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The Completion of the Manifold of Riemannian Metrics

by

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THE COMPLETION OF THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. We give a description of the completion of the manifold of all smooth Riemannian metrics on a fixed smooth, closed, finite-dimensional, orientable manifold with respect to a natural metric called the L^2 metric. The primary motivation for studying this problem comes from Teichmüller theory, where similar considerations lead to a completion of the well-known Weil-Petersson metric. We give an application of the main theorem to the completions of Teichmüller space with respect to a class of metrics that generalize the Weil-Petersson metric.

Contents

1. Introduction

This is the second in a pair of papers studying the metric geometry of the Fréchet manifold $\mathcal{M}$ of all smooth Riemannian metrics on a smooth, closed, finite-dimensional, orientable manifold M. The manifold $\mathcal{M}$ carries a natural weak Riemannian metric called the L^2 metric, defined in the

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next section. In the first paper [2], we showed that the L^2 metric induces a metric space structure on $\mathcal{M}$ (a nontrivial statement for weak Riemannian metrics; see Section 2.1.3). In this paper, we will give the following description of the metric completion $\overline{\mathcal{M}}$ of $\mathcal{M}$ with respect to the L^2 metric:

Theorem. *Let $\mathcal{M}_f$ denote the space of all measurable, symmetric, finite-volume $(0,2)$-tensor fields on M that induce a positive semidefinite scalar product on each tangent space of M. For $g_0, g_1 \in \mathcal{M}_f$ and $x \in M$, we say $g_0 \sim g_1$ if the following statement holds almost surely:*

$$g_0(x) \neq g_1(x) \iff g_0(x), g_1(x) \text{ are not positive definite.}$$

Then there is a natural identification $\overline{\mathcal{M}} \cong \mathcal{M}_f/\sim$.

Note that while $\mathcal{M}$ is a space of smooth objects, we must add in points corresponding to extremely degenerate objects in order to complete it. This is a reflection of the fact that the L^2 metric is a *weak* rather than a *strong* Riemannian metric. That is, the topology it induces on the tangent spaces—the L^2 topology—is weaker than the C^∞ topology coming from the manifold structure. In essence, the incompleteness of the tangent spaces then carries over to the manifold itself.

The manifold of Riemannian metrics—along with geometric structures on it—has been considered in several contexts. It originally arose in general relativity [4], and was subsequently studied by mathematicians [5, 7, 8]. In particular, the Riemannian geometry of the L^2 metric is well understood—its curvature, geodesics, and Jacobi fields are explicitly known. The metric geometry of the L^2 metric, though, was not as clear up to this point, and this paper seeks to illuminate one aspect of that.

Our motivation for studying the completion of $\mathcal{M}$—besides the intrinsic interest to Riemannian geometers of studying this important deformation space—came largely from Teichmüller theory. If the base manifold M is a closed Riemann surface of genus larger than one, the work of Fischer and Tromba [22] gives an identification of the Teichmüller space of M with $\mathcal{M}_{-1}/\mathcal{D}_0$, where $\mathcal{M}_{-1} \subset \mathcal{M}$ is the submanifold of hyperbolic metrics and $\mathcal{D}_0$ is the group of diffeomorphisms of M that are homotopic to the identity, acting on $\mathcal{M}_{-1}$ by pull-back. The L^2 metric restricted to $\mathcal{M}_{-1}$ descends to the Weil-Petersson metric on Teichmüller space, and its completion consists of adding in points corresponding to certain cusped hyperbolic metrics. The action of the mapping class group on Teichmüller space extends to this completion, and the quotient is homeomorphic to the Deligne-Mumford compactification of the moduli space of M. In Section 6, inspired by [9, 10], we generalize the Weil-Petersson metric and use the above theorem to formulate a condition on the completion of these generalized Weil-Petersson metrics.

The paper is organized as follows:

In Section 2, we recall the necessary background on the manifold of metrics, the L^2 metric, and completions of metric spaces. We also review some nonstandard geometric notions and fix notation and conventions for the paper.

In Section 3, we complete what we call amenable subsets of $\mathcal{M}$. They are defined in such a way that we can show that the completion of these subsets with respect to the L^2 *metric* on the subset is the same as with respect to the L^2 *norm* on $\mathcal{M}$ (this will be made precise below). This completion is the first step in a bootstrapping process of understanding the full completion.

In Section 4, we introduce a notion called *ω-convergence* for Cauchy sequences in $\mathcal{M}$ that describes how a Cauchy sequence converges to an element of $\mathcal{M}_f/\sim$. It is a kind of pointwise a.e.-convergence—except on a subset where the sequence degenerates in a certain way, where no convergence can be demanded of Cauchy sequences. We then use the results of Section 3 to show that this convergence notion gives an injective map from the completion of $\mathcal{M}$ into $\mathcal{M}_f/\sim$. To do this, we need to show two things. First, we prove that every Cauchy sequence in $\mathcal{M}$ has an ω-convergent subsequence. Second, we show in two theorems that two Cauchy sequences are equivalent (in the sense of the completion of a metric space) if and only if they ω-subconverge to the same limit. This section comprises the most technically challenging portion of the paper. It also contains the following result, which is in our eyes one of the most unexpected and striking of the paper:

Proposition. *Suppose that* $g_0, g_1 \in \mathcal{M}$*, and let* $E := \operatorname{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$*. Let* d *be the Riemannian distance function of the* L^2 *metric* $(\cdot, \cdot)$*. Then there exists a constant* $C(n)$ *depending only on* $n := \dim M$ *such that*

$$d(g_0, g_1) \leq C(n) \left(\sqrt{\operatorname{Vol}(E, g_0)} + \sqrt{\operatorname{Vol}(E, g_1)} \right).$$

In particular, we have

$$\operatorname{diam}\left(\{\tilde{g} \in \mathcal{M} \mid \operatorname{Vol}(M, \tilde{g}) \leq \delta\}\right) \leq 2C(n)\sqrt{\delta}.$$

The surprising thing about this proposition is that it says that two metrics can vary wildly, but as long as they do so on a set that has small volume with respect to each, they are close together in the L^2 metric.

In Section 5, we complete the proof of the main result. This is done by continuing the bootstrapping process begun in Section 3 to see that the map defined in Section 4 is in fact surjective. That is, we prove in stages that there are Cauchy sequences ω-converging to elements in ever larger subsets of $\mathcal{M}_f/\sim$.

In Section 6, we give the application to the geometry of Teichmüller space that was mentioned above.

Several different types of sequences and convergence notions enter into this work. For the reader's convenience, we have included an appendix which summarizes the relationships between these different concepts.

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2. Preliminaries

2.1. **The Manifold of Metrics.** For the entirety of the paper, let M denote a fixed closed, orientable, n-dimensional C^∞ manifold.

The basic facts about the manifold of Riemannian metrics given in this section can be found in [3, §2.5]

We denote by S^2T^*M the vector bundle of symmetric $(0,2)$ tensors over M, and by $\mathcal{S}$ the Fréchet space of C^∞ sections of S^2T^*M. The space $\mathcal{M}$ of Riemannian metrics on M is an open subset of $\mathcal{S}$, and hence it is trivially a Fréchet manifold, with tangent space at each point canonically identified with $\mathcal{S}$. (For a detailed treatment of Fréchet manifolds, see, for example, [11]. For a more thorough treatment of the differential topology and geometry of $\mathcal{M}$, see [5].)

2.1.1. *The* L^2 *Metric.* $\mathcal{M}$ carries a natural Riemannian metric $(\cdot, \cdot)$, called the L^2 *metric*, induced by integration from the natural scalar product on S^2T^*M. Given any $g \in \mathcal{M}$ and $h, k \in \mathcal{S} \cong T_g\mathcal{M}$, we define

$$(h, k)_g := \int_M \operatorname{tr}_g(hk)\, d\mu_g.$$

Here, $\operatorname{tr}_g(hk)$ is given in local coordinates by $\operatorname{tr}(g^{-1}hg^{-1}k) = g^{ij}h_{il}g^{lm}k_{jm}$, and μ_g denotes the volume form induced by g.

Throughout the paper, we use the notation d for the distance function induced from $(\cdot,\cdot)$ by taking the infimum of the lengths of paths between two given points.

The L^2 metric is a weak Riemannian metric, which means that its induced topology on the tangent spaces of $\mathcal{M}$ is weaker than the manifold topology. This leads to some phenomena that are unfamiliar from the world of finite-dimensional Riemannian geometry, or even strong Riemannian metrics on Hilbert manifolds. For instance, the L^2 metric does not *a priori* induce a metric space structure on $\mathcal{M}$. In [2], we nevertheless showed directly that $(\mathcal{M}, d)$ is a metric space, but other strange phenomena occur—for instance, the metric space topology of $(\mathcal{M}, d)$ is weaker than the manifold topology of $\mathcal{M}$. Indeed, in Lemma 5.11 we will see the following: When considered as a subset of its completion, $\mathcal{M}$ contains no open d-ball around *any* point! For more information on weak Riemannian metrics, see [3, §2.4], [2, §3]

The basic Riemannian geometry of $(\mathcal{M}, (\cdot,\cdot))$ is relatively well understood. For example, it is known that the sectional curvature of $\mathcal{M}$ is nonpositive [7, Cor. 1.17], and the geodesics of $\mathcal{M}$ are known explicitly [7, Thm. 2.3], [8, Thm. 3.2].

We will also consider related structures restricted to a point $x \in M$. Let $\mathcal{S}_x := S^2 T_x^* M$ denote the vector space of symmetric $(0,2)$-tensors at x, and let $\mathcal{M}_x \subset \mathcal{S}_x$ denote the open subset of tensors inducing a positive definite scalar product on $T_x M$. Then $\mathcal{M}_x$ is an open submanifold of $\mathcal{S}_x$, and its tangent space at each point is canonically identified with $\mathcal{S}_x$. For each $g \in \mathcal{M}_x$, we define a scalar product $\langle \cdot,\cdot \rangle_g$ on $T_g \mathcal{M}_x \cong \mathcal{S}_x$ by setting, for all $h, k \in \mathcal{S}_x$,

$$\langle h, k \rangle_g := \operatorname{tr}_g(hk).$$

Then $\langle \cdot,\cdot \rangle$ defines a Riemannian metric on the finite-dimensional manifold $\mathcal{M}_x$.

For each $g \in \mathcal{M}$, we denote the L^2 norm induced by g on $\mathcal{S}$ with $\| \cdot \|_g$, that is, $\|h\|_g := \sqrt{(h,h)_g}$. For any $g_0, g_1 \in \mathcal{M}$, the norms $\| \cdot \|_{g_0}$ and $\| \cdot \|_{g_1}$ are equivalent [17, §IX.2]. (As Ebin pointed out [5, §4], this statement even holds if g_0 and g_1 are only required to be continuous.)

2.1.2. *Geodesics.* As noted above, the geodesic equation of $\mathcal{M}$ can be solved explicitly. We will not need the full expression for an arbitrary geodesic for our purposes, but rather only for very special geodesics.

We denote by $\mathcal{P} \subset C^\infty(M)$ the group of strictly positive smooth functions on M. This is a Fréchet Lie group that acts on $\mathcal{M}$ by pointwise multiplication. For any $g_0 \in \mathcal{M}$, the next proposition gives the geodesics of the orbit $\mathcal{P} \cdot g_0$.

Proposition 2.1 ([7, Prop. 2.1])**.** *The geodesic in $\mathcal{M}$ starting at $g_0 \in \mathcal{M}$ with initial tangent vector ρg_0, where $\rho \in C^\infty(M)$, is given by*

$$g_t = \left(1 + n\frac{t}{4}\rho\right)^{4/n} g_0. \tag{2.1}$$

In particular, $\mathcal{P} \cdot g_0$ is a totally geodesic submanifold, and the exponential mapping $\exp_{g_0}$ is a diffeomorphism from the open set $U \cdot g_0 \subset T_{g_0}(\mathcal{P} \cdot g_0)$—where U is the set of functions satisfying $\rho > -4/n$—onto $\mathcal{P} \cdot g_0$.

2.1.3. *Metric Space Structures on $\mathcal{M}$.* In [2], we proved the following theorem:

Theorem 2.2. $(\mathcal{M}, d)$*, where d is the distance function of the L^2 metric, is a metric space.*

As mentioned above, the fact that the L^2 metric is a weak Riemannian metric means that general theorems imply only that d is a pseudometric. In fact, there are examples [14, 15] of weak Riemannian metrics where the induced distance between any two points is always zero!

To prove Theorem 2.2, we defined a function on $\mathcal{M} \times \mathcal{M}$ that was manifestly a metric (in the sense of metric spaces) and showed that this metric bounded d from below in some way. We also showed that the function $\mathcal{M} \to \mathbb{R}$ sending a metric to the square root of its total volume is Lipschitz with respect to d. These results will play a role in what is to come, so we review them here, along with the relevant definitions.

First, we have the lemma on Lipschitz continuity of the square root of the volume function.

Lemma 2.3 ([2, Lemma 17])**.** *Let $g_0, g_1 \in \mathcal{M}$. Then for any measurable subset $Y \subseteq M$,*

$$\left| \sqrt{\operatorname{Vol}(Y, g_1)} - \sqrt{\operatorname{Vol}(Y, g_0)} \right| \leq \frac{\sqrt{n}}{4} d(g_0, g_1).$$

Next, we define the metric on $\mathcal{M}$ that was mentioned above and state the lower bound it provides on d.

Definition 2.4. For each $x \in M$, consider $\mathcal{M}_x = \{\tilde{g} \in \mathcal{S}_x \mid \tilde{g} > 0\}$ (cf. Section 2.1.1). For any fixed $g \in \mathcal{M}$, define a Riemannian metric $\langle \cdot, \cdot \rangle^0$ on $\mathcal{M}_x$ by

$$\langle h, k \rangle^0_{\tilde{g}} = \operatorname{tr}_{\tilde{g}}(hk) \det g(x)^{-1} \tilde{g} \qquad \forall h, k \in T_{\tilde{g}} \mathcal{M}_x \cong \mathcal{S}_x.$$

We denote by θ^g_x the Riemannian distance function of $\langle \cdot, \cdot \rangle^0$.

Note that θ^g_x is automatically positive definite, since it is the distance function of a Riemannian metric on a finite-dimensional manifold. By integrating it in x, we can pass from a metric on $\mathcal{M}_x$ to a function on $\mathcal{M} \times \mathcal{M}$ as follows:

Lemma 2.5 ([2, Lemma 20, 21])**.** *For any measurable $Y \subseteq M$, define a function $\Theta_Y : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ by*

$$\Theta_Y(g_0, g_1) = \int_Y \theta^g_x(g_0(x), g_1(x)) \, d\mu_g(x).$$

Then Θ_Y does not depend upon the choice of metric g used to define θ^g_x. Furthermore, Θ_Y is a pseudometric on $\mathcal{M}$, and Θ_M is a metric. Finally, if $Y_0 \subseteq Y_1$, then $\Theta_{Y_0}(g_0, g_1) \leq \Theta_{Y_1}(g_0, g_1)$ for all $g_0, g_1 \in \mathcal{M}$.

The lower bound on d is the following:

Proposition 2.6 ([2, Prop. 22])**.** *For any $Y \subseteq M$ and $g_0, g_1 \in \mathcal{M}$, we have the following inequality:*

$$\Theta_Y(g_0, g_1) \leq d(g_0, g_1) \left(\sqrt{n} \, d(g_0, g_1) + 2\sqrt{\operatorname{Vol}(M, g_0)} \right).$$

In particular, Θ_Y is a continuous pseudometric (w.r.t. d).

2.2. **Completions of Metric Spaces.** To fix notation and recall a few elementary points, we briefly review the completion of a metric space. We will simply state the definition and explore a couple of consequences of it, then give an alternative, equivalent viewpoint for path metric spaces.

The *precompletion* of (X, δ) is the set $\overline{(X, \delta)}^{\mathrm{pre}}$, usually just denoted by $\overline{X}^{\mathrm{pre}}$, consisting of all Cauchy sequences of X, together with the distance function

$$\delta(\{x_k\}, \{y_k\}) := \lim_{k \to \infty} \delta(x_k, y_k).$$

(We denote the distance function of the precompletion of a space using the same symbol as for the space itself; which distance function is meant will always be clear from the context.)

The *completion* of (X, δ) is a quotient space of $\overline{X}^{\mathrm{pre}}$, $\overline{X} := \overline{X}^{\mathrm{pre}}/\sim$, where $\sim$ is the equivalence relation defined by

$$\{x_k\} \sim \{y_k\} \iff \delta(\{x_k\}, \{y_k\}) = 0. \tag{2.2}$$

Note that if $\{x_k\}$ is a Cauchy sequence in X and $\{x_{k_l}\}$ is a subsequence, then clearly $\{x_{k_l}\} \sim \{x_k\}$. Thus, given an element of the precompletion of X, we can always pass to a subsequence and still be talking about the same element of the completion.

Recall that a *path metric space* is a metric space for which the distance between any two points coincides with the infimum of the lengths of rectifiable curves joining the two points. (We will also call a rectifiable curve a *finite-length path* or simply a *finite path*.)

The following theorem describes the completion of a path metric space in terms of finite paths. Its proof is straightforward.

Theorem 2.7 ([3, Thm. 2.2]). *Let (X, δ) be a path metric space. Then the following description of the completion of (X, δ) is equivalent to the definition given above, in the sense that there exists a homeomorphism between the two completions.*

Define the precompletion $\overline{X}^{\mathrm{pre}}$ of X to be the set of rectifiable curves $\alpha : (0, 1] \to X$. It carries the pseudometric

$$\delta(\alpha_0, \alpha_1) := \lim_{t \to 0} \delta(\alpha_0(t), \alpha_1(t)). \tag{2.3}$$

Then the completion of (X, δ) is $\overline{X} := \overline{X}^{\mathrm{pre}} / \sim$, where $\alpha_0 \sim \alpha_1 \Longleftrightarrow \delta(\alpha_0, \alpha_1) = 0$.

2.3. **Geometric Preliminaries.** We now give some of the nonstandard geometric facts that we will need, in order to fix notation and recall the relevant notions.

2.3.1. *Sections of the Endomorphism Bundle of M.* Given a section H of the endomorphism bundle of M, the determinant, trace, and eigenvalues of H are well-defined functions over M. Furthermore, if H is measurable/continuous/smooth, then the determinant and trace will be so as well, since they are smooth functions from the space of $n \times n$ matrices into $\mathbb{R}$.

The following proposition allows us to characterize positive definite and positive semidefinite $(0, 2)$-tensors.

Proposition 2.8 ([12, Thm. 7.2.1]). *A symmetric $n \times n$ matrix T is positive definite (resp. positive semidefinite) if and only if all eigenvalues of T are positive (resp. nonnegative).*

In particular, if T is positive definite (resp. positive semidefinite), then $\det T > 0$ (resp. $\det T \geq 0$). If T is positive semidefinite but not positive definite, then $\det T = 0$.

We also need a result on the eigenvalues of a section of the endomorphism bundle.

Lemma 2.9 ([3, Lemma 2.11]). *Let h be any continuous, symmetric $(0, 2)$-tensor field. Suppose g is a Riemannian metric on M, and let H be the $(1, 1)$-tensor field obtained from h by raising an index using g. (That is, locally $H^i_j = g^{ik} h_{kj}$.) Then H is a continuous section of the endomorphism bundle $\mathrm{End}(M)$. Denote by $\lambda^H_{\min}(x)$ the smallest eigenvalue of $H(x)$. We have that*

1. *$\lambda^H_{\min}$ is a continuous function and*
2. *if h is positive definite, then $\min_{x \in M} \lambda^H_{\min}(x) > 0$.*

Furthermore, if $\lambda^H_{\max}(x)$ denotes the largest eigenvalue of $H(x)$, then $\lambda^H_{\max}$ is a continuous and hence bounded function.

Proof. From the min-max theorem [19, Thm. XIII.1], one can show that the map $A \mapsto \lambda^A_{\min}$ (resp. $A \mapsto \lambda^A_{\max}$) is a concave (resp. convex) function from the space of $n \times n$ symmetric matrices to $\mathbb{R}$. Thus both mappings are continuous [20, Thm. 10.1]. The proof of the continuity of $\lambda^H_{\min}$ and $\lambda^H_{\max}$ then follows via a standard argument using compactness of the sphere bundle $SM \subset TM$.

The bound on the minimal eigenvalue follows from continuity and Proposition 2.8. □

2.3.2. *Lebesgue Measure on Manifolds.* The concept of Lebesgue measurability carries over from $\mathbb{R}^n$ to smooth (or even topological) manifolds by simply declaring a subset to be Lebesgue measurable if its restriction to coordinates is Lebesgue measurable. Since the transition functions are smooth, this is independent of the chosen coordinates. Transition functions will also map nullsets to nullsets, so this notion is well-defined.

Convention 2.10. Whenever we refer to a measure-theoretic concept on M, we implicitly mean that we work with Lebesgue measure or Lebesgue sets, unless we explicitly state otherwise.

With Lebesgue measurable sets well-defined, the concept of a measurable function or a measurable map between manifolds is also well-defined. We can also speak about Lebesgue measures—for example, any nonnegative n-form μ on M with measurable coefficients induces a Lebesgue measure on M.

It is not hard to see that the same relation between Lebesgue measurable sets and Borel measurable sets that holds on $\mathbb{R}^n$ [18, §3.11] also holds on M. Namely, any Lebesgue measurable set E can be decomposed as $E = F \cup G$, where F is Borel measurable and G is a Lebesgue-nullset.

2.4. **Notation and Conventions.** Before we begin with the main body of the work, we will describe all nonstandard notation and conventions that will be used throughout the text.

The first thing we do is fix a reference metric, with respect to which all standard concepts will be defined.

Convention 2.11. For the remainder of the paper, we fix an element $g \in \mathcal{M}$. Whenever we refer to the L^p norm, L^p topology, L^p convergence etc., we mean that induced by g unless we explicitly state otherwise. The designation nullset refers to Lebesgue measurable subsets of M that have zero measure with respect to μ_g. If we say that something holds almost everywhere, we mean that it holds outside of a μ_g-nullset.

If we have a tensor $h \in \mathcal{S}$, we denote by the capital letter H the tensor obtained by raising an index with g, i.e., locally $H^i_j := g^{ik} h_{kj}$. Given a point $x \in M$ and an element $a \in \mathcal{M}_x$, the capital letter A means the same—i.e., we assume some coordinates and write $A = g(x)^{-1} a$, though for readability we will generally omit x from the notation.

Next, we'll fix an atlas of coordinates on M that is convenient to work with.

Definition 2.12. We call a finite atlas of coordinates $\{(U_\alpha, \phi_\alpha)\}$ for M *amenable* if for each U_α, there exist a compact set K_α and a different coordinate chart (V_α, ψ_α) (which does not necessarily belong to $\{(U_\alpha, \phi_\alpha)\}$) such that

$$U_\alpha \subset K_\alpha \subset V_\alpha \quad \text{and} \quad \phi_\alpha = \psi_\alpha | U_\alpha.$$

Convention 2.13. For the remainder of this paper, we work over a fixed amenable coordinate atlas $\{(U_\alpha, \phi_\alpha)\}$ for all computations and concepts that require local coordinates.

The next lemma heuristically says the following: in amenable coordinates, the coordinate representations of a smooth metric are somehow "uniformly positive definite". Additionally, the coefficients satisfy a uniform upper bound.

Lemma 2.14. *For any metric $\tilde{g} \in \mathcal{M}$, there exist constants $\delta(\tilde{g}) > 0$ and $C(\tilde{g}) < \infty$, depending only on $\tilde{g}$, with the property that for any α, any $x \in U_\alpha$, and $1 \leq i, j \leq n$,*

$$|\tilde{g}_{ij}(x)| \leq C(\tilde{g}) \text{ and } \lambda^{\tilde{G}}_{\min}(x) \geq \delta(\tilde{g}), \tag{2.4}$$

where we of course mean the value of $\tilde{g}_{ij}(x)$ in the chart (U_α, ϕ_α).

Proof. Note that Definition 2.12 implies that $\phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ is a relatively compact subset of $\psi_\alpha(V_\alpha)$. Thus, the proof of the first inequality is immediate and the second is clear from Lemma 2.9. □

Remark 2.15. The estimate $|\tilde{g}_{ij}(x)| \leq C(\tilde{g})$ also implies an upper bound in terms of $C(\tilde{g})$ on $\det \tilde{g}(x)$. This is clear from the fact that the determinant is a homogeneous polynomial in $\tilde{g}_{ij}(x)$ with $n!$ terms and coefficients ± 1.

The main point of using a fixed amenable coordinate atlas is the following: it gives us an easily understood and uniform—but nevertheless coordinate-dependent—notion of how "large" or "small" a metric is. The dependence of this notion on coordinates is perhaps somewhat dissatisfying at first glance, but it should be seen as merely an aid in our quest to prove statements that are, indeed, invariant in nature.

It is necessary to introduce somewhat more general objects than Riemannian metrics in this paper:

Definition 2.16. Let $\tilde{g}$ be a section of S^2T^*M. Then $\tilde{g}$ is called a *(Riemannian) semimetric* if it induces a positive semidefinite scalar product on T_xM for each $x \in M$.

We now make a couple of definitions on semimetrics and sequences of metrics.

Definition 2.17. Let $\tilde{g}$ be a semimetric on M (which we do not assume to be even measurable). We define the set

$$X_{\tilde{g}} := \{x \in M \mid \tilde{g}(x) \text{ is not positive definite}\} \subset M,$$

which we call the *deflated set* of $\tilde{g}$.

We call $\tilde{g}$ *bounded* if there exists a constant C such that

$$|\tilde{g}_{ij}(x)| \leq C$$

for a.e. $x \in M$ and all $1 \leq i, j \leq n$. Otherwise $\tilde{g}$ is called *unbounded*.

Definition 2.18. Let $\{g_k\} \subset \mathcal{M}$ be any sequence. We define the set

$$D_{\{g_k\}} := \{x \in M \mid \forall \delta > 0,\ \exists k \in \mathbb{N} \text{ s.t. } \det G_k(x) < \delta\},$$

which we call the *deflated set* of $\{g_k\}$.

The last definition we need in this vein distinguishes smooth metrics from (possibly nonsmooth) semimetrics.

Definition 2.19. A semimetric $\tilde{g}$ is called *degenerate* if $\tilde{g} \notin \mathcal{M}$ and *nondegenerate* if $\tilde{g} \in \mathcal{M}$.

Note that any measurable semimetric $\tilde{g}$ on M induces a nonnegative measure on M that is absolutely continuous with respect to the fixed volume form μ_g.

A measurable Riemannian semimetric $\tilde{g}$ on M gives rise to an "L^2 scalar product" on measurable functions in the following way. For any two functions ρ and σ on M, we define, as usual, $\mu_{\tilde{g}} := \sqrt{\det \tilde{g}}\, dx^1 \cdots dx^n$ and

$$(\rho, \sigma)_{\tilde{g}} := \int_M \rho\sigma\, \mu_{\tilde{g}}. \tag{2.5}$$

(We denote this by the same symbol as the L^2 scalar product on $\mathcal{S}$; which is meant will always be clear from the context.) We put "L^2 scalar product" in quotation marks because unless we put specific conditions on ρ, σ, and $\tilde{g}$, (2.5) is not guaranteed to be finite. It suffices, for example, to demand that ρ and σ are bounded and that the total volume $\mathrm{Vol}(M, \tilde{g}) = \int_M \mu_{\tilde{g}}$ of $\tilde{g}$ is finite. As in the case of the L^2 scalar product on $\mathcal{S}$, if g_0 and g_1 are both continuous metrics, then $(\cdot, \cdot)_{g_0}$ and $(\cdot, \cdot)_{g_1}$ are equivalent scalar products on $C^\infty(M)$. Therefore they induce the same topology, which we call the L^2 *topology*.

3. Amenable Subsets

We begin the study of the completion of $\mathcal{M}$ in this section, by first completing very special subsets of $\mathcal{M}$ called *amenable subsets* (defined below). The main result of the section is that the completion of such a subset with respect to d coincides with the completion with respect to the L^2 norm on $\mathcal{S}$, the vector space in which $\mathcal{M}$ resides.

3.1. **Amenable Subsets and their Basic Properties.** For the following definition, recall that we work over an amenable atlas (cf. Definition 2.12).

Definition 3.1. We call a subset $\mathcal{U} \subset \mathcal{M}$ *amenable* if $\mathcal{U}$ is convex and we can find constants $C, \delta > 0$ such that for all $\tilde{g} \in \mathcal{U}$, $x \in M$ and $1 \leq i, j \leq n$,

$$\lambda_{\min}^{\tilde{G}}(x) \geq \delta$$

(where we recall that $\tilde{G} = g^{-1}\tilde{g}$, with g our fixed metric) and

$$|\tilde{g}_{ij}(x)| \leq C.$$

Remark 3.2. We make a couple of remarks about the definition:

(1) The requirement that $\mathcal{U}$ is convex is technical, and is there to insure that we can consider simple, straight-line paths between points of $\mathcal{U}$ to estimate the distance between them.
(2) Recall that the function sending a matrix to its minimal eigenvalue is concave (cf. the proof of Lemma 2.9). Also, the absolute value function on $\mathbb{R}$ is convex by the triangle inequality. Therefore, the two bounds given in Definition 3.1 are compatible with the requirement of convexity.

One useful property the metrics $\tilde{g}$ of an amenable subset have is that the Radon-Nikodym derivatives $(\mu_{\tilde{g}}/\mu_g)$, with respect to the reference volume form μ_g, are bounded away from zero and infinity independently of $\tilde{g}$.

Lemma 3.3. *Let $\mathcal{U}$ be an amenable subset. Then there exists a constant $K > 0$ such that for all $\tilde{g} \in \mathcal{U}$,*

$$\frac{1}{K} \leq \left(\frac{\mu_{\tilde{g}}}{\mu_g}\right) \leq K \tag{3.1}$$

Proof. First, we note that

$$\left(\frac{\mu_{\tilde{g}}}{\mu_g}\right) = \det \tilde{G} \quad \text{and} \quad \left(\frac{\mu_{\tilde{g}}}{\mu_g}\right)^{-1} = \left(\frac{\mu_g}{\mu_{\tilde{g}}}\right) = (\det \tilde{G})^{-1}.$$

So the bounds (3.1) are equivalent to upper bounds on both $\det \tilde{G}$ and $(\det \tilde{G})^{-1}$.

Now, if the eigenvalues of $\tilde{G}$ are $\lambda_1^{\tilde{G}}, \dots, \lambda_n^{\tilde{G}}$, then

$$\det \tilde{G} = \lambda_1^{\tilde{G}} \cdots \lambda_n^{\tilde{G}} \geq \left(\lambda_{\min}^{\tilde{G}}\right)^n \geq \delta^n,$$

where δ is the constant guaranteed by the fact that $\tilde{g} \in \mathcal{U}$. This allows us to bound $(\det \tilde{G})^{-1}$ from above.

To bound $\det \tilde{G}$ from above, it is sufficient to bound the absolute value of the coefficients of $\tilde{G} = g^{-1}\tilde{g}$ from above. But bounds on the coefficients of $\tilde{g}$ are already assured by the fact that $\tilde{g} \in \mathcal{U}$, and bounds on the coefficients of g^{-1} are guaranteed by the fact that g^{-1} is a fixed, smooth cometric on M. So we are finished. ∎

Amenable subsets guarantee good behavior of the norms on $\mathcal{S}$ that are defined by their members—namely, the norms are in some sense "uniformly equivalent". More precisely, we have:

Lemma 3.4. *Let $\mathcal{U} \subset \mathcal{M}$ be an amenable subset. Then there exists a constant K such that for all pairs $g_0, g_1 \in \mathcal{U}$ and all $h \in \mathcal{S}$,*

$$\frac{1}{K} \|h\|_{g_1} \leq \|h\|_{g_0} \leq K \|h\|_{g_1}.$$

Proof. We will show that the norm of each $\tilde{g} \in \mathcal{U}$ is equivalent to that of the fixed reference metric g with a fixed constant K. This is equivalent to the following statement. Let

$$T_{\tilde{g}} : (S^2T^*M, \langle \cdot, \cdot \rangle_{\tilde{g}}) \to (S^2T^*M, \langle \cdot, \cdot \rangle_g)$$

be the identity mapping on the level of sets, sending the bundle S^2T^*M with the Riemannian structure $\langle \cdot, \cdot \rangle_{\tilde{g}}$ to itself with the Riemannian structure $\langle \cdot, \cdot \rangle_g$. Let $N(T_{\tilde{g}})(x)$ be the operator norm of $T_{\tilde{g}}(x) : \mathcal{S}_x \to \mathcal{S}_x$, and let $N(T_{\tilde{g}}^{-1})(x)$ be defined similarly. Then the statement on norms holds if and only if there are constants K_0 and K_1 such that

$$N(T_{\tilde{g}})(x)^2, N(T_{\tilde{g}}^{-1})(x)^2 \leq K_0 \quad \text{and} \quad (\mu_g/\mu_{\tilde{g}}), (\mu_{\tilde{g}}/\mu_g) \leq K_1.$$

But $N(T_{\tilde{g}})$ and $N(T_{\tilde{g}}^{-1})$ are continuous functions on the compact manifold M for fixed $\tilde{g}$. Secondly, we notice that $N(T_{\tilde{g}})(x)$ and $N(T_{\tilde{g}}^{-1})(x)$ depend only on the coordinate representations of $\tilde{g}(x)$ and $g(x)$. But $\tilde{g}(x)$ and $g(x)$ can only range over a compact subset of the space of positive definite $n \times n$ symmetric matrices, because $\tilde{g}$ ranges over an amenable subset. This implies the existence of K_0.

The existence of K_1 is immediately implied by Lemma 3.3. □

Lemma 3.3 immediately implies that the function $\tilde{g} \mapsto \operatorname{Vol}(M, \tilde{g})$ is bounded when restricted to any amenable subset. Recalling the form of the estimate in Proposition 2.6 then shows the following lemma.

Lemma 3.5. *Let $\mathcal{U}$ be an amenable subset and $g \in \mathcal{M}$. Then there exists a constant V such that for any $g_0, g_1 \in \mathcal{U}$ and $Y \subset M$,*

$$\Theta_Y(g_0, g_1) \leq 2d(g_0, g_1) \left(\frac{2\sqrt{n}}{4} d(g_0, g_1) + \sqrt{V} \right).$$

More precisely, $V = \sup_{\tilde{g} \in \mathcal{U}} \operatorname{Vol}(M, \tilde{g})$, which is finite by the discussion preceding the lemma.

3.2. **The Completion of $\mathcal{U}$ with Respect to d and $\|\cdot\|_g$.** We are now ready to prove a result that, in particular, implies equivalence of the topologies defined by d and $\|\cdot\|_g$ on an amenable subset $\mathcal{U}$.

Theorem 3.6. *Consider the L^2 topology on $\mathcal{M}$ induced from the scalar product $(\cdot, \cdot)_g$ (where g is fixed). Let $\mathcal{U} \subset \mathcal{M}$ be any amenable subset.*

Then the L^2 topology on $\mathcal{U}$ coincides with the topology induced from the restriction of the Riemannian distance function d of $\mathcal{M}$ to $\mathcal{U}$.

Additionally, the following holds:

1. *There exists a constant K such that $d(g_0, g_1) \leq K \|g_1 - g_0\|_g$ for all $g_0, g_1 \in \mathcal{U}$.*
2. *For any $\epsilon > 0$, there exists $\delta > 0$ such that if $d(g_0, g_1) < \delta$, then $\|g_0 - g_1\|_g < \epsilon$.*

Proof. To prove (1), consider the linear path g_t from g_0 to g_1. Note that we can clearly find an amenable subset $\mathcal{U}'$ containing $\mathcal{U}$ and g. We then have

$$L(g_t) = \int_0^1 \left\| (g_t)' \right\|_{g_t} dt = \int_0^1 \|g_1 - g_0\|_{g_t} dt \leq \int_0^1 K \|g_1 - g_0\|_g \, dt = K \|g_1 - g_0\|_g, \tag{3.2}$$

where K is the constant associated to $\mathcal{U}'$ guaranteed by Lemma 3.4. Since $d(g_0, g_1) \leq L(g_t)$ and the constant K depends only on the set $\mathcal{U}$, this inequality is shown.

Since $\mathcal{M}_x$ is a finite-dimensional Riemannian manifold, the topology induced from θ_x^g is the same as the manifold topology, which in turn is given by any norm on $\mathcal{S}_x$. For instance this norm is given by the scalar product $\langle \cdot, \cdot \rangle_{g(x)}$ on $\mathcal{S}_x$, which we recall is given by

$$\langle h, k \rangle_{g(x)} = \operatorname{tr}_{g(x)}(hk) \tag{3.3}$$

for $h, k \in \mathcal{S}_x$. That these two topologies are the same implies, in particular, that for all $\zeta > 0$ and $\tilde{g} \in \mathcal{M}_x$, we can find $\kappa > 0$ such that

$$B_{\tilde{g}}^{\theta_x^g}(\zeta) \subset B_{\tilde{g}}^{\langle \cdot,\cdot \rangle_{g(x)}}(\kappa),$$

where the above denotes the θ_x^g-ball of radius ζ around $\tilde{g}$ and the $\langle \cdot,\cdot \rangle_{g(x)}$-ball of radius κ around $\tilde{g}$, respectively.

Now, for $x \in M$ and $\tilde{g} \in \mathcal{M}$, we define a function $\eta_{x,\tilde{g}}(\zeta)$ by

$$\eta_{x,\tilde{g}}(\zeta) := \inf \left\{ \kappa \in \mathbb{R} \mid B_{\tilde{g}(x)}^{\theta_x^g}(\zeta) \subset B_{\tilde{g}(x)}^{\langle \cdot,\cdot \rangle_{g(x)}}(\kappa) \right\}$$

Then, because of the smooth dependence of $\langle \cdot,\cdot \rangle^0$ and $\langle \cdot,\cdot \rangle_{g(x)}$ on x, $\eta_{x,\tilde{g}}(\zeta)$ is continuous separately in x and $\tilde{g}$. If we define

$$\mathcal{U}_x := \{ \hat{g}(x) \mid \hat{g} \in \mathcal{U} \},$$

then $\mathcal{U}_x$ is a relatively compact subset of $\mathcal{M}_x$, since $\mathcal{U}$ is amenable. Since M is also compact, for any fixed $\zeta > 0$, we can define a function

$$\eta(\zeta) := \sup_{x \in M,\ \tilde{g} \in \mathcal{U}} \eta_{x,\tilde{g}}(\zeta) < \infty.$$

It follows from the definition that $\eta(\zeta) \to 0$ for $\zeta \to 0$.

Because of the relative compactness of $\mathcal{U}_x$ for each $x \in M$, together with compactness of M, there exists a constant C_0 such that $\theta_x^g(g_0(x), g_1(x)) \leq C_0$ for all $g_0, g_1 \in \mathcal{U}$ and $x \in M$. This implies immediately that

$$\Theta_M(g_0, g_1) = \int_M \theta_x^g(g_0(x), g_1(x))\, \mu_g(x) \leq C_0 \operatorname{Vol}(M, g).$$

Now, choose $\zeta > 0$ small enough that $\eta(\zeta) < \epsilon / \sqrt{2 \operatorname{Vol}(M, g)}$.

By Lemma 3.5, there exists a constant V such that

$$\Theta_M(g_0, g_1) \leq 2d(g_0, g_1) \left(\frac{2\sqrt{n}}{4} d(g_0, g_1) + \sqrt{V} \right) \tag{3.4}$$

for all $g_0, g_1 \in \mathcal{U}$.

Choose δ small enough that

$$2\delta \left(\frac{2\sqrt{n}}{4} \delta + \sqrt{V} \right) < \frac{\epsilon^2 \zeta}{2\eta(C_0)^2}.$$

We claim that $d(g_0, g_1) < \delta$ implies that $\|g_1 - g_0\|_g < \epsilon$. Note that the choices of ζ and C_0 were made independently of g_0 and g_1, hence δ is independent of g_0 and g_1, as required.

We define two closed subsets of M by

$$M_+ := \{ x \in M \mid \theta_x^g(g_0(x), g_1(x)) \geq \zeta \},$$
$$M_- := \{ x \in M \mid \theta_x^g(g_0(x), g_1(x)) \leq \zeta \}.$$

From (3.4) and our choice of δ, we have that

$$\zeta \operatorname{Vol}(M_+, g) = \zeta \int_{M_+} \mu_g \leq \int_{M_+} \theta_x^g(g_0(x), g_1(x))\, \mu_g(x) \leq \Theta_M(g_0, g_1) < \frac{\epsilon^2 \zeta}{2\eta(C_0)^2}. \tag{3.5}$$

implying $\operatorname{Vol}(M_+, g_0) < \epsilon^2 / 2\eta(C_0)^2$.

From the definitions of M_- and η, we have that

$$\sqrt{\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)}} \leq \eta(\zeta)$$

on M_-. From $\theta_x^g(g_0(x), g_1(x)) \leq C_0$, we have that

$$\sqrt{\langle g_1(x) - g_0(x), g_1(x) - g_0(x) \rangle_{g(x)}} \leq \eta(C_0)$$

on all of M, and in particular on M_+. Using this, we compute

$$\|g_1 - g_0\|_g^2 = \int_M \langle g_1(x) - g_0(x), g_1(x) - g_0(x)\rangle_{g(x)} \, \mu_g(x) \leq \eta(\zeta)^2 \int_{M_-} \mu_g + \eta(C_0)^2 \int_{M_+} \mu_g$$
$$< \eta(\zeta)^2 \operatorname{Vol}(M, g) + \eta(C_0)^2 \frac{\epsilon^2}{2\eta(C_0)^2} < \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.$$

This proves the second statement. □

Theorem 3.6 will give us our first result regarding the completion of $\mathcal{M}$. First, though, we need to make some definitions and prove a statement about metric spaces.

Definition 3.7. If $\mathcal{U} \subset \mathcal{M}$ is any subset, we denote by $\mathcal{U}^0$ the L^2-completion of $\mathcal{U}$ (that is, the completion of $\mathcal{U}$ with respect to $\| \cdot \|_g$).

Let's look back at Theorem 3.6 again. The first statement says that for any amenable subset $\mathcal{U}$ and any $g \in \mathcal{M}$, d is Lipschitz continuous with respect to $\| \cdot \|_g$ when viewed as a function on $\mathcal{U} \times \mathcal{U}$. The second statement says that $\| \cdot \|_g$ is uniformly continuous on $\mathcal{U} \times \mathcal{U}$ with respect to d. To put this knowledge to good use, we will need the following lemma:

Lemma 3.8. *Let X be a set, and let two metrics, d_1 and d_2, be defined on X. Denote by $\phi : (X, d_1) \to (X, d_2)$ the map which is the identity on the level of sets, i.e., ϕ simply maps $x \mapsto x$. Finally, denote by $\overline{X}^1$ and $\overline{X}^2$ the completions of X with respect to d_1 and d_2, respectively.*

If both ϕ and ϕ^{-1} are uniformly continuous, then there is a natural homeomorphism between $\overline{X}^1$ and $\overline{X}^2$.

Proof. The proof follows in a straightforward manner from the definition of the completion of a metric space from Section 2.2, and the fact that a uniformly continuous function maps Cauchy sequences to Cauchy sequences. The natural homeomorphism is given by the unique uniformly continuous extension of ϕ to $\overline{X^1}$. □

Now, Theorem 3.6 and Lemma 3.8 immediately imply

Theorem 3.9. *Let $\mathcal{U}$ be an amenable subset. Then we can identify $\overline{\mathcal{U}}$, the completion of $\mathcal{U}$ with respect to d, with $\mathcal{U}^0$, in the sense of Lemma 3.8. We can make the natural homeomorphism $\overline{\mathcal{U}} \to \mathcal{U}^0$ into an isometry by placing a metric on $\mathcal{U}^0$ defined by*

$$d(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1),$$

where $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $\mathcal{U}$ that L^2-converge to g_0 and g_1, respectively.

We have thus found a nice description of the completion of very special subsets of $\mathcal{M}$. As already discussed, our plan now is to start removing the nice properties that allowed us to understand amenable subsets so clearly, advancing through the completions of ever larger and more generally defined subsets of $\mathcal{M}$. Before that, though, we need to study general Cauchy sequences in $\mathcal{M}$ more closely in the next section.

4. Cauchy sequences and ω-convergence

In this chapter, we introduce and study a fundamental notion of convergence of our own invention for d-Cauchy sequences in $\mathcal{M}$. We call this *ω-convergence*, and its importance is made clear through two theorems we will prove, an existence and a uniqueness result. The existence result, proved in Section 4.1, says that every d-Cauchy sequence has a subsequence that ω-converges to a measurable semimetric, which we will then show has finite total volume. The uniqueness result, proved in Section 4.3, is that two ω-convergent Cauchy sequences in $\mathcal{M}$ are equivalent (in the sense of (2.2)) if and only if they have the same ω-limit. These results allow us to identify an equivalence class of

d-Cauchy sequences with the unique ω-limit that its representatives subconverge to, and thus give a geometric meaning to points of $\overline{\mathcal{M}}$.

4.1. **Existence of the ω-Limit.** We begin this section with an important estimate and some examples, followed by the definition of ω-convergence and some of its basic properties. After that, we start on the existence proof by showing a pointwise version, i.e., an analogous result on $\mathcal{M}_x$. Finally, we globalize this pointwise result to show the existence of an ω-convergent subsequence for any Cauchy sequence in $\mathcal{M}$.

4.1.1. *Volume-Based Estimates on d and Examples.* The following surprising proposition shows us that two metrics that differ only on a subset with small (intrinsic) volume are close with respect to d.

Proposition 4.1. *Suppose that $g_0, g_1 \in \mathcal{M}$, and let $E := \operatorname{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such that*

$$d(g_0, g_1) \leq C(n) \left(\sqrt{\operatorname{Vol}(E, g_0)} + \sqrt{\operatorname{Vol}(E, g_1)} \right). \tag{4.1}$$

In particular, we have

$$\operatorname{diam}\left(\{\tilde{g} \in \mathcal{M} \mid \operatorname{Vol}(M, \tilde{g}) \leq \delta\}\right) \leq 2C(n)\sqrt{\delta}.$$

Proof. The second statement follows immediately from the first, so we only prove the first.

The heuristic idea is the following. We want to construct a family of paths with three pieces, depending on a real parameter s, such that the metrics do not change on $M \setminus E$ as we travel along the paths. Therefore, we pretend that we can restrict all calculations to E. On E, the first piece of the path is the straight line from g_0 to sg_0 for some small positive number s. It is easy to compute a bound for the length of this path based on $\operatorname{Vol}(E, g_0)$. The second piece is the straight line from sg_0 to sg_1, which, as we will see, has length approaching zero for $s \to 0$. The last piece is the straight line from sg_1 to g_1, which again has length bounded from above by an expression involving $\operatorname{Vol}(E, g_1)$.

Our job is to now take this heuristic picture, which uses paths of L^2 metrics, and construct a family of paths of *smooth* metrics that captures the essential properties.

For each $k \in \mathbb{N}$ and $s \in (0, 1]$, we define three families of metrics as follows. Choose closed sets $F_k \subseteq E$ and open sets U_k containing E such that $\operatorname{Vol}(U_k, g_i) - \operatorname{Vol}(F_k, g_i) \leq 1/k$ for $i = 0, 1$. (This is possible because the Lebesgue measure is regular.) Let $f_{k,s} \in C^\infty(M)$ be functions with the following properties:

(1) $f_{k,s}(x) = s$ if $x \in F_k$,
(2) $f_{k,s}(x) = 1$ if $x \notin U_k$ and
(3) $s \leq f_{k,s}(x) \leq 1$ for all $x \in M$.

Now, for $t \in [0, 1]$, define

$$\hat{g}^{k,s}_t := ((1-t) + tf_{k,s})g_0, \qquad \bar{g}^{k,s}_t := f_{k,s}((1-t)g_0 + tg_1), \qquad \tilde{g}^{k,s}_t := ((1-t) + tf_{k,s})g_1.$$

We view these as paths in t depending on the family parameter s. Furthermore, we define a concatenated path

$$g^{k,s}_t := \hat{g}^{k,s}_t * \bar{g}^{k,s}_t * (\tilde{g}^{k,s}_t)^{-1},$$

where of course the inverse means we run through the path backwards. It is easy to see that $g^{k,s}_0 = g_0$ and $g^{k,s}_1 = g_1$ for all s.

We now investigate the lengths of each piece of $g_t^{k,s}$ separately, starting with that of $\hat{g}_t^{k,s}$. Recalling that by Convention 2.11, $G_0 = g^{-1} g_0$, we compute

$$\begin{aligned} L(\hat{g}_t^{k,s}) &= \int_0^1 \left(\int_M \operatorname{tr}_{((1-t)+tf_{k,s})g_0} \left(((f_{k,s}-1)g_0)^2 \right) \sqrt{\det \left(((1-t)+tf_{k,s})G_0 \right)} \, \mu_g \right)^{1/2} dt \\ &= \int_0^1 \left(\int_{U_k} ((1-t)+tf_{k,s})^{\frac{n}{2}-2} \operatorname{tr}_{g_0} \left(((1-f_{k,s})g_0)^2 \right) \sqrt{\det G_0} \, \mu_g \right)^{1/2} dt. \end{aligned}$$

since $\det(\lambda A) = \lambda^{n/2} \det A$ for any $n \times n$-matrix A and $\lambda \in \mathbb{R}$. Note that in the last line, we only integrate over U_k, which is justified by the fact that $1 - f_{k,s} = 0$ on $M \setminus U_k$. Since $s > 0$, it is easy to see that

$$(1 - f_{k,s})^2 \leq (1-s)^2 < 1,$$

from which we can compute the estimate

$$L(\hat{g}_t^{k,s}) < \int_0^1 \left(n \int_{U_k} ((1-t)+tf_{k,s})^{\frac{n}{2}-2} \, \mu_{g_0} \right)^{1/2} dt.$$

Now, to estimate this, we note that for $n \geq 4$, $\frac{n}{2} - 2 \geq 0$ and therefore $f_{k,s} \leq 1$ implies that

$$L(\hat{g}_t^{k,s}) < \sqrt{n \operatorname{Vol}(U_k, g_0)}. \tag{4.2}$$

For $1 \leq n \leq 3$, $\frac{n}{2} - 2 < 0$ and therefore one can compute that $f_{k,s} \geq s > 0$ implies

$$((1-t)+tf_{k,s})^{\frac{n}{2}-2} \leq (1-t)^{\frac{n}{2}-2}.$$

In this case, then,

$$L(\hat{g}_t^{k,s}) < \sqrt{n \operatorname{Vol}(U_k, g_0)} \int_0^1 (1-t)^{\frac{n}{4}-1} \, dt, \tag{4.3}$$

and the integral term is finite since $\frac{n}{4} - 1 > -1$. Furthermore, the value of this integral depends only on n. Putting together (4.2) and (4.3) therefore gives

$$L(\hat{g}_t^{k,s}) \leq C(n) \sqrt{\operatorname{Vol}(U_k, g_0)}, \tag{4.4}$$

where $C(n)$ is a constant depending only on n.

In exact analogy, we can show that the same estimate holds with g_1 in place of g_0.

Next, we look at the second piece of $g_t^{k,s}$. Here we have, using that $g_1 - g_0 = 0$ on $M \setminus E$,

$$\begin{aligned} \left\| (\bar{g}_t^{k,s})' \right\|_{\bar{g}_t^{k,s}}^2 &= \int_M \operatorname{tr}_{f_{k,s}((1-t)g_0+tg_1)} \left((f_{k,s}(g_1-g_0))^2 \right) \sqrt{\det \left(f_{k,s}((1-t)G_0+tG_1) \right)} \, \mu_g \\ &= \int_E f_{k,s}^{n/2} \operatorname{tr}_{(1-t)g_0+tg_1} \left((g_1-g_0)^2 \right) \sqrt{\det((1-t)G_0+tG_1)} \, \mu_g. \end{aligned}$$

Since $f_{k,s}(x) = s$ if $x \in F_k$, and $f_{k,s}(x) \leq 1$ for all $x \in M$, it follows from the above that

$$\begin{aligned} \left\| (\bar{g}_t^{k,s})' \right\|_{\bar{g}_t^{k,s}}^2 \leq s^{n/2} &\int_{F_k} \operatorname{tr}_{(1-t)g_0+tg_1} \left((g_1-g_0)^2 \right) \sqrt{\det((1-t)G_0+tG_1)} \, \mu_g \\ &+ \int_{E \setminus F_k} \operatorname{tr}_{(1-t)g_0+tg_1} \left((g_1-g_0)^2 \right) \sqrt{\det((1-t)G_0+tG_1)} \, \mu_g \end{aligned}$$

For each fixed t and k, the first term in the above clearly goes to zero as $s \to 0$. By our assumption on the sets F_k, the second term goes to zero as $k \to \infty$ for each fixed t (it does not depend on s at all). But since t only ranges over the compact interval $[0, 1]$ and all terms in the integrals depend smoothly on t, both of these convergences are uniform in t. From this, it is easy to see that

$$\lim_{k \to \infty} \lim_{s \to 0} L(\bar{g}_t^{k,s}) = 0. \tag{4.5}$$

Combining these considerations gives the desired estimate. □

As the following examples show, Proposition 4.1 implies that we cannot expect a Cauchy sequence in $\mathcal{M}$ to converge pointwise over subsets of M with volume that vanishes in the limit. Indeed, we cannot control its behavior at all.

Example 4.2. Consider the case where M is a two-dimensional torus, $M = T^2$. On the standard chart for T^2 ($[0,1] \times [0,1]$ with opposite edges identified), define the following sequences of flat metrics:

$$g_k^1 := \begin{pmatrix} 1 & 0 \\ 0 & k^{-1} \end{pmatrix}, \qquad g_k^2 := \begin{pmatrix} k^{-1} & 0 \\ 0 & k^{-1} \end{pmatrix}, \qquad g_k^3 := \begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-2kt} \end{pmatrix}, \qquad g_k^4 := \begin{pmatrix} |\cos k| & 0 \\ 0 & k^{-1} \end{pmatrix}.$$

Since $\mathrm{Vol}(T^2, g_k^i) \to 0$ for all $i = 1, 2, 3, 4$, Proposition 4.1 implies that each of these sequences is d-Cauchy, and all are equivalent. Yet in terms of pointwise convergence, $\{g_k^1\}$ converges to a circle (one dimension of T^2 collapses), $\{g_k^2\}$ converges to a point (both dimensions collapse), $\{g_k^3\}$ "converges" to a circle of infinite radius (in terms of Gromov-Hausdorff convergence, for example, it converges to the real line), and $\{g_k^4\}$ does not converge at all.

4.1.2. *ω-Convergence and its Basic Properties.* In this subsection, we give a convergence notion suited to the completion of $\mathcal{M}$, in that it allows sequences to behave badly on sets that collapse in the limit.

First, though, recall that we define general measure-theoretic notions (e.g., the notion of something holding almost everywhere, or a.e.) using the fixed reference metric g (cf. Convention 2.11). Furthermore, we need one definition before that of ω-convergence.

Definition 4.3. We denote by $\mathcal{M}_m$ the set of all measurable semimetrics on M. That is, $\mathcal{M}_m$ is the set of all sections of S^2T^*M that have measurable coefficients and that induce a positive semidefinite scalar product on T_xM for each $x \in M$.

Define an equivalence relation "$\sim$" on $\mathcal{M}_m$ by $g_0 \sim g_1$ if and only if

(1) their deflated sets X_{g_0} and X_{g_1} differ at most by a nullset, and
(2) $g_0(x) = g_1(x)$ for a.e. $x \in M \setminus (X_{g_0} \cup X_{g_1})$.

We denote the quotient space of $\mathcal{M}_m$ by

$$\widehat{\mathcal{M}_m} := \mathcal{M}_m / \sim.$$

Definition 4.4. Let $\{g_k\}$ be a sequence in $\mathcal{M}$, and let $[g_\infty] \in \widehat{\mathcal{M}_m}$. Recall that we denote the deflated set of the sequence $\{g_k\}$ by $D_{\{g_k\}}$ and the deflated set of an individual semimetric $\tilde{g}$ by $X_{\tilde{g}}$ (cf. Definitions 2.17 and 2.18). We say that $\{g_k\}$ *ω-converges* to $[g_\infty]$ if for every representative $\tilde{g}_\infty \in [g_\infty]$, the following holds:

(1) $\{g_k\}$ is d-Cauchy,
(2) X_{g_∞} and $D_{\{g_k\}}$ differ at most by a nullset,
(3) $g_k(x) \to g_\infty(x)$ for a.e. $x \in M \setminus D_{\{g_k\}}$, and
(4) $\sum_{k=1}^\infty d(g_k, g_{k+1}) < \infty$.

We call $[g_\infty]$ the *ω-limit* of the sequence $\{g_k\}$ and write $g_k \xrightarrow{\omega} [g_\infty]$.

More generally, if $\{g_k\}$ is a d-Cauchy sequence containing a subsequence that ω-converges to $[g_\infty]$, then we say that $\{g_k\}$ *ω-subconverges* to $[g_\infty]$.

Condition (1) in the definition is simply there for convenience, so we don't have to repeatedly assume that a sequence is ω-convergent *and* Cauchy. Condition (4) is technical and will aid us in proofs. Conceptually, it means that we can find paths α_k connecting g_k to g_{k+1} such that the concatenated path $\alpha_1 * \alpha_2 * \cdots$ has finite length. If $\{g_k\}$ is d-Cauchy, then this can always be achieved by passing to a subsequence. (We remark here, however, that these two conditions are not independent. In fact, (4) implies (1).)

Note that condition (3) implies that if $g_k \xrightarrow{\omega} [g_\infty]$, then for all $x \in M \setminus D_{\{g_k\}}$ and all $\tilde{g}_\infty \in [g_\infty]$, there exists some $\delta(x) > 0$ such that

$$\det g_k(x) \geq \delta(x)$$

for all $k \in \mathbb{N}$ and in every chart of an amenable atlas that contains x.

We now move on to proving some properties of ω-convergence. We first state an entirely trivial consequence of Definitions 4.3 and 4.4.

Lemma 4.5. *Let $[g_\infty] \in \mathcal{M}$, and let $\{g_k\}$ be a sequence in $\mathcal{M}$. Suppose that for one given representative $g_\infty \in [g_\infty]$, $\{g_k\}$ together with g_∞ satisfies conditions (1)–(4) of Definition 4.4. Then these conditions are also satisfied for $\{g_k\}$ together with every other representative of $[g_\infty]$.*

Therefore, if can we verify these conditions for one representative of an equivalence class, this already implies $\{g_k\} \xrightarrow{\omega} [g_\infty]$.

We can thus consistently say that $\{g_k\}$ ω-converges to an individual semimetric $g_\infty \in \mathcal{M}_m$ if the two together satisfy conditions (1)–(4) of Definition 4.4.

The next property of ω-convergence is obvious from property (2) of Definition 4.4.

Lemma 4.6. *If $\{g_k^0\}$ and $\{g_k^1\}$ both ω-converge to the same element $[g_\infty] \in \widehat{\mathcal{M}_m}$, then $\{g_k^0\}$ and $\{g_k^1\}$ have the same deflated set, up to a nullset.*

Recall that the main goal of this section is to show that each Cauchy sequence in $\mathcal{M}$ has an ω-convergent subsequence. To do this, we will first prove a pointwise result in the following subsection.

4.1.3. *(Riemannian) Metrics on $\mathcal{M}_x$ Revisited.* In order to more closely study the metric Θ_M on $\mathcal{M}$, we now take a closer look at the Riemannian metrics $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle^0$ (see Section 2.1.1 and Definition 2.4, respectively) that we have defined on the finite-dimensional manifold $\mathcal{M}_x$. The relationship between the two is quite simple:

$$\langle h, k \rangle^0_{\tilde{g}} = \langle h, k \rangle_{\tilde{g}} \det \tilde{G} \quad \text{for all } \tilde{g} \in \mathcal{M}_x \text{ and } h, k \in T_{\tilde{g}} \mathcal{M}_x \cong \mathcal{S}_x. \tag{4.6}$$

Thus, we will first study the simpler Riemannian metric $\langle \cdot, \cdot \rangle$ and find out what properties of $\langle \cdot, \cdot \rangle^0$ we can deduce in this way. Despite their close relationship, their qualitative properties are very different—in particular, $\mathcal{M}_x$ is complete with respect to $\langle \cdot, \cdot \rangle$. We will show this using a simplified version of the analogous computations for $(\cdot, \cdot)$ on $\mathcal{M}$ carried out in [7, Thm. 2.3].

Before we start, let's clear up some notation.

Definition 4.7. By d_x, we denote the distance function induced on $\mathcal{M}_x$ by $\langle \cdot, \cdot \rangle$. We denote the $\langle \cdot, \cdot \rangle$-length of a path a_t in $\mathcal{M}_x$ by $L^{\langle \cdot, \cdot \rangle}(a_t)$ and the $\langle \cdot, \cdot \rangle^0$-length by $L^{\langle \cdot, \cdot \rangle^0}(a_t)$.

Now we compute the Christoffel symbols.

Proposition 4.8. *Let h and k be constant vector fields on $\mathcal{M}_x$, and denote the Levi-Civita connection of $\langle \cdot, \cdot \rangle$ by ∇. Then the Christoffel symbols of $\langle \cdot, \cdot \rangle$ are given by*

$$\Gamma(h, k) = \nabla_h k|_{\tilde{g}} = -\frac{1}{2} \left(h \tilde{g}^{-1} k + k \tilde{g}^{-1} h \right).$$

Proof. All computations are done at the base point $\tilde{g}$, which we will omit from the notation for convenience. Let ℓ be any other constant vector field on $\mathcal{M}_x$. Using the Koszul formula, we can compute that

$$2 \langle \nabla_h k, \ell \rangle = h \langle k, \ell \rangle + k \langle \ell, h \rangle - \ell \langle h, k \rangle. \tag{4.7}$$

Using the fact that the derivative of the map $\hat{g} \mapsto \hat{g}^{-1}$ at the point $\tilde{g}$ is given by $a \mapsto -\tilde{g}^{-1} a \tilde{g}^{-1}$, one can compute that

$$h \langle k, \ell \rangle = - \operatorname{tr} \left((\tilde{g}^{-1} h \tilde{g}^{-1} k)(\tilde{g}^{-1} \ell) \right) - \operatorname{tr} \left((\tilde{g}^{-1} k)(\tilde{g}^{-1} h \tilde{g}^{-1} \ell) \right).$$

Repeating the same computation for the other permutations and substituting the results into (4.7) then gives the result. ∎

Using this, it is a relatively simple matter to solve the geodesic equation of $\langle \cdot, \cdot \rangle$.

Proposition 4.9. *The geodesic g_t in $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ with initial data g_0, g_0' is given by*

$$g_t = g_0 e^{t g_0^{-1} g_0'}.$$

In particular, $(\mathcal{M}_x, d_x)$ is a complete metric space.

Proof. Let $a_t := g_t'$. Since g_t is a geodesic, we have

$$0 = \nabla_{a_t} a_t = a_t' + \Gamma(a_t, a_t) = a_t' - a_t g_t^{-1} a_t \tag{4.8}$$

by Proposition 4.8. Now, multiplying (4.8) on the left by g_t^{-1} gives

$$(g_t^{-1} a_t)' = 0.$$

Thus $g_t^{-1} g_t'$ is constant, or $\log(g_t)' = g_t^{-1} g_t' \equiv g_0^{-1} g_0'$. The geodesic equation now follows, and since $g_0 e^{t g_0^{-1} g_0'} \in \mathcal{M}_x$ for all $t \in \mathbb{R}$, the Hopf-Rinow theorem implies that $(\mathcal{M}_x, d_x)$ is complete. □

We now want to use Proposition 4.9 and (4.6) to characterize θ_x^g-Cauchy sequences in $\mathcal{M}_x$. First, though, we need a lemma that is the pointwise version of Lemma 2.3. The proof is completely analogous to that of Lemma 2.3, and so we omit it.

Lemma 4.10. *Let $a_0, a_1 \in \mathcal{M}_x$. Then*

$$\left| \sqrt{\det A_1} - \sqrt{\det A_0} \right| \leq \frac{\sqrt{n}}{2} \theta_x^g(a_0, a_1).$$

(Recall Convention 2.11 for the definitions of A_i.)

Proposition 4.11. *Let a_k be a θ_x^g-Cauchy sequence. Then either*

1. $\det A_k \to 0$ *for $k \to \infty$, or*
2. *there exist constants $C, \eta > 0$ such that $|(a_k)_{ij}| \leq C$ and $\det A_k \geq \eta$ for all $1 \leq i, j \leq n$ and $k \in \mathbb{N}$.*

Proof. Keeping Lemma 4.10 in mind, it is more convenient to work with the square root of the determinant. This is, of course, completely equivalent for our purposes.

Now, by Lemma 4.10, the map $a \mapsto \sqrt{\det A}$ is θ_x^g-Lipschitz. Since a_k is θ_x^g-Cauchy, $L := \lim_{k\to\infty} \sqrt{\det A_k}$ is well-defined.

If for every $\eta > 0$, there exists k such that $\sqrt{\det A_k} \leq \eta$, then clearly $L = 0$.

It remains to show that if there exist i and j such that for all $C > 0$, there is a k such that $|(a_k)_{ij}| > C$, then $\sqrt{\det A_k} \to 0$. We will assume that $L > 0$ and show a contradiction.

Let's say that we are given $b_0, b_1 \in \mathcal{M}_x$ with $\det B_0, \det B_1 \geq \delta$. Let

$$L_{-\delta} := \inf \left\{ L^{\langle \cdot,\cdot \rangle^0}(b_t) \mid b_t \text{ is a path from } b_0 \text{ to } b_1 \text{ with } \det B_t \leq \delta/2 \text{ for some } t \in (0,1) \right\},$$

$$L_{+\delta} := \inf \left\{ L^{\langle \cdot,\cdot \rangle^0}(b_t) \mid b_t \text{ is a path from } b_0 \text{ to } b_1 \text{ with } \det B_t \geq \delta/2 \text{ for all } t \in [0,1] \right\}.$$

It is easy to see that $\theta_x^g(b_0, b_1) = \min(L_{-\delta}, L_{+\delta})$. Now let b_t be a path as in the definition of $L_{-\delta}$, and assume $\tau \in (0,1)$ is such that $\det B_\tau \leq \delta/2$. Then using Lemma 4.10, we have

$$\begin{aligned} L^{\langle \cdot,\cdot \rangle^0}(b_t) &\geq \frac{\sqrt{n}}{2} \left| \sqrt{\det B_0} - \sqrt{\det B_\tau} \right| + \frac{\sqrt{n}}{2} \left| \sqrt{\det B_1} - \sqrt{\det B_\tau} \right| \\ &\geq \sqrt{n} \left(\sqrt{\delta} - \sqrt{\frac{\delta}{2}} \right) = \sqrt{n} \left(1 - \frac{1}{\sqrt{2}} \right) \sqrt{\delta}. \end{aligned}$$

Therefore $L_{-\delta} \geq \sqrt{n}(1 - 1/\sqrt{2})\sqrt{\delta}$. Then, if b_t is a path as in the definition of $L_{+\delta}$, we have

$$L(b_t) = \int_0^1 \sqrt{\langle b_t', b_t' \rangle^0}\, dt = \int_0^1 \sqrt{\langle b_t', b_t' \rangle \det B_t}\, dt \geq \sqrt{\frac{\delta}{2}} \int_0^1 \sqrt{\langle b_t', b_t' \rangle}\, dt \geq \sqrt{\frac{\delta}{2}}\, d_x(b_0, b_1).$$

This gives $L_{+\delta} \geq \sqrt{\delta/2}\, d_x(b_0, b_1)$. Putting this together, we get that

$$\theta^g_x(b_0, b_1) \geq \min\{\sqrt{n}(1 - 1/\sqrt{2})\sqrt{\delta}, \sqrt{\delta/2}\, d_x(b_0, b_1)\} \tag{4.9}$$

whenever $\det B_0, \det B_1 \geq \delta$.

Now, let's apply the considerations of the last paragraph to the problem at hand. Let i and j be, as above, the indices for which $|(a_k)_{ij}|$ is unbounded, and choose a subsequence, which we again denote by a_k, such that $|(a_k)_{ij}| \geq k$ for all $k \in \mathbb{N}$. Passing to this subsequence does not change the limit $\lim_{k\to\infty} \sqrt{\det A_k}$.

Next, choose $K \in \mathbb{N}$ such that $k \geq K$ implies $\sqrt{\det A_k} \geq L/2$ and $k, l \geq K$ implies $\theta^g_x(a_k, a_l) \leq \frac{1}{2}\sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}$. The latter assumption is possible since a_k is Cauchy. By (4.9), if $k \geq K$, we also have

$$\theta^g_x(a_K, a_k) \geq \min\{\sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}, \sqrt{L/4}\, d_x(a_K, a_k)\}.$$

But $\theta^g_x(a_K, a_k) \geq \sqrt{n}(1 - 1/\sqrt{2})\sqrt{L/2}$ violates our assumptions on K. Furthermore, $d_x(a_K, a_k) \to \infty$ since $|(a_k)_{ij}| \to \infty$ and $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$ is complete. Therefore, if $\theta^g_x(a_K, a_k) \geq \sqrt{L/4}\, d_x(a_K, a_k)$ for all k, then our assumptions on K are violated as well. Thus we have achieved the desired contradiction. □

Since for every pair of constants $C, \eta > 0$, the set of elements $\tilde{g}$ of $\mathcal{M}_x$ with $|\tilde{g}_{ij}| \leq C$ and $\det \tilde{G} \geq \eta$ for all $1 \leq i, j \leq n$ is compact, we immediately get the following corollary of Proposition 4.11:

Corollary 4.12. *Let $\{g_k\}$ be a θ^g_x-Cauchy sequence. Then either*

1. $\det G_k \to 0$ *for $k \to \infty$, or*
2. *there exists an element $g_\infty \in \mathcal{M}_x$ such that $g_k \to g_\infty$, with convergence in the manifold topology of $\mathcal{M}_x$.*

This is essentially the pointwise equivalent of ω-convergence. In the next subsection, we will globalize this result. Before we do that, though, we use this opportune moment to prove two last pointwise results, which will be useful in Section 4.3. The first is the pointwise analog of Proposition 4.1. Again, the proof is analogous to that of Proposition 4.1, so we omit it.

Proposition 4.13. *Let $\tilde{g}, \hat{g} \in \mathcal{M}_x$. Then there exists a constant $C'(n)$, depending only on n, such that*

$$\theta^g_x(\tilde{g}, \hat{g}) \leq C'(n) \left(\sqrt{\det \tilde{G}} + \sqrt{\det \hat{G}} \right).$$

The last pointwise result we need combines Corollary 4.12 and Proposition 4.13 to give a description of the completion of the metric space $(\mathcal{M}_x, \theta^g_x)$.

Theorem 4.14. *For any given $x \in M$, let $\mathrm{cl}(\mathcal{M}_x)$ denote the closure of $\mathcal{M}_x \subset \mathcal{S}_x$ with regard to the natural topology. Then $\mathrm{cl}(\mathcal{M}_x)$ consists of all positive semidefinite $(0,2)$-tensors at x. Let us denote the boundary of $\mathcal{M}_x$, as a subspace of $\mathcal{S}_x$, by $\partial\mathcal{M}_x$.*

Then the completion of $(\mathcal{M}_x, \theta^g_x)$ can be identified with the quotient of the space $\mathrm{cl}(\mathcal{M}_x)$ where $\partial\mathcal{M}_x$ has been identified to a single point. The distance function is given by

$$\theta^g_x(g_0, g_1) = \lim_{k\to\infty} \theta^g_x(g^0_k, g^1_k),$$

where $\{g^0_k\}$ and $\{g^1_k\}$ are any sequences in $\mathcal{M}_x$ converging (in the topology of $\mathcal{S}_x$) to g_0 and g_1, respectively.

Proof. Let $\{g_k\}$ be any sequence in $\mathcal{M}_x$. By Corollary 4.12, if $\{g_k\}$ is Cauchy then either $g_k \to g_\infty \in \mathcal{M}_x$ (with convergence in the topology of $\mathcal{S}_x$), or $\det G_k \to 0$. In fact, by the equivalence of the topologies on $\mathcal{M}_x$ inherited from $\mathcal{S}_x$ and θ^g_x, it is easy to see that the converse holds as well. Furthermore, two Cauchy sequences in $\mathcal{M}_x$ that converge to distinct elements of $\mathcal{M}_x$ are inequivalent Cauchy sequences.

By Proposition 4.13, all sequences with $\det G_k \to 0$ are equivalent Cauchy sequences, and so they are identified in $\overline{(\mathcal{M}_x, \theta^g_x)}$. ☐

4.1.4. *The Existence Proof.* We now wish to globalize Corollary 4.12 to characterize d-Cauchy sequences, using Proposition 2.6 to reduce questions about d to questions about the simpler metric Θ_M.

Lemma 4.15. *Let $\{g_k\}$ be a Cauchy sequence in $\mathcal{M}$. By passing to a subsequence if necessary, we can assume that*

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.$$

Then the following holds:

$$\sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty.$$

Furthermore, define functions Ω and Ω_N for each $N \in \mathbb{N}$ by

$$\Omega_N := \sum_{k=1}^{N} \theta^g_x(g_k(x), g_{k+1}(x)), \quad \Omega := \sum_{k=1}^{\infty} \theta^g_x(g_k(x), g_{k+1}(x)).$$

Then Ω is a.e. finite, $\Omega \in L^1(M, g)$ and $\Omega_N \xrightarrow{L^1} \Omega$. Furthermore, by definition, Ω_N converges to Ω pointwise.

Proof. The first statement is clear, as is the statement that $\Omega_N \to \Omega$ pointwise. So we move on to the other statements.

Lemma 2.3 implies that $\sqrt{\operatorname{Vol}(M, g_k)}$ is a Cauchy sequence in $\mathbb{R}$. Therefore it is bounded, and we can find a constant V such that $\sqrt{\operatorname{Vol}(M, g_k)} \leq V$ for all k. Thus, by Proposition 2.6,

$$\Theta_M(g_k, g_{k+1}) \leq 2d(g_k, g_{k+1}) \left(\frac{\sqrt{n}}{2} d(g_k, g_{k+1}) + V \right).$$

But for large k, since $\{g_k\}$ is Cauchy, we must have $d(g_k, g_{k+1}) \leq 1$, so

$$\Theta_M(g_k, g_{k+1}) \leq \sqrt{n}\, d(g_k, g_{k+1})^2 + V d(g_k, g_{k+1}) \leq (\sqrt{n} + V) d(g_k, g_{k+1}).$$

The first statement is now immediate.

We can then compute

$$\int_M \Omega\, \mu_g = \int_M \left(\sum_{k=1}^{\infty} \theta^g_x(g_k, g_{k+1}) \right) \mu_g = \sum_{k=1}^{\infty} \int_M \theta^g_x(g_k, g_{k+1})\, \mu_g = \sum_{k=1}^{\infty} \Theta_M(g_k, g_{k+1}) < \infty,$$

where we have used the monotone convergence theorem of Lebesgue and Levi [1, Thm. 2.8.2] to exchange the infinite sum and the integral. Finiteness follows from the first part of the lemma. This proves that Ω is a.e. finite and $\Omega \in L^1(M, g)$. It remains to show that $\Omega_N \xrightarrow{L^1} \Omega$. But this is now immediate from [18, Thm. 8.5.1], which states that if $1 \leq p < \infty$, $f_i \to f$ a.e. and $\|f_i\|_p \to \|f\|_p$, then $f_i \xrightarrow{L^p} f$. ☐

Using this lemma, we can globalize Corollary 4.12.

Proposition 4.16. *Let $\{g_k\}$ be a Cauchy sequence in $\mathcal{M}$ such that*

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty.$$

Then for a.e. $x \in M$, $\{g_k(x)\}$ is θ^g_x-Cauchy and either:

(1) $\det G_{t_k}(x) \to 0$ *for* $k \to \infty$, *or*
(2) $g_k(x)$ *is a convergent sequence in* $\mathcal{M}_x$.

Furthermore, (1) *holds for a.e.* $x \in D_{\{g_k\}}$, *and* (2) *holds for a.e.* $x \in M \setminus D_{\{g_k\}}$.

Proof. By our assumption, all the conclusions of Lemma 4.15 hold. In particular, $\Omega_N \to \Omega$ pointwise and Ω is a.e. finite. Therefore, for a.e. $x \in M$,

$$\sum_{k=1}^{\infty} \theta_x^g(g_k(x), g_{k+1}(x)) = \Omega(x) < \infty. \tag{4.10}$$

From this, it is immediate that $\{g_k(x)\}$ is θ_x^g-Cauchy. The remaining results follow from Corollary 4.12. ☐

This proposition essentially delivers us the proof of the existence result.

Theorem 4.17. *For every Cauchy sequence* $\{g_k\}$, *there exists an element* $[g_\infty] \in \widehat{\mathcal{M}_m}$ *and a subsequence* $\{g_{k_l}\}$ *such that* $\{g_{k_l}\}$ ω*-converges to* $[g_\infty]$.

Explicitly, $[g_\infty]$ *is the unique equivalence class containing the element* $g_\infty \in \mathcal{M}_m$ *defined as follows. At points* $x \in M$ *where* $\{g_{k_l}(x)\}$ *is* θ_x^g*-Cauchy,*

(1) $g_\infty(x) := 0$ *for* $x \in D_{\{g_{k_l}\}}$ *and*
(2) $g_\infty(x) := \lim g_{k_l}(x)$ *for* $x \in M \setminus D_{\{g_{k_l}\}}$.

At points $x \in M$ *where* $\{g_{k_l}(x)\}$ *is not* θ_x^g*-Cauchy, we set* $g_\infty(x) := 0$.

Proof. Let $\{g_{k_l}\}$ be a subsequence of $\{g_k\}$ such that

$$\sum_{l=1}^{\infty} d(g_{k_l}, g_{k_{l+1}}) < \infty.$$

Then $\{g_{k_l}\}$ satisfies properties (1) and (4) of Definition 4.4, as well as the hypotheses of Corollary 4.16. Thus $\{g_{k_l}\}$ is a.e. θ_x^g-Cauchy, and so g_∞ is defined a.e. by the two conditions given above. From this, it is immediate that $\{g_{k_l}\}$ together with g_∞ also satisfies properties (2) and (3) of Definition 4.4. Thus, $\{g_{k_l}\}$ ω-converges to g_∞, and by Lemma 4.5 it therefore ω-converges to $[g_\infty]$—provided we can show that $g_\infty \in \mathcal{M}_m$. But g_∞ is clearly a semimetric, and it is the pointwise limit of the measurable semimetrics $\chi(M \setminus D_{\{g_{k_l}\}})g_{k_l}$ (one can easily construct the set $D_{\{g_{k_l}\}}$ from countable unions and intersections of open sets, so it is measurable). Therefore g_∞ itself is measurable. ☐

Knowing now that the ω-limit of a Cauchy sequence of $\mathcal{M}$ exists (after passing to a subsequence), we go further into the properties of ω-convergence.

4.2. ω-Convergence and the Concept of Volume.

In this brief subsection, we wish to prove that the volumes of measurable subsets behave well under ω-convergence. Specifically, we want to show that if $\{g_k\}$ ω-converges to $[g_\infty]$ and $Y \subseteq M$ is measurable, then for any representative $g_\infty \in [g_\infty]$,

$$\mathrm{Vol}(Y, g_k) \to \mathrm{Vol}(Y, g_\infty). \tag{4.11}$$

To see that the above expression is well-defined, note that given any two representatives $g_\infty^0, g_\infty^1 \in [g_\infty]$, we have that $\mu_{g_\infty^0} = \mu_{g_\infty^1}$ as measures—it is clear from Definition 4.3 that $\mu_{g_\infty^0}$ and $\mu_{g_\infty^1}$ can differ at most on a nullset. Thus $\mathrm{Vol}(Y, g_\infty^0) = \mathrm{Vol}(Y, g_\infty^1)$.

The proof of (4.11) is achieved via the Lebesgue dominated convergence theorem with the help of the next two lemmas.

Lemma 4.18. *Let* $\{g_k\}$ ω*-converge to* $g_\infty \in \mathcal{M}_m$. *Then*

$$\left(\frac{\mu_{g_k}}{\mu_g} \right) \xrightarrow{\text{a.e.}} \left(\frac{\mu_{g_\infty}}{\mu_g} \right).$$

Proof. We first prove that for a.e. $x \in D_{\{g_k\}}$,

$$\left(\frac{\mu_{g_k}}{\mu_g}\right) = \det G_k(x) \to 0 = \det G_\infty = \left(\frac{\mu_{g_\infty}}{\mu_g}\right)$$

as $k \to \infty$. By the definition of the deflated set, for every $x \in D_{\{g_k\}}$ and $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\det G_k(x) < \epsilon$. But we also know from Proposition 4.16 and property (4) of Definition 4.4 that $\{g_k(x)\}$ is θ^g_x-Cauchy for a.e. $x \in M$. Hence, by Lemma 4.10, $\left\{\sqrt{\det G_k(x)}\right\}$ is a Cauchy sequence in $\mathbb{R}$ at such points. Therefore it has a limit, and this limit must be 0.

Now, for a.e. $x \in M \setminus D_{\{g_k\}}$, $g_k(x) \to g_\infty(x)$. Since the determinant is a continuous map from the space of $n \times n$ matrices into $\mathbb{R}$, this immediately implies that $\det G_k(x) \to \det G_\infty(x)$ for a.e. $x \in M \setminus D_{\{g_k\}}$. □

Our next task is to find an L^1 function that dominates (μ_{g_k}/μ_g).

Lemma 4.19. *Let $\{g_k\}$ be a Cauchy sequence such that*

$$\sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty,$$

and let Ω be the function of Lemma 4.15. Then

$$\left(\frac{\mu_{g_k}}{\mu_g}\right)(x) \leq \frac{\sqrt{n}}{2}\Omega(x) + \left(\frac{\mu_{g_1}}{\mu_g}\right)(x)$$

for a.e. $x \in M$ and all $k \in \mathbb{N}$.

Proof. Fix some k for the moment. By Proposition 4.16, $\{g_k(x)\}$ is θ^g_x-Cauchy for a.e. $x \in M$. Let $x \in M$ be a point where this holds. Then by Lemma 4.10, the triangle inequality, and the definitions of Ω_N and Ω, we have

$$\begin{aligned}\left|\sqrt{\det G_k} - \sqrt{\det G_1}\right| &\leq \frac{\sqrt{n}}{2}\theta^g_x(g_k, g_1) \leq \frac{\sqrt{n}}{2}\sum_{m=1}^{k-1}\theta^g_x(g_m, g_{m+1}) \\ &= \frac{\sqrt{n}}{2}\Omega_{k-1}(x) \leq \frac{\sqrt{n}}{2}\Omega(x).\end{aligned}$$

The result is now immediate. □

Now, since μ_{g_1} is smooth, it has finite volume, implying that $(\mu_{g_1}/\mu_g) \in L^1(M, g)$. We have already seen in Lemma 4.15 that $\Omega \in L^1(M, g)$. Therefore Lemmas 4.18 and 4.19 allow us to apply the Lebesgue dominated convergence theorem to obtain:

Theorem 4.20. *Let $\{g_k\}$ ω-converge to $g_\infty \in \mathcal{M}_m$, and let $Y \subseteq M$ be any measurable subset. Then* $\mathrm{Vol}(Y, g_k) \to \mathrm{Vol}(Y, g_\infty)$.

An immediate corollary of this theorem and Lemma 2.3 is that the total volume of the ω-limit is finite:

Corollary 4.21. *If g_∞ is the ω-limit of a sequence $\{g_k\}$ in $\mathcal{M}$, then* $\mathrm{Vol}(M, g_\infty) < \infty$. *That is, $g_\infty \in \mathcal{M}_f$.*

Furthermore, as we might have suspected from the beginning, the volume of the deflated set $D_{\{g_k\}}$ of an ω-convergent sequence vanishes in the limit. This is because $\mathrm{Vol}(D_{\{g_k\}}, g_\infty) = 0$.

Corollary 4.22. *Let $\{g_k\}$ ω-converge to $g_\infty \in \mathcal{M}_f$. Then* $\mathrm{Vol}(D_{\{g_k\}}, g_k) \to 0$.

Since we now know the volume of an ω-limit is finite, we can refine Theorem 4.17:

Theorem 4.23. *For every Cauchy sequence $\{g_k\}$, there exists an element $[g_\infty] \in \widehat{\mathcal{M}_f}$ such that $\{g_k\}$ ω-subconverges to $[g_\infty]$.*

4.3. **Uniqueness of the ω-Limit.** The goal of this section is to prove the uniqueness of the ω-limit in the sense mentioned in the introduction to the chapter: we will show that two ω-convergent Cauchy sequences in $\mathcal{M}$ are equivalent if and only if they have the same ω-limit. We prove each direction in a separate subsection.

4.3.1. *First Uniqueness Result.* We first prove the statement that if two ω-convergent Cauchy sequences are equivalent, then their ω-limits agree. To do so, we will extend the pseudometric Θ_Y (cf. Definition 2.5) to the precompletion of $\mathcal{M}$. For this, we need an easy lemma.

Lemma 4.24. *Let $Y \subseteq M$ be measurable. If $\{g_k\}$ is a d-Cauchy sequence, then it is also Θ_Y-Cauchy.*

Proof. As noted in the proof of Lemma 4.15, since $\{g_k\}$ is d-Cauchy, the sequence $\sqrt{\operatorname{Vol}(M, g_k)}$ in $\mathbb{R}$ is bounded, so Proposition 2.6 gives the result easily. ☐

Now we give the extension of Θ_Y mentioned above.

Proposition 4.25. *Let $Y \subseteq M$ be measurable. Then the pseudometric Θ_Y on $\mathcal{M}$ can be extended to a pseudometric on $\overline{\mathcal{M}}^{\mathrm{pre}}$, the precompletion of $\mathcal{M}$, via*

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) := \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1) \tag{4.12}$$

This pseudometric is weaker than d in the sense that $d(\{g_k^0\}, \{g_k^1\}) = 0$ implies $\Theta_Y(\{g_k^0\}, \{g_k^1\}) = 0$ for any Cauchy sequences $\{g_k^0\}$ and $\{g_k^1\}$. More precisely, we have

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) \leq d(\{g_k^0\}, \{g_k^1\}) \left(\sqrt{n}\, d(\{g_k^0\}, \{g_k^1\}) + 2\sqrt{\operatorname{Vol}(M, g_0)} \right), \tag{4.13}$$

where g_0 is any element of $\mathcal{M}_f$ with $g_k^0 \xrightarrow{\omega} [g_0]$.

Furthermore, if $\{g_k^0\}$ and $\{g_k^1\}$ are sequences in $\mathcal{M}_V$ that ω-converge to g_0 and g_1, respectively, then the formula

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) = \int_Y \theta_x^g(g_0(x), g_1(x))\, \mu_g(x) \tag{4.14}$$

holds for all $g_0, g_1 \in \mathcal{M}_V$.

Remark 4.26. In (4.13), we choose any ω-limit of $\{g_k^0\}$. The existence of such a limit has already been proved, but not its uniqueness. On the other hand, if $\tilde{g}_0$ is a different ω-limit of $\{g_k^0\}$, Theorem 4.20 guarantees that $\operatorname{Vol}(M, \tilde{g}_0) = \operatorname{Vol}(M, g_0)$. Therefore, the estimate (4.13) is independent of the choice of ω-limit.

Proof of Proposition 4.25. The construction of a pseudometric on the precompletion of a metric space can be carried over to the case where we begin with a pseudometric space. Therefore, the limit in (4.12) is well-defined due to the fact that $\{g_k^0\}$ and $\{g_k^1\}$ are Cauchy sequences with respect to Θ_Y, and (4.12) indeed defines a pseudometric.

The inequality (4.13) is proved via the following simple computation, which uses (4.12), Proposition 2.6, and Theorem 4.20:

$$\begin{aligned}
\Theta_Y(\{g_k^0\}, \{g_k^1\}) &= \lim_{k \to \infty} \Theta_Y(g_k^0, g_k^1) \leq \lim_{k \to \infty} d(g_k^0, g_k^1) \left(\sqrt{n}\, d(g_k^0, g_k^1) + 2\sqrt{\operatorname{Vol}(M, g_k^0)} \right) \\
&= d(\{g_k^0\}, \{g_k^1\}) \left(\sqrt{n}\, d(\{g_k^0\}, \{g_k^1\}) + 2\sqrt{\operatorname{Vol}(M, g_0)} \right).
\end{aligned}$$

As for the last statement, note first that $\theta_x^g(g_0(x), g_1(x))$ is well-defined by Theorem 4.14, since g_0 and g_1 are positive semidefinite tensors at each point $x \in M$. To prove (4.14), we will first use Fatou's Lemma to show that $\theta_x^g(g_0(x), g_1(x))$ is integrable. We will then use this to apply the Lebesgue dominated convergence theorem.

By Proposition 4.16, for a.e. $x \in M$, $\{g_k^0(x)\}$ and $\{g_k^1(x)\}$ are θ_x^g-Cauchy. At such points, by definition,

$$\theta_x^g(g_0(x), g_1(x)) = \lim_{k \to \infty} \theta_x^g(g_k^0(x), g_k^1(x)). \tag{4.15}$$

So defining

$$f_k(x) := \theta_x^g(g_k^0(x), g_k^1(x)), \quad f(x) := \theta_x^g(g_0(x), g_1(x)),$$

we have $f_k \to f$ a.e.

Now, note that

$$\Theta_Y(g_k^0, g_k^1) = \int_Y f_k(x)\, \mu_g(x).$$

We have already seen that $\lim_{k\to\infty} \Theta_Y(g_k^0, g_k^1)$ exists, so $\{\Theta_Y(g_k^0, g_k^1)\}$ is in particular a bounded sequence of real numbers. Thus

$$\sup_k \int_Y f_k(x)\, \mu_g(x) = \sup_k \Theta_Y(g_k^0, g_k^1) < \infty,$$

where we have used Fatou's lemma.

Now we wish to verify the assumptions of the Lebesgue dominated convergence theorem for f_k and f. We note that for each $l > k$, the triangle inequality gives

$$f_k(x) \leq \sum_{m=k}^{l-1} \theta_x^g(g_m^0(x), g_{m+1}^0(x)) + \theta_x^g(g_l^0(x), g_l^1(x)) + \sum_{m=k}^{l-1} \theta_x^g(g_m^1(x), g_{m+1}^1(x))$$

Starting the sum above at $m = 1$ instead of $m = k$ and taking the limit $l \to \infty$ gives, for a.e. $x \in M$,

$$f_k(x) \leq \sum_{m=1}^{\infty} \theta_x^g(g_m^0(x), g_{m+1}^0(x)) + f(x) + \sum_{m=1}^{\infty} \theta_x^g(g_m^1(x), g_{m+1}^1(x)),$$

where we have used (4.15). Now we claim that the right-hand side of the above inequality is L^1-integrable. We already showed f is integrable using Fatou's Lemma. As for the two infinite sums, they are each also integrable by Lemma 4.15 and ω-convergence of g_k^i, $i = 0, 1$ (specifically, property (4) of Definition 4.4 and Lemma 4.15). Thus each f_k is bounded a.e. by an L^1 function not depending on k.

Knowing all of this, we can apply the Lebesgue dominated convergence theorem to show

$$\Theta_Y(\{g_k^0\}, \{g_k^1\}) = \lim_{k\to\infty} \Theta_Y(g_k^0, g_k^1) = \lim_{k\to\infty} \int_Y f_k\, \mu_g = \int_Y f\, \mu_g = \int_Y \theta_x^g(g_0(x), g_1(x))\, \mu_g(x),$$

which completes the proof. □

With this proposition, proving the first uniqueness result becomes a relatively simple matter.

Theorem 4.27. *Let two ω-convergent sequences $\{g_k^0\}$ and $\{g_k^1\}$, with ω-limits $[g_0]$ and $[g_1]$, respectively, be given. If g_k^0 and g_k^1 are equivalent, i.e., if*

$$\lim_{k\to\infty} d(g_k^0, g_k^1) = 0,$$

then $[g_0] = [g_1]$.

Proof. Suppose the contrary; then for any representatives $g_0 \in [g_0]$ and $g_1 \in [g_1]$, one of two possibilities holds:

1. X_{g_0} and X_{g_1} differ by a set of positive measure, or
2. $X_{g_0} = X_{g_1}$, up to a nullset, but g_0 and g_1 differ on a set E with $E \cap (X_{g_0} \cup X_{g_1}) = \emptyset$ and $\mathrm{Vol}(E, g) > 0$, where g is our fixed metric.

We will show that neither of these possibilities can actually occur.

To rule out (1), let $X_i := D_{\{g^i_k\}}$ denote the deflated set of the sequence $\{g^i_k\}$ for $i = 0, 1$. Then we claim $X_0 = X_1$, up to a nullset. If this is not true, then by swapping the two sequences if necessary, we see that $Y := (X_0 \setminus X_1)$ has positive volume with respect to g_1 and zero volume with respect to g_0. (Y is simply the set on which $\{g^0_k\}$ deflates and $\{g^1_k\}$ doesn't.) But then by Lemma 2.3,

$$\lim_{k\to\infty} d(g^0_k, g^1_k) \geq \lim_{k\to\infty} \sqrt{\mathrm{Vol}(Y, g^1_k)} = \sqrt{\mathrm{Vol}(Y, g_1)} > 0,$$

where we have used Theorem 4.20. This contradicts the assumptions of the theorem, so in fact $X_0 = X_1$ up to a nullset. Since by property (2) of Definition 4.4 $X_{g_i} = D_{\{g^i_k\}}$ up to a nullset as well, (1) cannot hold.

So suppose that (2) holds. Note that on E, g_0 and g_1 are both positive definite. Since E has positive g-volume, we can conclude from Proposition 4.25 (specifically (4.14)) that $\Theta_E(\{g^0_k\}, \{g^1_k\}) > 0$. But then this and (4.13) also imply that

$$\lim_{k\to\infty} d(g^0_k, g^1_k) = d(\{g^0_k\}, \{g^1_k\}) > 0.$$

This contradicts the assumptions of the theorem, and so (2) cannot hold either. □

4.3.2. *Second Uniqueness Result.* Our goal in this subsection is to prove the following statement: up to equivalence, there is only one d-Cauchy sequence ω-converging to a given element of $\widehat{\mathcal{M}_f}$. That is, if we have two sequences $\{g^0_k\}, \{g^1_k\}$ that both ω-converge to the same $[g_\infty] \in \widehat{\mathcal{M}_f}$, then

$$d(\{g^0_k\}, \{g^1_k\}) = \lim_{k\to\infty} d(g^0_k, g^1_k) = 0.$$

We will first prove the above statement for sequences that remain within a given amenable subset $\mathcal{U}$, and will then use this to extend the proof to arbitrary sequences.

Proposition 4.28. *Let $\mathcal{U}$ be an amenable subset, and let $\mathcal{U}^0$ be the L^2-completion of $\mathcal{U}$. If two sequences $\{g^0_k\}$ and $\{g^1_k\}$ in $\mathcal{U}$ both ω-converge to $[g_\infty] \in \widehat{\mathcal{M}_f}$, then $\{g^0_k\}$ and $\{g^1_k\}$ are equivalent. That is,*

$$\lim_{k\to\infty} d(g^0_k, g^1_k) = 0.$$

Furthermore, up to differences on a nullset, $[g_\infty]$ only contains one representative, g_∞, and $\{g^0_k\}$ and $\{g^1_k\}$ both L^2-converge to g_∞. In particular, $g_\infty \in \mathcal{U}^0$.

Proof. Note that Definition 3.1 of an amenable subset implies that the deflated sets of $\{g^0_k\}$ and $\{g^1_k\}$ are empty. Therefore, all representatives of $[g_\infty]$ differ at most by a nullset, and property (3) of Definition 4.4 implies that $g^0_k, g^1_k \xrightarrow{\text{a.e.}} g_\infty$.

Since all g^0_k and g^1_k satisfy the same bounds a.e. in each coordinate chart, it is easy to see that the set

$$\{|(g^k_l)_{ij}|^2 \mid 1 \leq i, j \leq n,\ k \in \mathbb{N}\}$$

is equicontinuous at $\emptyset$ [18, Dfn. 8.5.2] in each coordinate chart for both $l = 0$ and $l = 1$. Therefore, we can conclude from [18, Thm. 8.5.14, Thm. 8.3.3] that $\{g^0_k\}$ and $\{g^1_k\}$ converge in L^2 to g_∞, proving the second statement. This also implies that $\lim_{k\to\infty} \left\| g^1_k - g^0_k \right\|_g = 0$. But now, invoking Theorem 3.6 gives $\lim_{k\to\infty} d(g^0_k, g^1_k) = 0$. □

The next lemma establishes the strong correspondence between L^2- and ω-convergence within amenable subsets.

Lemma 4.29. *Let $\mathcal{U} \subset \mathcal{M}$ be amenable, and let $\tilde{g} \in \mathcal{U}^0$. Then for any sequence $\{g_k\}$ in $\mathcal{U}$ that L^2-converges to $\tilde{g}$, there exists a subsequence $\{g_{k_l}\}$ that ω-converges to $\tilde{g}$.*

In particular, for any element $\tilde{g} \in \mathcal{U}^0$, we can always find a sequence in $\mathcal{U}$ that both L^2- and ω-converges to $\tilde{g}$.

Proof. Let $\{g_k\}$ be any sequence L^2-converging to $\tilde{g} \in \mathcal{U}^0$. Then $\tilde{g}$ together with any subsequence of $\{g_k\}$ already satisfies properties (1) and (2) of Definition 4.4. This is clear from Theorem 3.6 and Definition 3.1 of an amenable subset. (Property (2) is empty here, as $\{g_k\}$ has empty deflated set by the definition of an amenable subset.) Since $\{g_k\}$ is d-Cauchy by Theorem 3.6, it is also easy to see that there is a subsequence $\{g_{k_m}\}$ of $\{g_k\}$ satisfying property (4) of ω-convergence.

To verify property (3), note that L^2-convergence of $\{g_{k_m}\}$ implies that there exists a subsequence $\{g_{k_l}\}$ of $\{g_{k_m}\}$ that converges to $\tilde{g}$ a.e. [18, Thm. 8.5.14, Thm. 8.3.6]. □

Given the results that we have so far, we can give an alternative description of the completion of an amenable set using ω-convergence instead of L^2-convergence.

Proposition 4.30. *Let $\mathcal{U} \subset \mathcal{M}$ be an amenable subset. Then the completion $\overline{\mathcal{U}}$ of $\mathcal{U}$ as a metric subspace of $\mathcal{M}$ can be identified with $\mathcal{U}^0$, the L^2 completion of $\mathcal{U}$, using ω-convergence. That is, there is a natural bijection between $\overline{\mathcal{U}}$ and $\mathcal{U}^0$ given by identifying each equivalence class of Cauchy sequences $[\{g_k\}]$ with the unique element of $\mathcal{U}^0$ that they ω-subconverge to.*

Proof. The existence result (Theorem 4.17) the first uniqueness result (Theorem 4.27) and Proposition 4.28 together imply that for every equivalence class $[\{g_k\}]$ of d-Cauchy sequences in $\mathcal{U}$, there is a unique L^2 metric $g_\infty \in \mathcal{U}^0$ such that every representative of $[\{g_k\}]$ ω-subconverges to g_∞, and that the representatives of a different equivalence class cannot also ω-subconverge to g_∞. This gives us the map from $\overline{\mathcal{U}}$ to $\mathcal{U}^0$ and shows that it is injective. Furthermore, by Lemma 4.29, there is a sequence in $\mathcal{U}$ ω-subconverging to every element of $\mathcal{U}^0$. Thus, this map is also surjective. □

With this identification, we can define a metric on $\mathcal{U}^0$ by declaring the bijection of the previous proposition to be an isometry. The result is the following:

Definition 4.31. Let $\mathcal{U}$ be an amenable subset. By $d_{\mathcal{U}}$, we denote the metric on the completion of $\mathcal{U}$, which we identify with the L^2-completion $\mathcal{U}^0$ via Proposition 4.30. Thus, for $g_0, g_1 \in \mathcal{U}^0$ and any sequences $g_k^0 \xrightarrow{\omega} g_0$, $g_k^1 \xrightarrow{\omega} g_1$, we have

$$d_{\mathcal{U}}(g_0, g_1) = \lim_{k \to \infty} d(g_k^0, g_k^1).$$

Note that by the preceding results, we can equivalently define $d_{\mathcal{U}}$ by assuming that $\{g_k^0\}$ and $\{g_k^1\}$ L^2-converge to g_0 and g_1, respectively.

The next lemma, the proof of which is immediate, shows that the metric $d_{\mathcal{U}}$ is nicely compatible with the metric d.

Lemma 4.32. *Let $\mathcal{U} \subset \mathcal{M}$ be amenable, and suppose $g_0, g_1 \in \mathcal{U}$ and $g_2 \in \mathcal{U}^0$. Then*

(1) $d(g_0, g_1) = d_{\mathcal{U}}(g_0, g_1)$, *and*
(2) $d(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_2) + d_{\mathcal{U}}(g_2, g_1)$.

With a little bit of effort, we can use previous results to extend Proposition 4.1, a statement about $\mathcal{M}$, to the completion of an amenable subset. We first prove a very special case in a lemma, followed by the full result.

Lemma 4.33. *Let $\mathcal{U}$ be any amenable subset and $g^0, g^1 \in \mathcal{U}$. Let $C(n)$ be the constant of Proposition 4.1, and let $E \subseteq M$ be measurable. Then*

$$d_{\mathcal{U}}(g^0, \chi(M \setminus E) g^0 + \chi(E) g^1) \leq C(n) \left(\sqrt{\mathrm{Vol}(E, g^0)} + \sqrt{\mathrm{Vol}(E, g^1)} \right)$$

Proof. For each $k \in \mathbb{N}$, choose closed subsets F_k and open subsets U_k such that $F_k \subseteq E \subseteq U_k$ and $\mathrm{Vol}(U_k, g) - \mathrm{Vol}(F_k, g) \leq 1/k$. Furthermore, choose functions $f_k \in C^\infty(M)$ satisfying

(1) $0 \leq f_k(x) \leq 1$ for all $x \in M$,
(2) $f_k(x) = 1$ for $x \in F_k$ and
(3) $f_k(x) = 0$ for $x \notin U_k$.

Then it is not hard to see that the sequence defined by

$$g_k := (1 - f_k)g^0 + f_k g^1$$

L^2-converges to $\chi(M \setminus E)g^0 + \chi(E)g^1$, so in particular

$$d_{\mathcal{U}}(g^0, \chi(M \setminus E)g^0 + \chi(E)g^1) = \lim_{k\to\infty} d(g^0, g_k). \tag{4.16}$$

Furthermore, since g^0 and all g_k are smooth, Proposition 4.1 gives

$$d(g^0, g_k) \leq C(n)\left(\sqrt{\mathrm{Vol}(U_k, g^0)} + \sqrt{\mathrm{Vol}(U_k, g_k)}\right). \tag{4.17}$$

By our assumptions on the sets U_k, it is clear that $\mathrm{Vol}(U_k, g^0) \to \mathrm{Vol}(E, g^0)$. So if we can show that $\mathrm{Vol}(U_k, g_k) \to \mathrm{Vol}(E, g^1)$, then (4.16) and (4.17) combine to give the desired result.

Now, because $g_k = g^1$ on F_k, we have

$$\mathrm{Vol}(U_k, g_k) = \int_{F_k} \mu_{g^1} + \int_{U_k \setminus F_k} \mu_{g_k}.$$

The first term converges to $\mathrm{Vol}(E, g^1)$ for $k \to \infty$ by the definition of F_k. We claim that the second term converges to zero. Note that since the bounds of Definition 3.1 are pointwise convex, we can enlarge $\mathcal{U}$ to an amenable subset containing g_k for each $k \in \mathbb{N}$. (By the definition, each g_k is, at each point $x \in M$, a sum $(1 - s)g^0(x) + sg^1(x)$ with $0 \leq s \leq 1$.) Therefore, by Lemma 3.3, there exists a constant K such that

$$\left(\frac{\mu_{g_k}}{\mu_g}\right) \leq K.$$

But using this, our claim is clear from the assumptions on U_k and F_k. $\square$

Theorem 4.34. *Let $\mathcal{U}$ be any amenable subset with L^2-completion $\mathcal{U}^0$. Suppose that $g_0, g_1 \in \mathcal{U}^0$, and let $E := \mathrm{carr}(g_1 - g_0) = \{x \in M \mid g_0(x) \neq g_1(x)\}$. Then there exists a constant $C(n)$ depending only on $n = \dim M$ such that*

$$d_{\mathcal{U}}(g_0, g_1) \leq C(n)\left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)}\right).$$

In particular, we have

$$\mathrm{diam}_{\mathcal{U}}\left(\{\tilde{g} \in \mathcal{U}^0 \mid \mathrm{Vol}(M, \tilde{g}) \leq \delta\}\right) \leq 2C(n)\sqrt{\delta}.$$

Proof. Using Lemma 4.29, choose any two sequences $\{g_k^0\}$ and $\{g_k^1\}$ in $\mathcal{U}$ that both L^2- and ω-converge to g_0 and g_1, respectively. Then by the triangle inequality and Lemma 4.32(1), for each $k \in \mathbb{N}$,

$$d_{\mathcal{U}}(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_k^0) + d(g_k^0, g_k^1) + d_{\mathcal{U}}(g_k^1, g_1). \tag{4.18}$$

By Theorem 3.6, the first and last terms above approach zero as $k \to \infty$. Furthermore, we claim that the middle term satisfies

$$\lim_{k\to\infty} d(g_k^0, g_k^1) \leq C(n)\left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)}\right),$$

which would complete the proof.

By the triangle inequality (2) of Lemma 4.32, we have

$$d(g_k^0, g_k^1) \leq d_{\mathcal{U}}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) + d_{\mathcal{U}}(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1). \tag{4.19}$$

By Lemma 4.33 and Theorem 4.20, we can conclude

$$\lim_{k\to\infty} d_{\mathcal{U}}(g_k^0, \chi(M \setminus E)g_k^0 + \chi(E)g_k^1) \leq C(n)\left(\sqrt{\mathrm{Vol}(E, g_0)} + \sqrt{\mathrm{Vol}(E, g_1)}\right).$$

Therefore, if we can show that the second term of (4.19) converges to zero as $k \to \infty$, then we will have the desired result. But $\{g_k^0\}$ L^2-converges to g_0 and $\{g_k^1\}$ L^2-converges to g_1. Additionally, $\chi(M \setminus E)g_0 = \chi(M \setminus E)g_1$. Therefore, Definition 4.31 implies that

$$\lim_{k \to \infty} d_{\mathcal{U}}(\chi(M \setminus E)g_k^0 + \chi(E)g_k^1, g_k^1) = 0,$$

which is what was to be shown. □

Next, we need another technical result that will help us in extending the second uniqueness result from amenable subsets to all of $\mathcal{M}$.

Proposition 4.35. *Say $g_0 \in \mathcal{M}$ and $h \in \mathcal{S}$, and let $E \subseteq M$ be any open set. Define an L^2 tensor $g_1 \in \mathcal{S}^0$ by $g_1 := g_0 + h^0$, where $h^0 := \chi(E)h$. Assume that we can find an amenable subset $\mathcal{U}$ such that $g_1 \in \mathcal{U}^0$. Finally, define a path g_t of L^2 metrics by $g_t := g_0 + th^0$, $t \in [0,1]$.*

Then without loss of generality (by enlarging $\mathcal{U}$ if necessary), $g_t \in \mathcal{U}^0$ for all t, so in particular $d_{\mathcal{U}}(g_0, g_1)$ is well-defined. Furthermore,

$$d_{\mathcal{U}}(g_0, g_1) \leq L(g_t) := \int_0^1 \left\| h^0 \right\|_{g_t} dt, \tag{4.20}$$

i.e., the length of g_t, when measured in the naive way, bounds $d_{\mathcal{U}}(g_0, g_1)$ from above.

Lastly, suppose that on E, the metrics g_t, $t \in [0,1]$, all satisfy the bounds

$$|(g_t)_{ij}(x)| \leq C \quad \text{and} \quad \lambda_{\min}^{G_t}(x) \geq \delta$$

for some $C, \delta > 0$, all $1 \leq i,j \leq n$ and a.e. $x \in E$. (That this is satisfied for some C and δ is guaranteed by $g_t \in \mathcal{U}^0$.) Then there is a constant $K = K(C, \delta)$ such that

$$d_{\mathcal{U}}(g_0, g_1) \leq K \left\| h^0 \right\|_g .$$

Proof. The existence of the enlarged amenable subset $\mathcal{U}$ is clear from the construction of g_t. So we turn to the proof of (4.20).

Let any $\epsilon > 0$ be given. By Theorem 3.6, we can choose $\delta > 0$ such that for any $\tilde{g}_0, \tilde{g}_1 \in \mathcal{U}$, $\|\tilde{g}_1 - \tilde{g}_0\|_g < \delta$ implies $d(\tilde{g}_0, \tilde{g}_1) < \epsilon$.

Next, for each $k \in \mathbb{N}$, we choose closed sets $F_k \subseteq E$ and open sets $U_k \supseteq E$ with the property that $\mathrm{Vol}(U_k, g) - \mathrm{Vol}(F_k, g) < 1/k$. Given this, let's even restrict ourselves to k large enough that

$$\|\chi(U_k \setminus F_k)h\|_g < \min\{\delta, \epsilon\}. \tag{4.21}$$

We then choose $f_k \in C^\infty(M)$ satisfying

(1) $f_k(x) = 1$ if $x \in F_k$,
(2) $f_k(x) = 0$ if $x \notin U_k$ and
(3) $0 \leq f_k(x) \leq 1$ for all $x \in M$,

The first consequence of our assumptions above is

$$\|g_1 - (g_0 + f_k h)\|_g \leq \|\chi(U_k \setminus F_k)h)\|_g < \delta. \tag{4.22}$$

The second inequality is (4.21), and the first inequality holds for two reasons. First, on both F_k and $M \setminus U_k$, $g_0 + f_k h = g_0 + \chi(F_k)h = g_1$. Second, on $U_k \setminus F_k$, $g_1 - (g_0 + f_k h) = (1 - f_k)h$, and by our third assumption on f_k, $0 \leq 1 - f_k \leq 1$. Now, inequality (4.22) allows us to conclude, by our assumption on δ, that

$$d_{\mathcal{U}}(g_0 + f_k h, g_1) < \epsilon. \tag{4.23}$$

Since by the triangle inequality

$$d_{\mathcal{U}}(g_0, g_1) \leq d_{\mathcal{U}}(g_0, g_0 + f_k h) + d_{\mathcal{U}}(g_0 + f_k h, g_1) < d_{\mathcal{U}}(g_0, g_0 + f_k h) + \epsilon,$$

we must now get some estimates on $d_{\mathcal{U}}(g_0, g_0 + f_k h)$ to prove (4.20).

To do this, define a path g_t^k in $\mathcal{M}$, for $t \in [0,1]$, by $g_t^k := g_0 + t f_k h$. Then we have, as is easy to see,

$$d(g_0, g_0 + f_k h) \leq L(g_t^k) = \int_0^1 \|f_k h\|_{g_t^k} \, dt \tag{4.24}$$

This is almost what we want, but we first have to replace $f_k h$ with $h^0 = \chi(E)h$. Also note that the L^2 norm in (4.24) is that of g_t^k. To put this in a form useful for proving (4.20), we therefore also have to to replace g_t^k with g_t.

Using the facts that on F_k, $f_k h = h^0$ and $g_t^k = g_t$, as well as that $f_k = 0$ on $M \setminus U_k$, we can write

$$\|f_k h\|_{g_t^k}^2 = \int_{F_k} \operatorname{tr}_{g_t}\left((h^0)^2\right) \mu_{g_t} + \int_{U_k \setminus F_k} \operatorname{tr}_{g_t^k}\left((f_k h)^2\right) \mu_{g_t^k}. \tag{4.25}$$

For the first term above, we clearly have

$$\int_{F_k} \operatorname{tr}_{g_t}\left((h^0)^2\right) \mu_{g_t} \leq \left\|h^0\right\|_{g_t}^2. \tag{4.26}$$

As for the second term, it can be rewritten and estimated by

$$\int_{U_k \setminus F_k} \operatorname{tr}_{g_t^k}\left((f_k h)^2\right) \mu_{g_t^k} = \|\chi(U_k \setminus F_k) f_k h\|_{g_t^k} \leq \|\chi(U_k \setminus F_k) h\|_{g_t^k},$$

where the inequality follows from our third assumption on f_k above. Now, recall that g_t is contained within an amenable subset $\mathcal{U}$. It is possible to enlarge $\mathcal{U}$, without changing the property of being amenable, so that $\mathcal{U}$ contains g_t^k for all $t \in [0,1]$ and all $k \in \mathbb{N}$. Therefore, by Lemma 3.4, there exists a constant $K' = K'(\mathcal{U}, g_0, g_1)$—i.e., K' does not depend on k—such that

$$\|\chi(U_k \setminus F_k) h\|_{g_t^k} \leq K' \|\chi(U_k \setminus F_k) h\|_g.$$

But by (4.21), we have that $\|\chi(U_k \setminus F_k) h\|_g < \epsilon$. Combining this with (4.25) and (4.26), we therefore get

$$\|f_k h\|_{g_t^k}^2 \leq \left\|h^0\right\|_{g_t} + K' \epsilon.$$

The above inequality, substituted into (4.24), gives

$$d(g_0, g_1^k) \leq \int_0^1 \left(\left\|h^0\right\|_{g_t} + K' \epsilon\right) dt = L(g_t) + K' \epsilon.$$

The final step in the proof is then to estimate, using the above inequality and (4.23), that

$$d_{\mathcal{U}}(g_0, g_1) \leq d(g_0, g_1^k) + d_{\mathcal{U}}(g_1^k, g_1) < L(g_t) + (1 + K')\epsilon.$$

Since ϵ was arbitrary and K' is independent of k, we are finished with the proof of (4.20).

Finally, the third statement follows from the following estimate, which is proved in exactly the same way as Lemma 3.4:

$$\left\|h^0\right\|_{g_t} = \left(\int_E \operatorname{tr}_{g_t}\left(h^2\right) \mu_{g_t}\right)^{1/2} \leq K(C, \delta) \left\|h^0\right\|_g.$$

□

With Theorem 4.34 and Proposition 4.35 as part of our toolbox, we are now ready to take on the proof of the second uniqueness result in its full generality.

So let two d-Cauchy sequences $\{g_k^0\}$ and $\{g_k^1\}$, as well as some $g_\infty \in \mathcal{M}_f$, be given. Suppose further that $\{g_k^0\}$ and $\{g_k^1\}$ both ω-converge to g_∞ for $k \to \infty$. We will prove that

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0. \tag{4.27}$$

The heuristic idea of our proof is very simple, which is belied by the rather technical nature of the rigorous proof. The point, though, is essentially that for all $l \in \mathbb{N}$, we break M up into two sets,

E_l and $M \setminus E_l$. The set E_l has positive volume with respect to g_∞, but $\{g_k^0\}$ and $\{g_k^1\}$ L^2-converge to g_∞ on E_l, so the contribution of E_l to $d(g_k^0, g_k^1)$ vanishes in the limit $k \to \infty$. The set $M \setminus E_l$ contains the deflated sets of $\{g_k^0\}$ and $\{g_k^1\}$, so the sequences need not converge on $M \setminus E_l$. However, we choose things such that $\mathrm{Vol}(M \setminus E_l, g_\infty)$ vanishes in the limit $l \to \infty$, so that Proposition 4.1 implies that the contribution of $M \setminus E_l$ to $d(g_k^0, g_k^1)$ vanishes after taking the limits $k \to \infty$ and $l \to \infty$ in succession.

The rigorous proof is achieved in three basic steps, which we will describe after some brief preparation.

For each $l \in \mathbb{N}$, let

$$E_l := \left\{ x \in M \;\middle|\; \det g_k^i(x) > \frac{1}{l},\ \left|(g_k^i)_{rs}(x)\right| < l\ \forall i = 0, 1;\ k \in \mathbb{N};\ 1 \le r, s \le n \right\}, \tag{4.28}$$

where these local notions are of course defined with respect to our fixed amenable atlas (cf. Convention 2.13), and the inequalities in the definition should hold in each chart containing the point x in question. Thus, E_l is a set over which the sequences g_i^k neither deflate nor become unbounded. We first note that for each $k \in \mathbb{N}$, there exists an amenable subset $\mathcal{U}_k$ such that the metrics

$$g_k^0,\ g_k^1 \text{ and } g_k^0 + \chi(E_l)(g_k^1 - g_k^0)$$

are contained in $\mathcal{U}_k^0$. This is possible due to smoothness of g_k^0 and g_k^1, as well as pointwise convexity of the bounds of Definition 3.1.

The steps in our proof are the following. We will show first that

$$\lim_{k\to\infty} d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) = 0 \tag{4.29}$$

for all fixed $l \in \mathbb{N}$. Second,

$$\lim_{k\to\infty} d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \le 2C(n)\sqrt{\mathrm{Vol}(M \setminus E_l, g_\infty)} \tag{4.30}$$

for all fixed $k \in \mathbb{N}$ (where $C(n)$ is the constant from Theorem 4.34). And third,

$$\lim_{l\to\infty} \mathrm{Vol}(E_l, g_\infty) = \mathrm{Vol}(M, g_\infty). \tag{4.31}$$

Since the triangle inequality of Lemma 4.32(2) implies that

$$d(g_k^0, g_k^1) \le d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) + d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1)$$

for all $l \in \mathbb{N}$, taking the limits $k \to \infty$ followed by $l \to \infty$ of both sides then gives (4.27).

We now prove each of (4.29), (4.30) and (4.31) in its own lemma.

Lemma 4.36.

$$\lim_{k\to\infty} d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) = 0$$

Proof. We know that

$$g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0) \in \mathcal{U}_k^0,$$

where $\mathcal{U}_k$ is an amenable subset. Therefore, for each fixed $k \in \mathbb{N}$, Proposition 4.35 applies to give

$$d_{\mathcal{U}_k}(g_k^0, g_k^0 + \chi(E_l)(g_k^1 - g_k^0)) \le K_l \left\| \chi(E_l)(g_k^1 - g_k^0) \right\|_g, \tag{4.32}$$

where K_l is some constant depending only on l. (That the constant only depends on l is the result of the fact that g_k^0 and g_k^1 satisfy the bounds given in (4.28) on E_l, which only depend on l.)

Now, recalling the definition (4.28) of E_l, we note that for all $1 \le i, j \le n$ and all $k \in \mathbb{N}$, we have $|(g_k^1)_{ij}(x) - (g_k^0)_{ij}(x)|^2 \le 4l^2$ for $x \in E_l$, and hence the family of (local) functions

$$\{\chi(E_l)((g_k^1)_{ij} - (g_k^0)_{ij}) \mid 1 \le i, j \le n,\ k \in \mathbb{N}\}$$

is equicontinuous at $\emptyset$. Furthermore, since property (3) of Definition 4.4 implies that $\chi(E_l) g_a^k \to \chi(E_l) g_\infty$ a.e. for $a = 0, 1$, we have that $\chi(E_l)(g_k^1 - g_k^0) \to 0$ a.e. Therefore, as in the proof of Proposition 4.28, we have that

$$\left\| \chi(E_l)(g_k^1 - g_k^0) \right\|_g \to 0$$

for $k \to \infty$. Together with (4.32), this implies the result immediately. □

Lemma 4.37.

$$\lim_{k \to \infty} d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq 2C(n) \sqrt{\operatorname{Vol}(M \setminus E_l, g_\infty)}$$

Proof. First note that $g_k^1 = g_k^0 + \chi(E_l)(g_k^1 - g_k^0)$ on E_l. Therefore, by Theorem 4.34,

$$d_{\mathcal{U}_k}(g_k^0 + \chi(E_l)(g_k^1 - g_k^0), g_k^1) \leq C(n) \left(\sqrt{\operatorname{Vol}(M \setminus E_l, g_k^0)} + \sqrt{\operatorname{Vol}(M \setminus E_l, g_k^1)} \right).$$

But now the result follows immediately from Theorem 4.20, since $\operatorname{Vol}(M \setminus E_l, g_i^k) \to \operatorname{Vol}(M \setminus E_l, g_\infty)$ for $i = 0, 1$. □

Lemma 4.38.

$$\lim_{l \to \infty} \operatorname{Vol}(E_l, g_\infty) = \operatorname{Vol}(M, g_\infty).$$

Proof. Recall that $X_{g_\infty} \subseteq M$ denotes the deflated set of g_∞, i.e., the set where g_∞ is not positive definite. This set has volume zero w.r.t. g_∞, since $\mu_{g_\infty} = 0$ a.e. on X_{g_∞}. Therefore $\operatorname{Vol}(M, g_\infty) = \operatorname{Vol}(M \setminus X_{g_\infty}, g_\infty)$.

We note that $\chi(E_l)$ converges a.e. to $\chi(M \setminus X_{g_\infty})$ and that $\chi(E_l)(x) \leq 1$ for all $x \in M$. Since g_∞ has finite volume, the constant function 1 is integrable w.r.t. μ_{g_∞}, and therefore the Lebesgue dominated convergence theorem implies that

$$\lim_{l \to \infty} \operatorname{Vol}(E_l, g_\infty) = \lim_{l \to \infty} \int_M \chi(E_l)\, \mu_{g_\infty} = \int_M \chi(M \setminus X_{g_\infty})\, \mu_{g_\infty} = \operatorname{Vol}(M \setminus X_{g_\infty}, g_\infty).$$

□

As already noted, Lemmas 4.36, 4.37 and 4.38 combine to give the desired result. We summarize what we have just proved in a theorem.

Theorem 4.39. *Let $[g_\infty] \in \widehat{\mathcal{M}_f}$. Suppose we have two sequences $\{g_k^0\}$ and $\{g_k^1\}$ with $g_k^0, g_k^1 \xrightarrow{\omega} [g_\infty]$ for $k \to \infty$. Then*

$$\lim_{k \to \infty} d(g_k^0, g_k^1) = 0,$$

that is, $\{g_k^0\}$ and $\{g_k^1\}$ are equivalent in the precompletion $\overline{\mathcal{M}}^{\mathrm{pre}}$ of $\mathcal{M}$.

As we have already discussed, combining this theorem with the existence result (Theorem 4.23) and the first uniqueness result (Theorem 4.27) gives us an identification of $\overline{\mathcal{M}}$ with a subset of $\widehat{\mathcal{M}_f}$:

Definition 4.40. Denote by $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ the map sending an equivalence class of Cauchy sequences to the unique element of $\widehat{\mathcal{M}_f}$ that all of its representatives ω-subconverge to.

In the next section, we will prove that $\overline{\mathcal{M}}$ is actually identified with all of $\widehat{\mathcal{M}_f}$, i.e., Ω is surjective. We note here that for the purpose of studying $\overline{\mathcal{M}}$ we can now use Ω to drop the distinction between an ω-convergent sequence and the element of $\widehat{\mathcal{M}_f}$ that it converges to. We will employ this trick in what follows to simplify formulas and proofs.

5. The completion of $\mathcal{M}$

In this section, our previous efforts come to fruition and we are able to complete our description of $\overline{\mathcal{M}}$ by proving, in Section 5.3, that the map $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ defined in the previous chapter is a bijection.

Section 5.1 provides some necessary preparation for the surjectivity proof by going into more depth on the behavior of volume forms under ω-convergence. After this, Section 5.2 presents a partial result on the image of Ω. Namely, we show that all equivalence classes of measurable, bounded semimetrics (cf. Definition 2.17) are contained in $\Omega(\overline{\mathcal{M}})$. This marks the final preparation we need to prove the main result.

5.1. **Measures induced by measurable semimetrics.**

For use in Section 5.3, we need to record a couple of properties of the measure $\mu_{\tilde{g}}$ induced by an element $\tilde{g} \in \mathcal{M}_f$.

The first property is continuity of the norms of continuous functions under ω-convergence. It follows immediately from Theorem 4.20 and the Portmanteau theorem [21, Thm. 8.1]:

Lemma 5.1. *Let $\tilde{g} \in \mathcal{M}_f$, and let $\rho \in C^0(M)$ be any continuous function. If the sequence $\{g_k\}$ ω-converges to $\tilde{g}$, then μ_{g_k} converges weakly to $\mu_{\tilde{g}}$, so in particular*

$$\lim_{k\to\infty} \|\rho\|_{g_k} = \|\rho\|_{\tilde{g}} .$$

The next fact we need is that if $\tilde{g} \in \mathcal{M}_f$, i.e., $\tilde{g}$ is a measurable, finite-volume semimetric, then the set of C^∞ functions is dense in $L^p(M, \tilde{g})$ for $1 \leq p < \infty$, just as in the case of a smooth volume form.

To prove this claim, we first state a fact about measures on $\mathbb{R}^n$. One can prove it almost identically to [1, Cor. 4.2.2], where the statement is made for Borel measures. To prove it for Lebesgue measures, one must simply approximate Lebesgue-measurable sets by Borel-measurable sets using the discussion of Section 2.3.2.

Theorem 5.2. *Let a nonnegative measure ν on the algebra of Lebesgue sets in $\mathbb{R}^n$ be bounded on bounded sets. Then the class $C_0^\infty(\mathbb{R}^n)$ of smooth functions with bounded support is dense in $L^p(\mathbb{R}^n, \nu)$, $1 \leq p < \infty$.*

Now, since any $\tilde{g} \in \mathcal{M}_f$ has finite volume, its induced measure $\mu_{\tilde{g}}$ clearly satisfies the hypotheses of the theorem in any coordinate chart. Therefore, we have:

Corollary 5.3. *If $\tilde{g} \in \mathcal{M}_f$, then $C^\infty(M)$ is dense in $L^p(M, \tilde{g})$.*

5.2. **Bounded semimetrics.**

In this section, we go one step further in our understanding of the injection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ that was introduced in Definition 4.40. Specifically, we want to see that the image $\Omega(\overline{\mathcal{M}})$ contains all equivalence classes of bounded, measurable semimetrics (cf. Definition 2.17).

Our strategy for proving this is to first prove the fact for smooth semimetrics by showing that for any smooth semimetric g_0, there is a finite path g_t, $t \in (0, 1]$, in $\mathcal{M}$ with $\lim_{t\to 0} g_t = g_0$ (where we take the limit in the C^∞ topology of $\mathcal{S}$). Then, if we simply let t_k be any monotonically decreasing sequence converging to zero, it is trivial to show $g_{t_k} \xrightarrow{\omega} g_0$ for $k \to \infty$. We then use this to handle the general, nonsmooth case.

5.2.1. *Paths to the boundary.*

Before we get into the proofs, we put ourselves in the proper setting, for which we first need to introduce the notion of a quasi-amenable subset. These are defined by weakening the requirements for an amenable subset (cf. Definition 3.1), giving up the condition of being "uniformly inflated":

Definition 5.4. We call a subset $\mathcal{U} \subset \mathcal{M}$ *quasi-amenable* if $\mathcal{U}$ is convex and we can find a constant C such that for all $\tilde{g} \in \mathcal{U}$, $x \in M$ and $1 \leq i, j \leq n$,

$$|\tilde{g}_{ij}(x)| \leq C. \tag{5.1}$$

We also define $\partial\mathcal{M}$ to be the boundary of $\mathcal{M}$ as a topological subset of $\mathcal{S}$. Thus, it consists of all smooth semimetrics that somewhere fail to be positive definite.

Let $\mathcal{U}$ be any quasi-amenable subset, and denote by $\mathrm{cl}(\mathcal{U})$ the closure of $\mathcal{U}$ in the C^∞ topology of $\mathcal{S}$. Thus, $\mathrm{cl}(\mathcal{U})$ may contain some smooth semimetrics. One can easily see that any $g_0 \in \partial\mathcal{M}$ is contained in $\mathrm{cl}(\mathcal{U})$ for an appropriate quasi-amenable subset $\mathcal{U}$.

Now, suppose some $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ is given, and let $g_1 \in \mathcal{U}$ have the property that $h := g_1 - g_0 \in \mathcal{M}$. Exploiting the linear structure of $\mathcal{M}$, we define the simplest path imaginable from g_0 to g_1:

$$g_t := g_0 + th. \tag{5.2}$$

Then by the convexity of $\mathcal{U}$, g_t is a path $(0,1] \to \mathcal{U}$ with limit (in the topology of $\mathcal{S}$) as $t \to 0$ equal to g_0.

Recall that the length of g_t is given by

$$L(g_t) = \int_0^1 \left(\int_M \mathrm{tr}_{g_t}((g_t')^2)\, \mu_{g_t} \right)^{1/2} dt = \int_0^1 \left(\int_M \mathrm{tr}_{g_t}(h^2) \sqrt{\det(g^{-1}g_t)}\, \mu_g \right)^{1/2} dt \tag{5.3}$$

To prove that g_t is a finite path, we must therefore estimate the inner integrand. This will follow from pointwise estimates combined with a compactness/continuity argument.

5.2.2. *Pointwise estimates.* Let $A = (a_{ij})$ and $B = (b_{ij})$ be real, symmetric $n \times n$ matrices, with $A_t := A + tB$ for $t \in (0,1]$. We will assume that $B > 0$ and that $A \geq 0$. (In this scheme, A and B play the role of $g_0(x)$ and $h(x)$, respectively, at some point $x \in M$.) Furthermore, we fix an arbitrary matrix C that is invertible and symmetric (this plays the role of $g(x)$).

To get a pointwise estimate on $\mathrm{tr}_{g_t}(h^2)\sqrt{\det g^{-1}g_t}$, we need to estimate $\mathrm{tr}_{A_t}(B^2)\sqrt{\det(C^{-1}A_t)}$. We prove the desired estimate in two lemmas.

For any symmetric matrix D, let $\lambda^D_{\min} = \lambda^D_1 \leq \cdots \leq \lambda^D_n = \lambda^D_{\max}$ be its eigenvalues numbered in increasing order.

Lemma 5.5.

$$\begin{aligned} \lambda^{A_t}_{\min} &\geq \lambda^A_{\min} + t\lambda^B_{\min} \\ \lambda^{A_t}_{\max} &\leq \lambda^A_{\max} + t\lambda^B_{\max} \leq \lambda^A_{\max} + \lambda^B_{\max} \end{aligned}$$

Proof. Immediate from the concavity/convexity of the minimal/maximal eigenvalue (cf. the proof of Lemma 2.9). □

Lemma 5.6.

$$\mathrm{tr}_{A_t}\left(B^2\right)\sqrt{\det C^{-1}A_t} \leq \frac{n\left(\lambda^B_{\max}\right)^2\left(\lambda^{A_t}_{\max}\right)^{\frac{n-1}{2}}}{\sqrt{\det C}\left(\lambda^B_{\min}\right)^{3/2}} \frac{1}{t^{3/2}}$$

Proof. We focus on the trace term first.

Since B is a symmetric matrix, there exists a basis for which B is diagonal, so that $B = \mathrm{diag}(\lambda^B_1, \ldots, \lambda^B_n)$. In this basis, if we denote A_t^{-1} by (a_t^{ij}), then we have

$$\mathrm{tr}\left(\left(A_t^{-1}B\right)^2\right) = \sum_{ij} a_t^{ij}\lambda^B_j a_t^{ji}\lambda^B_i \leq \left(\lambda^B_{\max}\right)^2 \sum_{ij}\left(a_t^{ij}\right)^2 = \left(\lambda^B_{\max}\right)^2 \mathrm{tr}\left(A_t^{-2}\right), \tag{5.4}$$

where we have used the symmetry of A_t^{-1}.

We note that the trace of the square of a matrix is given by the sum of the squares of its eigenvalues. Therefore,

$$\operatorname{tr}\left(A_t^{-2}\right) = \sum_i \left(\lambda_i^{A_t}\right)^{-2} \leq n \left(\lambda_{\min}^{A_t}\right)^{-2}. \tag{5.5}$$

This takes care of the trace term.

For the determinant term, we clearly have $\det A_t \leq \lambda_{\min}^{A_t} (\lambda_{\max}^{A_t})^{n-1}$. Combining this, equations (5.4) and (5.5), the estimate of Lemma 5.5, and the fact that $\lambda_{\min}^{A} \geq 0$ (as $A \geq 0$) now implies the result. ☐

5.2.3. *Finiteness of $L(g_t)$.* We want to use the pointwise estimate of Lemma 5.6 to prove the main result of the section.

It is clear that to pass from the pointwise result of Lemma 5.6 to a global result, we will have to estimate the maximum and minimum eigenvalues of h, as well as the maximum eigenvalue of g_t. We begin by noting that since we work over an amenable coordinate atlas (cf. Definition 2.12), all coefficients of h, g and g_0 are bounded in absolute value. Therefore, so are their determinants. In particular, since $g > 0$ and $h > 0$, we can assume that $\det g \geq C_0$ and $C_1 \geq \det h \geq C_2$ over each chart of the amenable atlas for some constants $C_0, C_1, C_2 > 0$.

Lemma 5.7. *The quantities $\lambda_{\max}^h$ and $\lambda_{\max}^{g_t}$, as local functions on each coordinate chart, are uniformly bounded, say $\lambda_{\max}^h(x) \leq C_3$ and $\lambda_{\max}^{g_t}(x) \leq C_4$ for all x and t.*

Proof. Note that g_t lies in the quasi-amenable subset $\mathcal{U}$ for all t, so we have upper bounds (in absolute value) on the coefficients of g_t and h that are uniform in x and t. Thus, the bounds on their maximal eigenvalues follow straighforwardly from the min-max theorem [19, Thm. XIII.1]. ☐

Lemma 5.8. *The quantity $\lambda_{\min}^h$, as a function over each coordinate chart, is uniformly bounded away from 0, say $\lambda_{\min}^h \geq C_5 > 0$.*

Proof. We clearly have $\det h(x) \leq \lambda_{\min}^h(x) \lambda_{\max}^h(x)^{n-1}$. Therefore, by Lemma 5.7 and the discussion before it, $\lambda_{\min}^h(x) \geq \lambda_{\max}^h(x)^{1-n} \det h(x) \geq C_3^{1-n} C_2 =: C_5$. ☐

Theorem 5.9. *Define a path g_t as in* (5.2). *Then*

$$L(g_t) < \infty.$$

Proof. At each point $x \in M$, we use Lemma 5.6 to see

$$\operatorname{tr}_{g_t(x)}(h(x)^2) \sqrt{\det(g(x)^{-1} g_t(x))} \leq \frac{1}{\sqrt{C_0}} \frac{C_3^2}{C_5^{3/2}} C_4^{\frac{n-1}{2}} \frac{1}{t^{3/2}} =: \frac{C_6}{t^{3/2}}. \tag{5.6}$$

The result then follows from (5.3) and the integrability of $t^{-3/4}$. ☐

5.2.4. *Bounded, nonsmooth semimetrics.* We now move on to showing that the equivalence class of any bounded semimetric, not just smooth ones, is contained in $\Omega(\overline{\mathcal{M}})$.

So far, we know from Proposition 4.30 that the equivalence class of any measurable metric that can be obtained as the L^2 limit of a sequence of metrics from an amenable subset belongs to $\Omega(\overline{\mathcal{M}})$. We also know from the preceding arguments that any smooth semimetric $\tilde{g}$ is in the image of Ω. Given the remarks at the end of Section 4, we can therefore unambiguously write things like $d(g_0, g_1)$—where g_0 and g_1 are known to belong to the image of Ω—in place of expressions involving sequences ω-converging to g_0 and g_1.

To begin proving the result on bounded, nonsmooth semimetrics, we want to prove a result about quasi-amenable subsets that is a generalization of Theorem 3.6—weakened so that it still applies for these more general subsets. First, though, we need to prove a couple of lemmas.

Lemma 5.10. *Let $\mathcal{U} \subset \mathcal{M}$ be quasi-amenable. Recall that we denote the closure of $\mathcal{U}$ in the C^∞ topology of $\mathcal{S}$ by $\mathrm{cl}(\mathcal{U})$, and we denote the boundary of $\mathcal{M}$ in the C^∞ topology of $\mathcal{S}$ by $\partial\mathcal{M}$. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that $d(g_0, g_0 + \delta g) < \epsilon$ for all $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$.*

Proof. For any $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$, we consider the path $g_t := g_0 + th$, where $h := \delta g$ and $t \in (0, 1]$. The proof consists of reexamining the estimates of Theorem 5.9 and showing that they only depend on upper bounds on the entries of g_0 (and g, but we get these automatically when we work over an amenable atlas), and that the bound on the length of g_t goes to zero as $\delta \to 0$.

So, recall the main estimate (5.6) of Theorem 5.9:

$$\operatorname{tr}_{g_t(x)}(h(x)^2)\sqrt{\det(g(x)^{-1}g_t(x))} \leq \frac{n\left(\lambda^h_{\max}(x)\right)^2 \left(\lambda^{g_t}_{\max}(x)\right)^{\frac{n-1}{2}}}{\sqrt{\det g(x)}\left(\lambda^h_{\min}(x)\right)^{3/2}} \frac{1}{t^{3/2}}.$$

Since $\det g(x)$ is constant w.r.t. δ, we ignore this term. By Lemma 5.5,

$$\lambda^{g_t}_{\max}(x) \leq \lambda^{g_0}_{\max}(x) + \lambda^h_{\max}(x) = \lambda^{g_0}_{\max}(x) + \delta\lambda^g_{\max}(x).$$

Therefore, using the same arguments as in Lemma 5.7, $\lambda^{g_t}_{\max}(x)$ is bounded from above, uniformly in x and t, by a constant that decreases as δ decreases. Furthermore, this constant does not depend on our choice of $g_0 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$, since the proof of Lemma 5.7 depended only on uniform upper bounds on the entries of g_0, and we are guaranteed the same upper bounds on all elements of $\mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ since $\mathcal{U}$ is quasi-amenable.

We now focus our attention on the term

$$\frac{\left(\lambda^h_{\max}(x)\right)^2}{\left(\lambda^h_{\min}(x)\right)^{3/2}} = \frac{\left(\delta\lambda^g_{\max}(x)\right)^2}{\left(\delta\lambda^g_{\min}(x)\right)^{3/2}} = \frac{\left(\lambda^g_{\max}(x)\right)^2}{\left(\lambda^g_{\min}(x)\right)^{3/2}}\sqrt{\delta}.$$

This expression clearly goes to zero as $\delta \to 0$. Therefore, we have shown that the constant C_6 in the estimate (5.6) depends only on the choice of $\mathcal{U}$ and δ, and that $C_6 \to 0$ as $\delta \to 0$. The result now follows. ☐

The next lemma implies, in particular, that $\partial\mathcal{M}$ is *not closed* in the L^2 topology of $\mathcal{S}$, nor is it in the topology of d on $\Omega(\overline{\mathcal{M}})$. It also implies that around any point in $\mathcal{M}$, there exists no L^2- or d-open neighborhood.

Lemma 5.11. *Let $\mathcal{U} \in \mathcal{M}$ be any quasi-amenable subset. Then for all $\epsilon > 0$, there exists a function $\rho_\epsilon \in C^\infty(M)$ with the properties that for all $g_1 \in \mathcal{U}$,*

1. *$\rho_\epsilon g_1 \in \partial\mathcal{M}$,*
2. *$\rho_\epsilon(x) \leq 1$ for all $x \in M$,*
3. *$\|g_1 - \rho_\epsilon g_1\|_g < \delta$ and*
4. *$d(g_1, \rho_\epsilon g_1) < \epsilon$.*

Proof. Let $x_0 \in M$ be any point, and for each $n \in \mathbb{N}$, choose a function $\rho_n \in C^\infty(M)$ satisfying

1. $\rho_n(x_0) = 0$,
2. $0 \leq \rho_n(x) \leq 1$ for all $x \in M$ and
3. $\rho_n \equiv 1$ outside an open set Z_n with $\mathrm{Vol}(Z_n, g) \leq 1/n$.

Then clearly $\|g_1 - \rho_n g_1\|_g \to 0$ as $n \to \infty$, and this convergence is uniform in g_1 because of the upper bounds guaranteed by the fact that $g_1 \in \mathcal{U}$. Using arguments similar to those in the last lemma, we can also see that the length of the path $g^n_t := \rho_n g_1 + t(g_1 - \rho_n g_1)$ converges to zero as $n \to \infty$. Therefore, choosing n large enough gives the desired function. ☐

The next theorem is the desired analog of Theorem 3.6. Note that only one half of Theorem 3.6 holds in this case, and even this is proved only in a weaker form.

Theorem 5.12. *Let $\mathcal{U} \subset \mathcal{M}$ be quasi-amenable. Then for all $\epsilon > 0$, there exists $\delta > 0$ such that if $g_0, g_1 \in \mathrm{cl}(\mathcal{U})$ with $\|g_0 - g_1\|_g < \delta$, then $d(g_0, g_1) < \epsilon$.*

Proof. First, we enlarge $\mathcal{U}$ if necessary to include *all* metrics satisfying the bound given in Definition 5.4. This enlarged $\mathcal{U}$ is then clearly convex by the triangle inequality for the absolute value, and hence it is still a quasi-amenable subset.

Now, let $\epsilon > 0$ be given. We prove the statement first for $g_0, g_1 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$, then use this to prove the general case.

By Lemma 5.10, we can choose $\delta_1 > 0$ such that $d(\hat{g}, \hat{g} + \delta_1 g) < \epsilon/3$ for all $\hat{g} \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$. We define an amenable subset of $\mathcal{M}$ by

$$\mathcal{U}' := \{\hat{g} + \delta_1 g \mid \hat{g} \in \mathcal{U}\}.$$

Lemma 5.5 implies that this set is indeed amenable. Now, by Theorem 3.6, there exists $\delta > 0$ such that if $\tilde{g}_0, \tilde{g}_1 \in \mathcal{U}'$ with $\|\tilde{g}_0 - \tilde{g}_1\|_g < \delta$, then $d(\tilde{g}_0, \tilde{g}_1) < \epsilon/3$. Let $g_0, g_1 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ be such that $\|g_0 - g_1\|_g < \delta$. If we define $\tilde{g}_i := g_i + \delta_1 g$ for $i = 1, 2$, then it is clear that $\|\tilde{g}_0 - \tilde{g}_1\|_g = \|g_0 - g_1\|_g < \delta$. Given this and the definition of δ_1, we have

$$d(g_0, g_1) \leq d(g_0, \tilde{g}_0) + d(\tilde{g}_0, \tilde{g}_1) + d(\tilde{g}_1, g_1) < \epsilon.$$

Now we prove the general case. Let $\epsilon > 0$ be given. By the special case we just proved, we can choose $\delta > 0$ such that if $\tilde{g}_0, \tilde{g}_1 \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ with $\|\tilde{g}_0 - \tilde{g}_1\|_g < \delta$, then $d(\tilde{g}_0, \tilde{g}_1) < \epsilon/3$. Let $g_0, g_1 \in \mathcal{U}$ be any elements with $\|g_0 - g_1\|_g < \delta$. By Lemma 5.11 and our enlargement of $\mathcal{U}$, we can choose a function $\rho \in C^\infty(M)$ such that for $i = 0, 1$,

(1) $\rho g_i \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$,
(2) $\rho(x) \leq 1$ for all $x \in M$, and
(3) $d(g_i, \rho g_i) < \epsilon/3$.

(If $g_i \in \mathrm{cl}(\mathcal{U}) \cap \partial\mathcal{M}$ for both $i = 1$ and 2, we might as well just choose $\rho \equiv 1$.) In particular, the second property of ρ implies that

$$\|\rho g_1 - \rho g_0\|_g \leq \|g_1 - g_0\|_g < \delta.$$

Then we immediately get

$$d(g_0, g_1) \leq d(g_0, \rho g_0) + d(\rho g_0, \rho g_1) + d(\rho g_1, g_1) < \epsilon.$$

This proves the general case and thus the theorem. ☐

Using the relationship between d and $\|\cdot\|_g$ determined in Theorem 5.12, we can prove the following.

Proposition 5.13. *Let $[\tilde{g}] \in \widehat{\mathcal{M}_f}$ be an equivalence class of bounded, measurable semimetrics. Then for any representative $\tilde{g} \in [\tilde{g}]$, there exists a sequence $\{g_k\}$ in $\mathcal{M}$ that both L^2- and ω-converges to $\tilde{g}$. Thus $[\tilde{g}] \in \Omega(\overline{\mathcal{M}})$.*

Moreover, suppose $\tilde{g} \in \mathcal{U}^0$ for some quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$. Then for any sequence $\{g_l\}$ in $\mathcal{U}$ that L^2-converges to $\tilde{g}$, $\{g_l\}$ is d-Cauchy and there exists a subsequence $\{g_k\}$ that also ω-converges to $\tilde{g}$.

Proof. It is clear that for every bounded representative $\tilde{g} \in [\tilde{g}]$, we can find a quasi-amenable subset $\mathcal{U} \subset \mathcal{M}$ such that $\tilde{g} \in \mathcal{U}^0$. Thus, there exists a sequence $\{g_l\}$ that L^2-converges to $\tilde{g}$. It is d-Cauchy by Theorem 5.12. We wish to show that it contains a subsequence $\{g_k\}$ that also ω-converges to $\tilde{g}$, so we still need to verify properties (2)–(4) of Definition 4.4.

By passing to a subsequence, we can assume that property (4) is satisfied for $\{g_l\}$. Property (3) is verified in the same way as in the proof of Lemma 4.29. That is, L^2-convergence of $\{g_l\}$ implies that there exists a subsequence $\{g_k\}$ of $\{g_l\}$ that converges to $\tilde{g}$ a.e. Finally, a.e.-convergence of $\{g_k\}$ to $\tilde{g}$ and continuity of the determinant function imply that property (2) holds. ☐

Thus, like we did for more restricted types of metrics before, this proposition allows us to cease to distinguish between bounded semimetrics and sequences ω-converging to them.

5.3. **Unbounded metrics and the proof of the main result.** Up to this point, we have an injection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$, and we have determined that the image $\Omega(\overline{\mathcal{M}})$ contains all equivalence classes containing bounded semimetrics. In this section, we prove that Ω is surjective.

Theorem 5.14. *Let any $[\tilde{g}] \in \widehat{\mathcal{M}_f}$ be given. Then there exists a sequence $\{g_k\}$ in $\mathcal{M}$ such that*

$$g_k \xrightarrow{\omega} [\tilde{g}].$$

Proof. In view of Proposition 5.13, it remains only to prove this for the equivalence class of a measurable, unbounded semimetric $\tilde{g} \in \mathcal{M}_f$.

Given any element $\hat{g} \in \mathcal{M}_f$, we can define $\exp_{\hat{g}}$ on tensors of the form $\sigma\hat{g}$, where σ is any function, purely algebraically. We simply set

$$\exp_{\hat{g}}(\sigma\hat{g}) := \left(1 + \frac{n}{4}\sigma\right)^{4/n}\hat{g}, \tag{5.7}$$

so that the expression coincides with the usual one if $\hat{g} \in \mathcal{M}$ and $\sigma \in C^\infty(M)$ with $\sigma > -\frac{4}{n}$ (cf. Theorem 2.1). If σ is additionally measurable, then $\exp_{\hat{g}}(\sigma\hat{g})$ will also be measurable.

Now, let $\tilde{g} \in \mathcal{M}_f$. Then we can find a measurable, positive function ξ on M such that $g_0 := \xi\tilde{g}$ is a bounded semimetric. A simple calculation using the finite volume of $\tilde{g}$ shows that $\rho := \xi^{-1} \in L^{n/2}(M, g_0)$.

Define the map ψ by $\psi(\sigma) := \exp_{g_0}(\sigma g_0)$, and let

$$\lambda := \frac{4}{n}\left(\rho^{n/4} - 1\right). \tag{5.8}$$

Then clearly $\psi(\lambda) = \rho g_0 = \tilde{g}$. Moreover, we claim that $\lambda \in L^2(M, g_0)$ and hence, by Corollary 5.3, we can find a sequence $\{\lambda_k\}$ of smooth functions that converge in $L^2(M, g_0)$ to λ . That $\lambda \in L^2(M, g_0)$ follows from two facts. First, $\rho \in L^{n/2}(M, g_0)$, implying that $\rho^{n/4} \in L^2(M, g_0)$. Second, finite volume of g_0 implies that the constant function $1 \in L^2(M, g_0)$ as well.

Since $\lambda_k \to \lambda$ in $L^2(M, g_0)$, as in the proof of Lemma 4.29 we can pass to a subsequence and assume that $\lambda_k \to \lambda$ pointwise a.e., where we note that here, "almost everywhere" means with respect to μ_{g_0}. With respect to the fixed, smooth, strictly positive volume form μ_g, this actually means that $\lambda_k(x) \to \lambda(x)$ for a.e. $x \in M \setminus X_{g_0}$, since X_{g_0} is a nullset with respect to μ_{g_0}. Note also that $X_{g_0} = X_{\tilde{g}}$, since we assumed that the function ξ is positive. Therefore $\lambda_k(x) \to \lambda(x)$ for a.e. $x \in M \setminus X_{\tilde{g}}$.

Furthermore, since from (5.8) and positivity of ξ it is clear that $\lambda > -\frac{4}{n}$, we can choose the sequence $\{\lambda_k\}$ such that $\lambda_k > -\frac{4}{n}$ for all $k \in \mathbb{N}$. This implies, in particular, that $X_{\psi(\lambda_k)} = X_{g_0} = X_{\tilde{g}}$, which is easily seen from (5.7).

We make one last assumption on the sequence $\{\lambda_k\}$. Namely, by passing to a subsequence, we can assume that

$$\sum_{k=1}^{\infty} \|\lambda_{k+1} - \lambda_k\|_{g_0} < \infty. \tag{5.9}$$

Now, we claim that

$$d(\psi(\sigma), \psi(\tau)) \leq \sqrt{n}\, \|\tau - \sigma\|_{g_0} \tag{5.10}$$

for all $\sigma, \tau \in C^\infty(M)$ with $\sigma, \tau > -\frac{4}{n}$. We delay the proof of this statement to Lemma 5.16 below and first finish the proof of the theorem.

We wish to construct a sequence that ω-converges to $\tilde{g}$ using the sequence $\{\psi(\lambda_k)\}$. We can't use $\{\psi(\lambda_k)\}$ directly, since it is a sequence in $\Omega(\overline{\mathcal{M}})$, not $\mathcal{M}$ itself. So we first verify the properties of ω-convergence for $\{\psi(\lambda_k)\}$ and then construct a sequence in $\mathcal{M}$ that approximates $\{\psi(\lambda_k)\}$ well enough that it still satisfies all the conditions for ω-convergence.

Since the sequence $\{\lambda_k\}$ is convergent in $L^2(M, g_0)$, it is also Cauchy in $L^2(M, g_0)$. Using the inequality (5.10), it is then immediate that $\{\psi(\lambda_k)\}$ is a Cauchy sequence in $(\Omega(\overline{\mathcal{M}}), d)$. This verifies property (1) of ω-convergence (cf. Definition 4.4).

We next verify property (3). Note that $X_{\tilde{g}} \subseteq D_{\{\psi(\lambda_k)\}}$, since we have already shown that $X_{\psi(\lambda_k)} = X_{\tilde{g}}$. (Keep in mind here the subtle point that $X_{\psi(\lambda_k)}$ is the deflated set of the *individual* semimetric $\psi(\lambda_k)$, while $D_{\{\psi(\lambda_k)\}}$ is the deflated set of the *sequence* $\{\psi(\lambda_k)\}$. Refer to Definitions 2.17 and 2.18 for details.) The inclusion implies that

$$M \setminus D_{\{\psi(\lambda_k)\}} \subseteq M \setminus X_{\tilde{g}},$$

so it suffices to show that $\psi(\lambda_k)(x) \to \tilde{g}(x)$ for a.e. $x \in M \setminus X_{\tilde{g}}$. But this is clear from the fact, proved above, that $\lambda_k(x) \to \lambda(x)$ for a.e. $x \in M \setminus X_{\tilde{g}}$.

To verify property (2), we claim that $D_{\{\psi(\lambda_k)\}} = X_{\tilde{g}}$, up to a nullset. In the previous paragraph, we already showed that $X_{\tilde{g}} \subseteq D_{\{\psi(\lambda_k)\}}$. Furthermore, for a.e. $x \in M \setminus X_{\tilde{g}}$, $\{\psi(\lambda_k)(x)\}$ converges to $\tilde{g}(x)$, which is positive definite, so for a.e. $x \in M \setminus X_{\tilde{g}}$, $\lim \det \psi(\lambda_k) > 0$. This immediately implies that $D_{\{\psi(\lambda_k)\}} \subseteq X_{\tilde{g}}$, up to a nullset.

The last property to verify is (4). But this is immediate from (5.9) and (5.10).

So we have shown that $\{\psi(\lambda_k)\}$ satisfies the properties of ω-convergence, save that it is a sequence of measurable semimetrics, rather than a sequence of smooth metrics as required. To get a sequence in $\mathcal{M}$ that ω-converges to $\tilde{g}$, recall that each of the functions λ_k is smooth and therefore bounded, and also that g_0 is a bounded, measurable semimetric. Therefore, for each fixed $k \in \mathbb{N}$, $\psi(\lambda_k)$ is also a bounded, measurable semimetric, and so by Proposition 5.13 we can find a sequence $\{g^k_l\}$ in $\mathcal{M}$ that ω-converges to $\psi(\lambda_k)$ for $l \to \infty$. By a standard diagonal argument, it is then possible to select $l_k \in \mathbb{N}$ for each $k \in \mathbb{N}$ such that the sequence $\{g^k_{l_k}\}$ ω-converges to $\tilde{g}$ for $k \to \infty$. Thus we have found the desired sequence.

It still remains to prove (5.10). The following two lemmas do this and thus complete the proof of the theorem. □

Lemma 5.15. *Let $\tilde{g} \in \mathcal{M}$. If $\sigma, \tau \in C^\infty(M)$ satisfy $\sigma, \tau > -\frac{4}{n}$, then*

$$d(\exp_{\tilde{g}}(\sigma\tilde{g}), \exp_{\tilde{g}}(\tau\tilde{g})) \leq \sqrt{n}\, \|\tau - \sigma\|_{\tilde{g}}\,.$$

Proof. Let $\hat{g} := \exp_{\tilde{g}}(\sigma\tilde{g})$. We first note that $\mathcal{P} \cdot \tilde{g} = \mathcal{P} \cdot \hat{g}$. Therefore, by Proposition 2.1, there is a neighborhood $V \in C^\infty(M)$ such that $\exp_{\hat{g}} : V \cdot \hat{g} \to \mathcal{P} \cdot \hat{g}$ is a diffeomorphism.

Now

$$d(\exp_{\tilde{g}}(\sigma\tilde{g}), \exp_{\tilde{g}}(\tau\tilde{g})) \leq \left\| \exp_{\hat{g}}^{-1} \exp_{\tilde{g}}(\tau\tilde{g}) \right\|_{\hat{g}}, \tag{5.11}$$

since the right-hand side is the length of a radial geodesic emanating from $\exp_{\tilde{g}}(\sigma\tilde{g})$ and ending at $\exp_{\tilde{g}}(\tau\tilde{g})$. Showing that the right-hand side of (5.11) is equal to $\sqrt{n}\,\|\tau - \sigma\|$ is a straightforward computation using (2.1). □

Lemma 5.16. *Let g_0 and ψ be as in the proof of Theorem 5.14. If $\sigma, \tau \in C^\infty(M)$ satisfy $\sigma, \tau > -\frac{4}{n}$, then*

$$d(\psi(\sigma), \psi(\tau)) \leq \sqrt{n}\, \|\tau - \sigma\|_{g_0}\,.$$

Proof. Since g_0 is bounded, we can find a quasi-amenable subset $\mathcal{U}$ such that $g_0 \in \mathcal{U}^0$, i.e., such that g_0 belongs to the completion of $\mathcal{U}$ with respect to $\|\cdot\|_g$. Using Proposition 5.13, choose a sequence $\{g_k\}$ in $\mathcal{U}$ that both L^2- and ω-converges to g_0. For each $k \in \mathbb{N}$, define a map ψ_k by $\psi_k(\kappa) := \exp_{g_k}(\kappa g_k)$.

By the triangle inequality, we have

$$d(\psi(\sigma), \psi(\tau)) \leq d(\psi(\sigma), \psi_k(\sigma)) + d(\psi_k(\sigma), \psi_k(\tau)) + d(\psi_k(\tau), \psi(\tau)) \tag{5.12}$$

for each k. But since $g_k \in \mathcal{M}$, Lemma 5.15 applies to give

$$d(\psi_k(\sigma), \psi_k(\tau)) \leq \sqrt{n} \, \|\tau - \sigma\|_{g_k} \stackrel{k\to\infty}{\longrightarrow} \sqrt{n} \, \|\tau - \sigma\|_{g_0} , \tag{5.13}$$

where the convergence follows from Lemma 5.1. By (5.12) and (5.13), if we can show that

$$d(\psi(\sigma), \psi_k(\sigma)) \to 0 \quad \text{and} \quad d(\psi_k(\tau), \psi(\tau)) \to 0, \tag{5.14}$$

then we are finished. But it is not hard to show that $\psi_k(\sigma)$ L^2-converges to $\psi(\sigma)$, which then implies (5.14) by Proposition 5.13. ☐

From the results of Section 4, we already know that the map $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$ is an injection. Theorem 5.14 now states that this map is a surjection as well. Thus, we have already proved the main result of this paper, which we state again here in full detail.

Theorem 5.17. *There is a natural identification of $\overline{\mathcal{M}}$, the completion of $\mathcal{M}$ with respect to the L^2 metric, with $\widehat{\mathcal{M}_f}$, the set of measurable semimetrics with finite volume on M modulo the equivalence given in Definition 4.3.*

This identification is given by a bijection $\Omega : \overline{\mathcal{M}} \to \widehat{\mathcal{M}_f}$, where we map an equivalence class $[\{g_k\}]$ of d-Cauchy sequences to the unique element of $\widehat{\mathcal{M}_f}$ that all of its members ω-subconverge to. This map is an isometry if we give $\widehat{\mathcal{M}_f}$ the metric $\bar{d}$ defined by

$$\bar{d}([g_0], [g_1]) := \lim_{k\to\infty} d(g_k^0, g_k^1)$$

where $\{g_k^0\}$ and $\{g_k^1\}$ are any sequences in $\mathcal{M}$ ω-subconverging to $[g_0]$ and $[g_1]$, respectively.

Here, we briefly note what geometric notions are well-defined for elements of $\widehat{\mathcal{M}_f}$. Given an equivalence class $[\tilde{g}] \in \widehat{\mathcal{M}_f}$, the metric space structure of different representatives may differ—e.g., if $M = T^2$, the torus, then the equivalence class of the zero metric also contains a geometric circle, where only one dimension has collapsed. On the other hand, since representatives of a given equivalence class in $\widehat{\mathcal{M}_f}$ all have equal induced measures, things like L^p spaces of functions are well-defined for an equivalence class, as they are the same across all representatives. But even more is true—C^k and H^s spaces of sections of fiber bundles can be defined Therefore, an equivalence class doesn't induce a well-defined scalar product on a vector bundle at any individual point, but the integral of the scalar product does not depend on the chosen representative.

To end this section, we remark that one might hope that Theorem 5.17 would give some information on the completion of the space $\mathcal{M}/\mathcal{D}$ of *Riemannian structures* with respect to the distance that the L^2 metric induces on it. (Here, $\mathcal{D}$ is the group of orientation-preserving diffeomorphisms of M acting by pull-back.) $\mathcal{M}/\mathcal{D}$ is the moduli space of Riemannian metrics, and hence is of great intrinsic interest to geometers. The problem here is that the proof of Theorem 5.17 does not indicate which degenerations of Cauchy sequences of metrics arise from "vertical" degenerations—that is, sequences $\{\varphi_n^* \tilde{g}\}$, where $\{\varphi_n\} \subset \mathcal{D}$ is a degenerating sequence of diffeomorphisms—and "horizontal" degenerations—that is, sequences of metrics that can be connected by horizontal paths. (See [7, §3] for a discussion of horizontal and vertical paths on $\mathcal{M}$.) Of course, only horizontal degenerations are relevant for the quotient. So there is some work remaining to do in order to understand the completion of $\mathcal{M}/\mathcal{D}$. We hope to investigate these questions in a future paper.

6. Application to Teichmüller Theory

In this section, we describe an application of our main theorem to the theory of Teichmüller space. Teichmüller space was historically defined in the context of complex manifolds, but the work of Fischer and Tromba translates this original picture into the context of Riemannian geometry, using the manifold of metrics [22]. We outline this construction of Teichmüller space in the first subsection, then define the much-studied Weil-Petersson metric. In the second subsection, we prove a result on the completions of a class of metrics that generalize the Weil-Petersson metric.

6.1. **The Weil-Petersson Metric on Teichmüller Space.**

Convention 6.1. For the remainder of the paper, let our base manifold M be a smooth, closed, oriented, two-dimensional manifold of genus $p \geq 2$.

Convention 6.2. In this chapter, we abandon Convention 2.11. That is, when we write g for a metric in $\mathcal{M}$, we no longer assume that this is fixed, but allow g to vary arbitrarily.

We have already noted that the group $\mathcal{P}$ acts on $\mathcal{M}$ by pointwise multiplication, and it turns out that the quotient $\mathcal{M}/\mathcal{P}$ is a smooth Fréchet manifold. Let $\mathcal{D}$ be the Fréchet Lie group of orientation-preserving diffeomorphisms of M, and let $\mathcal{D}_0 \subset \mathcal{D}$ be the subgroup of diffeomorphisms homotopic to the identity. Then both $\mathcal{D}$ and $\mathcal{D}_0$ act on $\mathcal{M}$ and $\mathcal{M}/\mathcal{P}$ by pull-back. Let $\mathcal{T}$ denote the Teichmüller space of M, $\mathcal{R}$ the Riemann moduli space of M, and $MCG := \mathcal{D}/\mathcal{D}_0$ the *mapping class group* of M. Then there are identifications

$$\mathcal{T} \cong {}^{(\mathcal{M}/\mathcal{P})}\big/_{\mathcal{D}_0}, \qquad \mathcal{R} \cong {}^{\mathcal{T}}\big/_{MCG} \cong {}^{(\mathcal{M}/\mathcal{P})}\big/_{\mathcal{D}},$$

where the first identification is a diffeomorphism. Note that Teichmüller space finite-dimensional.

By the Poincaré uniformization theorem, there exists a unique hyperbolic metric (one with scalar curvature -1) in each conformal class $[g] \in \mathcal{M}/\mathcal{P}$. Furthermore, one can show that the subset $\mathcal{M}_{-1} \subset \mathcal{M}$ of hyperbolic metrics is a smooth Fréchet submanifold. Therefore, $\mathcal{M}_{-1}$ is the image of a smooth section of the principal $\mathcal{P}$-bundle $\mathcal{M} \to \mathcal{M}/\mathcal{P}$. It is easy to see that $\mathcal{M}_{-1}$ is $\mathcal{D}$-invariant, and therefore

$$\mathcal{T} \cong {}^{\mathcal{M}_{-1}}\big/_{\mathcal{D}_0}, \qquad \mathcal{R} \cong {}^{\mathcal{M}_{-1}}\big/_{\mathcal{D}},$$

where the first identification is again a diffeomorphism. We denote by $\pi : \mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$ the projection.

It is not hard to see that the L^2 metric $(\cdot,\cdot)$ on $\mathcal{M}$ is $\mathcal{D}$-invariant, so it descends to a MCG-invariant Riemannian metric, also denoted $(\cdot,\cdot)$, on the quotient $\mathcal{M}_{-1}/\mathcal{D}_0$. This metric is called the *Weil-Petersson metric*. (It differs from the usual definition of the Weil-Petersson metric by a constant scalar factor; cf. [22, §2.6].) With these definitions, we see that $(\mathcal{M}_{-1}, (\cdot,\cdot)) \to (\mathcal{M}_{-1}/\mathcal{D}_0, (\cdot,\cdot))$ is a weak Riemannian principal $\mathcal{D}_0$-bundle—that is, at each point $g \in \mathcal{M}_{-1}$, the differential $D\pi(g)$ of the projection is an isometry when restricted to the horizontal tangent space at g.

6.2. **Generalized Weil-Petersson Metrics.**

We now wish to generalize the construction of the Weil-Petersson metric by selecting a different section of $\mathcal{M} \to \mathcal{M}/\mathcal{P}$. In fact, we will simultaneously consider all smooth sections $\mathcal{N}$ with the property that they are $\mathcal{D}$-invariant, which we require so that we still have diffeomorphisms $\mathcal{T} \cong \mathcal{N}/\mathcal{D}_0$ and $\mathcal{R} \cong \mathcal{N}/\mathcal{D}$. This idea is directly inspired by [9] and [10], though our metrics on Teichmüller space differ from theirs.

Definition 6.3. We call a smooth, $\mathcal{D}$-invariant section of $\mathcal{M} \to \mathcal{M}/\mathcal{P}$ a *modular section.* Given a modular section $\mathcal{N} \subset \mathcal{M}$, we call the quotients $\mathcal{N}/\mathcal{D}_0$ and $\mathcal{N}/\mathcal{D}$ the *$\mathcal{N}$-model of Teichmüller space* and the *$\mathcal{N}$-model of moduli space*, respectively.

For the remainder of the talk, let $\mathcal{N}$ be an arbitrary modular section. It is not hard to see that $\mathcal{N} \cong \mathcal{M}_{-1}$ via a $\mathcal{D}$-equivariant diffeomorphism, so in fact we do have the desired diffeomorphisms $\mathcal{T} \cong \mathcal{N}/\mathcal{D}_0$ and $\mathcal{R} \cong \mathcal{N}/\mathcal{D}$.

Modular sections other than $\mathcal{M}_{-1}$ of course exist—for example, there are the Bergman and Arakelov metrics (see [10, §1] for details). We briefly describe the Bergman metric on a Riemann surface here. Recall that conformal structures (elements of $\mathcal{M}/\mathcal{P}$) are in one-to-one correspondence with complex structures on the surface, so we can work with these instead. Let c be a complex structure on M. Then the space of holomorphic one-forms on (M, c) has complex dimension p, the genus of M [6, Prop. III.2.7]. Let $\theta_1, \ldots, \theta_p$ be an L^2-orthonormal basis of this space. That is,

$$\frac{i}{2} \int_M \theta_j \wedge \overline{\theta_k} = \delta_{jk}.$$

The Bergman metric is defined by

$$g_B := \frac{1}{p} \sum_{i=1}^{p} \theta_i \bar{\theta}_j.$$

It is clear from this construction that the set of all Bergman metrics is indeed a modular section.

As in the case of the section $\mathcal{M}_{-1}$, the L^2 metric on $\mathcal{M}$ projects to an MCG-invariant metric on $\mathcal{N}/\mathcal{D}_0$. We call this metric the *generalized Weil-Petersson metric* on the $\mathcal{N}$-model of Teichmüller space. As in the case of the bundle $\mathcal{M}_{-1} \to \mathcal{M}_{-1}/\mathcal{D}_0$, these metrics turn the bundle $\mathcal{N} \to \mathcal{N}/\mathcal{D}_0$ into a weak Riemannian principal $\mathcal{D}_0$-bundle.

Theorem 6.4. *For any C^1 path $\gamma : [0,1] \to \mathcal{N}/\mathcal{D}_0$ and any $g \in \pi_{\mathcal{N}}^{-1}(\gamma(0))$, there exists a unique horizontal lift $\tilde{\gamma} : [0,1] \to \mathcal{N}$ with $\tilde{\gamma}(0) = g$.*

Furthermore, $L(\tilde{\gamma}) = L(\gamma)$ and $\tilde{\gamma}$ has minimal length among the class of curves whose image projects to γ under $\pi_{\mathcal{N}}$.

Proof. The existence of horizontal lifts is not usually guaranteed on Fréchet manifolds, but since $\mathcal{N}/\mathcal{D}_0$ is finite-dimensional, the horizontal tangent space of $\mathcal{N}$ is finite-dimensional at each point. Therefore, integral curves of horizontal vector fields exist (cf. [16, Thm. 7.2 and Dfn. 5.6ff]).

An alternative proof, one which does not rely on the existence theory of solutions to ODEs in Fréchet spaces, is given in [3, Thm. 6.16].

Minimality of $\tilde{g}$ can be easily shown using the fact that $\mathcal{N} \to \mathcal{N}/\mathcal{D}_0$ is a weak Riemannian principal bundle. □

The next theorem applies the paper's main to the completion of $\mathcal{N}/\mathcal{D}_0$ with respect to a generalized Weil-Petersson metric. In the following, we denote the distance function of $(\mathcal{N}, (\cdot, \cdot))$ by $d_{\mathcal{N}}$.

Theorem 6.5. *Let $\{[g_k]\}$ be a Cauchy sequence in the $\mathcal{N}$-model of Teichmüller space, $\mathcal{N}/\mathcal{D}_0$, with respect to the generalized Weil-Petersson metric. Then there exist representatives $\tilde{g}_k \in [g_k]$ and an element $[g_\infty] \in \widehat{\mathcal{M}_f}$ such that $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence that ω-subconverges to $[g_\infty]$.*

Furthermore, if $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are equivalent Cauchy sequences in $\mathcal{N}/\mathcal{D}_0$, then there exist representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$, as well as an element $[g_\infty] \in \widehat{\mathcal{M}_f}$, such that $\{\tilde{g}_k^0\}$ and $\{\tilde{g}_k^1\}$ are $d_{\mathcal{N}}$-Cauchy sequences that both ω-subconverge to $[g_\infty]$.

Finally, if $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are inequivalent Cauchy sequences in $\mathcal{N}/\mathcal{D}_0$, then there exists no choice of representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$ such that $\{\tilde{g}_k^0\}$ and $\{\tilde{g}_k^1\}$ ω-subconverge to the same element of $\widehat{\mathcal{M}_f}$.

Proof. The first claim would follow from Theorem 5.17 if we could show that there are representatives $\tilde{g}_k \in [g_k]$ such that $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence, since this implies that it is also a d-Cauchy sequence. So this is what we will show.

Let's denote the distance function induced by the generalized Weil-Petersson metric on $\mathcal{N}/\mathcal{D}_0$ by δ. For each $k \in \mathbb{N}$, let $\gamma_k : [0,1] \to \mathcal{N}/\mathcal{D}_0$ be any path from $[g_k]$ to $[g_{k+1}]$ such that

$$L(\gamma_k) \leq 2\delta([g_k], [g_{k+1}]).$$

For any $\tilde{g}_1 \in \pi_{\mathcal{N}}^{-1}([g_1])$, let $\tilde{\gamma}_1$ be the horizontal lift of γ_1 to $\mathcal{N}$ with $\tilde{\gamma}_1(0) = \tilde{g}_1$ which is guaranteed by Theorem 6.4. Then clearly $\tilde{g}_2 := \gamma_2(1) \in \pi_{\mathcal{N}}^{-1}([g_2])$. Furthermore,

$$d_{\mathcal{N}}(\tilde{g}_1, \tilde{g}_2) \leq L(\tilde{\gamma}_1) = L(\gamma_1) \leq 2\delta([g_1], [g_2]).$$

We repeat this process, i.e., let $\tilde{\gamma}_2$ be the unique horizontal lift of γ_2 with $\tilde{\gamma}_2(0) = \tilde{g}_2$, and set $\tilde{g}_3 := \tilde{\gamma}_2(1)$, etc. In this way, we get a sequence of representatives $\tilde{g}_k \in [g_k]$ such that for each $k \in \mathbb{N}$,

$$d_{\mathcal{N}}(\tilde{g}_k, \tilde{g}_{k+1}) \leq 2\delta([g_k], [g_{k+1}]).$$

Thus, since $\{[g_k]\}$ is a Cauchy sequence, $\{\tilde{g}_k\}$ is a $d_{\mathcal{N}}$-Cauchy sequence, as was to be shown.

The proof of the second statement is similar.

To prove the last statement, note that since $\{[g_k^0]\}$ and $\{[g_k^1]\}$ are inequivalent, we have

$$\lim_{k\to\infty} \delta([g_k^0], [g_k^1]) > 0.$$

Thus by Theorem 6.4, no matter what representatives $\tilde{g}_k^0 \in [g_k^0]$ and $\tilde{g}_k^1 \in [g_k^1]$ we choose,

$$\lim_{k\to\infty} d_{\mathcal{N}}(\tilde{g}_k^0, \tilde{g}_k^1) \geq \epsilon > 0.$$

So Theorem 5.17 implies the statement immediately. □

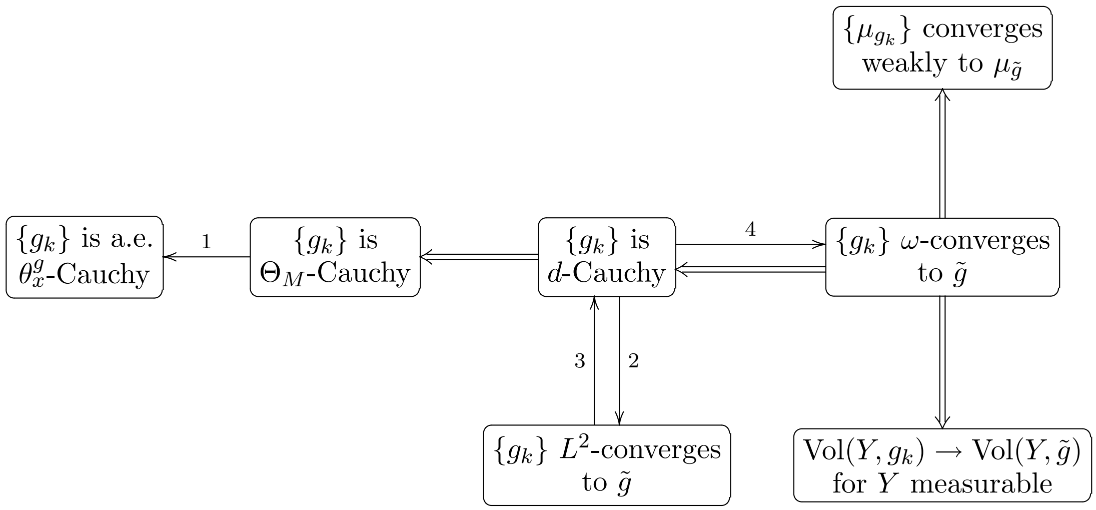
Remark 6.6. We note that the choice of $[g_\infty]$ in Theorem 6.5 is in some sense unique up to $\mathcal{D}_0$-equivalence. Namely, say in the proof of the first statement we were to have chosen a different sequence of paths γ_k^0 connecting $[g_k]$ to $[g_{k+1}]$. Since Teichmüller space is known to be contractible, the horizontal lifts $\tilde{\gamma}_k$ and $\tilde{\gamma}_k^0$ have the same endpoints $\tilde{g}_{k+1} \in [g_{k+1}]$. Thus, they determine the same $d_{\mathcal{N}}$-Cauchy sequence, implying the only choice we made in the proof that matters was that of $\tilde{g}_1 \in [g_1]$.

Theorem 6.5 generalizes what is known about the completion of the Weil-Petersson metric [13], which is completed by adding in certain cusped hyperbolic surfaces—which in particular can be viewed as elements of $\widehat{\mathcal{M}_f}$. However, they are only degenerate along a set of disjoint simple closed geodesics—connecting nicely with the complex-analytic viewpoint of degeneration—whereas elements of $\widehat{\mathcal{M}_f}$ can be degenerate over an arbitrary subset of M. With more investigation and perhaps appropriate conditions on the section $\mathcal{N}$, we expect that the statement can be considerably improved.

Despite the shortcomings of the above result, we see it as quite useful, as it gives relatively strong information about a new class of metrics on Teichmüller space—namely, that their completions can consist only of finite-volume metrics. Furthermore, it illustrates the potential for applications of our main theorem and provides a starting point for further investigations.

Appendix. Relations between Various Notions of Convergence and Cauchy Sequences

In the following chart, we illustrate the relationships between the different notions of Cauchy and convergent sequences on $\mathcal{M}$. We let $\{g_k\}$ be a sequence in $\mathcal{M}$ and $\tilde{g} \in \mathcal{M}_f$. A double arrow ("$\Longrightarrow$") between two statements means that the one implies the other. A single arrow ("$\longrightarrow$") means that one statement implies the other, assuming the condition that is listed below the chart.



(1) After passing to a subsequence
(2) If there exists an amenable subset $\mathcal{U}$ such that $\{g_k\} \subset \mathcal{U}$, then there exists some $\tilde{g} \in \mathcal{U}^0$ such that the implication holds
(3) If there exists a quasi-amenable subset $\mathcal{U}$ such that $\{g_k\} \subset \mathcal{U}$
(4) After passing to a subsequence, there exists some $\tilde{g} \in \mathcal{M}_f$ such that the implication holds

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