.e., C^∞ sections of the line bundle $\Lambda^n T^*M$), and therefore its tangent spaces are canonically identified with $\Omega^n(M)$.

Given any volume form $\mu \in \mathcal{V}$ and any n -form $\alpha \in \Omega^n(M)$, there exists a unique C^∞ function, denoted by (α/μ) , such that

$$(1) \quad \alpha = \left(\frac{\alpha}{\mu} \right) \mu.$$

If α is also a smooth volume form, then (α/μ) is additionally a strictly positive function. If g_0 and g_1 are two Riemannian metrics on M , then locally, the Radon-Nikodym derivative of μ_{g_1} with respect to μ_{g_0} is given locally by

$$(2) \quad \left(\frac{\mu_{g_1}}{\mu_{g_0}} \right) = \sqrt{\frac{\det g_1}{\det g_0}} = \sqrt{\det(g_0^{-1}g_1)}.$$

Now, for any fixed volume form $\mu \in \mathcal{V}$, let $\mathcal{M}_\mu \subset \mathcal{M}$ denote the set of metrics inducing the volume μ . It is a smooth submanifold of \mathcal{M} [6, §8] with tangent space

$$T_g\mathcal{M}_\mu = \{h \in \mathcal{S} \mid \text{tr}_g h = 0\}.$$

This follows from the fact that the differential of the map $g \mapsto \mu_g$ at the point g_0 is $h \mapsto \frac{1}{2} \text{tr}_{g_0}(h)\mu_{g_0}$ [3, Lemma 2.38].

We now define a map

$$i_\mu : \mathcal{V} \times \mathcal{M}_\mu \rightarrow \mathcal{M} \\ (\nu, g) \mapsto \left(\frac{\nu}{\mu} \right)^{2/n} g.$$

It is not hard to see that $\mu_{i_\mu(\nu, g)} = \nu$, as well as that i_μ is a diffeomorphism.

\mathcal{M}_μ inherits a Riemannian metric as a submanifold of $(\mathcal{M}, (\cdot, \cdot))$. We can also pull back the L^2 metric on \mathcal{M} to \mathcal{V} via i_μ , namely by choosing any $g \in \mathcal{M}_\mu$ and noting that $i_\mu|_{\mathcal{V} \times \{g\}}$ is an embedding of \mathcal{V} into \mathcal{M} . A relatively straightforward computation then shows that this pull-back metric is given by

$$(3) \quad ((\alpha, \beta))_\nu = \frac{4}{n} \int_M \left(\frac{\alpha}{\nu} \right) \left(\frac{\beta}{\nu} \right) d\nu, \quad \alpha, \beta \in \Omega^n(M) \cong T_\nu\mathcal{V}.$$

Thus, this metric is independent of our choices of g and μ . In fact, it is just the constant factor $\frac{4}{n}$ times the most obvious Riemannian metric on \mathcal{V} . Note that it is not hard to see that

$$i_\mu(\mathcal{V} \times \{g\}) = \mathcal{P} \cdot g,$$

where \mathcal{P} is the group of strictly positive functions on M , acting on \mathcal{M} by pointwise multiplication. Furthermore, the tangent space to $i_\mu(\mathcal{V} \times \{g\})$ at ρg , for any $\rho \in \mathcal{P}$, is given by $C^\infty(M) \cdot g$.

2.3. Geodesics. As noted above, the geodesic equation of \mathcal{M} can be solved explicitly (see [8, Thm. 2.3], [9, Thm. 3.2]). We will not need it in full generality here, but we note that from the geodesic equation, one can deduce that \exp_g is not surjective for any $g \in \mathcal{M}$. For a nice and explicit presentation of this fact, see [9, §3]

We will need the geodesics associated to the product manifold structure of \mathcal{M} . They are given in the following two propositions.

Proposition 1 ([8, Prop. 2.1]). *The geodesic in \mathcal{V} starting at ν_0 with initial tangent α is given by*

$$\nu_t = \left(1 + \frac{t}{2} \left(\frac{\alpha}{\nu_0}\right)\right)^2 \nu_0.$$

As a result, for every $g_0 \in \mathcal{M}$, the exponential mapping \exp_{g_0} is a diffeomorphism from an open set $U \subset T_{g_0}(\mathcal{P} \cdot g_0)$ onto $\mathcal{P} \cdot g_0$.

Furthermore, if $\mu \in \mathcal{V}$ and $g \in \mathcal{M}_\mu$, then $i_\mu(\mathcal{V} \times \{g\})$ is a totally geodesic submanifold.

Proposition 2 ([6, Thm. 8.9] and [8, Prop. 1.27, Prop. 2.2]). *The submanifold \mathcal{M}_μ is not totally geodesic. However, it is a globally symmetric space, and the geodesic starting at g_0 with initial tangent $g'_0 = h$ is given by*

$$g_t = g_0 \exp(tH),$$

where $H := g_0^{-1}h$.

In particular, \mathcal{M}_μ is geodesically complete, and \exp_g is a diffeomorphism from $T_g\mathcal{M}_\mu$ to \mathcal{M}_μ for any $g \in \mathcal{M}_\mu$.

As we will see in Section 4, Propositions 1 and 2 are sufficient to prove that \mathcal{V} and \mathcal{M}_μ , together with the Riemannian metrics induced from the L^2 metric on \mathcal{M} , are metric spaces. However, to prove that the L^2 metric on \mathcal{M} itself induces a metric space structure, the geodesic equation is insufficient due to the non-surjectivity of the exponential mapping.

3. WEAK RIEMANNIAN MANIFOLDS

On a finite-dimensional Riemannian manifold (N, γ) —one modeled on \mathbb{R}^n —the tangent space $T_x N$ is, via a choice of coordinates, isomorphic to \mathbb{R}^n for each $x \in N$. The equivalence of all scalar products on \mathbb{R}^n implies that the scalar product induced by the Riemannian metric, when viewed as a scalar product on \mathbb{R}^n , is equivalent to the Euclidean scalar product. In particular, it induces the standard topology on \mathbb{R}^n . One can then deduce that the induced distance function d_γ of γ is a metric (in the sense of metric spaces), and the metric topology of (N, d_γ) coincides with the manifold topology of N .

In the case of an infinite-dimensional Riemannian manifold (N, γ) modeled on a Hilbert or Fréchet space E , we cannot necessarily say that the scalar product induced by γ on a tangent space $T_x N$ is equivalent to the Hilbert space scalar product of E . Therefore, the topology that γ induces on $T_x N$ may differ from the topology of E . Thus, we can distinguish two types of Riemannian metrics on a Fréchet (or, as a special case, Hilbert) manifold. We call γ a *strong Riemannian metric* if it induces the model space topology on each tangent space, and a *weak Riemannian metric* if it induces a weaker topology. In the case of a proper Fréchet space (where the topology does not come from any single norm), only weak Riemannian metrics are possible. This cements their importance in global analysis.

The subtle but important distinction between weak and strong Riemannian metrics leads to a vast gulf in the two theories one can develop around each structure. For a strong Riemannian metric, one can reproduce most of the important results in finite-dimensional Riemannian geometry. For example, the Levi-Civita connection, geodesics, and the exponential mapping exist. A strong Riemannian metric induces a distance function that gives a metric space structure on the manifold. In addition, the metric topology agrees with the manifold topology.

None of the above-mentioned results hold in general for weak Riemannian manifolds, though they can be directly shown for many important examples.

In this section, we will give some basic results on the distance function of a weak Riemannian manifold. We have not found these results formally recorded anywhere, though they may be known to experts in the field. Our approach essentially follows that of [11, §1.8], which treats the case of strong Riemannian manifolds. We have made the necessary adjustments to the results and proofs so that they hold in the weak case.

3.1. (Weak) Riemannian Fréchet manifolds. A Riemannian metric on a Fréchet manifold is defined exactly analogously to one on a finite-dimensional manifold, modulo the distinction between weak and strong metrics mentioned above.

Recall that on a Banach manifold N modeled on a Banach space E , each tangent space $T_x N$ is naturally isomorphic to the model space E , the isomorphism being given by any choice of coordinates around x (this choice is, of course, usually very non-canonical). The same holds true for Fréchet manifolds, and we keep this in mind as we make the following definition.

Definition 3. Let N be a Fréchet manifold modeled on a Fréchet space E . A *Riemannian metric* γ on N is a choice of scalar product $\gamma(x)$ on $T_x N$ for each $x \in N$, such that for each $x \in N$, the following holds:

- (1) γ is smooth in the sense that if U is any open neighborhood of x and V, W are vector fields defined on U , then $\gamma(\cdot)(V, W) : U \rightarrow \mathbb{R}$ is a smooth local function;
- (2) $\gamma(x)$ is a continuous (i.e., bounded) bilinear mapping; and
- (3) $\gamma(x)$ is positive definite on $T_x N$.

Furthermore, γ is called

- (1) *strong* if the topology induced by γ coincides with the topology of the model space E ; and
- (2) *weak* otherwise, i.e., if the topology induced by γ is weaker than the model space topology.

The pair (N, γ) is called a *Riemannian Fréchet manifold*.

To put it another way, (N, γ) is a strong Riemannian Fréchet manifold if its tangent spaces are complete with respect to γ , and it is weak if the tangent spaces are incomplete with respect to γ .

The first definition of weak Riemannian Hilbert manifolds that we know of (though our knowledge is surely incomplete) is in [6], the paper that founded the study of the geometry of the manifold of metrics. The generalization to weak Riemannian Fréchet manifolds is natural and has been used in several works. In no particular order, here is a list of papers that consider weak Riemannian manifolds (specifically, those that explicitly deal with the questions posed by “weakness” and are not mentioned elsewhere in this paper): [1, 4, 7, 12, 13, 16, 17, 18, 19]. We have made no attempt to make this list complete—it is simply a smattering of examples.

Let (N, γ) be a Riemannian Fréchet manifold. Just as in the case of finite-dimensional Riemannian manifolds, we can use γ to define a distance between points of N by taking the infima of lengths of paths. It is then clear that this distance function is a *pseudometric*, i.e., it satisfies all the properties of a metric except that the distance between distinct points need not be positive.

As mentioned above, if γ is strong, then the distance function is in fact a metric. On the other hand, there are examples [14, 15] of weak Riemannian manifolds where the induced distance between *any* two points is zero! So there is no hope of a general theorem like in the strong case.

3.2. The exponential mapping and distance function on a weak Riemannian manifold.

The problem in proving a general theorem on the distance function on a weak Riemannian manifold is the following: for each point x on a strong Riemannian manifold (N, γ) , a neighborhood of $0 \in T_x N$ contains an open γ -ball of some sufficiently small radius. However, if (N, γ) is a weak Riemannian manifold, since the topology induced by γ is weaker than the manifold topology of $T_x N$, an open neighborhood of 0 (in the manifold topology) *need not necessarily* contain any open γ -balls.

This phenomenon does indeed occur—it occurs in the examples of [14, 15] mentioned above, and it in fact also occurs for the manifold of Riemannian metrics [9, Rmk. 3.5].

The first result we can indeed show for weak Riemannian metrics is familiar from finite-dimensional Riemannian geometry. The proof carries over, so we omit it here (or refer the reader to [3, Prop. 2.23]).

Proposition 4. *Let (N, γ) be a weak Riemannian manifold on which the Levi-Civita connection exists. Let $p \in N$ and $v \in T_p N$, and suppose that v is in the domain of \exp_p . Then the geodesic $\alpha(t) := \exp_p(tv)$, $t \in [0, 1]$, has length $\|v\|_\gamma$.*

Unfortunately, we cannot prove much more that is useful about weak Riemannian manifolds without first making a couple of assumptions on the exponential mapping. Basically, we want it to exist and to be a diffeomorphism between some open sets—so we'll have to assume that as well. (Here again, there are examples [4, §1] where the exponential mapping exists but is not a local diffeomorphism.) The next bit of terminology incorporates this, and also adds one technical detail that we'll soon need.

Definition 5. We call a weak Riemannian manifold (N, γ) *normalizable at x* if there are open neighborhoods $U_x \subseteq T_x N$ and $V_x \subseteq N$ of 0 and x , respectively, such that

- (1) U_x is convex and
- (2) the exponential mapping \exp_x exists and is a C^1 -diffeomorphism between U_x and V_x .

Note that the neighborhoods U_x and V_x are required to be open in the manifold topology of N . We do not require that U_x be open in the topology induced by γ .

We call (N, γ) *normalizable* if it is normalizable at each $x \in N$.

Definition 6. Let (N, γ) be a weak Riemannian manifold and let $x \in N$. We denote by $S_x N \subset T_x N$ the unit sphere, i.e.,

$$S_x N = \{v \in T_x N \mid \|v\|_\gamma = 1\}.$$

For the rest of this section, let (N, γ) be a weak Riemannian manifold that is normalizable at a point $x \in N$, and retain the notation of Definition 5.

Lemma 7. *The function*

$$\begin{aligned} R : T_x N \setminus \{0\} &\rightarrow \mathbb{R}_{\geq 0} \\ v &\mapsto \sup\{r \in \mathbb{R}_{\geq 0} \mid r \cdot v \in U_x\}, \end{aligned}$$

is continuous.

Proof. R is the multiplicative inverse of the Minkowski functional of U_x , so the result follows from [20, II.1.6] by the convexity and openness of U_x . \square

The following lemma shows that the exponential mapping of a weak Riemannian manifold that is normalizable at x is defined on some nonzero vector pointing in each direction in $T_x N$.

Lemma 8. *For each $v \in T_x N \setminus \{0\}$, $R(v) > 0$.*

Proof. Let $v \in T_x N$ be given. Since $T_x N$ with its manifold topology is a topological vector space and U_x is a neighborhood of the origin, there is some $\epsilon > 0$ such that $\epsilon \cdot v \in U_x$. \square

Remark 9. Lemma 8 does not imply that $R(v)$ is uniformly bounded away from zero, even if we restrict the domain of R to $S_x N$ at each $x \in N$.

The next two propositions allow us to control the lengths of paths based at x , provided they lie within the neighborhood U_x . We will remark below on how these statements have been weakened from the strong case.

Proposition 10. *Let $r(s) \cdot v(s) \in U_x$, $s \in [0, 1]$, be a path in U_x such that $v(s) \in S_x N$, $r(s) \in \mathbb{R}_{\geq 0}$. (That is, we express the path in polar coordinates.) We define a path α by $\alpha(s) := \exp_x(r(s)v(s))$, $s \in [0, 1]$. Then*

$$L(\alpha) \geq |r(1) - r(0)|,$$

with equality if and only if $v(s)$ is constant and $r'(s) \geq 0$.

Proof. By Lemmas 7 and 8, as well as the compactness of $[0, 1]$, there exist $\epsilon, \delta > 0$ such that if

$$(s, t) \in U_{\epsilon, \delta} := \{(s, t) \in \mathbb{R}^2 \mid s \in [0, 1], t \in [-\epsilon, r(s) + \delta]\},$$

then $t \cdot v(s) \in U_x$.

We define a one-parameter family of paths in N by

$$c_s(t) := \exp(t \cdot v(s)), \quad (s, t) \in U_{\epsilon, \delta}$$

Note that for each fixed s , the path $t \mapsto c_s(t)$ is a geodesic with

$$(4) \quad \|\partial_t c_s(t)\|_\gamma \equiv \|\partial_t c_s(0)\|_\gamma = \|v(s)\|_\gamma = 1.$$

Note also that the image of the family of paths $c(\cdot)$ is a singular surface in N parametrized by the coordinates (s, t) .

Keeping this in mind, we compute

$$(5) \quad \begin{aligned} \partial_t \gamma(\partial_s c_s(t), \partial_t c_s(t)) &= \gamma\left(\frac{\nabla}{\partial t} \partial_s c_s(t), \partial_t c_s(t)\right) + \gamma\left(\partial_s c_s(t), \frac{\nabla}{\partial t} \partial_t c_s(t)\right) \\ &= \gamma\left(\frac{\nabla}{\partial s} \partial_t c_s(t), \partial_t c_s(t)\right) \\ &= \frac{1}{2} \partial_s \gamma(\partial_t c_s(t), \partial_t c_s(t)) \\ &= 0. \end{aligned}$$

Here, the second line holds because

- s and t are coordinate functions, and hence (covariant) derivatives in the two directions commute, and
- $t \mapsto c_s(t)$ is a geodesic, hence $\frac{\nabla}{\partial t} \partial_t c_s(t) = 0$.

The last line follows directly from (4).

From (5), we immediately see that

$$\gamma(\partial_s c_s(t), \partial_t c_s(t))$$

is independent of t . However, we also have that $c_s(0) = x$ for all s , implying that $\partial_s c_s(0) = 0$, thus

$$0 = \gamma(\partial_s c_s(0), \partial_t c_s(0)) = \gamma(\partial_s c_s(t), \partial_t c_s(t))$$

for all t . That is, $\partial_s c_s(t)$ and $\partial_t c_s(t)$ are orthogonal for all s and t .

We now estimate:

$$\begin{aligned} \|\alpha'(s)\|_\gamma^2 &= \left\| \frac{d}{ds} c_s(r(s)) \right\|_\gamma^2 \\ &= \|\partial_s c_s(r(s)) + r'(s) \partial_r c_s(r(s))\|_\gamma^2 \\ &= \|\partial_s c_s(r(s))\|_\gamma^2 + |r'(s)|^2 \|\partial_r c_s(r(s))\|_\gamma^2 \\ &\geq |r'(s)|^2. \end{aligned}$$

Here, in the third line, we have used orthogonality of $\partial_s c_s(t)$ and $\partial_r c_s(t)$. In the last line, we have used (4). Note that equality holds if and only if $\|\partial_s c_s(r(s))\|_\gamma \equiv 0$.

Finally, we see that

$$L(\alpha) = \int_0^1 \|\alpha'(s)\|_\gamma ds \geq \int_0^1 |r'(s)| ds \geq \left| \int_0^1 r'(s) ds \right| = |r(1) - r(0)|,$$

which proves the desired inequality. We note that the first inequality is an equality if and only if $\|\partial_s c_s(r(s))\|_\gamma \equiv 0$ (see the previous paragraph) and the second inequality is an equality if and only if $r'(s) \geq 0$ for all s . \square

Proposition 11. *Suppose $y \in V_x$ with $\exp_x^{-1}(y) = v$. Then the path*

$$\alpha : [0, 1] \rightarrow V_x, \quad \alpha(t) = \exp_x(t \cdot v)$$

satisfies $L(\alpha) = \|v\|_\gamma$, and α is of minimal length among all paths in V_x from x to y . Furthermore, α is the unique minimal path (up to reparametrization) in V_x from x to y .

Remark 12. Note that we will only show that α is minimal only among paths (or geodesics) in V_x , not all paths (or geodesics) in N . In particular, we cannot conclude from Proposition 11 that $d_\gamma(x, y) = L(\alpha)$, as is done in the case of a strong Riemannian manifold, where the neighborhood V_x contains a d_γ -ball of positive radius.

Proof. The equality $L(\alpha) = \|v\|_\gamma$ holds by Proposition 4.

A path $\eta(s)$, $s \in [0, 1]$, in V_x from x to y corresponds via \exp_x^{-1} to a path $r(s) \cdot v(s)$ in U_x with $v(s) \in S_x N$, $r(0) = 0$ and $r(1) \cdot v(1) = v$, implying $|r(1)| = \|v\|_\gamma$. By Proposition 10, we therefore have that

$$(6) \quad L(\eta) \geq \|v\|_\gamma = L(\alpha),$$

immediately implying minimality of α .

Let equality hold in (6). Again by Proposition 10, this implies $v(s)$ is constant and $r'(s) \geq 0$ for all s . However, this means that η is just a reparametrization of α , proving the second statement. \square

As an obvious result of Proposition 11, we get the following criterion for a weak Riemannian manifold to be a metric space. It requires rather strong assumptions which could probably be weakened significantly, but it will be sufficient for some purposes that we have in mind.

Theorem 13. *Let (N, γ) be a weak Riemannian manifold. Suppose that for some $x \in N$, the exponential mapping \exp_x is a diffeomorphism between an open (in the manifold topology) neighborhood U_x of $0 \in T_x N$ and N .*

Then (N, d_γ) , where d_γ is the Riemannian distance function of γ , is a metric space.

Proof. Let $y \in N$. It remains to show that if $y \neq x$, then $d(x, y) > 0$. But if $\exp_x^{-1}(y) = v$, then Proposition 11 shows that the shortest path from x to y in N is $\exp_x(t \cdot v)$, which has length $\|v\|$. Therefore $d(x, y) = \|v\| > 0$. \square

By Propositions 1 and 2, we see that \mathcal{V} and \mathcal{M}_μ satisfy the prerequisites of the theorem, so we get the following two corollaries.

Corollary 14. *The weak Riemannian metric $((\cdot, \cdot))$ (cf. (3)) induces a metric space structure on \mathcal{V} .*

Corollary 15. *The L^2 metric on \mathcal{M} induces a metric space structure on the submanifold \mathcal{M}_μ .*

$(\mathcal{M}, (\cdot, \cdot))$ itself certainly does not satisfy the prerequisites of Theorem 13, as is explicitly shown in [9, §3]. Therefore, we will have to use a different strategy in this case.

4. \mathcal{M} IS A METRIC SPACE

Proving that d is a metric will be done by finding a manifestly positive-definite metric (in the sense of metric spaces) on \mathcal{M} that in some way bounds the d -distance between two points from below, implying that it is positive. First, though, we will prove a preliminary result that bounds from below the distance between metrics with inducing differing volume forms.

4.1. Lipschitz continuity of the square root of the volume. We wish to show Lipschitz continuity of the square root of the volume function on \mathcal{M} . In fact, the following lemma shows that for any measurable $Y \subseteq M$, the function defined by

$$\tilde{g} \mapsto \sqrt{\text{Vol}(Y, \tilde{g})}$$

is Lipschitz with respect to d . Using this as a first step to proving that d is a metric takes its inspiration from [14, §3.3].

Lemma 16. *Let $g_0, g_1 \in \mathcal{M}$. Then for any measurable subset $Y \subseteq M$,*

$$\left| \sqrt{\text{Vol}(Y, g_1)} - \sqrt{\text{Vol}(Y, g_0)} \right| \leq \frac{\sqrt{n}}{4} d(g_0, g_1).$$

Proof. Let $g_t, t \in [0, 1]$, be any path from g_0 to g_1 , and define $h_t := g'_t$. We compute

$$\begin{aligned} \partial_t \text{Vol}(Y, g_t) &= \partial_t \int_Y \mu_{g_t} = \int_Y \partial_t \mu_{g_t} = \int_Y \frac{1}{2} \text{tr}_{g_t}(h_t) \mu_{g_t} \\ (7) \quad &\leq \left(\int_Y \mu_{g_t} \right)^{1/2} \left(\frac{1}{4} \int_Y \text{tr}_{g_t}(h_t)^2 \mu_{g_t} \right)^{1/2} \\ &\leq \frac{1}{2} \sqrt{\text{Vol}(Y, g_t)} \left(\int_M \text{tr}_{g_t}(h_t)^2 \mu_{g_t} \right)^{1/2}, \end{aligned}$$

where the first line follows from [3, Lemma 2.38], the second line follows from Hölder's inequality, and the last line from the nonnegativity of $\text{tr}_{g_t}(h_t)^2$. Now, let A and B be any $n \times n$ matrices, and denote their traceless parts by A^T and B^T , respectively. We then have the formula

$$\begin{aligned} (8) \quad \text{tr}(AB) &= \text{tr} \left(\left(A^T + \frac{1}{n} \text{tr}(A) I \right) \left(B^T + \frac{1}{n} \text{tr}(B) I \right) \right) \\ &= \text{tr}(A^T B^T) + \frac{1}{n} \text{tr}(A) \text{tr}(B). \end{aligned}$$

The second line follows from the fact that traceless and pure trace matrices are orthogonal in the scalar product defined by $\text{tr}(AB)$. We have also used $\text{tr} I = n$.

Using (8) with the g_t -trace and $A = B = h_t$, we see that

$$\text{tr}_{g_t}(h_t^2) = \text{tr}_{g_t}((h_t^T)^2) + \frac{1}{n} \text{tr}_{g_t}(h_t)^2,$$

implying

$$\text{tr}_{g_t}(h_t)^2 = n \left(\text{tr}_{g_t}(h_t^2) - \text{tr}_{g_t}((h_t^T)^2) \right) \leq n \text{tr}_{g_t}(h_t^2),$$

since $\text{tr}_{g_t}((h_t^T)^2) \geq 0$. Applying this to (7) gives

$$(9) \quad \begin{aligned} \partial_t \text{Vol}(Y, g_t) &\leq \frac{1}{2} \sqrt{\text{Vol}(Y, g_t)} \left(n \int_M \text{tr}_{g_t}(h_t^2) \mu_{g_t} \right)^{1/2} \\ &\leq \frac{\sqrt{n}}{2} \sqrt{\text{Vol}(Y, g_t)} \|h_t\|_{g_t}. \end{aligned}$$

We next compute

$$(10) \quad \begin{aligned} \sqrt{\text{Vol}(Y, g_1)} - \sqrt{\text{Vol}(Y, g_0)} &= \int_0^1 \partial_t \sqrt{\text{Vol}(Y, g_t)} dt = \int_0^1 \frac{1}{2} \frac{\partial_t \text{Vol}(Y, g_t)}{\sqrt{\text{Vol}(Y, g_t)}} dt \\ &\leq \int_0^1 \frac{\sqrt{n}}{4} \|h_t\|_{g_t} dt = \frac{\sqrt{n}}{4} L(g_t), \end{aligned}$$

where the inequality follows from (9). Since this holds for all paths from g_0 to g_1 , and we can repeat the computation with g_0 and g_1 interchanged, it implies the result immediately. \square

We note that Lemma 16 in particular gives a positive lower bound on the distance between two metrics in \mathcal{M} that have different total volumes—so we must now deal with the case where the two metrics have the same total volume.

4.2. A (positive-definite) metric on \mathcal{M} . For the remainder of the paper, we fix an arbitrary reference metric $g \in \mathcal{M}$.

We begin by defining a function on $\mathcal{M} \times \mathcal{M}$ and showing that it is indeed a metric.

Definition 17. Consider $\mathcal{M}_x = \{\tilde{g} \in \mathcal{S}_x \mid \tilde{g} > 0\}$ (cf. Section 2.1). Define a Riemannian metric $\langle \cdot, \cdot \rangle^0$ on \mathcal{M}_x given by

$$\langle h, k \rangle_{\tilde{g}}^0 = \text{tr}_{\tilde{g}}(hk) \det g(x)^{-1} \tilde{g} \quad \forall h, k \in T_{\tilde{g}} \mathcal{M}_x \cong \mathcal{S}_x.$$

(Recall that $g \in \mathcal{M}$ is our fixed reference element.) We denote by θ_x^g the Riemannian distance function of $\langle \cdot, \cdot \rangle^0$.

Note that θ_x^g is automatically positive definite, since it is the distance function of a Riemannian metric on a finite-dimensional manifold. By integrating it in x , we can pass from a metric on \mathcal{M}_x to a function on $\mathcal{M} \times \mathcal{M}$ as follows:

Definition 18. For any measurable $Y \subseteq M$, define a function $\Theta_Y : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ by

$$\Theta_Y(g_0, g_1) = \int_Y \theta_x^g(g_0(x), g_1(x)) \mu_g(x).$$

We have omitted the metric g from the notation for Θ_Y . The next lemma justifies this choice.

Lemma 19. Θ_Y does not depend on the choice of $g \in \mathcal{M}$ in the above definition. That is, if we choose any other $\tilde{g} \in \mathcal{M}$ and define $\langle \cdot, \cdot \rangle^0$ and $\theta_x^{\tilde{g}}$ with respect to this new reference metric, then

$$\int_Y \theta_x^g(g_0(x), g_1(x)) \mu_g(x) = \int_Y \theta_x^{\tilde{g}}(g_0(x), g_1(x)) \mu_{\tilde{g}}(x)$$

Proof. Let $\tilde{g} \in \mathcal{M}$ be any other metric. Recall that θ_x^g was the distance function associated to the Riemannian metric $\langle \cdot, \cdot \rangle^0$ on \mathcal{M}_x , and the metric g enters in the definition of this Riemannian metric. Take a path $g_t(x)$ in \mathcal{M}_x . For now, let's put g and \tilde{g} back in the notation, so that we can write formulas unambiguously. For example, if we use g to define $\langle \cdot, \cdot \rangle^0$, we write $L_g(g_t(x))$ for the length of $g_t(x)$ w.r.t. $\langle \cdot, \cdot \rangle^0$; if we use \tilde{g} in the definition, we write $L_{\tilde{g}}(g_t(x))$ for the length; and similarly for other notation.

Using the definitions of Θ_Y^g and θ_x^g , where infima are always taken over paths $g_t(x)$ from $g_0(x)$ to $g_1(x)$, and where $h_t(x) := g_t(x)'$, we can compute:

$$\begin{aligned} \Theta_Y^g(g_0, g_1) &= \int_Y \theta_x^g(g_0(x), g_1(x)) \mu_g(x) \\ &= \int_Y \left(\inf \int_0^1 \sqrt{\langle g_t(x)', g_t(x)' \rangle_{g_t(x)}^0} dt \right) \mu_g(x) \\ &= \int_Y \left(\inf \int_0^1 \sqrt{\text{tr}_{g_t(x)}(h_t(x)^2) \frac{\det g_t(x)}{\det g(x)}} dt \right) \sqrt{\det g(x)} dx^1 \cdots dx^n \\ &= \int_Y \left(\inf \int_0^1 \sqrt{\text{tr}_{g_t(x)}(h_t(x)^2) \frac{\det g_t(x)}{\det \tilde{g}(x)}} dt \right) \sqrt{\det \tilde{g}(x)} dx^1 \cdots dx^n \\ &= \Theta_Y^{\tilde{g}}(g_0, g_1), \end{aligned}$$

where the last line follows from running the first lines of the computation through in reverse. \square

Lemma 20. *Let any $Y \subseteq M$ be given. Then Θ_Y is a pseudometric on \mathcal{M} , and Θ_M is a metric (in the sense of metric spaces).*

Furthermore, if $Y_1 \subset Y_2$, then $\Theta_{Y_1}(g_0, g_1) \leq \Theta_{Y_2}(g_0, g_1)$ for all $g_0, g_1 \in \mathcal{M}$.

Proof. Nonnegativity, vanishing distance for equal elements, symmetry and the triangle inequality are clear from the corresponding properties for θ_x^g .

That Θ_M is positive definite is also not hard to prove. Since θ_x^g is a metric on \mathcal{M}_x , $\theta_x^g(g_0(x), g_1(x))$ is positive whenever $g_0(x) \neq g_1(x)$. But since g_0 and g_1 are smooth metrics, if they differ at a point, they differ over an open neighborhood of that point. Hence the integral of $\theta_x^g(g_0(x), g_1(x))$ must be positive.

The second statement follows immediately from nonnegativity of θ_x^g . \square

4.3. Proof of the main result. We have set up everything we need to prove the main result of this section—that d is a metric. To do this, we use Lemma 16 in order to control the volume of the metrics making up a path in terms of the length of that path, combined with a Hölder's inequality argument, and show that the pseudometrics Θ_Y provide a lower bound for the distance between elements of \mathcal{M} as measured by d .

Proposition 21. *For any $Y \subseteq M$ and $g_0, g_1 \in \mathcal{M}$, we have the following inequality:*

$$\Theta_Y(g_0, g_1) \leq d(g_0, g_1) \left(\sqrt{n} d(g_0, g_1) + 2\sqrt{\text{Vol}(M, g_0)} \right).$$

In particular, Θ_Y is a continuous pseudometric (w.r.t. d).

Proof. By Lemma 20, we need only prove the inequality for $Y = M$, and then it follows for any subset.

We can clearly find a path g_t from g_0 to g_1 with $L(g_t) \leq 2d(g_0, g_1)$. Then for any $\tau \in [0, 1]$, we get

$$2d(g_0, g_1) \geq L(g_t) \geq L(g_t|_{[0, \tau]}) \geq d(g_0, g_\tau) \geq \frac{4}{\sqrt{n}} \left| \sqrt{\text{Vol}(M, g_\tau)} - \sqrt{\text{Vol}(M, g_0)} \right|,$$

where the last inequality is Lemma 16. In particular, we get

$$(11) \quad \sqrt{\text{Vol}(M, g_\tau)} \leq \sqrt{\text{Vol}(M, g_0)} + \frac{\sqrt{n}}{2} d(g_0, g_1) =: V$$

for all $\tau \in [0, 1]$.

To find the length of g_t , we first integrate $\langle g_t', g_t' \rangle$ over $x \in M$, then take the square root, and finally integrate over t . Ideally, we would wish to change the order of integration, so that we first

integrate over t , then over x . We cannot do this exactly, but we can bound the computation of the length from below by an expression where we integrate in the opposite order, and this expression will involve θ_x^g and Θ_M . So let's see how this works.

Let $h_t := g_t'$. From Hölder's inequality,

$$\int_M \sqrt{\operatorname{tr}_{g_t}(h_t^2)} d\mu_{g_t} \leq \left(\int_M d\mu_{g_t} \right)^{1/2} \left(\int_M \operatorname{tr}_{g_t}(h_t^2) d\mu_{g_t} \right)^{1/2},$$

which gives

$$\begin{aligned} (12) \quad \|h_t\|_{g_t} &= \left(\int_M \operatorname{tr}_{g_t}(h_t^2) d\mu_{g_t} \right)^{1/2} \geq \frac{1}{\sqrt{\operatorname{Vol}(M, g_t)}} \int_M \sqrt{\operatorname{tr}_{g_t}(h_t^2)} d\mu_{g_t} \\ &\geq \frac{1}{V} \int_M \sqrt{\operatorname{tr}_{g_t}(h_t^2)} d\mu_{g_t}, \end{aligned}$$

where we have also used (11). To remove the t -dependence from the volume element, we use

$$\mu_{g_t} = \frac{\sqrt{\det g_t}}{\sqrt{\det g}} \mu_g = \sqrt{\det g^{-1} g_t} \mu_g.$$

We then rewrite (12) as

$$(13) \quad \|h_t\|_{g_t} \geq \frac{1}{V} \int_M \sqrt{\operatorname{tr}_{g_t}(h_t^2) \det g^{-1} g_t} \mu_g = \frac{1}{V} \int_M \sqrt{\langle h_t(x), h_t(x) \rangle_{g_t(x)}^0} \mu_g(x),$$

where we have used the Riemannian metric $\langle \cdot, \cdot \rangle^0$ on \mathcal{M}_x (cf. Definition 17).

Since we have removed the t -dependence from the measure above, we can change the order of integration in the calculation of the length of g_t :

$$\begin{aligned} L(g_t) &= \int_0^1 \|h_t\|_{g_t} dt \geq \frac{1}{V} \int_0^1 \int_M \sqrt{\langle h_t(x), h_t(x) \rangle_{g_t(x)}^0} \mu_g(x) dt \\ &= \frac{1}{V} \int_M \int_0^1 \sqrt{\langle h_t(x), h_t(x) \rangle_{g_t(x)}^0} dt \mu_g(x). \end{aligned}$$

Now we concentrate on the t -integral in the expression above. Since $g_t(x)$ is a path in \mathcal{M}_x from $g_0(x)$ to $g_1(x)$ with tangents $h_t(x)$, the t -integral is actually the length of $g_t(x)$ with respect to $\langle \cdot, \cdot \rangle^0$. But by definition, this length is bounded from below by $\theta_x^g(g_0(x), g_1(x))$. Therefore, we get the estimate

$$L(g_t) \geq \frac{1}{V} \int_M \theta_x^g(g_0(x), g_1(x)) \mu_g(x) = \frac{1}{V} \Theta_M(g_0, g_1).$$

But now the result is immediate given (11) and the fact that we have assumed $L(g_t) \leq 2d(g_0, g_1)$. \square

The previous proposition gives us the main result of this paper immediately. Since Θ_M is a (positive-definite) metric by Lemma 20, $\Theta_M(g_0, g_1) > 0$ for any $g_0 \neq g_1$. From this, Proposition 21 immediately implies that $d(g_0, g_1) > 0$ as well. Since we have already mentioned that the distance function induced by a weak Riemannian manifold is automatically a pseudometric, we have proved:

Theorem 22. *(\mathcal{M}, d), where d is the distance function induced from the L^2 metric (\cdot, \cdot) , is a metric space.*

Remark 23. As already noted, in [15], Michor and Mumford studied weak Riemannian manifolds where the induced distance between all points is zero. They also showed that the sectional curvature is unbounded from above at each point—thus, the vanishing of the distance function can be related to the fact that the space curls up on itself arbitrarily tightly. In a sense, our theorem confirms this “mechanism” for allowing the induced distance to vanish, since as we already noted in Section 2.1, the sectional curvature of $(\mathcal{M}, (\cdot, \cdot))$ is bounded from above—indeed, it is nonpositive—preventing the pathologies found by Michor and Mumford.

Theorem 22 is a first step in investigating the metric geometry of \mathcal{M} —that is, it is certainly a prerequisite for \mathcal{M} itself (rather than the quotient space where elements at distance zero are identified) to have any metric geometry at all. In a forthcoming paper [2], motivated by the appearance of $(\mathcal{M}, (\cdot, \cdot))$ in Teichmüller theory and the important results in that field concerning the completion of the Weil-Petersson metric, we will continue this study by giving a description of the completion of (\mathcal{M}, d) .

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