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On the orders of periodic symplectomorphisms of 4-manifolds

by

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Abstract. In this paper we investigate generalizations of the classical Hurwitz theorem concerning bound of the order of automorphism group of a Riemann surface of genus at least two to smooth 4-manifolds. In particular, it is shown that for a simply connected symplectic 4-manifold (X,ω) with $b_2^+>1$ and $[\omega]\in H^2(X;\mathbb{Q})$, the order of a periodic symplectomorphism of prime order is bounded from above by a constant C, which depends on ω in a rather unstable way.

1. Introduction

A classical theorem of Hurwitz says that, for a complex curve Σ of genus $g\geq 2$, the order of its automorphism group $\mathrm{Aut}(\Sigma)$ satisfies the following topological bound:

$$|\mathrm{Aut}(\Sigma)|\leq 84(g-1)=42 \deg K_\Sigma.$$

Various attempts have been made to generalize this result to higher-dimensional projective varieties. For a minimal smooth projective surface X of general type, Xiao obtained the optimal result in [27, 28] that $|\mathrm{Aut}(X)|\leq 42^2\, c_1(K_X)^2$, after a series of earlier work by Andreotti [1], Howard and Sommese [14], Corti [9], Huckleberry and Sauer [15] and Xiao [26]. For dimensions greater than 2, see recent work of D.-Q. Zhang [30] and the references therein.

The purpose of this paper is to investigate generalizations of Hurwitz's theorem to finite automorphism groups of smooth 4-manifolds. For simplicity, we shall restrict our consideration to simply connected 4-manifolds and cyclic automorphism groups of prime order. More precisely, we initiate an investigation of the following question.

Main Question: With certain exceptions, for a given simply connected smooth 4-manifold X, does there exist a constant $C>0$, such that there are no nontrivial smooth $\mathbb{Z}_p$-actions of prime order on X provided that $p>C$?

In the above question we have excluded consideration of topological actions. This is because of the following theorem of Edmonds [10]: *For any simply connected 4-manifold X and prime number $p>3$, there exists a homologically trivial, locally linear, pseudofree topological $\mathbb{Z}_p$-action on X.*

The first issue is to understand what would be the exceptional smooth 4-manifolds that one has to exclude from consideration. Obviously, if a 4-manifold admits a smooth

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circle action, there is no chance for such a constant C to exist which bounds the order p. The question is: if the 4-manifold does not admit a smooth circle action, to what extent the answer to the Main Question is affirmative. In this regard, a theorem of Baldridge [3] gives a useful criterion for smooth circle actions: Let X be a 4-manifold admitting a smooth circle action with nonempty fixed point set. Then X has vanishing Seiberg-Witten invariant when $b_2^+ > 1$, and when $b_2^+ = 1$ and X is symplectic, X is diffeomorphic to a rational or ruled surface. Note that any circle action on a simply connected 4-manifold must have a fixed point by the Lefschetz fixed point theorem.

Let's first look at holomorphic actions on Kähler surfaces — these are the primary examples of smooth actions on 4-manifolds. We observe

Theorem 1.1. *Let X be the underlying smooth 4-manifold of a simply connected compact Kähler surface which does not admit any smooth circle actions. Then given any complex structure on X, there exists a constant $C > 0$ such that there are no nontrivial holomorphic $\mathbb{Z}_p$-actions of prime order on X provided that $p > C$. (Here C depends only on the Euler characteristic and the signature of X.)*

Proof. (*A sketch.*) First of all, any $\mathbb{Z}_p$-action of prime order has to be homologically trivial (in integral coefficients) provided that the order p is greater than the Euler characteristic of X, cf. Lemma 2.1. So without loss of generality, we assume that the actions are homologically trivial. Given such a holomorphic $\mathbb{Z}_p$-action, note that the (-1)-curves (if there are any) must be invariant under the $\mathbb{Z}_p$-action. By equivariantly blowing down the (-1)-curves, we can further assume without loss of generality that X is minimal.

According to the Enriques-Kodaira Classification, X is either a $K3$ surface, a properly elliptic surface, or a surface of general type. With this understood, Theorem 1.1 follows readily from Xiao's generalization of Hurwitz's theorem when X is a surface of general type, and in the case of $K3$ surfaces or properly elliptic surfaces, it follows from the following homological rigidity theorem of Peters in [24]: a holomorphic action which is trivial on $H_*(X;\mathbb{Z})$ must be trivial.

□

We formulate the symplectic analog of Theorem 1.1 in the following conjecture.

Conjecture 1.2. *Let X be a simply connected symplectic 4-manifold. Suppose X does not admit any smooth circle actions. Then given any symplectic structure ω on X, there exists a constant $C > 0$ such that there are no nontrivial symplectic $\mathbb{Z}_p$-actions of prime order on X provided that $p > C$.*

The main theorem of this paper gives some evidence for the above conjecture.

Theorem 1.3. (*Main Theorem*) *Let X be a simply connected symplectic 4-manifold with $b_2^+ > 1$. Then given any symplectic structure ω with $[\omega] \in H^2(X;\mathbb{Q})$, there exists a constant $C > 0$ such that there are no nontrivial symplectic $\mathbb{Z}_p$-actions of prime order on X provided that $p > C$.*

We should point out that, unlike in Theorem 1.1, the constant C in the Main Theorem not only depends on the Euler characteristic and the signature of X, but also on the symplectic structure ω (in an interesting way). More precisely, since the class $[\omega]$ is rational, we let N_ω be the smallest positive integer such that $[N_\omega \omega]$ is an integral class. Then the constant C in the Main Theorem depends on

$$C_\omega \equiv N_\omega c_1(K) \cdot [\omega],$$

where K is the canonical bundle of (X, ω). Since $c_1(K)$ is a Seiberg-Witten basic class of X, the constant C depends on the smooth structure of X as well. Note that the constant C_ω depends on $[\omega]$ in a rather unstable fashion because of the factor N_ω, therefore one can not remove the assumption that $[\omega]$ is rational by simply perturbing ω into one which is of rational class.

Example 1.4. The following construction shows that, in the most general form of Conjecture 1.2, the constant C should depend on the pairing $c_1(K) \cdot [\omega]$ (for simplicity, we assume $[\omega]$ is integral so that $N_\omega = 1$).

Let X_0 be the smooth rational elliptic surface given by the Weierstrass equation

$$y^2 z = x^3 + v^5 z^3.$$

For any prime number $p \geq 5$, one can define an order-p automorphism g of X_0 as follows (cf. [29]):

$$g : (x, y, z; v) \mapsto (\mu_p^{-5} x, y, \mu_p^{-15} z; \mu_p^6 v), \quad \mu_p \equiv \exp(2\pi i/p).$$

Then g preserves the elliptic fibration and leaves exactly the two singular fibers (at $v = 0$ and $v = \infty$) invariant. Now we fix an integral Kähler form ω on X_0. Let X_p be the symplectic 4-manifold obtained from X_0 by performing knot surgery (cf. [13]) using the trefoil knot on p copies of regular fibers of the elliptic fibration which are invariant under the order-p automorphism g. Then X_p is homeomorphic to X_0, and the canonical class of X_p is given by the formula

$$c_1(K_{X_p}) = (2p - 1) \cdot [T],$$

where $[T]$ is the fiber class of the elliptic fibration which pairs positively with the Kähler form ω. It is clear that X_p inherits a periodic symplectomorphism of order p. Since $c_1(K_{X_p}) \cdot [\omega] > 0$ and $c_1(K_{X_p})^2 = 0$, X_p is not rational or ruled, cf. [18]. By Baldridge's theorem [3], X_p does not admit any smooth circle actions.

We thus obtained, for any prime number $p \geq 5$, a symplectic 4-manifold X_p homeomorphic to the rational elliptic surface, which admits no smooth circle actions but has a periodic symplectomorphism of order p. Observe that the order p satisfies

$$p \leq \frac{1}{2}(c_1(K_{X_p}) \cdot [\omega] + 1).$$

Note that in the above example, the manifold X_p has $b_2^+ = 1$. It is unclear, however, that with the condition $b_2^+ > 1$, whether the constant C can be made independent of C_ω. In particular, the symplectic analog of Xiao's theorem remains open. More precisely, assuming $b_2^+ > 1$, $c_1(K)^2 > 0$ and (X, ω) minimal, it is not known whether the constant C in the Main Theorem can be replaced by a multiple of $c_1(K)^2$. Notice that the construction in Example 1.4 does not extend to the $b_2^+ > 1$ case by simply replacing the rational elliptic surface with some other elliptic surfaces with $b_2^+ > 1$

because of Theorem 1.1. On the other hand, there are simply connected symplectic 4-manifolds (X,ω) with $b_2^+>1$, which have the same topological type, but the constant C_ω can be made arbitrarily large.

For the smooth analog of Conjecture 1.2, the examples X_p in Example 1.4 also indicate that the constant C should depend on the smooth structure in general.

In the examples X_p, the underlying topological 4-manifold does admit a smooth structure which supports a smooth circle action, i.e., the rational elliptic surface X_0. What if we only consider 4-manifolds which do not admit smooth circle actions for any smooth structures? By a theorem of Atiyah and Hirzebruch [2], a spin 4-manifold with non-zero signature does not admit any smooth circle actions.

Question 1.5. Let X be a simply connected smoothable 4-manifold with even intersection form and non-zero signature. Does there exist a constant $C>0$ depending on the topological type of X only, such that for any prime number $p>C$, there are no $\mathbb{Z}_p$-actions on X which are smooth with respect to some smooth structure ?

We remark that Question 1.5 is particularly interesting in the case of $K3$ surfaces. It is a long-standing problem as whether the $K3$ surface (with the standard smooth structure) admits any smooth finite group actions which are homologically trivial. It is well-known that there are no such holomorphic actions, and recently it was shown that there are no such symplectic finite group actions as well, cf. [6]. Since for $p>23$, any $\mathbb{Z}_p$-action of prime order on a homotopy $K3$ surface is homologically trivial, we see that Question 1.5 is related to the above homological rigidity problem of smooth actions on the $K3$ surface (or more generally, on a homotopy $K3$ surface).

Next we discuss the main ideas and ingredients in the proof of the Main Theorem. Let (M,ω) be a symplectic 4-manifold and G be a finite group acting on M smoothly and effectively which preserves the symplectic form ω. Denote by $b_G^{2,+}$ the dimension of the maximal subspace of $H^2(M;\mathbb{R})$ over which the cup-product is positive and the induced action of G is trivial. Then an equivariant version of Taubes' theorem $SW\Rightarrow Gr$ (cf. [25]) may be applied to (M,ω) provided that $b_G^{2,+}\geq 2$. More precisely, when $b_G^{2,+}\geq 2$, the G-equivariant Seiberg-Witten invariant is well-defined and is non-zero for the G-equivariant canonical bundle K_ω. This implies that, for any $r>0$, the r-version of Taubes' perturbed Seiberg-Witten equations has a solution $((\alpha,\beta),a)$ which is fixed under the action of G. Letting $r\to\infty$ as usual, the zero set $\alpha^{-1}(0)$ converges to a finite set of J-holomorphic curves $\{C_i\}$, such that $c_1(K_\omega)=\sum_i n_iC_i$ for some integers $n_i>0$. Here J is any fixed choice of G-equivariant, ω-compatible almost complex structure.

Since α is fixed under G, it follows easily that the set $\cup_i C_i$ is G-invariant, and furthermore, $\cup_i C_i$ contains all the fixed points of G except for those isolated ones at which the representation of G on the complex tangent space has determinant 1. This in principle allows one to analyze the action of G near its fixed point set, and sometimes even the induced representation of G on the second cohomology — the two crucial pieces of informations about the action of G on M — by looking at the restriction of the G-action in a neighborhood of $\cup_i C_i$. The main difficulty lies in the fact that in general not much can be said about the structure of the set $\cup_i C_i$. Unlike the non-equivariant case where $\{C_i\}$ can be made disjoint and embedded for a generic choice of J, in the presence of group actions the set $\cup_i C_i$ could be very complicated in general even with a choice of generic equivariant J, cf. [4]. The only exceptional case is when (M, ω) is minimal and $c_1(K_\omega)^2 = 0$. This was explored in [6] in investigating the homological rigidity of symplectic finite group actions. For further applications concerning group actions and exotic smooth structures, see [7, 8].

With the preceding understood, the proof of the Main Theorem relies in a crucial way on the following technical lemma. Recall from [25], Section 5(e), that for any point $x \in \cup_i C_i$, and for any embedded J-holomorphic disk D such that $D \cap (\cup_i C_i) = \{x\}$, a local intersection number $\mathrm{int}_D(x)$ is defined. (Note that it was shown in [25] that such embedded J-holomorphic disks D exist in abundance.)

Lemma 1.6. *Let $1 \neq g \in G$ and $x \in \cup_i C_i$ such that $g \cdot x = x$. Suppose the action of g near x is given by*

$$g \cdot (z_1, z_2) = (\lambda^{m_1} z_1, \lambda^{m_2} z_2)$$

in an ω-compatible local complex coordinate system (z_1, z_2) centered at x, where $\lambda = \exp(2\pi i/m)$ with $m \equiv order(g)$, and $0 \leq m_1, m_2 < m$. Suppose further that $x \in \alpha^{-1}(0)$ for all $r > 0$. Then there exist non-negative integers a_1, a_2 with $a_1 + a_2 > 0$ satisfying the congruence relation

$$(a_1 + 1)m_1 + (a_2 + 1)m_2 = 0 \pmod{m},$$

such that

$$int_D(x) \geq a_1 + a_2$$

for any embedded J-holomorphic disk D.

Remark 1.7. When the representation of g on the complex tangent space of x has determinant $\neq 1$, i.e., when $m_1 + m_2 \neq 0 \pmod{m}$, the assumption that $x \in \alpha^{-1}(0)$ for all $r > 0$ is automatically satisfied. This is because when $m_1 + m_2 \neq 0 \pmod{m}$, the representation of g on the fiber of the G-equivariant canonical bundle K_ω at x is non-trivial, so that $\alpha(x) = 0$ has to be true since α is fixed by g.

The following recipe will be used frequently in determining the local intersection number $\mathrm{int}_D(x)$: suppose a branch of $\cup_i C_i$ near x is parametrized by a holomorphic map over a neighborhood of $0 \in \mathbb{C}$ which is given in local coordinates by

$$z_1 = z^l, \; z_2 = cz^{l'} + \cdots \text{ (higher order terms)},$$

where $l' > l$ unless $l = 1$ and $c = 0$, and suppose the multiplicity of the branch is n, then the contribution of the branch to $\mathrm{int}_D(x)$ is equal to nl provided that the J-holomorphic disk D is not tangent to $z_2 = 0$ at x (i.e., the tangent space of the branch at x). See Theorem 7.1 in Micallef and White [22].

We end with a few remarks about the remaining case of Conjecture 1.2, i.e., when $b_2^+ = 1$. A simply connected, symplectic 4-manifold with $b_2^+ = 1$ is homeomorphic to a rational or ruled surface, and when the manifold does not admit a smooth circle action, the smooth structure must be an exotic one which is characterized by the condition $c_1(K) \cdot [\omega] > 0$, cf. [18]. Many symplectic exotic rational surfaces have been constructed recently following the work of Jongil Park [23].

Our proof of the Main Theorem breaks down in the case of $b_2^+ = 1$, even though the main line of arguments continues to work in this case. The missing ingredient is the equivariant version of Taubes' theorem, i.e., for any $r > 0$, the r-version of Taubes' perturbed Seiberg-Witten equations associated to the equivariant canonical bundle has a solution which is fixed under the group action. Notice that with the condition $c_1(K) \cdot [\omega] > 0$, one can argue using the wall-crossing formula that the (non-equivariant) r-version of Taubes' perturbed Seiberg-Witten equations associated to the square of the canonical bundle has a solution for sufficiently large $r > 0$ provided that the dimension of the corresponding Seiberg-Witten moduli space is non-negative (which is equivalent to $c_1(K)^2 \geq 0$). One could argue similarly using wall-crossing to get an equivariant version of this result which would be a good substitute of Taubes' theorem for our purpose, but unfortunitely the non-negativity of the dimension of the corresponding moduli space of equivariant Seiberg-Witten equations is much harder to verify; the calculation of the dimension requires knowledge about the group action near the fixed point set, which is not known a priori in general except for the case of a homology $\mathbb{CP}^2$ due to the work of Edmonds and Ewing [12].

Theorem 1.8. *Let X be a smooth 4-manifold which is a homology $\mathbb{CP}^2$. Then for any symplectic structure ω with $c_1(K) \cdot [\omega] > 0$, there exists a constant $C > 0$ such that there are no nontrivial symplectic $\mathbb{Z}_p$-actions of prime order on X for $p > C$.*

Remark 1.9. (1) Theorem 1.8 holds true more generally when X is only a $\mathbb{Q}$-homology $\mathbb{CP}^2$ provided that the $\mathbb{Z}_p$-actions are pseudofree, i.e., having only isolated fixed points. The primary examples of a symplectic $\mathbb{Q}$-homology $\mathbb{CP}^2$ with $c_1(K) \cdot [\omega] > 0$ are complex surfaces of general type with $p_g = 0$ and $c_1^2 = 9$, known as fake projective planes. Many of them have nontrivial automorphism groups which always give pseudofree actions, cf. [17].

(2) Since $b_2 = b_2^+ = 1$ in this case, one can always rescale ω so that $[\omega]$ is a generater of $H^2(X)$. With this choice of ω, $N_\omega = 1$ and $c_1(K) \cdot [\omega] = 3$, so that the constant C in Theorem 1.8 is in fact independent of ω. It is an interesting problem to find out the optimal value of C in Theorem 1.8.

The organization of this paper is as follows. Section 2 consists of a set of preliminary lemmas preparing for the proof of the Main Theorem. In particular, it contains the proof of Lemma 1.6. Section 3 is devoted to the proof of the Main Theorem, and the proof of Theorem 1.8 is given in Section 4.

2. Preliminary lemmas

Lemma 2.1. *Let X be a simply connected 4-manifold. Then for any prime number $p > \chi(X)$ (the Euler characteristic of X), a locally linear topological $\mathbb{Z}_p$-action on X is necessarily homologically trivial.*

Proof. First of all, since $p > \chi(X) \geq 2$, p must be odd and the $\mathbb{Z}_p$-action must be orientation preserving. By a theorem of Kwasik and Schultz [16], the induced $\mathbb{Z}_p$ integral representation on $H^2(X)$ is decomposed into a direct sum

$$H^2(X) = \mathbb{Z}[\mathbb{Z}_p]^r \oplus \mathbb{Z}^t \oplus \mathbb{Z}[\mu_p]^s$$

for some integers $r, t, s \geq 0$, where the group ring $\mathbb{Z}[\mathbb{Z}_p]$ is the regular representation of $\mathbb{Z}$-rank p, $\mathbb{Z}$ is the trivial representation of $\mathbb{Z}$-rank 1, and $\mathbb{Z}[\mu_p]$ is the representation of cyclotomic type of $\mathbb{Z}$-rank $p-1$, which is the kernel of the augmentation homomorphism $\mathbb{Z}[\mathbb{Z}_p] \to \mathbb{Z}$. (Here $\mu_p \equiv \exp(2\pi i/p)$.) This gives rise to

$$b_2(X) = rp + t + s(p-1).$$

Now if $p > \chi(X) = b_2(X) + 2$, the only solution to the above equation is $r = s = 0$ and $t = b_2(X)$, which means that the induced action of $\mathbb{Z}_p$ on $H^2(X)$ is trivial.

□

Thus without loss of generality, we may assume the $\mathbb{Z}_p$-action on (X, ω) under consideration is homologically trivial by assuming that $p > \chi(X)$. Under this assumption, the only 2-dimensional components in the fixed point set are 2-spheres, cf. [11].

Lemma 2.2. *Let G be a finite group acting on a symplectic 4-manifold (M, ω), preserving the form ω and inducing a trivial action on $H^2(M)$. Then there exists a symplectic 4-manifold (M', ω'), which is a symplectic blowdown of (M, ω), with an induced G-action preserving the form ω' and inducing a trivial action on $H^2(M')$, such that for any G-equivariant, ω'-compatible almost complex structure J, M' contains no embedded J-holomorphic 2-spheres with self-intersection -1. Furthermore, if $[\omega]$ is rational, so is $[\omega']$, and one has $C_{\omega'} \leq C_\omega$.*

Proof. Suppose there exists a G-equivariant, ω-compatible almost complex structure J on M such that M contains an embedded J-holomorphic 2-sphere C with $C^2 = -1$. Since G acts trivially on $H^2(M)$, the class of $g \cdot C$ equals the class of C for any $g \in G$. This implies $g \cdot C = C$, i.e., C is invariant under G, because otherwise $(g \cdot C) \cdot C \geq 0$ by the positivity of intersections of J-holomorphic curves, which then contradicts the identity $(g \cdot C) \cdot C = C^2 = -1$.

We G-equivariantly blow down (M, ω) along C and obtain a symplectic 4-manifold (M', ω'), which inherits a symplectic G-action from (M, ω). Clearly the induced action of G on $H^2(M')$ is also trivial.

We shall prove that if $[\omega]$ is a rational class, so is $[\omega']$, and one has $C_{\omega'} \leq C_\omega$. To see this, let $x \in M'$ be the image of C under the blowing down. Then there exist small neighborhoods U of C in M and U' of x in M', such that $(M \backslash U, \omega)$ and $(M' \backslash U', \omega')$ are symplectomorphic, cf. [20]. It follows easily that $[N_\omega \omega']$ is an integral class in $H^2(M')$. (Here recall that N_ω is the smallest positive integer such that $[N_\omega \omega]$ is an integral class in $H^2(M)$.) This implies $N_{\omega'} \leq N_\omega$, and since $c_1(K_{M'}) \cdot [\omega'] < c_1(K_M) \cdot [\omega]$, one clearly has $C_{\omega'} \leq C_\omega$.

This process will terminate because $b_2(M') = b_2(M) - 1$. At the end we obtain a symplectic 4-manifold, still denoted by (M', ω'), such that for any G-equivariant, ω'-compatible almost complex structure J, M' contains no embedded J-holomorphic 2-spheres with self-intersection -1.

□

A natural question is whether the manifold (M', ω') is minimal. Recall that a symplectic 4-manifold is said to be minimal if there exist no embedded symplectic 2-spheres with self-intersection -1. The next lemma gives an answer to this question.

Lemma 2.3. *Let (M,ω) be a symplectic 4-manifold which is not rational or ruled. If there exists a ω-compatible almost complex structure J_0 such that M contains no embedded J_0-holomorphic 2-spheres with self-intersection -1, then (M,ω) is minimal.*

Proof. Suppose to the contrary that (M,ω) is not minimal, and let Σ be an embedded symplectic 2-sphere with self-intersection -1 in M. Then there exists a ω-compatible almost complex structure J' on M such that Σ is J'-holomorphic. On the other hand, by Corollary 3.3.4 in McDuff-Salamon [21], for any J an embedded J-holomorphic 2-sphere with self-intersection -1 is always a regular point in the corresponding moduli space of J-holomorphic curves, which is also the only point in the moduli space because of positivity of intersections of J-holomorphic curves. This implies that the Gromov invariant counting J-holomorphic 2-spheres in the class of Σ equals ± 1. In particular, there exists a finite set of J_0-holomorphic curves $\{\Gamma_j\}$ such that

$$[\Sigma]=\sum_j l_j\Gamma_j \text{ for some integers } l_j>0.$$

Consider first the case where $b_2^+>1$. By Taubes [25] the Gromov invariant $Gr(K_M)\neq 0$, so that there exists a finite set of J_0-holomorphic curves $\{C_i\}$ such that

$$c_1(K_M)=\sum_i n_iC_i \text{ for some integers } n_i>0.$$

Since $c_1(K_M)\cdot[\Sigma]=-1<0$, there exists a j such that $c_1(K_M)\cdot\Gamma_j<0$. Because of positivity of intersections of pseudo-holomorphic curves, there must be an i such that $\Gamma_j=C_i$. Now $c_1(K_M)\cdot C_i<0$ implies that $C_i^2\leq\frac{1}{n_i}c_1(K_M)\cdot C_i<0$, and by the adjunction inequality one has

$$0\leq \text{genus}(C_i)\leq C_i^2+c_1(K_M)\cdot C_i+2\leq(-1)+(-1)+2=0,$$

which implies that C_i is an embedded 2-sphere with self-intersection -1, contradicting the assumption on J_0. Hence (M,ω) must be minimal in this case.

The case of $b_2^+=1$ is slightly more involved. First, we symplectically blowdown (M,ω) to get a minimal symplectic 4-manifold (M',ω'). Then because (M,ω) is not rational or ruled, one must have $c_1(K_{M'})\cdot[\omega']>0$ and $c_1(K_{M'})^2\geq 0$, cf. [18]. With these conditions, one can show by a wall-crossing argument that the Gromov invariant $Gr(2K_{M'})\neq 0$. On the other hand, by the blowup formula of Gromov invariant [19], if we denote by $E_1,\cdots,E_n\in H^2(M)$ the exceptional divisors of the symplectic blowdown $\pi:M\rightarrow M'$ (i.e., the Poincaré duals of the symplectic (-1)-spheres in M), then

$$Gr(2K_M+\sum_{s=1}^n E_s)=Gr(\pi^*(2K_{M'})-\sum_{s=1}^n E_s)=Gr(2K_{M'})\neq 0.$$

Consequently, there exists a finite set of J_0-holomorphic curves $\{\hat{C}_k\}$ such that

$$c_1(2K_M+\sum_{s=1}^n E_s)=\sum_k \hat{n}_k\hat{C}_k \text{ for some integers } \hat{n}_k>0.$$

Finally, we observe that the Gromov invariant $Gr(-E_s) \neq 0$ for each s. (Note that we have proved this fact for the Poincaré dual of Σ.) Hence for each s, there exists a finite set of J_0-holomorphic curves $\{\Gamma_{js}\}$ such that

$$c_1(-E_s) = \sum_j l_{js}\Gamma_{js} \text{ for some integers } l_{js} > 0.$$

Putting these together, one has

$$c_1(2K_M) = \sum_k \hat{n}_k \hat{C}_k + \sum_{j,s} l_{js}\Gamma_{js} = \sum_i n_i C_i,$$

for a finite set of J_0-holomorphic curves $\{C_i\}$ and integers $n_i > 0$. Again, since $c_1(2K_M)\cdot[\Sigma] = -2 < 0$, there exists a j such that $c_1(2K_M)\cdot\Gamma_j < 0$. Then there must be an i such that $\Gamma_j = C_i$, and as we argued earlier, $c_1(K_M)\cdot C_i < 0$ implies that C_i is an embedded 2-sphere with self-intersection -1, contradicting the assumption on J_0. This proves that (M,ω) is also minimal in the case of $b_2^+ = 1$.

□

We remark that Lemma 2.3 is false if one drops the assumption that (M,ω) is not rational or ruled as shown by the following example: the Hirzebruch surface F_3 contains no (-1)-holomorphic curves but it is not minimal as a symplectic 4-manifold. (Thanks to Tian-Jun Li for pointing out an error in an earlier version of the paper and communicating this example to me.)

Let $G \equiv \mathbb{Z}_p$. With the preceding lemmas understood (i.e., Lemma 2.1, Lemma 2.2 and Lemma 2.3), it suffices, for the proof of the Main Theorem, to consider the following simplified version: (X,ω) is minimal and the $\mathbb{Z}_p$-action is trivial on $H^2(X)$ by assuming $p > \chi(X)$. (Note that in particular, $p \geq 5$.) Since $b_G^{2,+} = b_2^+ > 1$, the equivariant version of Taubes' theorem in [25] applies here, and as we have explained earlier in Section 1, after fixing a generic choice of G-equivariant, ω-compatible almost complex structure J, there is a finite set of J-holomorphic curves $\{C_i\}$, such that $c_1(K) = \sum_i n_i C_i$ for some integers $n_i > 0$. Furthermore, the set $\cup_i C_i$ is G-invariant, and $\cup_i C_i$ contains all the fixed points of G except for those isolated ones at which the representation of G on the complex tangent space has determinant 1. Notice that since (M,ω) is minimal, $c_1(K)\cdot C_i \geq 0$ and $c_1(K)\cdot C_i \geq C_i^2$ for any i. (We remark that by the equivariant symplectic neighborhood theorem, J may be taken to be integrable in a neighborhood of the fixed point set, even though J has to be chosen to be generic in order to rule out certain possibilities.)

Lemma 2.4. *(1) If there exists a C_i with genus$(C_i) \geq 2$, then $p \leq 82c_1(K)^2$.*
(2) If there exists a C_i with genus$(C_i) = 1$ and $c_1(K)\cdot C_i \geq 1$, then $p \leq c_1(K)^2$.

Proof. (1) First consider the case where C_i is invariant under G. Since the 2-dimensional fixed components of G are 2-spheres, the induced G-action on C_i is nontrivial. By Hurwitz's theorem, $p \leq 82(\text{genus}(C_i) - 1)$. On the other hand, by the adjunction inequality,

$$\text{genus}(C_i) - 1 \leq \frac{1}{2}(C_i^2 + c_1(K)\cdot C_i) \leq c_1(K)\cdot C_i \leq c_1(K)^2.$$

Hence $p \leq 82c_1(K)^2$ as claimed.

Now suppose C_i is not invariant under G. Then $g \cdot C_i \neq C_i$ for all $1 \neq g \in G$. This implies that

$$c_1(K)^2 \geq c_1(K) \cdot (\sum_{g \in G} g \cdot C_i) = p \cdot c_1(K) \cdot C_i \geq p,$$

because $c_1(K) \cdot C_i \geq \frac{1}{2}(C_i^2 + c_1(K) \cdot C_i) \geq \text{genus}(C_i) - 1 \geq 1$. The lemma follows.

(2) First consider the case where C_i is invariant under G. Since the 2-dimensional fixed components of G are 2-spheres, the induced G-action on C_i is nontrivial. Since we assume $p \geq 5$, and by assumption $\text{genus}(C_i) = 1$, we see immediately that C_i contains no fixed points of G. With this understood, the dimension of the moduli space of the corresponding G-invariant J-holomorphic curves at C_i, which is given by

$$2(-\frac{1}{p}c_1(K) \cdot C_i + 2(1 - \text{genus}(C_i/G))) = -\frac{2}{p}c_1(K) \cdot C_i,$$

is negative because of the assumption $c_1(K) \cdot C_i \geq 1$. Note that here since the G-action on C_i is free, $\text{genus}(C_i/G) = 1$. By choosing a generic G-equivariant J, this case can be ruled out. (Here and throughout the rest of the paper, moduli spaces of G-invariant pseudo-holomorphic curves in X are canonically identified with the corresponding moduli spaces of pseudo-holomorphic curves in the orbifold X/G.)

Suppose C_i is not invariant under G. Then

$$c_1(K)^2 \geq c_1(K) \cdot (\sum_{g \in G} g \cdot C_i) = p \cdot c_1(K) \cdot C_i \geq p$$

as claimed. This finishes the proof of the lemma.

□

Lemma 2.5. *For any i, if genus$(C_i) = 1$ and $c_1(K) \cdot C_i = 0$, then C_i is an embedded torus of self-intersection 0, which is disjoint from the rest of the set of J-holomorphic curves $\{C_i\}$.*

Proof. Since $0 = c_1(K) \cdot C_i \geq C_i^2$, we have, by the adjunction inequality, that

$$0 \geq C_i^2 + c_1(K) \cdot C_i \geq \text{genus}(C_i) - 1 = 0,$$

which implies that $C_i^2 = c_1(K) \cdot C_i = 0$, and C_i is embedded. If C_i is not disjoint from the rest of the set of J-holomorphic curves $\{C_i\}$, we would have $c_1(K) \cdot C_i > C_i^2 = 0$, which is a contradiction. Hence the lemma.

□

Since we assume $p \geq 5$, the curves C_i as described in Lemma 2.5 will not contain any fixed points of G. Furthermore, they will not make any contributions to the calculation of either $c_1(K) \cdot C_i$ for any i, or $c_1(K)^2$. So this kind of J-holomorphic curves will play no role in our argument, and henceforth for simplicity we simply assume they do not exist.

With the preceding understood, without loss of generality we may assume that each curve in the set $\{C_i\}$ is a 2-sphere, i.e., $\text{genus}(C_i) = 0$.

Lemma 2.6. *If there exists a C_i (with genus$(C_i) = 0$) which is not invariant under G, then $p \leq c_1(K)^2$.*

Proof. Since C_i is not invariant under G, $g \cdot C_i \neq C_i$ for any $1 \neq g \in G$. On the other hand, G acts trivially on $H^2(X)$, so that $C_i^2 = (g \cdot C_i) \cdot C_i \geq 0$.

If $c_1(K) \cdot C_i \geq 1$, we have as before that

$$c_1(K)^2 \geq c_1(K) \cdot (\sum_{g \in G} g \cdot C_i) = p \cdot c_1(K) \cdot C_i \geq p.$$

If $c_1(K) \cdot C_i = 0$, then $C_i^2 = 0$ as well, which implies that $(g \cdot C_i) \cdot C_i = 0$ for all $g \in G$. In particular $g \cdot C_i$ and C_i are disjoint for any $1 \neq g \in G$, so that C_i contains no fixed points of G. The dimension of the moduli space of the corresponding J-holomorphic curves at C_i is given by

$$d = 2(-c_1(K) \cdot C_i + 2(1 - \text{genus}(C_i)) - 3) = -2,$$

so that by choosing a generic G-equivariant J, such a C_i does not exist.

$\square$

Lemma 2.7. *Suppose there exist i, j such that $C_i \neq C_j$ and C_i and C_j intersect at a point which is not fixed under G. Then $p \leq 2 + 2c_1(K)^2$.*

Proof. Suppose both of C_i, C_j are invariant under G; otherwise the lemma follows from the previous lemma. Without loss of generality, we assume $n_i \geq n_j$. Set $\delta = c_1(K) \cdot C_j$. Then

$$\delta \geq (n_i C_i + n_j C_j) \cdot C_j \geq n_j (C_i \cdot C_j + C_j^2) \geq n_j (p + C_j^2).$$

Here we used $C_i \cdot C_j \geq p$, which follows from the fact that the set $C_i \cup C_j$ is G-invariant and C_i, C_j intersect at a point not fixed under G. This gives $\delta \geq \frac{1}{n_j}\delta \geq p + C_j^2$.

Now by the adjunction inequality, we obtain

$$(\delta - p) + \delta + 2 \geq C_j^2 + c_1(K) \cdot C_j + 2 \geq 0,$$

which gives rise to $p \leq 2 + 2\delta \leq 2 + 2c_1(K)^2$, as we claimed.

$\square$

We end this section with the

Proof of Lemma 1.6

The local intersection number $\text{int}_D(x)$ is defined to be the limit

$$\text{int}_D(x) = \lim_{n \to \infty} \int_D \frac{i}{2\pi} F_{a_n}$$

for a sequence of solutions $((\alpha_n, \beta_n), a_n)$ to the r_n-version of the Taubes' perturbed Seiberg-Witten equations, where $r_n \to \infty$ as $n \to \infty$, cf. Proposition 5.6 in [25]. In Lemma 5.8 of [25], Taubes gave a lower bound for $\text{int}_D(x)$ which takes the form

$$\int_D \frac{i}{2\pi} F_{a_n} \geq m_0 + z_3(4^{-n} + \rho^2),$$

where $\rho > 0$ can be taken arbitrarily small, and m_0 is a positive integer (see (5.19) and (5.20) in [25]). Here z_3 is an independent constant. Clearly,

$$\text{int}_D(x) \geq m_0.$$

To explain m_0, recall that by our assumption, $x \in \alpha_n^{-1}(0)$ for all n. Fix a Gaussian coordinate system at x and pull back the solutions $((\alpha_n, \beta_n), a_n)$ to the Gaussian system. After rescaling by a factor $\sqrt{r_n}$, the solutions converge in C^∞-topology over compact subsets to a solution $((\alpha_0, 0), a_0)$ to the $r = 1$ version of the Taubes' perturbed Seiberg-Witten equations on $\mathbb{C}^2$. Moreover, the $U(1)$-connection a_0 defines a holomorphic structure on the trivial complex line bundle over $\mathbb{C}^2$ of which α_0 is a holomorphic section. Finally, $\alpha_0^{-1}(0)$ is the zero set of a polynomial on $\mathbb{C}^2$. With the preceding understood, the number m_0 is the local contribution at $0 \in \mathbb{C}^2$ to the intersection number of any complex line in $\mathbb{C}^2$ with $\alpha_0^{-1}(0)$.

Now write $\alpha_0 = f(z_1, z_2) \cdot s$, where s is a non-zero holomorphic section, and

$$f(z_1, z_2) = \sum_{i=1}^{N} c_i z_1^{a_{1,i}} z_2^{a_{2,i}} + \cdots \text{(higher order terms)}.$$

Here $a \equiv a_{1,i} + a_{2,i} > 0$ which is independent of $i = 1, 2, \cdots, N$. Then the above interpretation of m_0 shows that $m_0 \geq a$. On the other hand, the representation of g on the fiber of the G-equivariant canonical bundle is given by multiplication by $\lambda^{-(m_1+m_2)}$, where $\lambda = \exp(2\pi i/m)$, and $m \equiv \text{order}(g)$. Apparently $g \cdot s = \lambda^{-(m_1+m_2)} s$ and $g \cdot \alpha_0 = \alpha_0$, which implies that

$$f(g \cdot (z_1, z_2)) = \lambda^{-(m_1+m_2)} \cdot f(z_1, z_2).$$

The above equation gives the congruence relation

$$a_{1,i} m_1 + a_{2,i} m_2 = -(m_1 + m_2) \pmod{m}, \quad \forall i = 1, 2, \cdots, N.$$

The lemma follows easily by taking (a_1, a_2) to be any of the $(a_{1,i}, a_{2,i})$'s. $\square$

3. Proof of main theorem

With the preliminary lemmas proved in the previous section, we may assume without loss of generality that the curves in the set $\{C_i\}$ satisfy:

(1) each C_i is a 2-sphere, which may be singular or immersed;
(2) each C_i is G-invariant, either being fixed by G or containing ≤ 2 fixed points;
(3) two distinct C_i, C_j intersect only at fixed points of G;
(4) each C_i is embedded away from the fixed points of G.

Here for the last condition, (4), if there is a C_i which is not embedded away from the fixed points of G, we obtain a bound for p by the adjunction inequality: let y_k, $k = 1, 2, \cdots, p$, be a subset of singular points of C_i which is invariant under G and denote by δ_{y_k} the contribution of y_k to the adjunction inequality, then $\delta_{y_k} \geq 2$ and

$$c_1(K)^2 + 1 \geq c_1(K) \cdot C_i + 1 \geq \frac{1}{2}(C_i^2 + c_1(K) \cdot C_i) + 1 \geq \frac{1}{2} \sum_{k=1}^{p} \delta_{y_k} \geq p.$$

Before we start, it is useful to make observation of the following fact.

Lemma 3.1. *$\sum_i n_i \leq C_\omega$, where $c_1(K) = \sum_i n_i C_i$. In particular, $n_i \leq C_\omega$ for each i.*

Proof. $c_1(K) \cdot [\omega] = \sum_i n_i \omega(C_i) \geq \sum_i n_i \cdot \frac{1}{N_\omega}$, from which the lemma follows. $\square$

The following lemma eliminates the case of non-pseudofree actions.

Lemma 3.2. *If there exists a C_i which is fixed under G. Then $p \leq 1 + C_\omega$.*

Proof. Let n_i be the multiplicity of C_i. We pick a point $x \in C_i$ such that x does not lie in any other $C_j \neq C_i$. Let D be a J-holomorphic disk intersecting C_i transversely and $D \cap (\cup_j C_j) = \{x\}$. Then the local intersection number

$$\mathrm{int}_D(x) = n_i,$$

cf. [25], Section 5. Let $g \in G$ be the element whose action near x is given $g \cdot (z_1, z_2) = (z_1, \mu_p z_2)$, where $\mu_p \equiv \exp(2\pi i/p)$. Then by Lemma 1.6, there exist non-negative integers a_1, a_2 satisfying $(a_1+1)\cdot 0 + (a_2+1)\cdot 1 \equiv 0 \pmod p$ (here $m_1 = 0$, $m_2 = 1$), such that $\mathrm{int}_D(x) \geq a_1 + a_2$. It follows that $n_i = \mathrm{int}_D(x) \geq a_2 \geq p-1$. This gives

$$p \leq 1 + n_i \leq 1 + C_\omega.$$

$\square$

We shall assume, in what follows, that the $\mathbb{Z}_p$-action is pseudofree.

Case (a): $c_1(K)^2 = 0$. First of all, notice that X has non-zero signature. Then according to Corollary B of [6], the $\mathbb{Z}_p$-action must be trivial unless $p = 1 \pmod 4$ or $p = 1 \pmod 6$. Moreover, from the proof of Corollary B, the following are also true: (i) when $p = 1 \pmod 4$, there must be C_i, C_j, both embedded, with $n_i = n_j$, such that $C_i^2 = C_j^2 = -2$ and C_i, C_j intersect at a fixed point x with tangency of order 2; (ii) when $p = 1 \pmod 6$, then either there is a C_i which is a 2-sphere with a cusp singularity x fixed by G, or there are 3 distinct embedded (-2)-spheres C_i, C_j, C_k intersecting transversely at a fixed point x of G. Furthermore, there are no fixed points where the representation of G on the complex tangent space has determinant 1. By Remark 1.7, Lemma 1.6 applies here to all the fixed points of G. We fix a J-holomorphic disk D whose tangent plane at x is different from that of any of the J-holomorphic curves in $\{C_i\}$. In this case, $\mathrm{int}_D(x)$ can be easily determined using Theorem 7.1 of Micallef and White [22].

In case (i), Lemma 1.6 gives us

$$C_\omega \geq n_i + n_j = \mathrm{int}_D(x) \geq a_1 + a_2,$$

where a_1, a_2 satisfy $(a_1+1)\cdot 1 + (a_2+1)\cdot 2 \equiv 0 \pmod p$. This implies that

$$p \leq 2(a_1 + a_2) + 3 \leq 2C_\omega + 3.$$

As for case (ii), in the former case of a cusp sphere, Lemma 1.6 gives us

$$2C_\omega \geq 2n_i = \mathrm{int}_D(x) \geq a_1 + a_2,$$

where a_1, a_2 satisfy $(a_1+1)\cdot 2 + (a_2+1)\cdot 3 \equiv 0 \pmod p$. This implies that

$$p \leq 3(a_1 + a_2) + 5 \leq 6C_\omega + 5.$$

In the latter case of (ii), Lemma 1.6 gives us

$$C_\omega \geq n_i + n_j + n_k = \mathrm{int}_D(x) \geq a_1 + a_2,$$

where a_1, a_2 satisfy $(a_1+1)\cdot 1 + (a_2+1)\cdot 1 \equiv 0 \pmod p$. This implies that

$$p \leq (a_1+a_2)+2 \leq C_\omega + 2.$$

The proof of the Main Theorem for the case where $c_1(K)^2 = 0$ follows.

Case (b): $c_1(K)^2 > 0$. We start with the following lemma.

Lemma 3.3. *By choosing a generic G-equivariant almost complex structure J, the set $\cup_i C_i$ contains no fixed points of G where the representation of G on the complex tangent space has determinant* 1.

Proof. Suppose x is such a fixed point, and $x \in C_0 \in \{C_i\}$. Let $f : \mathbb{S}^2 \to X$ be a J-holomorphic map parametrizing C_0, and let $t_1, t_2 \in \mathbb{S}^2$ be the two points mapped to fixed points under f such that $f(t_1) = x$. Note that t_1, t_2 are fixed under the induced action of G on $\mathbb{S}^2$. Let $g_1, g_2 \in G$ be the elements which act by a rotation of angle $2\pi/p$ near t_1, t_2 respectively. Moreover, suppose the actions of g_1, g_2 near the fixed points in X are given respectively by

$$g_i \cdot (z_1, z_2) = (\mu_p^{m_{i,1}} z_1, \mu_p^{m_{i,2}} z_2),\ i = 1, 2,$$

where $\mu_p = \exp(2\pi i/p), 0 < m_{i,1}, m_{i,2} < p$. Then the dimension of the moduli space of the corresponding G-invariant J-holomorphic curves at C_0 is

$$\begin{aligned} d &= 2(-\frac{1}{p}c_1(K)\cdot C_0 + 2 - \sum_{i=1}^{2} \frac{m_{i,1}+m_{i,2}}{p} - 1) \\ &= -2(\frac{1}{p}c_1(K)\cdot C_0 + \frac{m_{2,1}+m_{2,2}}{p}), \end{aligned}$$

see [4], p. 19. Here we used the fact that the representation of G on the complex tangent space of x has determinant 1, so that $m_{1,1}+m_{1,2} = p$. By choosing a generic G-equivariant J (cf. [4], Lemma 1.10), $d \geq 0$ if C_0 exists. But this is impossible because $c_1(K)\cdot C_0 \geq 0$.

∎

With the preceding lemma, Lemma 1.6 applies to any fixed point contained in $\cup_i C_i$ (cf. Remark 1.7).

Lemma 3.4. *If there exists a C_i which is not embedded, then*

$$p \leq max(16C_\omega^2, (5+2c_1(K)^2)^2, 4C_\omega^2(3+2c_1(K)^2)^2).$$

Proof. We first note that all C_i are embedded away from the fixed points of G. Fix any curve C_0 in the set $\{C_i\}$. Let $x \in C_0$ be a fixed point of G. We parametrize C_0 by a J-holomorphic map $f_0 : \mathbb{S}^2 \to X$, and suppose $0 \in \mathbb{S}^2$ is mapped to x under f_0. In a local complex coordinate system (z_1, z_2) centered at x, suppose f_0 is represented by a holomorphic map with z as a local coordinate centered at $0 \in \mathbb{S}^2$:

$$z_1 = z^{l_0}, z_2 = c_0 z^{l_0'} + \cdots \text{(higher order terms)},$$

where $l_0 < l_0'$ unless $c_0 = 0$ and $l_0 = 1$.

We first show that if $l_0 \geq 2$, then

$$p \leq \max(16C_\omega^2, (5+2c_1(K)^2)^2, 4C_\omega^2(3+2c_1(K)^2)^2).$$

Let $f_j : z \mapsto (z^{l_j}, c_j z^{l_j'} + \cdots)$, $j = 1, 2, \cdots, N$, be the holomorphic maps which parametrize all the branches of $\cup_i C_i$ near x other than the one parametrized by f_0 in a neighborhood of $0 \in \mathbb{S}^2$. Here for each j, $l_j < l_j'$ unless $c_j = 0$ and $l_j = 1$. If we fix a generator $g \in G$ and suppose the action of g near x is given by $g \cdot (z_1, z_2) = (\mu_p^m z_1, \mu_p^{m'} z_2)$, where $\mu_p = \exp(2\pi i/p)$ and $0 < m, m' < p$, then it follows easily that $l_j = k_j m, l_j' = k_j m' \pmod p$ for some k_j for all $0 \leq j \leq N$.

We assume $p \geq 16C_\omega^2$. There are two possibilities:

Case (i): $l_j \leq (2C_\omega)^{-1} \cdot \sqrt{p}$ for all $j = 0, 1, 2, \cdots, N$. Denote by n_j' the multiplicity of the branch parametrized by the map f_j, $j = 0, 1, 2, \cdots, N$. Then by Lemma 3.1 we have $\sum_{j=0}^N n_j' \leq 2C_\omega$. (Note that at most 2 branches lie in the same C_i.) We obtain

$$\sqrt{p} = 2C_\omega \cdot ((2C_\omega)^{-1} \cdot \sqrt{p}) \geq (\sum_{j=0}^N n_j') \cdot ((2C_\omega)^{-1} \cdot \sqrt{p}) \geq \sum_{j=0}^N n_j' l_j.$$

Now if we pick a J-holomorphic disk D whose tangent plane at x is different from that of any of the branches parametrized by f_j, $0 \leq j \leq N$, then by Theorem 7.1 in Micallef and White [22], $\mathrm{int}_D(x) = \sum_{j=0}^N n_j' l_j$. By Lemma 1.6, we obtain

$$\sqrt{p} \geq \mathrm{int}_D(x) \geq a_1 + a_2,$$

where a_1, a_2 satisfy $(a_1+1)l_j + (a_2+1)l_j' \equiv 0 \pmod p$, $0 \leq j \leq N$, because of the congruence relations $l_j = k_j m, l_j' = k_j m' \pmod p$ for some k_j for all $0 \leq j \leq N$. Particularly, we have

$$(\sqrt{p}+2)l_0' \geq (a_1+a_2+2)l_0' \geq (a_1+1)l_0 + (a_2+1)l_0' \geq p,$$

which implies $l_0' \geq (\sqrt{p}+2)^{-1}p \geq \sqrt{p} - 2$. On the other hand, by Theorem 7.3 in Micallef and White [22], the point $x \in C_0$ makes a local contribution of $\delta_x \geq (l_0 - 1)(l_0' - 1)$ to the adjunction inequality for C_0, which gives

$$2c_1(K)^2 \geq C_0^2 + c_1(K) \cdot C_0 \geq -2 + (l_0-1)(l_0'-1) \geq -2 + (\sqrt{p} - 3).$$

Note that here we used $l_0 \geq 2$. This implies that $p \leq (5 + 2c_1(K)^2)^2$.

Case (ii): there exists a $j = 0, 1, 2, \cdots, N$ such that $l_j \geq (2C_\omega)^{-1} \cdot \sqrt{p}$. Then $l_j \geq 2$ since $p \geq 16C_\omega^2$, and $l_j' > l_j \geq (2C_\omega)^{-1} \cdot \sqrt{p}$ for that j. Let C_i be the J-holomorphic curve which contains the branch parametrized by f_j near x. Then $x \in C_i$ makes a local contribution of $\delta_x \geq (l_j - 1)(l_j' - 1)$ to the adjunction inequality for C_i, which gives

$$2c_1(K)^2 \geq C_i^2 + c_1(K) \cdot C_i \geq -2 + (l_j - 1)(l_j' - 1) \geq -2 + ((2C_\omega)^{-1}\sqrt{p} - 1).$$

This implies that $p \leq 4C_\omega^2(3 + 2c_1(K)^2)^2$. Hence if $l_0 \geq 2$, one has

$$p \leq \max(16C_\omega^2, (5+2c_1(K)^2)^2, 4C_\omega^2(3+2c_1(K)^2)^2).$$

To finish the proof of the lemma, it remains to rule out the possibility that there is a $w \in \mathbb{S}^2$, $w \neq 0 \in \mathbb{S}^2$, such that $f_0(w) = f_0(0) = x$. Note that by the arguments in the previous paragraphs, we may assume that f_0 is embedded near both 0 and w.

Consider first the case where the tangent planes $(f_0)_*(T_0\mathbb{S}^2)$ and $(f_0)_*(T_w\mathbb{S}^2)$ intersect transversely at x. Suppose $g \in G$ is the element which acts near $0 \in \mathbb{S}^2$ as rotation by an angle of $2\pi/p$. Then g^{-1} acts near $w \in \mathbb{S}^2$ as rotation by an angle of $2\pi/p$. It follows that the action of g near x is given in local coordinates by $g \cdot (z_1, z_2) = (\mu_p z_1, \mu_p^{-1} z_2)$, where $\mu_p = \exp(2\pi i/p)$. But this has been ruled out by Lemma 3.3. Now if $(f_0)_*(T_0\mathbb{S}^2) = (f_0)_*(T_w\mathbb{S}^2)$, then $g = g^{-1}$ on $(f_0)_*(T_0\mathbb{S}^2) = (f_0)_*(T_w\mathbb{S}^2)$, which implies that $p = 2$. But we have assumed that $p \geq 5$.

This shows that if C_0 is not embedded near x, one has to have

$$p \leq \max(16C_\omega^2, (5 + 2c_1(K)^2)^2, 4C_\omega^2(3 + 2c_1(K)^2)^2).$$

$\square$

With the preceding lemma, we may assume in what follows that all C_i are embedded.

Lemma 3.5. *For any fixed point x of G, if there exist two distinct J-holomorphic curves C_i, C_j from the set $\{C_i\}$ such that C_i, C_j intersect at x non-transversely, then*

$$p \leq (3 + C_\omega)^2(c_1(K)^2 + 2).$$

Proof. First of all, we shall prove that for any $1 \neq g \in G$, if the action of g near x is given in local coordinates by $g \cdot (z_1, z_2) = (\mu_p^{m_1} z_1, \mu_p^{m_2} z_2)$, where $\mu_p = \exp(2\pi i/p)$ and $0 < m_1, m_2 < p$, then

$$\max(m_1, m_2) \geq (3 + C_\omega)^{-1} p.$$

To see this, if both m_1, m_2 are less than $(3+C_\omega)^{-1}p$, then by Lemma 3.1, Lemma 1.6,

$$\begin{aligned} (C_\omega + 2) \cdot (3 + C_\omega)^{-1} p &\geq (\sum_i n_i + 2) \cdot (3 + C_\omega)^{-1} p \\ &\geq (\mathrm{int}_D(x) + 2) \cdot (3 + C_\omega)^{-1} p \\ &\geq (a_1 + 1)m_1 + (a_2 + 1)m_2 \\ &\geq p, \end{aligned}$$

which is a contradiction. Here D is chosen such that it is not tangent to any of the curves in $\{C_i\}$ which contains x, and consequently, $\mathrm{int}_D(x) \leq \sum_i n_i$ by Theorem 7.1 in Micallef and White [22] (notice that we have assumed that each C_i is embedded).

With the preceding understood, since C_i, C_j intersect at x non-transversely, there exist local coordinates z_1, z_2 centered at x, such that locally C_i is given by $z_2 = 0$, and C_j is given by the graph of $z_2 = z_1^m + \cdots$ (higher order terms). Let $g \in G$ be the element which acts on C_i by a rotation of angle $2\pi/p$ near x. Then the action of g near x is given by $g \cdot (z_1, z_2) = (\mu_p z_1, \mu_p^m z_2)$. We have just shown that

$$m = \max(1, m) \geq (3 + C_\omega)^{-1} p,$$

which implies that $C_i \cdot C_j \geq m \geq (3 + C_\omega)^{-1} p$.

Now we write $c_1(K) \cdot C_i = \sum_{k \neq i} n_k C_k \cdot C_i + n_i C_i^2$, and with the adjunction inequality, we have

$$\sum_{k \neq i} n_k C_k \cdot C_i + (n_i + 1)C_i^2 + 2 \geq 0.$$

This gives rise to

$$C_i^2 \geq -\frac{1}{n_i+1}(2+\sum_{k\neq i} n_k C_k \cdot C_i).$$

Then we have

$$\begin{aligned} c_1(K)\cdot C_i &= \sum_{k\neq i} n_k C_k \cdot C_i + n_i C_i^2 \\ &\geq \sum_{k\neq i} n_k C_k \cdot C_i - \frac{n_i}{n_i+1}(2+\sum_{k\neq i} n_k C_k \cdot C_i) \\ &= \frac{1}{n_i+1}(\sum_{k\neq i} n_k C_k \cdot C_i) - \frac{2n_i}{n_i+1} \\ &\geq \frac{1}{n_i+1}\cdot C_j \cdot C_i - \frac{2n_i}{n_i+1} \\ &\geq \frac{1}{C_\omega+1}\cdot\frac{p}{3+C_\omega} - 2. \end{aligned}$$

This implies that $p \leq (3+C_\omega)^2(c_1(K)\cdot C_i + 2) \leq (3+C_\omega)^2(c_1(K)^2+2)$.

□

Corollary 3.6. *Suppose $p > \max(3+C_\omega, (3+C_\omega)^2(c_1(K)^2+2))$. Then for any fixed point x, there exist at most two distinct C_i, C_j containing x. Moreover, C_i, C_j intersect transversely at x.*

Proof. Since $p > 3+C_\omega$, we have $\max(m_1, m_2) \geq (3+C_\omega)^{-1}p > 1$ for any $1 \neq g \in G$ whose action is given in local coordinates by $g\cdot(z_1,z_2) = (\mu_p^{m_1}z_1, \mu_p^{m_2}z_2)$. It follows that the action of G at x has two distinct eigenvalues. If x is contained in more than two distinct J-holomorphic curves from the set $\{C_i\}$, there must be two distinct C_i, C_j intersecting non-transversely at x, which contradicts $p > (3+C_\omega)^2(c_1(K)^2+2)$.

□

With the preceding understood, we assume $p > \max(3+C_\omega, (3+C_\omega)^2(c_1(K)^2+2))$. Then for any C_i, there are 4 possibilities:

(1) C_i does not intersect with any other curves in $\{C_i\}$;
(2) C_i intersects with exactly one C_j at exactly one fixed point;
(3) C_i intersects with exactly one C_j at two fixed points;
(4) C_i intersects with two distinct C_j, C_k at two fixed points.

Note that since C_i is embedded, one has $c_1(K)\cdot C_i + C_i^2 + 2 = 0$.

Case (1): $n_iC_i^2 + C_i^2 + 2 = c_1(K)\cdot C_i + C_i^2 + 2 = 0$, which implies $n_i = 1$ and $C_i^2 = -1$. This contradicts the minimality of (X,ω).

Case (2): $n_iC_i^2 + n_j + C_i^2 + 2 = c_1(K)\cdot C_i + C_i^2 + 2 = 0$, which implies $C_i^2 = -1$ if $n_j < n_i$. Hence in this case, one must have $n_j \geq n_i$ by the minimality of (X,ω).

Case (3): $n_iC_i^2 + 2n_j + C_i^2 + 2 = c_1(K)\cdot C_i + C_i^2 + 2 = 0$, which implies $C_i^2 = -1$ if $n_j < n_i$. By the symmetry between i and j, we see that $n_i = n_j$, and $C_i^2 = C_j^2 = -2$. Moreover, $c_1(K)\cdot C_i = c_1(K)\cdot C_j = 0$.

Case (4): We assume that $n_i \geq n_j, n_k$. Then in this case,

$$n_i C_i^2 + n_j + n_k + C_i^2 + 2 = c_1(K) \cdot C_i + C_i^2 + 2 = 0,$$

which implies that $n_i = n_j = n_k$ and $C_i^2 = -2$. Moreover, $c_1(K) \cdot C_i = 0$.

From the preceding analysis, it is easily seen that Case (2) can not occur, and that for any C_i, $c_1(K) \cdot C_i = 0$. It follows that $c_1(K)^2 = \sum_i n_i c_1(K) \cdot C_i = 0$, which is a contradiction.

This completes the proof of the Main Theorem.

4. Proof of Theorem 1.8

Lemma 4.1. *Let (X, ω) be a symplectic homology $\mathbb{CP}^2$ with $c_1(K) \cdot [\omega] > 0$. Then any symplectic $\mathbb{Z}_p$-action of prime order on X must be pseudofree and p must be odd.*

Proof. First of all, we show that a smooth involution on a homology $\mathbb{CP}^2$ must have a 2-dimensional component in the fixed point set (cf. [11]). Suppose g is a smooth involution which has only isolated fixed points. Let Σ be a smoothly embedded surface in X which represents a generater of $H_2(X)$. By slightly perturbing Σ we may assume that Σ does not contain any fixed points of g, and furthermore, $g \cdot \Sigma$ and Σ intersect transversely. It is clear that the intersection points of $g \cdot \Sigma$ and Σ come in pairs, so that the intersection number $(g \cdot \Sigma) \cdot \Sigma = 0 \pmod 2$. However, since Σ represents a generater of $H_2(X)$ for a homology $\mathbb{CP}^2$, $(g \cdot \Sigma) \cdot \Sigma = 1 \pmod 2$, which is a contradiction.

Secondly, we show that any symplectic $\mathbb{Z}_p$-action on X must be pseudofree. Suppose C is a 2-dimensional component in the fixed point set. Then since the action is naturally homologically trivial, C must be an embedded 2-sphere (cf. [12]), which is also naturally symplectic. From $C \cdot [\omega] > 0$ and the assumption that $c_1(K) \cdot [\omega] > 0$, we see that $c_1(K) \cdot C > 0$ also. But this violates the adjunction inequality for C since we also have $C^2 > 0$. Hence the lemma.

□

With the preceding lemma, the following theorem of Edmonds and Ewing will play a crucial role in the proof of Theorem 1.8.

Theorem 4.2. (*Edmonds and Ewing,* [12]) *The fixed point set structure of a locally linear, pseudofree, topological $\mathbb{Z}_p$-action of odd order on a homology $\mathbb{CP}^2$ is the same as that of a linear action on $\mathbb{CP}^2$.*

More concretely, a locally linear, pseudofree, topological $\mathbb{Z}_p$-action of odd order has three fixed points $x_1, x_2, x_3 \in X$. Fix a generater g of the group. Then at each x_i, there is a pair of integers (a_i, b_i) (unordered) satisfying $0 < a_i, b_i < p$, such that the induced representation of g on the tangent space at x_i is given by

$$(z_1, z_2) \mapsto (\mu_p^{a_i} z_1, \mu_p^{b_i} z_2), \text{ where } \mu_p = \exp(2\pi i/p),$$

for some complex structure on the tangent space which is compatible with the orientation. Note that (a_i, b_i) is unique up to a change of sign, i.e., a change to $(p - a_i, p - b_i)$. (However, if requiring that the complex structure on the tangent space is ω-compatible, then (a_i, b_i) is uniquely determined.) With this understood, Theorem 4.2 says that $\{(a_i, b_i)\}$ is given by

$$(a, b), (p - a, b - a), (p - b, p + a - b)$$

for some $0 < a < b < p$.

With the preceding understood, Theorem 1.8 follows from the following proposition as we explained at the end of Section 1.

Proposition 4.3. *For sufficiently large $r > 0$, the r-version of Taubes' perturbed Seiberg-Witten equations associated to the square of the equivariant canonical bundle has a solution $((\alpha, \beta), a)$ which is fixed under the group action.*

Assume the proposition momentarily. Letting $r \to \infty$, the zero set $\alpha^{-1}(0)$ converges to a finite set of J-holomorphic curves $\{C_i\}$, such that $2c_1(K) = \sum_i n_i C_i$ for some integers $n_i > 0$. Moreover, $\cup_i C_i$ is invariant under the group action and contains all the fixed points except those x_i such that $2(a_i + b_i) = 0 \pmod p$. (Since p is odd, this is equivalent to $a_i + b_i = 0 \pmod p$.) With this understood, and with the congruence relation in Lemma 1.6 replaced by the following one

$$(a_1 + 2)m_1 + (a_2 + 2)m_2 = 0 \pmod m,$$

the same arguments for the proof of the Main Theorem, when applied to the set $\{C_i\}$ above, will yield a proof for Theorem 1.8. (Regarding Remark 1.7, the new "applicability" condition which ensures the hypothesis '$x \in \alpha^{-1}(0)$' in Lemma 1.6 is $2(m_1 + m_2) \neq 0 \pmod m$, but again, since $p = m$ is odd, this is equivalent to the original condition $m_1 + m_2 \neq 0 \pmod m$.)

The proof of Proposition 4.3 goes as follows. Since $b_G^{2,+} = 1$ in this case, the equivariant Seiberg-Witten invariant (which is simply the Seiberg-Witten invariant of the orbifold X/G) is well-defined only after specifying a choice of chambers. Let E be an equivariant complex line bundle over X. We denote by $SW^G(E)$ the equivariant Seiberg-Witten invariant defined using the associated r-version of Taubes' perturbed Seiberg-Witten equations with $r > 0$ sufficiently large. Then the wall-crossing formula gives

$$|SW^G(E) \pm SW^G(K - E)| = 1$$

provided that the formal dimension $d(E)$ of the equivariant Seiberg-Witten moduli space is non-negative. (Thanks to Tian-Jun Li for explaining this to me.)

We consider the case where $E = 2K$. Notice that $SW^G(K - E)$ must be zero, because otherwise by Taubes' $SW \Rightarrow Gr$ theorem in [25], $c_1(K - E) = c_1(-K)$ is represented by J-holomorphic curves which contradicts the assumption $c_1(K) \cdot [\omega] > 0$. It follows that $SW^G(2K) = \pm 1$ if $d(2K) \geq 0$.

A formula for $d(E)$ may be found in Appendix A of [5] (see also Lemma 3.3 in [4]). In the present case, we have

$$d(2K) = \frac{1}{p}(c_1(2K)^2 - c_1(2K) \cdot c_1(K) + \sum_{i=1}^{3} \sum_{x=1}^{p-1} \frac{2(\mu_p^{-2(a_i+b_i)x} - 1)}{(1 - \mu_p^{-a_i x})(1 - \mu_p^{-b_i x})}),$$

where $\mu_p = \exp(2\pi i/p)$.

Proposition 4.3 follows by showing that $d(2K)\geq 0$. In the calculation of $d(2K)$, the fact that $\{(a_i,b_i)\}$ is given by

$$(a,b),(p-a,b-a),(p-b,p+a-b)$$

for some $0<a<b<p$ plays a crucial role.

Lemma 4.4. *Let c,d be satisfying $0\leq c\leq p$, $0<d<p$, and let $\delta(c,d)$ be the unique solution to $c-d\delta=0 \pmod p$ for $0\leq\delta<p$. Then*

$$\sum_{x=1}^{p-1}\frac{2\mu_p^{cx}}{1-\mu_p^{-dx}}=p-1-2\delta(c,d).$$

Proof. Set $\phi_{c,d}(t)\equiv\sum_{x=1}^{p-1}\mu_p^{cx}(1-\mu_p^{-dx}t)^{-1}$. Then

$$\begin{aligned}
\phi_{c,d}(t) &= \sum_{x=1}^{p-1}\mu_p^{cx}\sum_{l=0}^{\infty}(\mu_p^{-dx}t)^l\\
&= \sum_{l=0}^{\infty}t^l(\sum_{x=1}^{p-1}\mu_p^{(c-dl)x})\\
&= \sum_{l=0}^{\infty}t^l(-1)+\sum_{l=0}^{\infty}t^{\delta(c,d)+pl}\cdot p\\
&= \frac{1}{t-1}+\frac{pt^{\delta(c,d)}}{1-t^p}\\
&= \frac{t^{p-1}+\cdots+1-pt^{\delta(c,d)}}{t^p-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{x=1}^{p-1}\frac{2\mu_p^{cx}}{1-\mu_p^{-dx}} &= 2\phi_{c,d}(1)\\
&= 2\cdot\frac{(t^{p-1}+\cdots+1-pt^{\delta(c,d)})'|_{t=1}}{(t^p-1)'|_{t=1}}\\
&= 2\cdot\frac{(p-1)+\cdots+1-p\delta(c,d)}{p}\\
&= p-1-2\delta(c,d).
\end{aligned}$$

$\square$

With the preceding lemma, we compute

$$\begin{aligned}\sum_{x=1}^{p-1}\frac{2(\mu_p^{-2(a_i+b_i)x}-1)}{(1-\mu_p^{-a_ix})(1-\mu_p^{-b_ix})} &= \sum_{x=1}^{p-1}\frac{2(\mu_p^{(a_i+b_i)x}-1)}{(1-\mu_p^{-a_ix})(1-\mu_p^{-b_ix})}\\ &= \sum_{x=1}^{p-1}\frac{2\mu_p^{(a_i+b_i)x}}{1-\mu_p^{-b_ix}}+\sum_{x=1}^{p-1}\frac{2\mu_p^{b_ix}}{1-\mu_p^{-a_ix}}\\ &= p-1-2\delta(a_i+b_i,b_i)+p-1-2\delta(b_i,a_i).\end{aligned}$$

Now without loss of generality, we assume

$$(a_1,b_1)=(a,b),(a_2,b_2)=(p-a,b-a),(a_3,b_3)=(p-b,p+a-b).$$

Then one can check directly that

$$(\delta(a_1+b_1,b_1)+\delta(b_3,a_3))b=2b \pmod p,$$

which implies that

$$\delta(a_1+b_1,b_1)+\delta(b_3,a_3)=\begin{cases}2 & \text{if } a+b=p\\ p+2 & \text{if } a+b\neq p.\end{cases}$$

Similarly,

$$(\delta(a_2+b_2,b_2)+\delta(a_3+b_3,b_3))(b-a)=3(b-a) \pmod p,$$

which implies that

$$\delta(a_2+b_2,b_2)+\delta(a_3+b_3,b_3)=\begin{cases}0 & \text{if } b=2a \text{ and } p+a=2b\\ 3 & \text{if } b=2a \text{ or } p+a=2b\\ p+3 & \text{if } b\neq 2a \text{ and } p+a\neq 2b,\end{cases}$$

and

$$(\delta(b_1,a_1)+\delta(b_2,a_2))a=a \pmod p,$$

which implies that $\delta(b_1,a_1)+\delta(b_2,a_2)=p+1$.

Finally, we note that $c_1(K)^2=9$, hence

$$\begin{aligned}d(2K) &= \frac{1}{p}(18+6(p-1)-2\sum_{i=1}^{3}(\delta(a_i+b_i,b_i)+\delta(b_i,a_i))\\ &\geq \frac{1}{p}(18+6(p-1)-2(3p+6))=0.\end{aligned}$$

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