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Shatalov-Sternin's construction of complex WKB solutions – a comment on geometric issues.

by

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Abstract

We study Shatalov-Sternin's proof of existence of resurgent solutions of a linear ODE and discuss the construction of analytic continuation to a common "Riemann surface" of all terms of the von Neumann series appearing in their proof. We give a detailed proof of a more modest statement in a special situation.

1 Introduction.

Resurgent analysis is a method of studying hyperasymptotic expansions

$$\sum_{k,j} e^{-c_k/h} a_{k,j} h^j, \quad h \to 0+ \tag{1}$$

and those of similar kind by treating such expansions as asymptotics obtained from a Laplace integral

$$\int_\gamma \Phi(s) e^{-s/h} ds, \tag{2}$$

where Φ is a ramified analytic function in the complex domain with a discrete set of singularities and γ is an infinite path on the Riemann surface of Φ. The crucial observation is that the terms of (1) can be recovered from studying the singularities of Φ, see [V83], [E], [CNP], [DP99], as well as [G] for this author's preferred terminology.

The methods of resurgent analysis have been used, in particular, to study asymptotics of solutions of linear ODE with a small parameter, especially the Schrödinger equation in the semiclassical approximation, see, e.g. [DDP97]. These applications justify the need to give a fully rigorous and detailed treatment of foundations of resurgent analysis, and the present work is a step in this direction.

Probably the most important of these foundational questions is the existence of resurgent solutions of a Schrödinger equation. More specifically, consider an equation of the type

$$-h^2 \partial_x^2 \phi(h,x) + V(x)\phi(h,x) = 0 \tag{3}$$

where x ranges over $\mathbb{C}$, h is a small complex asymptotic parameter, and $V(x)$ is an entire function often assumed to be a polynomial. The transformation (2) brings this equation to the form

$$-\partial_s^{-2}\partial_x^2\Phi(s,x) + V(x)\Phi(s,x) \;=\; 0 \tag{4}$$

which needs to be satisfied modulo functions that are entire with respect to s for every values of x for some ramified analytic function Φ. Since the beginnings of resurgent analysis in the early 1980s there has been no real doubt that (4) possesses two linearly independent (in an appropriate sense) solutions that are endlessly analytically continuable with respect to s and satisfy certain growth conditions at infinity.

This article arises from the author's attempt to understand the proof of this fact given in the book [ShSt], which we will recall in the next section. In this book, the solution Φ is represented as a sum of an infinite von Neumann series and it is claimed that 1) all term of the von Neumann series are defined on the same endlessly continuable Riemann surface (actually, a complex two-dimensional manifold), and 2) the series converges. We do not quite understand the convergence proof for reasons mentioned in the next section and plan to treat it elsewhere. Here we will explicitly construct a part of the Riemann surface in question.

Here is the main thing that the author has learned by working on this article. In the literature on resurgent analysis the function Φ is defined on a Riemann surface with a priori infinitely many sheets with no artificial boundaries, and this gives the theory much of its elegance and aesthetic appeal. For example, [V83] derives asymptotic connection formulae by considering how singularities of Φ are expected to behave on the first as well as on the second sheet of the Riemann surface. However, [1] if we are only interested in asymptotics of solutions of (3) in the form (1), it is enough to construct Φ on the whole of the first sheet and analytically continue it just a little bit beyond the cuts. It seems that Shatalov-Sternin's method allows us to do just that (see below for a precise statement); we need to make no reference to the further sheets of the Riemann surface.

Trying to understand [ShSt]'s construction of the Riemann surface, we took a case of a potential $V(x)$ roughly corresponding to an anharmonic oscillator, with two turning points x_1, x_2. We used the conclusions of the heuristic argument in [V83] to describe an open complex 2-dimensional manifold $\mathcal{S}$, on which one expects the solution $\Phi(s,x)$ to be defined. In [ShSt]'s proof Φ is expressed via a sum of a von Neumann series (8) of iterations of integral operators R_1, R_2 applied to some initial function. We covered $\mathcal{S}$ by open subsets and constructed an analytic continuation of the summands of the von Neumann series to each one of those, which led to consideration of some two hundred separate cases. Then we classified the arguments that we have been using and stated them as lemmas 5.2-5.21. Thus the proof of the existence of the analytic continuation of Φ to all points in $\mathcal{S}$ has become a list of references to these lemmas with a few additional comments.

The argument presented in this paper remains too combinatorial for the author's taste. However, we believe that a similar statement for more complicated potentials can be proven in essentially the same way using lemmas 5.2-5.21. Thus, although we do not give a proof of the general case, we offer a method that should in principle work for an arbitrary entire potential $V(x)$.

[1] The author read this idea in one of the versions of [DP99] that is no longer available on the Web

2 Shatalov-Sternin's construction.

Specializing the exposition of [ShSt], let us consider the one-dimensional Schrödinger equation

$$[-h^2\partial_x^2 + V(x)]\varphi(h,x) = 0, \tag{5}$$

where the variable x takes values in $\mathbb{C}$ and $V(x)$ is an entire function. We will be solving the Laplace-transformed equation

$$-\partial_s^{-2}\partial_x^2\Phi(s,x) + V(x)\Phi(s,x) \;=\; 0 \tag{6}$$

modulo functions that are entire with respect to s for every values of x, and we will be looking for its solution Φ in the class of ramified analytic function, i.e. an analytic function on some complex two-dimensional manifold projecting to $\mathbb{C}^2$ with coordinates (x,s).

Fix a point x_0 such that $V(x_0) \neq 0$ and a determination $p(x)$ of $\sqrt{V(x)}$ in a neighborhood of x_0. Let $p_1(x) = -p_2(x) = \sqrt{V(x)}$; let further $S(x) = \int_{x_0}^x p_j(y)dy$ and $S_j(x) = \int_{x_0}^x p_j(y)dy$, $j = 1,2$.

We will use the following two operations on the classes of ramified analytic functions modulo functions entire with respect to s:

$$\hat{h}\Phi(s,x) \;=\; \partial_s^{-1}\Phi(s,x),$$
$$e^{\hat{h}^{-1}a(x)}\Phi(s,x) \;=\; \Phi(s+a(x),x).$$

Define the operator R_j acting on (germ of) ramified analytic functions $G(s,x)$ by the formula

$$(R_jG)(s,x) = \int_{x_0}^x (D_1G)(s+S_j(x)-S_j(y),y)p_j(y)dy, \tag{7}$$

where D_1 stands for the derivative of the function with respect to the first argument. In [ShSt] this derivative is missing.

An equation

$$[\frac{1}{p_j(x)}\partial_s^{-1}\partial_x - 1]u(s,x) = b(s,x)$$

admits solutions of the form

$$u(s,x) \;=\; R_jb(s,x) + C(\hat{h})e^{\hat{h}^{-1}S_j}f(s)$$

where $f(s)$ is any (germ of) a ramified analytic function and $C(\hat{h})$ is a polynomial in $\hat{h}$, or, more generally, a convolution with a ramified analytic function $\hat{C}(s)$ in the sense usual in resurgent analysis.

Next [ShSt] proceed to formally find an operator $\tilde{Y}$ such that $\tilde{Y}f$ is a solution of (6) for any $f(s)$. In our situation, rewrite (6) as

$$\left(p^2(x)[-\frac{1}{p(x)}h\partial_x - 1] - hp'(x)\right)\left(\frac{1}{p(x)}h\partial_x - 1\right) - hp'(x) \;=\; -h^2\partial_x^2 + p^2(x)$$

and start looking for the operator $\tilde{Y}$ in the form

$$\tilde{Y} = R_1\tilde{Y}_1 - C_1(h)e^{\hat{h}^{-1}S_1}.$$

Substitution yields an operator equation

$$\left\{\left(p^2(x)[-\frac{1}{p(x)}h\partial_x - 1] - hp'(x)\right) - hp'(x)R_1\right\}\tilde{Y}_1 \;=\; -\hat{h}p'(x)C_1(h)e^{\hat{h}^{-1}S_1}.$$

We now look for the operator $\tilde{Y}_1$ in the form

$$\tilde{Y}_1 = R_2\tilde{Y}_2 - C_2(\hat{h})e^{\hat{h}^{-1}S_2}$$

which yields

$$\tilde{Y}_2 - \hat{h}\frac{p'(x)}{p^2(x)}\{R_2 + R_1R_2\}\tilde{Y}_2 \;=\; -h\frac{p'(x)}{p^2(x)}C_2(\hat{h})e^{\hat{h}^{-1}S_2} - \hat{h}\frac{p'(x)}{p^2(x)}R_1C_2(h)e^{\hat{h}^{-1}S_1} - \hat{h}\frac{p'(x)}{p^2(x)}C_1(h)e^{\hat{h}^{-1}S_1}.$$

Denoting

$$\mathcal{R}_2 = -\frac{p'(x)}{p^2(x)}(1 + R_1),$$

and putting for definiteness $C_1 = 1$ and $C_2 = 0$, we formally obtain

$$\tilde{Y} = R_1R_2\tilde{Y}_2 - e^{\hat{h}^{-1}\tilde{S}_1(x)},$$

$$\tilde{Y}_2 + \hat{h}\mathcal{R}_2R_2\tilde{Y}_2 = \hat{h}(-\frac{p'(x)}{p^2(x)})e^{\hat{h}^{-1}\tilde{S}_1(x)}.$$

The last equation has a formal solution

$$\tilde{Y}_2 \;=\; \sum_{j=0}^{\infty}(-1)^j\hat{h}^{j+1}[(-\frac{p'(x)}{p^2(x)})(1 + R_1)R_2]^jF_1(S_1(x),\hat{h})e^{\hat{h}^{-1}\tilde{S}_1(x)}. \tag{8}$$

In order to obtain a solution for (6), one can take $f(s) = s\text{Ln } s$ and consider the infinite series defining $\tilde{Y}f(s)$. The first task is to construct a "Riemann surface" – a two dimensional complex manifold on which all terms of the infinite series in the expression of $\tilde{Y}f(s)$ are defined. It is easy to see that an equivalent question is to construct a "Riemann surface" on which all functions

$$R_{j_k}...R_{j_2}R_{j_1}f(s + S(x)), \quad j_i = 1, 2, \quad k \geq 0$$

are defined. This is the question we are dealing with in this article. The second task would be to show that the infinite series converges on the Riemann surface. Unfortunately, a derivative in the integrand is missing in [ShSt]'s definition of operators R_j and we cannot suggest an easy way to repair their convergence argument, but hope to give an alternative proof elsewhere.

3 The example under consideration. Notation.

We will consider a region of the complex plane surrounding the following Stokes pattern similar to that of an anharmonic oscillator with two turning points and six unbounded Stokes curves. Let us now say this more formally.

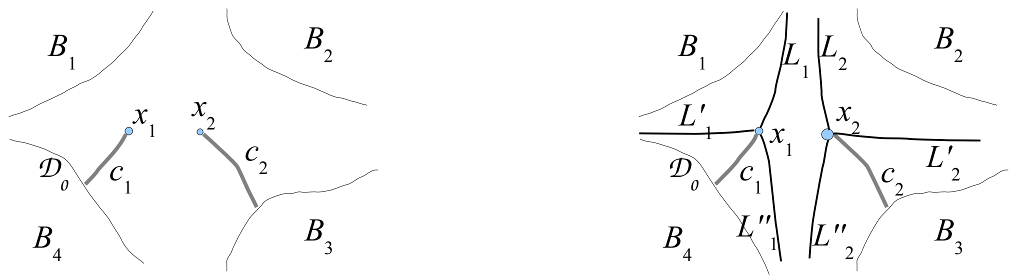

Figure 1

Let $V(x)$ be a function analytic on the closure of a domain $\mathcal{D}_0 \subset \mathbb{C}$ which is simply connected and such that $\mathbb{C}\backslash\mathcal{D}_0$ has four connected components B_1,..., B_4 numbered in a conterclockwise order. Let $V(x)$ has two zeros in $\mathcal{D}_0$ at points x_1 and x_2, and both zeros are simple. Draw two smooth curves c_1 and c_2 from the points x_1 and x_2 to the boundary components B_4 and B_3 respectively so that $\mathcal{D}_0\backslash(c_1 \cup c_2)$ is homeomorphic to a disc, fig.1, left. Fix a determination of $\sqrt{V(x)}$ on $\mathcal{D}_0\backslash(c_1 \cup c_2)$ and denote it $p(x)$.

Let L_1, L_1', L_1'' be curves satisfying Im $\int_{x_1}^x p(y)dy = 0$ for x on any of these curves. Suppose all these curves go off to infinity inside $\mathcal{D}_0\backslash(c_1 \cup c_2)$, L_1 between B_1 and B_2, L_1' between B_1 and B_4 and L_1'' between B_4 and B_3, the cut c_1 stays between L_1' and L_1'', and x_2 is in the region bounded by L_1', L_1'', ∂B_2, ∂B_3, fig. 1, right.

Similarly, let L_2, L_2', L_2'' be curves satisfying Im $\int_{x_2}^x p(y)dy = 0$ for x on any of these curves. Suppose all these curves go off to infinity inside $\mathcal{D}_0\backslash(c_1 \cup c_2)$, L_2 between B_1 and B_2, L_2' between B_2 and B_3 and L_2'' between B_2 and B_3, the cut c_2 stays between L_2' and L_2'', and x_1 is in the region bounded by L_2', L_2'', ∂B_1, ∂B_4.

The curves L_j, L_j', L_j'', $j = 1, 2$, as well as their preimages to the universal cover of $\mathcal{D}_0\backslash\{x_1, x_2\}$ are called *Stokes curves.*

Moreover, assume that Re $\int^x p(y)dy$ decreases along L_j and increases along L_j', L_j'' in the direction away from x_j, $j = 1, 2$.

Fix two numbers $\delta > 0$, $\varepsilon > 0$.

Let x_0 be a point in the part of $\mathcal{D}_0$ bounded by L_1, L_1', and ∂B_1. Let $S(x) = \int_{x_0}^x p(y)dy$ which is well-defined on the closure of $\mathcal{D}_0$ minus $c_1 \cup c_2$. This function can be analytically continued to $\mathcal{D}_0\backslash\{x_1, x_2\}$. Assume there is a constant $A' > 0$ such that: Im $S(x) < -\delta/2$ on ∂B_1, Im $S(x) - S(x_2) > A'/2$ on ∂B_2, also assume that Im $S(x) < 0$ for $x \in \partial B_4$ where the determination of $S(x)$ is obtained by going clockwise around x_1, and also that Im $S(x_1) + S(x_2) - S(x) > 0$ **or is it** < 0**?** on ∂B_3 where the determination of $S(x)$ obtained by going counterclockwise around x_2 is meant.

Now let us consider a subdomain $\mathcal{D}$ of $\mathcal{D}_0$ bounded by the curve Im $S(x) = -\delta/2$ in the "quadrant" defined by L_1 and L_1', bounded by Im $S(x) =$ Im $S(x_2) + A'/2$ in the "quadrant" defined by L_2 and L_2', bounded by Im $S(x) =$ Im $S(x_1) + S(x_2)$ in the "quadrant" defined by L_2' and L_2'', bounded by Im $S(x) = 0$ in the "quadrant" defined by L_1' and L_1'', where the same determinations of S are meant as in the previous paragraph.

Denote by L_0 the curve Im $S(x) = 0$ passing through x_0.

In the universal cover of $\mathcal{D}\backslash\{x_1, x_2\}$ (with base point x_0) consider preimages $\tilde{L}_1'$, $\tilde{L}_1''$, $\tilde{L}_2'$, $\tilde{L}_2''$ of

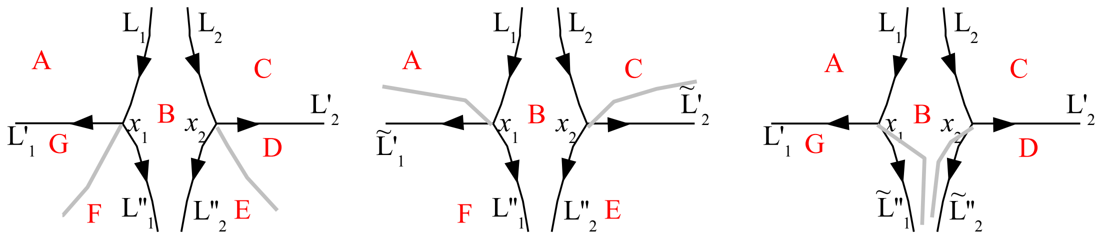

Figure 2: The arrow on the Stokes curves indincates the direction in which Re S grows

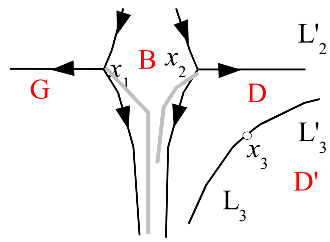

Figure 3

Stokes curves L_1', L_1'', L_2', L_2'' lying on further sheets, fig.2. Consider the subset $\tilde{\mathcal{D}}'$ of the universal cover of $\mathcal{D}\backslash\{x_1, x_2\}$ bounded by $\tilde{L}_1'$, $\tilde{L}_1''$, $\tilde{L}_2'$, $\tilde{L}_2''$. The curves $\tilde{L}_1'$, $\tilde{L}_1''$, $\tilde{L}_2'$, $\tilde{L}_2''$ will be called *external Stokes curves.*

In the subset of $\tilde{\mathcal{D}}'$ bounded by L_2' and $\tilde{L}_2''$ put a *generalized turning point* x_3 such that $S(x_3) = S(x_1)$, fig. 3, and draw two *generalized Stokes curves* L_3 and L_3'.

Let $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}'\backslash x_3$. This set will be split into *Stokes regions* A,B,C,D,D$'$,E,F,G. It is for x in this set $\tilde{\mathcal{D}}$ that we will be discussing the construction of the solution for the equation (6).

4 Structure of the Riemann surface.

From now on we assume that

$$\delta \ < \ \frac{1}{3}\text{Im } S(x_1); \qquad \delta < \frac{1}{3}\text{Im } [S(x_2) - S(x_1)] \tag{9}$$

In this section we will introduce the two-dimensional complex manifold $\mathcal{S}$ with a projection to $\mathbb{C}^2$ with coordinates (s, x) on which our ramified analytic functions $G(s, x)$ will be defined. We will describe $\mathcal{S}$ as the union of $\mathcal{S}' \cup \mathcal{S}''$by covering both $\mathcal{S}'$ and $\mathcal{S}''$ with charts; gluing maps will be obvious and not stated explicitly.

For $\underline{x} \in \tilde{\mathcal{D}}$ define the fiber of $\mathcal{S}'$ over $\underline{x}$, as on figures 4–11. As usually, the *canonical distance* between points $\underline{x}_1$ and $\underline{x}_2$ of $\tilde{\mathcal{D}}$ is min $\int_\pi |p(y)|ds$ where the minimum is taken over all paths π in $\tilde{\mathcal{D}}$ connecting $\underline{x}_1$ and $\underline{x}_2$.

On these pictures $d_1 = 2\text{Im } [S(\underline{x}) - S(x_1)]$ and $d_2 = 2\text{Im } [S(x_2) - S(\underline{x})]$, $d_{12} = 2\text{Im } [S(\underline{x}) + S(x_1) - S(x_2)]$, $d_{1+2} := 2\text{Im } [S(x_1) + S(x_2) - S(\underline{x})]$, $d_0 := 2\text{Im } S(x)$ for the determinations of S in the corresponding Stokes regions.

On these pictures the corresponding fibers are given as complex planes with a few singularities, locations of singularities are marked. There are two groups of singularities,

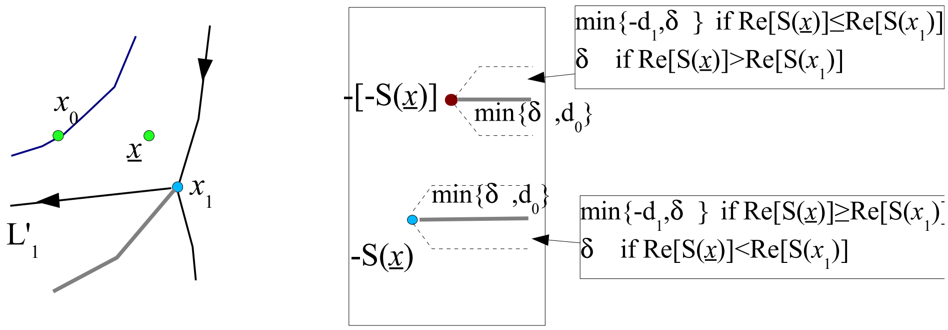

Figure 4: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region A below the curve L_0.

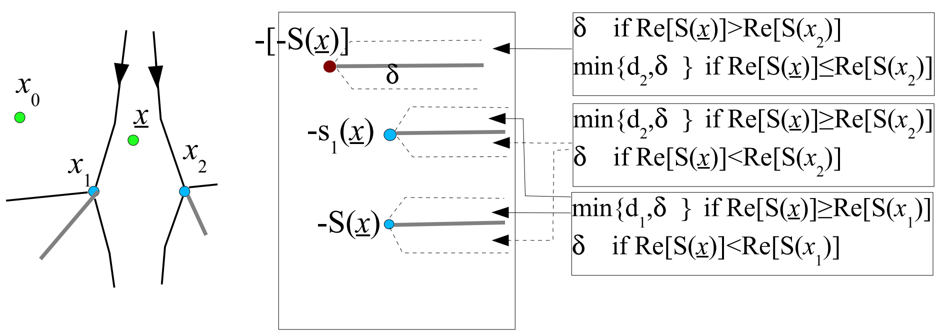

Figure 5: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region B.

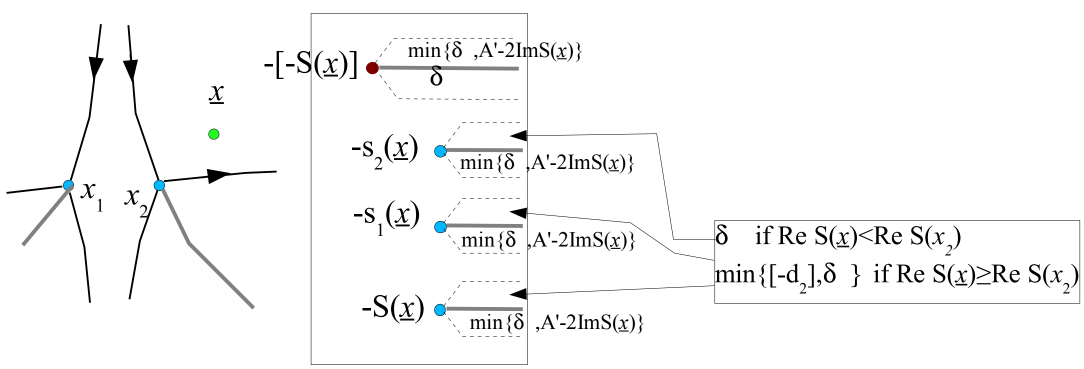

Figure 6: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region C.

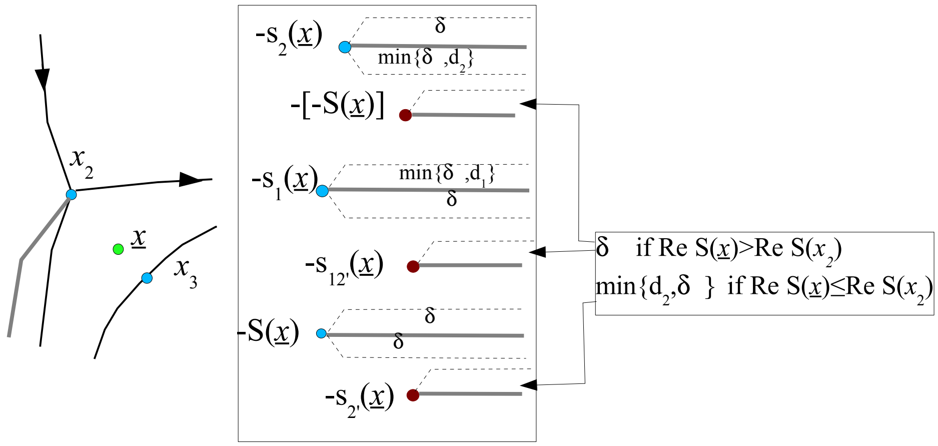

Figure 7: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region D.

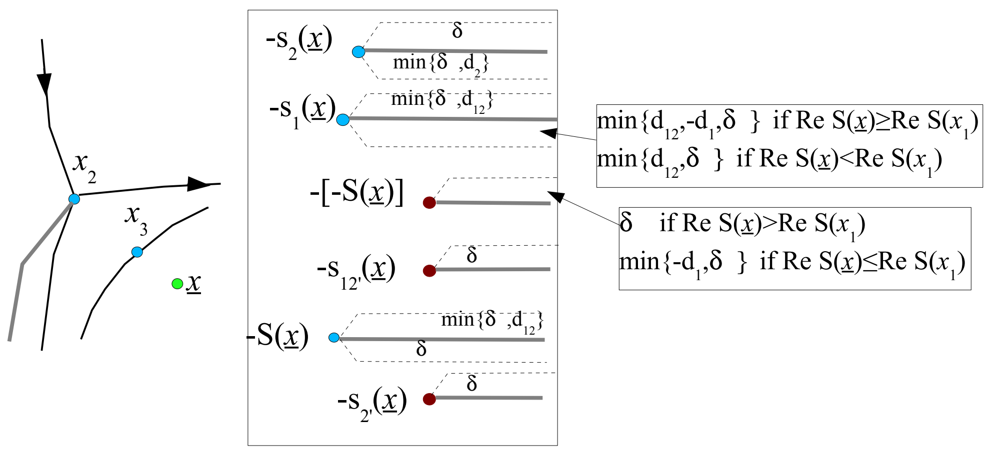

Figure 8: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region D$'$.

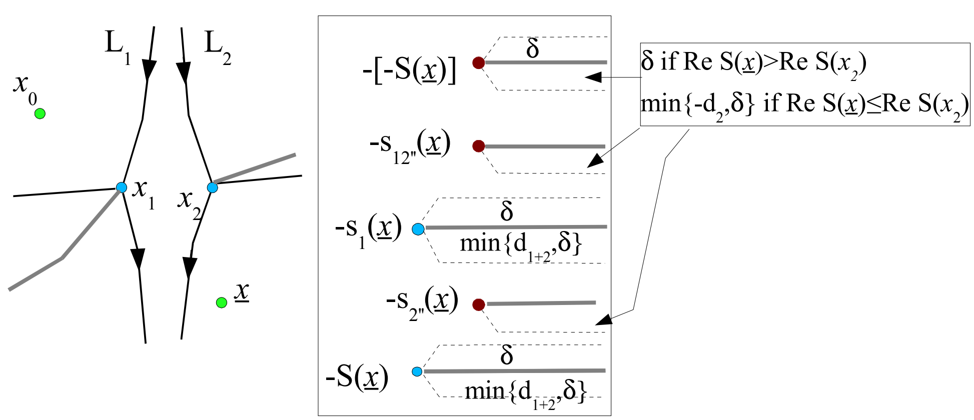

Figure 9: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region E.

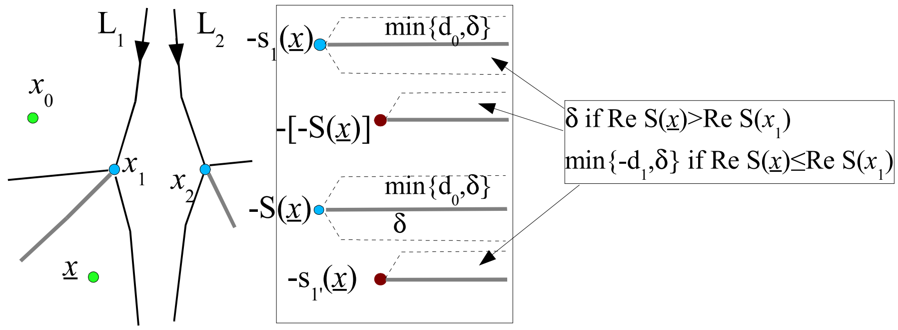

Figure 10: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region F.

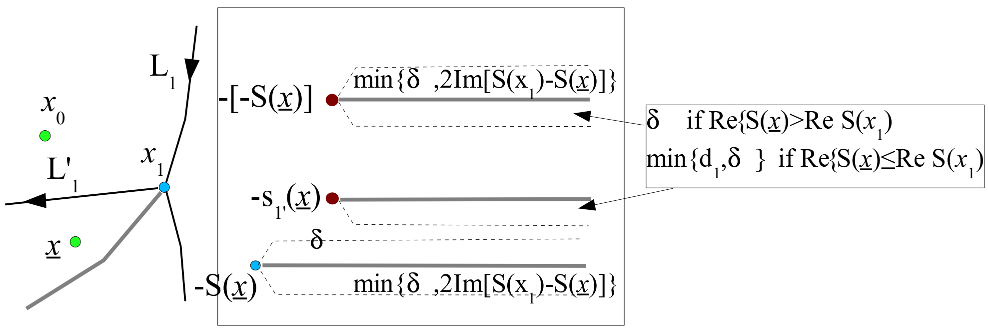

Figure 11: Fiber of $\mathcal{S}'$ over $\underline{x}$ when $\underline{x}$ is in the region G.

- "blue" singularities $-S(\underline{x})$, $-s_1(\underline{x}) = 2S(x_1) - S(\underline{x})$, $-s_2(\underline{x}) = 2S(x_2) - S(\underline{x})$
- "red" singularities $S(\underline{x})$, $-s_{2'}(\underline{x}) = -2S(x_2) + S(\underline{x})$, and $-s_{12'}(\underline{x}) = 2S(x_1) - 2S(x_2) + S(\underline{x})$, $-s_{2''}(\underline{x}) = -2S(x_2) + S(\underline{x})$, $-s_{12''}(\underline{x}) = 2S(x_1) - 2S(x_2) + S(\underline{x})$, $-s_{1'}(\underline{x}) = -2S(x_1) + S(\underline{x})$,

where every time we mean the determination of S relevant in the corresponding Stokes region. There are cuts introduced in the positive direction, and we attach *flaps* on both sides of most of the cuts, of sizes specified on the pictures.

The sizes of the flaps are chosen according to the following principle: they are δ except near the Stokes curves where another singularity approaches the cut on the second sheet from the given side, in which the flap is drawn right up to the singularity. Sometimes we construct only one flap which would be totally sufficient for the purposes of deforming the contour of the Laplace integral (2) as in [V83], page 218.

For $\underline{x}$ along (i.e. canonical distance $< \delta/2$ from) the curve L_0, internal Stokes curves L_j, L'_j, L''_j, $j = 1, 2$, generalized Stokes curves L_3 and L'_3 we will introduce more charts whose union will be $\mathcal{S}''$.

All the above mentioned curves are characterized by pairs of singularities one of which crosses the horizontal cut starting at the other singularity. For $\underline{x}$ along these curves, for every pair of such singularities we will add to $\mathcal{S}''$ four additional subsets. Depending on a situations, these subsets are either similar to the ones depicted on fig.13 for $\rho = 0$ but unbounded in the positive real direction, and the similar subsets but flipped over with respect to a horizontal line, or smaller subsets similar to the four bottom ones on the figure 16 if the corresponding singularity is a beginning of a cut that has only one flap in $\mathcal{S}'$.

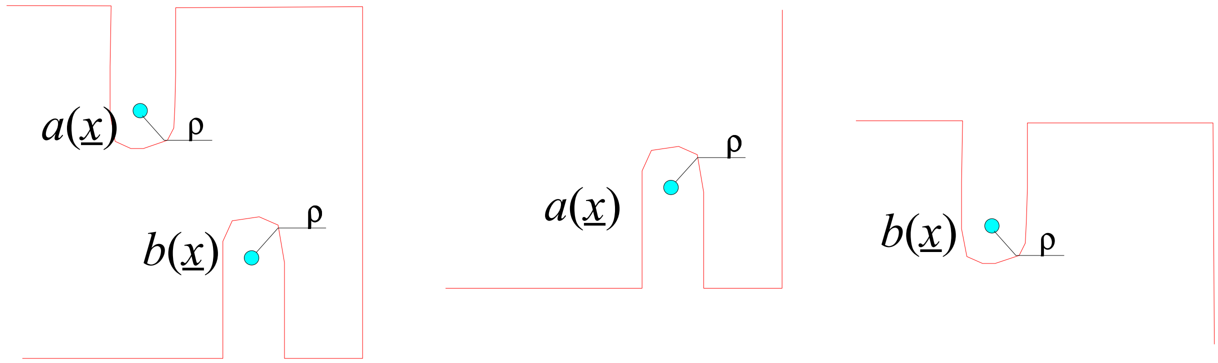

Figure 12**:** the strip between singularities $a(\underline{x})$ and $b(\underline{x})$ (left), the area above $a(\underline{x})$ (middle), and the area below $b(\underline{x})$ (bottom).

Obviously, the function $(s+S(x))\text{Ln}\,(s+S(x))$ has an analytic continuation to $\mathcal{S}$.

5 Construction of analytic continuations.

5.1 Statement of the main result and preliminary remarks

Theorem 5.1 *If $G(s,x)$ is defined on $\mathcal{S}$, then R_jG, $j=1,2$ have analytic continuations to $\mathcal{S}$.*

Proof. For every $\rho \in (0,\delta)$ and $N \in \mathbb{R}$ we will consider subsets in the fiber $\mathcal{S}_{\underline{x}}$ of $\mathcal{S}$ over $\underline{x}$ as shown on figure 12. Here $a(\underline{x})$ and $b(\underline{x})$ are two singularities of $\mathcal{S}_{\underline{x}}$. Each of these sets is unbounded in the negative real direction, is bounded by Re $s=N$ from the right, and are bounded in the vertical direction by Im $a(\underline{x})$ or Im $b(\underline{x})$ plus or minus the relevant flap size. From these sets we have cut out vertical slots that consist of a circle of radius ρ around the corresponding singularity and a rectangle with the base 2ρ and height equal to the relevant flap size. We will call these sets *the strip between singularities $a(\underline{x})$ and $b(\underline{x})$, the area above $a(\underline{x})$, and the area below $b(\underline{x})$*, respectively. Sometimes we will use other similar subsets of $\mathcal{S}_{\underline{x}}$ and hope that their description will not lead to any confusion. If cutting out the slots splits a strip into several connected pieces, we consider further the one that is unbounded in the $-\infty$ direction.

For $\underline{x}$ along a Stokes curve or a generalized Stokes curve (i.e. canonical distance $<\delta/2$ from it and so that Re $S(\underline{x}) \geq$ Re $S(x_i)$ or Re $S(\underline{x}) \leq$ Re $S(x_i)$, whichever is appropriate, where x_i is the turning point at the origin of the Stokes curve), or L_0, we will also consider four subsets of $\mathcal{S}_{\underline{x}}$ surrounding the two singularities one of which crosses the horizontal cut starting at the other when $\underline{x}$ lies on the curve in question. Two of these subsets are depicted on fig.13, and the other two are obtained from this figure by reflecting it with respect to a horizontal line.

We will prove existence of analytic continuation to every such strip and then send $\rho \to 0+$ and $N \to +\infty$. Thus the existence of analytic continuation to the whole $\mathcal{S}$ will follow.

Actually, the construction we are about to present proves a slightly weaker statement, namely, that the function $R_jG(s,x)$ can be analytically continued to a manifold $\mathcal{S}^{(\delta')}$ constructed analogously to $\mathcal{S}$ but with δ replaced by any positive $\delta' < \delta$. Since $\bigcup_{\delta'<\delta} \mathcal{S}^{(\delta')} = \mathcal{S}$, the theorem will be proven as stated. To simplify the exposition, we will keep this subtlety in mind for the rest of the proof but will not explicitly mention it again.

5.2 Strategy of the proof. Deformation of the integration path. Proof for the region A.

If $G(s,x)$ was an entire function, we could define $(R_jG)(s,\underline{x})$ by the formula

$$(R_jG)(s,\underline{x}) = \int_{x_0}^{\underline{x}} (D_1G)(s+S_j(\underline{x})-S_j(y),y)p_j(y)dy \tag{10}$$

where the integral is taken along any path from x_0 to x in $\tilde{\mathcal{D}}$. Since, however, $G(s,x)$ has singularities, we need to find for a given s a path $y(t)$ in $\tilde{\mathcal{D}}$ from x_0 to $\underline{x}$ such that the point $(s+S_j(\underline{x})-S_j(y(t)),y(t))$ stays in $\mathcal{S}$ and does not hit any of its singularities. We will from now on refer to $y(t)$ as the *integration path* and draw it green on our figures.

A moment of reflection shows that if $\underline{x}$ is contained in a compact subset $K \subset \tilde{\mathcal{D}}$ and $x_0 \in K$, then there is a constant $N_K \in \mathbb{R}$ such that any integration path $y(t)$ from x_0 to $\underline{x}$ contained in K will satisfy the desired property for any s with Re $s < N_K$.

We will occupy ourselves now with construction for each given $\underline{x}$ of integration paths $y(t)$ such that these paths can be lifted to $\mathcal{S}$ by the formula $(s+S_j(\underline{x})-S_j(y(t)),y(t))$ (in the sense that $(s+S_j(\underline{x})-S_j(y(t)),y(t))$ will stay in $\mathcal{S}$ and avoid its singularities) for any s in a subset of $\mathcal{S}_{\underline{x}}$ of the types shown on fig.12. If we use such a path $y(t)$ in the formula (10), we will obtain an analytic continuation of R_jG to the corresponding strip in $\mathcal{S}_{\underline{x}}$.

For $U \subset \mathcal{S}_{\underline{x}}$ let us try to construct an integration path $y(t)$ satisfying the above property for all $s \in U$. Suppose $S_j(x)+c$ is one of the singularities of $\mathcal{S}$, for some constant c. We want to make sure that $s+S_j(\underline{x})-S_j(y(t))$ avoids the singularity $S_j(y(t))+c$, i.e. we want the equality

$$2S_j(y(t)) \ = \ s+S_j(\underline{x})-c$$

to hold for no point $y(t)$ along the integration path and for no point $s \in U$. That is to say, we want the integration path $y(t)$ to avoid the set $S_j^{-1}\left(\frac{U+S_j(\underline{x})-c}{2}\right) \subset \tilde{\mathcal{D}}$. Care needs to be taken to keep track of the appropriate branches of the functions involved in this expression.

On our figures we will draw the contour of U in red and the contours of the sets of type $V = S_j^{-1}\left(\frac{U+S_j(\underline{x})-c}{2}\right)$ in purple.

When consructing integration paths $y(t)$ for R_jG, we found it convenient to construct the parallel transport of the set $U \in \mathcal{S}_{\underline{x}}$ by defining $U(y(t)) = U + S_j(\underline{x}) - S_j(y)$ (in terms of the projection to the complex s-plane). Then the set $U(y(t))$ and the singularities of type $-S_j(y)+const$ move along the path $y(t)$ parallel to each other, and the singularities of type $S_j(y)+const$ moving in the s-plane relative to $U(y)$. For this reason, we will call singularities of type $-S_j(y)+const$ *stationary* and the singularities of type $S_j(y)+const$ *moving* singularities. When the index j changes, the roles of moving and stationary singularities reverse. Sometimes we will symbolically draw the trajectories of the moving singularities in the s-plane relative to the set $U(y)$ in green.

Stokes regions have a natural partial order: we say that a Stokes region I is *closer* to x_0 than a Stokes region II, or that I *comes earlier* than II, if any path in $\tilde{\mathcal{D}}$ connecting II to x_0 passes through

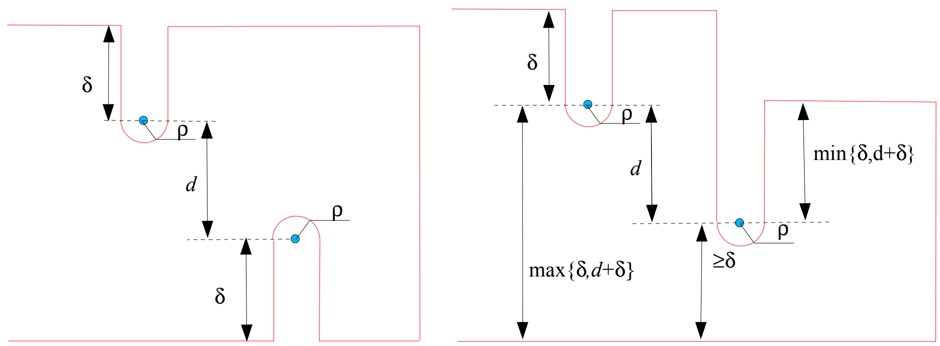

Figure 13**:** Strips in $\mathcal{S}_{\underline{x}}$ surrounding two singularities one of which crosses the horizontal cut starting from the other. Here d is theee difference between the imaginary parts of the singularities, $-\delta < d < \delta$.

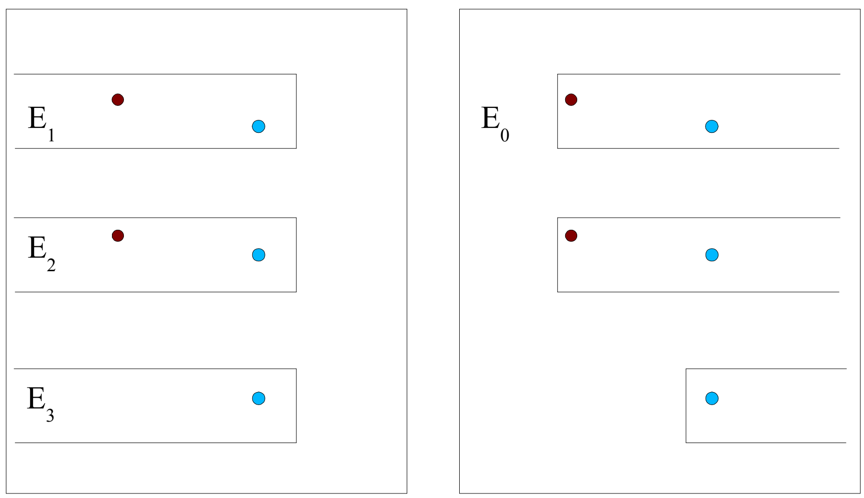

Figure 14

I. In the proof we will construct the integration path step by step, starting from $\underline{x}$ and leading it to an earlier Stokes region than region containing $\underline{x}$; we will explicitly construct integration paths from points of the region A to x_0.

In the proof we will separate the cases of $\underline{x}$ inside a Stokes region and $(s, \underline{x}) \in \mathcal{S}'$ and $\underline{x}$ along a Stokes curve, i.e. canonical distance $< \delta/2$ from the Stokes curve and $(s, \underline{x}) \in \mathcal{S}''$.

Along the internal Stokes curves and the generalized Stokes curves the first sheet will be thought of as a union of strips $E_1, E_2, ...$ containing singularities and the rest of the first sheet E_0, fig.14. Given a point $\underline{x}$ along the Stokes curve and an s belonging to E_0, we can draw an integration path on the x-plane perpendicular to the Stokes curve from $\underline{x}$ to some point in the earlier Stokes region and hence analytically continue $R_j G$ from the earlier Stokes region to the point $\underline{x}$.

We will adopt the following convention for the sizes of the strips cut out from E_0, fig.15.

The strips $E_1, E_2, ...$ around singularities will be split into charts symbolically denoted as on figure 16, with sizes given by fig.13.

The same conventions will apply along the generalized Stokes curves L_3 and L_3'.

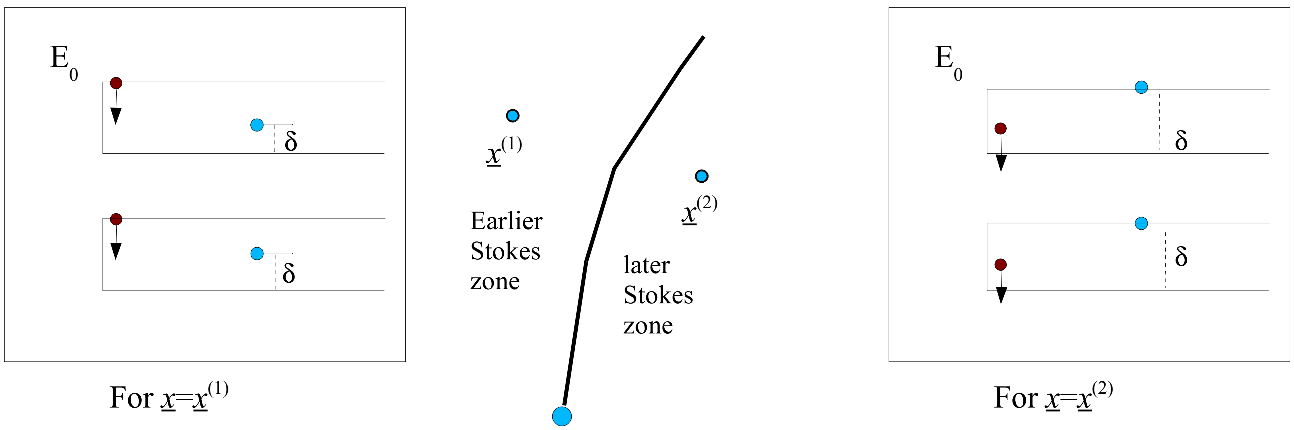

Figure 15: When $\underline{x}$ is in an "earlier" Stokes region, the gap goes from the Im of moving singularity to the distance δ on the opposite side of a stationary singularity. When we are in a "later" Stokes region, the gap goes from the stationary singularity distance δ in the direction of the moving singularity. If there is just a stationary singularity, then remove only the cut around this singularity. If there is just a moving singularity $s(x)$, cut out a strip of the same size as if $2s(x_t) - s(x)$ was a singularity, where x_t is the turning point – origin of the Stokes curve.

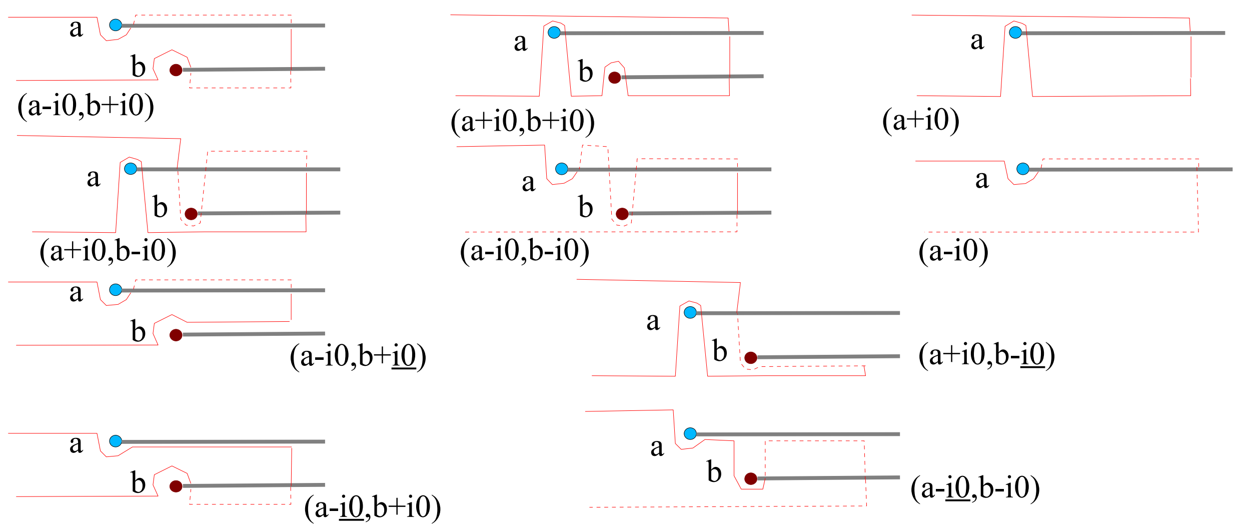

Figure 16: Manifold charts describing the Riemann surface $\mathcal{S}_{\underline{x}}$ near the place where two cuts cross. The first four pictures correspond to the situation when there are two singularities on the corresponding sheet, the two on the right – when there is only one singularity, the four on the bottom – special smaller charts which we will use when construction of a larger chart would involve crossing one of the lines $\tilde{L}_1',\tilde{L}_1'',\tilde{L}_2',\tilde{L}_2''$.

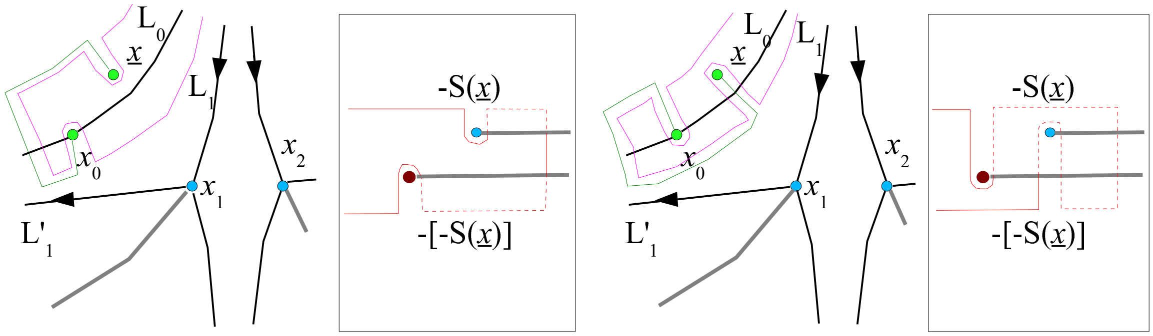

Figure 17

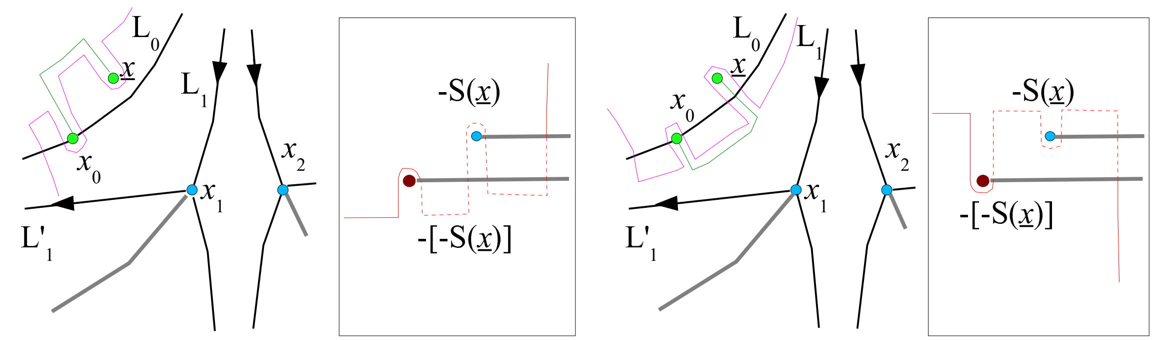

Figure 18

Now let us construct the integration paths for $\underline{x}$ along L_0 or for $\underline{x}$ in the region A and $(s, \underline{x}) \in \mathcal{S}'$. We will work with R_1G, the argument for R_2G being completely analogous.

If $\underline{x}$ is canonical distance $< \delta/2$ from the curve L_0, the corresponding part of $\mathcal{S}$ can be covered by open sets of four types (with specific sizes described earlier) shown on figures 17 and 18. These figures also show how to draw the integration path from x_0 to $\underline{x}$ avoiding the appropriate preimage in $\tilde{\mathcal{D}}$ of the given subsets of $\mathcal{S}_{\underline{x}}$. The figures show the case of Re $S(\underline{x}) \leq 0$, the case Re $S(\underline{x}) \geq 0$ can be treated analogously.

For $\underline{x}$ in the region A and $(s, \underline{x}) \in \mathcal{S}'$, cover $\mathcal{S}'$ by strips and draw the path of integration as shown on figure 19.

5.3 Continuation to the further Stokes regions – model cases.

The problem we are facing now is how to construct the integration path from $\underline{x}$ lying around Stokes curves or in one of the further Stokes regions to one of the earlier Stokes regions. We found that there is a finite number of typical arguments that allow us to do this, and we are presenting them in this subsection.

We want our constructions to be applicable to construction of both R_1G and R_2G, and so we will introduce here notation that is independent from that in the rest of the paper.

Let x_t be a turning point where the potential $V(x)$ has a simple zero. Three Stokes curves ℓ, ℓ', and ℓ'' defined by the condition Im $\int_{x_t}^{x} \sqrt{V(y)}dy = 0$ start from x_t. We also introduce a cut starting

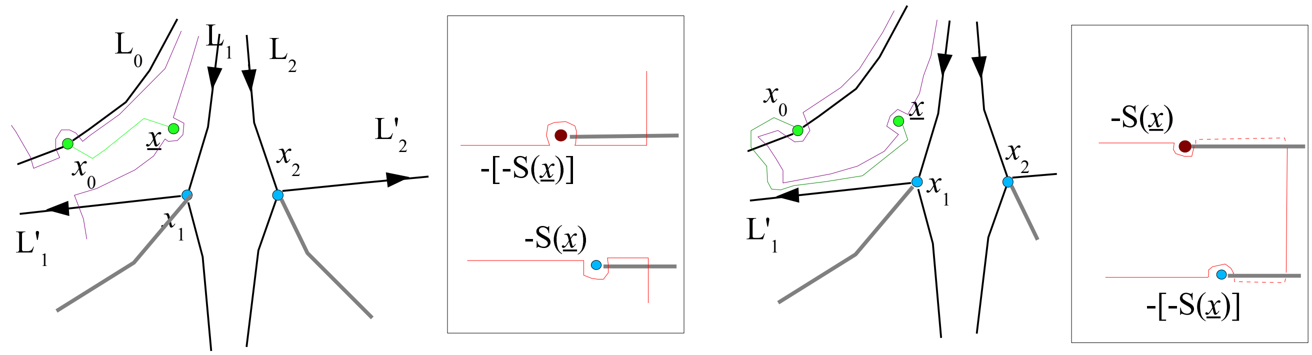

Figure 19

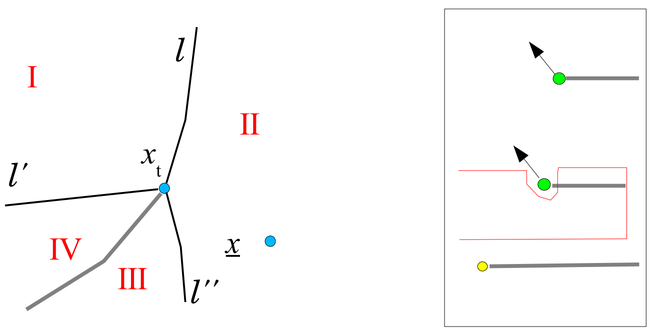

Figure 20

from x_t and making the function $S(x) = \int_{x_t}^{x} \sqrt{V(y)}dy$ univalued as shown on fig. 20. Thus we obtain four Stokes regions denoted I-IV (or sometimes slightly differently) in the vicinity of x_t. We will work with two groups of singularities – singularities located $S(x) + const$ and singularities located as $-S(x) + const$, respectively. We will not specify the index j in the definition of the operator R_j, but instead we will designate one group of singularities as "moving" and draw them green, and the other group as "stationary" and draw them yellow.

On the figure 20 the order of the Stokes regions I-II is clockwise with respect to the turning point; when we want to apply any of the lemmas of this subsection for the situation when this order should be counterclockwise, then the arguments that follow can be repeated with "top" and "bottom" reversed in the s-plane. Compare, e.g., the right and left sides of the figure 21.

We will work upon understanding that Stokes regions I-IV are contained in a domain similar to the domain $\tilde{\mathcal{D}}$ constructed above, but we will skip a detailed formulation here.

Construction of the integration path is trivial (i.e., this path is not subject to any conditions) in the following case:

Lemma 5.2 *Suppose s_0 is a stationary singularity, $E \subset \mathbb{C}$. We assume that the function G is constructed in a connected region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$ whose projection is contained in the regions I and II and the fiber over a point x is $s_0(x) + E$.*
AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.

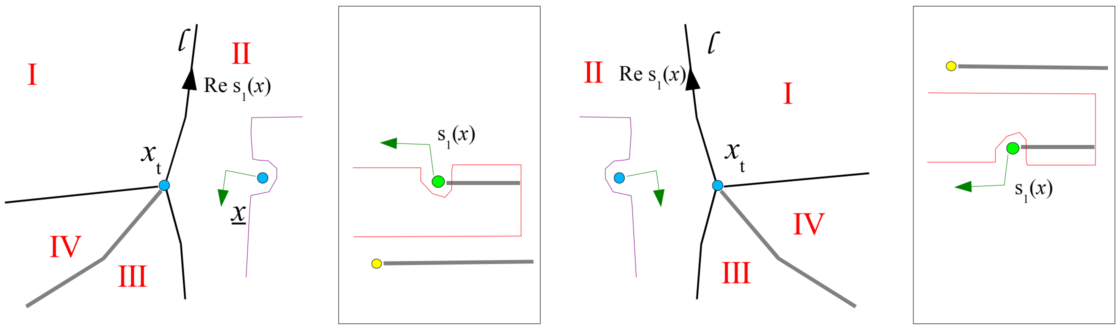

Figure 21

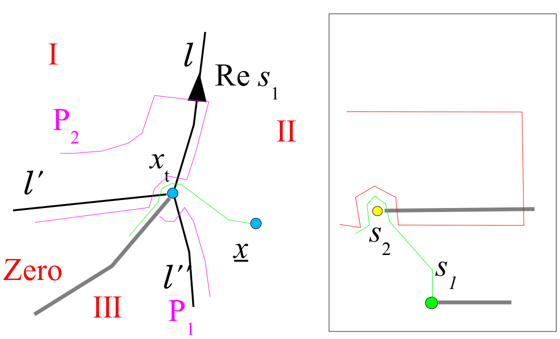

Figure 22**:** Lemma 5.3. Draw an integration path from $\underline{x}$ around x_t into the region Zero. The purple sets P_1, P_2 correspond to the condition that the integration path $y(t)$ lifted to $\mathcal{S}$ as $(s + S_j(\underline{x}) - S_j(y(t))$ hits the singularity $s_1(y(t))$ or the singularity $s_1(y(t))$ from the second sheet, respectively, for some value s in the the fiber $D_{\underline{x}}$ of D over $\underline{x}$.

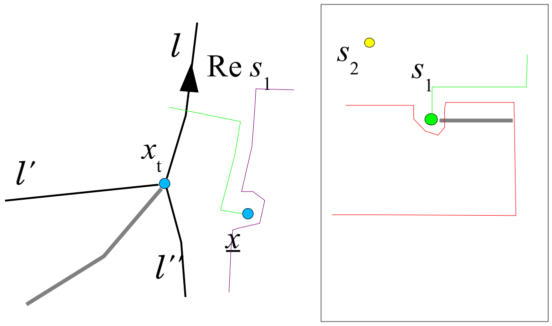

Figure 23: Lemma 5.4 – just move to the region I the most obvious way.

Lemma 5.3 *Suppose s_2 is a stationary and s_1 is a moving singularity and $s_2(x_t) = s_1(x_t)$, and* Re s_1 *grows along ℓ in the direction away from x_t. Let $A > \delta$, $B > 0$, $\text{Im } s_2(x_t) - s_1(x_t) > \delta$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all x such that $\text{Im } s_2(x) - s_1(x) < A$.*
- *for x in the region I, canonical distance $< \rho_0$ (for some $\rho_0 > 0$) from x_t;*
- *for x in the region Zero, canonical distance $< \delta/2$ from ℓ'*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by $\text{Re } s = N + \text{Re } s_2$ from the right;*
- *Upper boundary at $\text{Im } s_2 + B$.*
- *Lower boundary: (possibly with slots cut out around singularities)*
  - *at $\text{Im } s_2 - \delta$ a) when x is inside II distance $\geq \delta/2$ from ℓ, ℓ''; b) also in II distance $< \delta/2$ from ℓ if s_1 is not present in I.*
  - *in II distance $< \delta/2$ from ℓ when s_1 stays on the first sheet in I – according to fig.13;*
  - *in II distance $< \delta/2$ from ℓ'' and in the region Zero – at $\max\{\text{Im } s_2 - \delta, \text{Im } s_1\}$;*
  - *in I close to x_t – at $\text{Im } s_2 - \delta$;*

AND $R_j G$ is defined on all points of this chart that project to the region Zero,
THEN: $R_j G$ is also defined on the same set.

Proof follows from fig.22 □

Lemma 5.4 *Suppose s_2 is a stationary and s_1 is a moving singularity and $s_2(x_t) = s_1(x_t)$, and* Re s_1 *grows along ℓ in the direction away from x_t. Assume $A > \delta$, $B > 0$, $\text{Im } s_2(x_t) - s_1(x_t) > \delta$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all x such that $\text{Im } s_2(x) - s_1(x) < A$.*

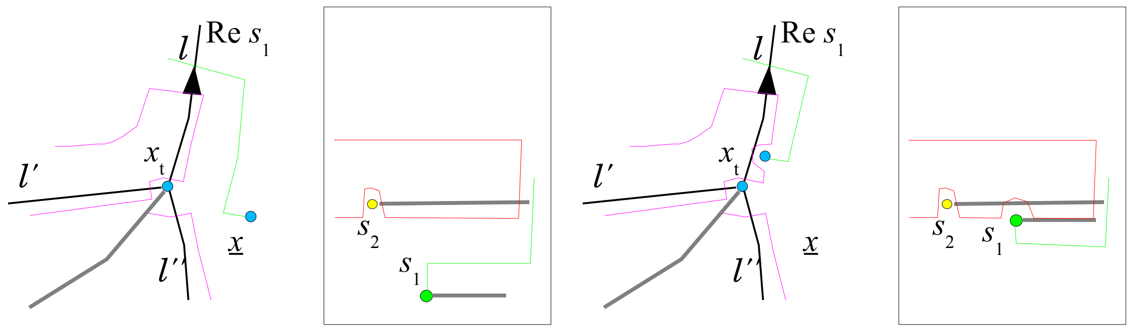

Figure 24**:** Lemma 5.5.

- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* Re $s = N +$ Re s_2 *from the right;*
- *Upper boundary: (possibly with slots cut out around singularities)*
  - *at* Im $s_1 + \delta$ *inside II distance $\geq \delta/2$ from ℓ, ℓ'';*
  - *in accordance with fig.13 distance $< \delta/2$ from ℓ, ℓ''.*
- *Lower boundary: at* Im $s_2 + B$.

AND $R_j G$ is defined on all points of this chart that project to the region I can.distance $> \delta/2$,
THEN: $R_j G$ is also defined on the same set.
The lemma also works when x_t is a generalized turning point.

Proof cf. fig. 23. □

Let s_3 be a moving singularity, s_2 are stationary; $s_2(x_t) = s_3(x_t)$.

Lemma 5.5 *Suppose s_2 is a stationary and s_1 is a moving singularity and $s_2(x_t) = s_1(x_t)$, and* Re s_1 *grows along ℓ in the direction away from x_t. Let $A > \delta$, $C > \delta$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all x such that* Im $s_2(x) - s_1(x) < C$.
- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ;*
- *for x in the region I, canonical distance $< \delta$ from the Stokes curve ℓ (i.e.* Im $s_1 - s_2 > -\delta$*) for which* Re $s_1 - s_2 > N$*;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* Re $s = N +$ Re s_2 *from the right;*

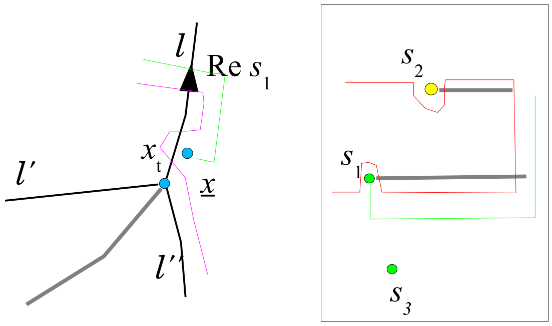

Figure 25**:** Lemma 5.6

- *Upper boundary at* $\mathrm{Im}\ s_2 + A$
- *Lower boundary: (possibly with slots cut out around singularities)*
  - *at* $\mathrm{Im}\ s_2 - \delta$ *a) when* x *is in* II *and* $\mathrm{Re}\ s_1 - s_2 \le N$*, or b) when* $\mathrm{Re}\ s_1 - s_2 \le N$*;*
  - *according to fig.13 when* x *is in* I*, distance* $\le \delta/2$ *from* ℓ *and* $\mathrm{Re}\ s_1 - s_2 \le N$*;*

AND R_jG *is defined on all points of this chart that project to the region* I*,*
THEN: R_jG *is also defined on the same set.*
The Lemma equally well works if x_t *is a generalized turning point.*

Proof: see fig. 24 □

Lemma 5.6 *Suppose* s_2 *is a stationary and* s_1 *is a moving singularity and* $s_2(x_t) = s_1(x_t)$*, and* $\mathrm{Re}\ s_1$ *grows along* ℓ *in the direction away from* x_t*. (Respectively, assume that there are also moving singularities* $s_3, ..., s_k$ *such that* $\mathrm{Im}\ s_1 - s_j > \delta$ *for all* $3 \le j \le k$*.) Let* $\delta < A$*. We assume that the function* G *is constructed in region* $D \subset \mathbb{C}_s \times \mathbb{C}_x$ *with the projection to* $\mathbb{C}_x$*:*

- *for* x *in the region II, for all* x *such that* $\delta < \mathrm{Im}\ s_2(x) - s_1(x) < A$*.*
- *for* x *in the regions I and II close to the Stokes curve* ℓ*, i.e.,* $|\mathrm{Im}\ s_2(x) - s_1(x)| < \delta$*, such that* $\mathrm{Re}\ s_1(x) - s_2(x) > N$ *(Respectively,* $\mathrm{Re}\ s_1 - s_2 > N$ *and* $\mathrm{Re}\ s_j - s_2 > N$ *for all* $3 \le j \le k$*)*

and the fiber over $x \in \mathbb{C}_x$ *as follows:*

- *unbounded in the* $-\infty$ *direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary: (with possibly a slot cut out around a singularity) at* $\mathrm{Im}\ s_2 + \delta$*;*
- *Lower boundary: (with possibly a slot cut out around a singularity)*
  - *for* $\mathrm{Im}\ s_2 - s_1 \ge \delta$ *and* $\mathrm{Re}\ s_1 - s_2 \le N$ *(respectively,* $\mathrm{Re}\ s_1 - s_2 \le N$ *or* $\mathrm{Re}\ s_j - s_2 \le N$ *for some* $3 \le j \le k$*) – at* $\min\{\mathrm{Im}\ s_1 - \delta, \mathrm{Im}\ s_2 - A\}$*;*
  - *for* $\mathrm{Re}\ s_1 - s_2 > N$ *(respectively,* $\mathrm{Re}\ s_1 - s_2 > N$ *and* $\mathrm{Re}\ s_j - s_2 > N$ *for all* $3 \le j \le k$*) – at* $\mathrm{Im}\ s_2 - A$*;*

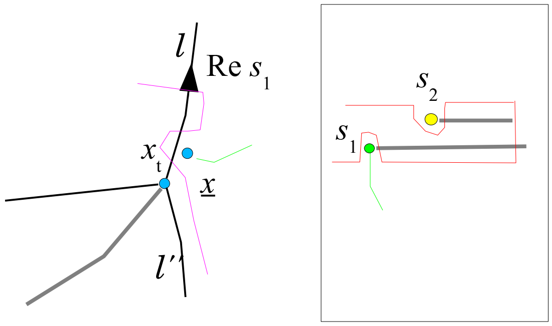

Figure 26: Lemma 26

– *for* Im $s_2 - s_1 < \delta$ – *at* Im $s_2 - A$*;*

AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.
This lemma also works if x_t is a generalized turning point.

PROOF: See Fig. 25. □

Lemma 5.7 *Suppose s_2 is a stationary and s_1 is a moving singularity and $s_2(x_t) = s_1(x_t)$, and* Re s_1 *grows along ℓ in the direction away from x_t. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all x such that* $\delta <$ Im $s_2(x) - s_1(x) < 2\delta$*;*
- *for x in the region I for* Im $s_1(x) - s_2(x) < \delta$*;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* Re $s =$ Re $s_2 + N$ *from the right;*
- *Upper boundary: (with possibly a slot cut out around a singularity) at* Im $s_2 + \delta$*;*
- *Lower boundary: (with possibly a slot cut out around a singularity) at* Im $s_1 - \delta$*.*

AND R_jG is defined on all points of this chart that project to the region II can. distance $\geq \delta/2$ from ℓ
THEN: R_jG is also defined on the same set.
This lemma also works if x_t is a generalized turning point.

PROOF: See Fig. 26. □

Lemma 5.8 *Suppose s_1, s_2 are moving and s_3 is a stationary singularity and $s_2(x_t) = s_3(x_t)$, and* Re s_1 *grows along ℓ in the direction away from x_t. Assume $B > A > \delta$, $B - A <$* Im $s_2 - s_1$*. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

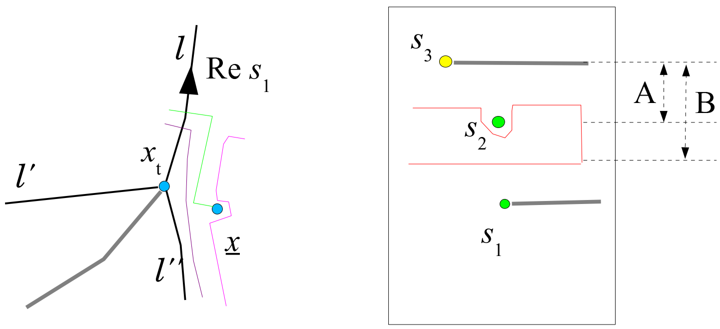

Figure 27: Lemma 5.8.

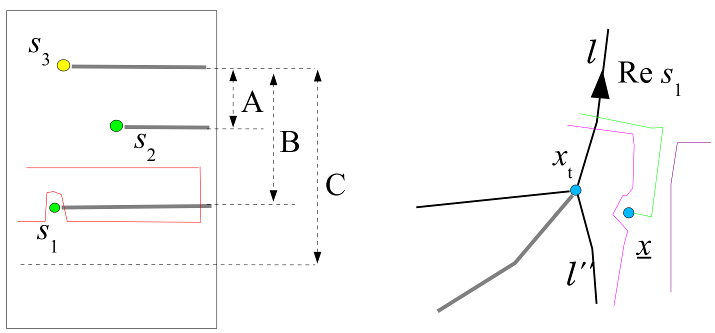

Figure 28: Lemma 5.9.

- *all x in the region II, for all $\underline{x}$ such that* $\mathrm{Im}\ s_3 - s_1 > B$ *and* $\mathrm{Im}\ s_3 - s_2 < A$*.*
- *also all x in the region II such that* $\mathrm{Im}\ s_3 - s_1 < B + \delta$ *and* $\mathrm{Re}\ s_2 - s_3 > N$
- *all x in the region I, canonical distance $< \delta/2$ and such that* $\mathrm{Re}\ s_2 - s_3 > N$

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_3 + N$ *from the right;*
- *Upper boundary at* $\mathrm{Im}\ s_3 - A + \delta$ *(possibly, with a slot cut out around a singularity)*
- *Lower boundary at* $\mathrm{Im}\ s_3 - B$

AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.
The lemma also applies when x_t is a generalized turning point.

PROOF: see fig. 27 □

Lemma 5.9 *Suppose s_1, s_2 are moving and s_3 is a stationary singularity and $s_2(x_t) = s_3(x_t)$, and* $\mathrm{Re}\ s_1$ *grows along ℓ in the direction away from x_t. Assume $C > B > A > 0$, $B - A <$* $\mathrm{Im}\ s_2 - s_1$*. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *all x in the region II, for all $\underline{x}$ such that* $\operatorname{Im} s_3 - s_1 < C$ *and* $\operatorname{Im} s_3 - s_2 > A$.
- *also all x in the region II such that* $\operatorname{Im} s_3 - s_1 < C$ *and* $\operatorname{Re} s_1 - s_3 > N$
- *all x in the region I, canonical distance $< \delta/2$ and such that* $\operatorname{Re} s_1 - s_3 > N$

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\operatorname{Re} s = \operatorname{Re} s_3 + N$ *from the right;*
- *Upper boundary at* $\operatorname{Im} s_3 - A$
- *Lower boundary at* $\max\{\operatorname{Im} s_3 - B - \delta, \operatorname{Im} s_3 - C\}$*(possibly, with a slot cut out around a singularity)*

AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.
The lemma also applies when x_t is a generalized turning point.

PROOF: see fig. 28 □

As before, assume $s_1(x_t) = s_2(x_t)$.

Lemma 5.10 *Suppose s_1 is a moving and s_3 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\operatorname{Re} s_1$ *decreases along ℓ in the direction away from x_t. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$ concentrated in II, in I near ℓ, in III near ℓ'' so that* $-\delta < \operatorname{Im} s_1(x) - s_2(x) < 2\delta$*, and the fiber over $x \in \mathbb{C}_x$ defined similar to fig.13, i.e.:*

- *unbounded in the $-\infty$ direction, bounded by* $\operatorname{Re} s = \operatorname{Re} s_2 + N$ *from the right;*
- *Upper boundary (with possibly a slot cut out around s_1) at* $\operatorname{Im} s_1 + \delta$*;*
- *Lower boundary: (with possibly a slot cut out around s_2) at* $\operatorname{Im} s_2 - \delta$*,*

AND R_jG is defined on all points of this chart that project to the subset of the region II with $\operatorname{Im} s_1 - s_2 > \delta$*,*
THEN: R_jG is also defined on the same set.

PROOF Draw the integration path by moving $\underline{x}$ further from the Stokes curve as shown on the figure fig.29. □

Lemma 5.11 *Suppose s_1 is a moving and s_2 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\operatorname{Re} s_1$ *decreases along ℓ in the direction away from x_t. Let $A > 0$, $\varepsilon > 0$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all $\underline{x}$ such that* $\operatorname{Im} s_1(x) - s_2(x) < A + \delta$.
- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

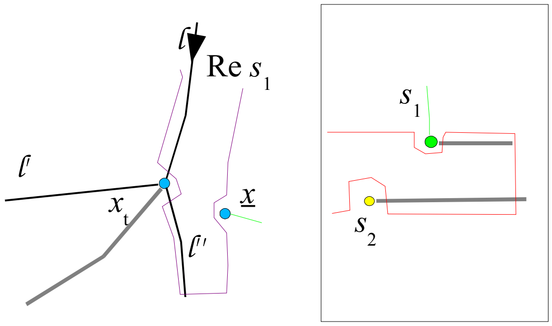

Figure 29**:** Lemma 5.10.

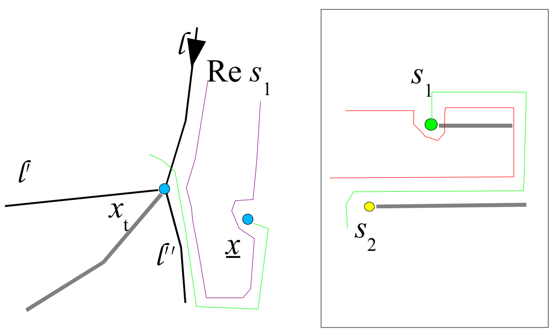

Figure 30**:** Lemma 5.11.

- *unbounded in the* $-\infty$ *direction, bounded by* $\text{Re } s = \text{Re } s_2 + N$ *from the right;*
- *Upper boundary: (with possibly a slot cut out around* s_1*)*
  - *at* $\text{Im } s_1 + \delta + \min\{A - \text{Im } (s_1 - s_2), 0\}$ *when* $\text{Re } s_1 - s_2 < N$ *and* x *is inside II distance* $> \delta$ *from* ℓ, ℓ''*;*
  - $\text{Im } s_2 + A + \delta$ *a) when* $\text{Re } s_1 - s_2 > N$*, or b) when* $\text{Im } s_1 - s_2 < \delta$ *(i.e. near* ℓ'' *in II or near* ℓ *in I or II);*
- *Lower boundary at* $\text{Im } s_2 + \varepsilon$*.*

AND $R_j G$ *is defined on all points of this chart that project to the region* I*,*
THEN: $R_j G$ *is also defined on the same set.*

Proof cf. fig. 30 □

Supplement to Lemma 5.11. *The same lemma also holds if we fix a* $\varepsilon' > 0$ *take the lower boundary to be* $\text{Im } s_2 - \varepsilon'$ *for* $\text{Re } s < \text{Re } s_2$ *and* $\text{Im } s_2 + \varepsilon$ *for* $\text{Re } s > \text{Re } s_2$*.*

Lemma 5.12 *Suppose* s_1 *is a moving and* s_2 *is a stationary singularity and* $s_1(x_t) = s_2(x_t)$*, and* $\text{Re } s_1$ *decreases along* ℓ *in the direction away from* x_t*. Suppose* $\delta < A$*. We assume that the function* G *is constructed in region* $D \subset \mathbb{C}_s \times \mathbb{C}_x$ *with the projection to* $\mathbb{C}_x$*:*

- *for* x *in the region II, for all* $\underline{x}$ *such that* $\text{Im } s_1(x) - s_2(x) < A$*.*

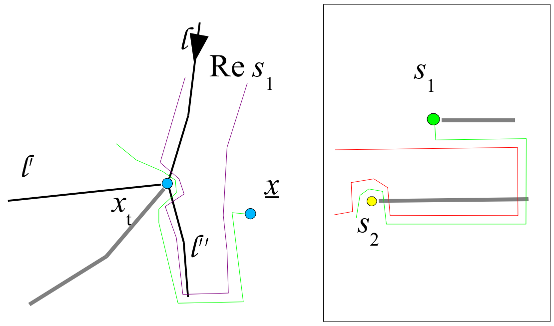

Figure 31: Lemma 5.12

- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ;*
- *for x in the region III, canonical distance $< \delta/2$ from the Stokes curve ℓ'';*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\mathrm{Re}\, s = \mathrm{Re}\, s_2 + N$ *from the right;*
- *Upper boundary:*
  - *at* $\mathrm{Im}\, s_1 - \delta$ *inside II when x is canonical distance $> \delta/2$ from ℓ, ℓ'' and* $\mathrm{Re}\, s_1 - s_2 < N$*;*
  - $\mathrm{Im}\, s_2 + A - \delta$ *a) when* $\mathrm{Re}\, s_1 - s_2 > N$, *or b) when* $\mathrm{Im}\, s_1 - s_2 < \delta$*;*
- *Lower boundary:*
  - $\mathrm{Im}\, s_2 - \delta$ *when* $\mathrm{Im}\, s_1 - s_2 > \delta$.
  - *according to fig.13 when x is in any of the regions I,II,III with canonical distance $\delta/2$ from ℓ, ℓ''.*

AND $R_j G$ is defined on all points of this chart that project to the region I,
THEN: $R_j G$ is also defined on the same set.

PROOF follows from fig.31. □

Lemma 5.13 *Suppose s_1 is a moving and s_2 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\mathrm{Re}\, s_1$ *decreases along ℓ in the direction away from x_t. Suppose $A \geq 0$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all $\underline{x}$ such that* $\mathrm{Im}\, s_1(x) - s_2(x) < A + \delta$.
- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ;*
- *for x in the region III, canonical distance $< \delta/2$ from the Stokes curve ℓ'';*

and the fiber over $x \in \mathbb{C}_x$ as follows:

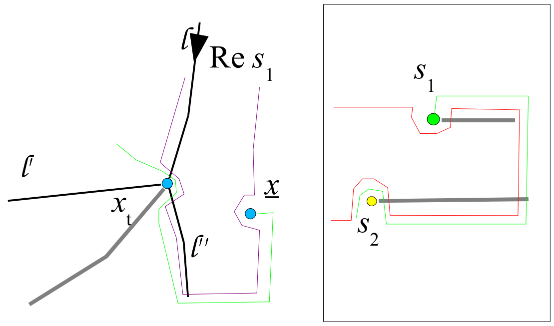

Figure 32**:** Lemma 5.13.

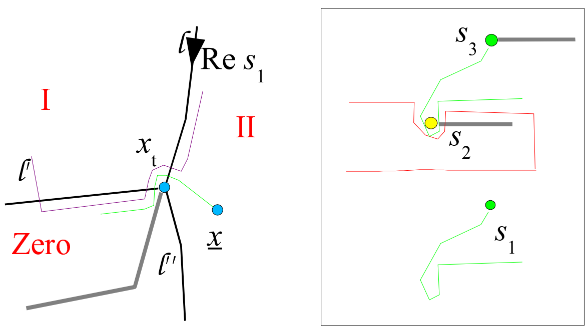

Figure 33**:** Lemma 5.14

- *unbounded in the* $-\infty$ *direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary: (with a possible slot for* s_1*)*
  - *at* $\mathrm{Im}\ s_1 + \min\{\delta, A - \mathrm{Im}\ (s_1 - s_2)\}$ *in region and* $\mathrm{Re}\ s_1 - s_2 < N$*;*
  - $\mathrm{Im}\ s_2 + A + \delta$ *a) when* $\mathrm{Re}\ s_1 - s_2 > N$*, or b) when* $\mathrm{Im}\ s_1 - s_2 < 0$ *(i.e. when* x *is in regions* I *or* III*);*
- *Lower boundary:*
  - $\mathrm{Im}\ s_2 - \delta$ *when* $\mathrm{Im}\ s_1 - s_2 > 0$.
  - *according to fig.13 when* x *is in any of the regions* I *or* III*.*

AND R_jG *is defined on all points of this chart that project to the region* I*,*
THEN: R_jG *is also defined on the same set.*

PROOF follows from fig.32. □

Lemma 5.14 *Suppose* s_1*,* s_3 *are moving and* s_2 *is a stationary singularity and* $s_3(x_t) = s_2(x_t)$*, and* $\mathrm{Re}\ s_1$ *decreases along* ℓ *in the direction away from* x_t*. Fix two constants* $A \in \mathbb{R}$*,* $C > 0$*. We assume that the function* G *is constructed in region* $D \subset \mathbb{C}_s \times \mathbb{C}_x$ *with the projection to* $\mathbb{C}_x$*:*

- *for* x *in the region II, for all* $\underline{x}$ *such that* $\mathrm{Im}\ s_3 - s_2 < A$*;*

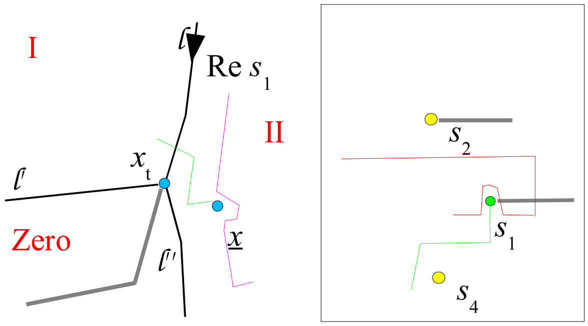

Figure 34: Lemma 5.15.

- *for x in the region I, canonical distance $< \rho_0$ from x_t (for any arbitrarily small $\rho_0 > 0$);*
- *for x in the region Zero – canonical distance $< \delta/2$ from the Stokes curve ℓ'.*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\text{Re } s = \text{Re } s_2 + N$ *from the right;*
- *Upper boundary consists of the following line segments, with slots cut out around singularities as explained earlier:*
  - *at* $\text{Im } s_2 + \delta$ *for x a) inside II distance $> \delta/2$ from ℓ, or inside II arbitrarily close to ℓ''; b) also in II canonical distance $< \delta/2$ from ℓ if there is no singularity s_3 in II confluent with s_2 at x_t; c) also inside I close to x_t;* [2]
  - *at* $\min\{\text{Im } s_2 + \delta, \text{Im } s_3\}$ *for* $\text{Re } s \to -\infty$, *and* $\text{Im } s_2 + \delta$ *for* $\text{Re } s > \text{Re } s_2$ *along ℓ'', when there is singularity s_3 in II confluent with s_2 at x_t;*
  - *for x in region Zero, at* $\text{Im } s_2 + \delta$ *for* $\text{Re } s < \text{Re } s_2$ *and* $\text{Im } s_3$ *for* $\text{Re } s > \text{Re } s_2$*;*
- *Lower boundary: at* $\text{Im } s_2 - C$*,* [3]

AND R_jG is defined on all points of this chart that project to the region Zero,
THEN: R_jG is also defined on the same set.

Proof. The integration path on figure 33 is the integration path from a point in region Zero to other values of $\underline{x}$ under consideration. □

Lemma 5.15 *Suppose s_1 is a moving and s_2, s_4 are stationary singularities and $s_1(x_t) = s_4(x_t)$, and* $\text{Re } s_1$ *decreases along ℓ in the direction away from x_t. Assume* $\text{Im }[s_2(x_t) - s_1(x_t)] - A > \delta$*. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all $\underline{x}$ such that* $\text{Im } s_2(x) - s_1(x) > A$*;*

[2]This means, intuitively, that there are no moving singularities between s_2 and s_3

[3]This means, in particular, that for any moving singularity s_1 below s_2 we have that $\text{Im } s_1 < \text{Im } s_2 - C$ in the for the values of x under consideration.

- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ, and such that* $\mathrm{Re}\ s_1(x) < \mathrm{Re}\ s_1(x_t)$*;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary at* $\mathrm{Im}\ s_4 + C$*;*
- *Lower boundary: (with slots cut out around singularities)*
  - *Inside II – at* $\mathrm{Im}\ s_1 - \delta$*;*
  - *Along ℓ (canonical distance $< \delta/2$ on both sides) – according to fig.13 if s_4 is present on the first sheet in I, or* $\mathrm{Im}\ s_1 - \delta$ *if s_4 is absent on the first sheet in I;*
  - *In the region II canonical distance $< \delta/2$ from ℓ'' (where* $\mathrm{Re}\ s_4 < \mathrm{Re}\ s_1$*: the lower boundary is at* $\mathrm{Im}\ s_4$ *for* $-\infty < \mathrm{Re}\ s < \mathrm{Re}\ s_4$ *and at* $\mathrm{Im}\ s_1 - \delta$ *for s in* $\mathrm{Re}\ s_4 < \mathrm{Re}\ s < N$*.*

AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.

PROOF See the integration path on figure 34. □

Lemma 5.16 *Suppose s_1 is a moving and s_2 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\mathrm{Re}\ s_1$ *decreases along ℓ in the direction away from x_t. Assume $C > \delta$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the regions I and II canonical distance $< \delta/2$ from ℓ, i.e. for all x such that* $\mathrm{Re}\ s_1 < \mathrm{Re}\ s_2$ *and* $|\mathrm{Im}\ s_2(x) - s_1(x)| < \delta$*;*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary at* $\mathrm{Im}\ s_2 + C$*;*
- *Lower boundary according to fig. 13*

AND R_jG is defined on all points of this chart that project to the region I,
THEN: R_jG is also defined on the same set.

PROOF see fig. 35. □

Lemma 5.17 *Suppose s_1 is a moving and s_2 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\mathrm{Re}\ s_1$ *decreases along ℓ in the direction away from x_t. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the regions I and II canonical distance $< \delta/2$ from ℓ, i.e. for all x such that* $\mathrm{Re}\ s_1 < \mathrm{Re}\ s_4$ *and* $|\mathrm{Im}\ s_4(x) - s_1(x)| < \delta$*;*

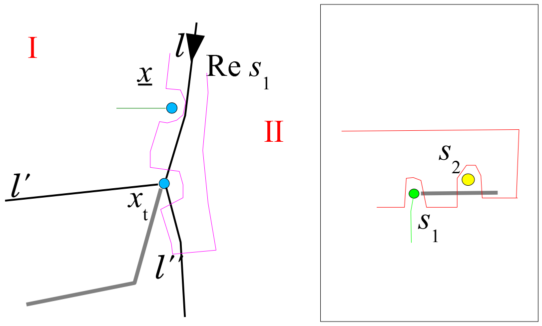

Figure 35**:** Lemma 5.16.

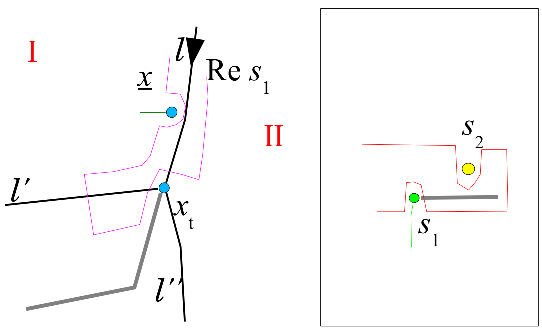

Figure 36**:** Lemma 5.17

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* Re $s = $ Re $s_2 + N$ *from the right;*
- *Upper boundary at* Im $s_4 + \delta$ *(with a slot cut out around the singularity);*
- *Lower boundary at* Im $s_1 - \delta$ *(with a slot cut out around the singularity)*

AND $R_j G$ is defined on all points of this chart that project to the region I,
THEN: $R_j G$ is also defined on the same set.

PROOF see fig. 36. □

Lemma 5.18 *Suppose s_1, s_2 are stationary and s_3 is a moving singularity and $s_3(x_t) = s_2(x_t)$, and* Re s_3 *decreases along ℓ in the direction away from x_t. Let $B > \delta$, $0 < A <$ Im $[s_2 - s_1] - \delta$ [4] We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the region II, for all $\underline{x}$ such that* Im $s_3(\underline{x}) - s_2(\underline{x}) < B$.
- *for x in the region I, canonical distance $< \rho_0$ (for some fixed $\rho_0 > 0$) from x_t;*

[4] I.e. if there is a moving singularity s_4 that appears from under the cut $[s_1, \infty)$ in the region IV, we leave a space for it between this cut and the red strip.

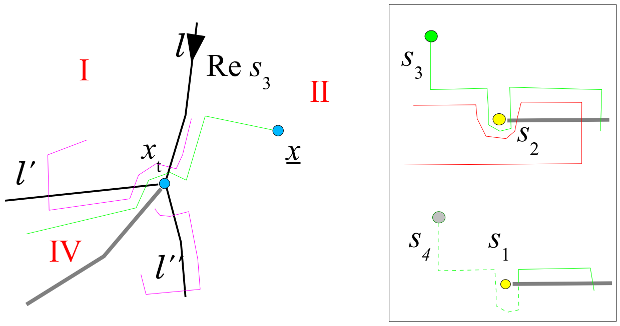

Figure 37**:** Lemma 5.18.

- *for x in the region I, canonical distance $< \delta/2$ from the Stokes curve ℓ and satisfying* $\mathrm{Re}\ s_3(x) - s_2(x) > N$*;*
- *for x in the region IV, canonical distance $< \delta/2$ from the Stokes curve ℓ';*

and the fiber over $x \in \mathbb{C}_x$ as follows:

- *unbounded in the $-\infty$ direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_1 + N$ *from the right;*
- *Upper boundary: (with slots cut out around singularities)*
  - *at* $\mathrm{Im}\ s_2 + \delta$ *a) inside II distance $> \delta/2$ from ℓ, ℓ''; b) in all x of our chart belonging to I.*
  - *for x in II along ℓ (i.e., canonical distance $< \delta/2$), and* $\mathrm{Re}\ s_3 < \mathrm{Re}\ s_2$ *the upper boundary is at* $\min\{\mathrm{Im}\ s_3, \mathrm{Im}\ s_2 + \delta\}$ *for* $\mathrm{Re}\ s < \mathrm{Re}\ s_2$ *and at* $\mathrm{Im}\ s_2 + \delta$ *for* $\mathrm{Re}\ s > \mathrm{Re}\ s_2$*;*
  - *according to fig. 13 a) in II along ℓ'', b) in IV along ℓ'.*
- *Lower boundary at* $\mathrm{Im}\ s_2 - A$*.*

AND R_jG is defined on all points of this chart that project to the region I outside of the ρ_0 neighborhood of x_t,
THEN: R_jG is also defined on the same set.

PROOF by fig.37 □

Lemma 5.19 *Suppose s_1 is a moving and s_2 is a stationary singularity and $s_1(x_t) = s_2(x_t)$, and* $\mathrm{Re}\ s_1$ *decreases along ℓ in the direction away from x_t. Let $B > \delta$. We assume that the function G is constructed in region $D \subset \mathbb{C}_s \times \mathbb{C}_x$ with the projection to $\mathbb{C}_x$:*

- *for x in the regions I and II along ℓ so that* $-\delta < \mathrm{Im}\ s_1(\underline{x}) - s_2(\underline{x})| < 2\delta$*.*
- *for x in the region I, canonical distance $< \delta$ from the Stokes curve ℓ' (i.e.* $\mathrm{Im}\ s_2 - s_1 < 2\delta$*) and satisfying* $\mathrm{Re}\ s_1(x) - s_2(x) > N$*;*
- *for x in the regions I and IV, canonical distance $< \delta/2$ from the Stokes curve ℓ';*

and the fiber over $x \in \mathbb{C}_x$ as follows:

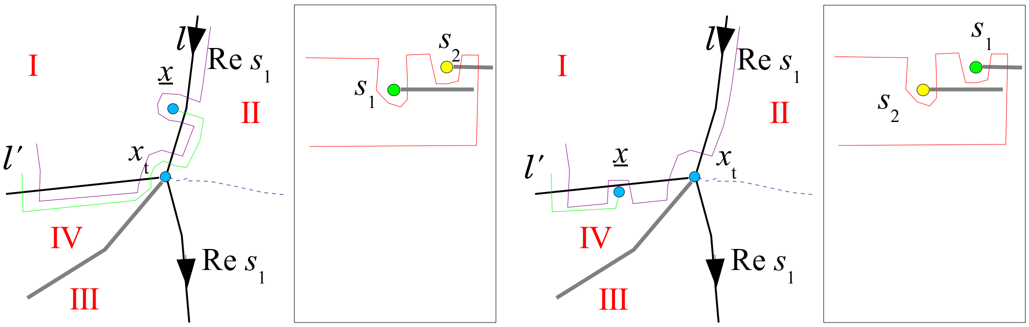

Figure 38: Lemma 5.19.

- *unbounded in the* $-\infty$ *direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary: (with slots cut out around singularities)*
  - *at* $\min\{\mathrm{Im}\ s_1 + \delta, \mathrm{Im}\ s_2 + 2\delta$ *for* $\mathrm{Re}\ s < \mathrm{Re}\ s_2$ *and at* $\min\{\mathrm{Im}\ s_1 + \delta, \mathrm{Im}\ s_2 + \delta\}$ *for* $\mathrm{Re}\ s > \mathrm{Re}\ s_2$*.*
  - *according to fig. 13 in I and IV can. distance* $< \delta/2$ *from* ℓ'*.*
  - *at* $\mathrm{Im}\ s_2 + \delta$ *for* $\mathrm{Im}\ s_1 - s_2 > N$*.*
- *Lower boundary at* $\mathrm{Im}\ s_2 - B$*.*

AND R_jG *is defined on all points of this chart that project to the region I satisfying* $\mathrm{Im}\ s_1 - s_2 > \delta$*, THEN:* R_jG *is also defined on the same set.*

PROOF: Draw the integration integration path as on fig. 38. □

Lemma 5.20 *Suppose* s_1 *is a moving and* s_2 *is a stationary singularity and* $s_1(x_t) = s_2(x_t)$*, and* $\mathrm{Re}\ s_1$ *decreases along* ℓ *in the direction away from* x_t*. Let* $B > \delta$*. We assume that the function* G *is constructed in region* $D \subset \mathbb{C}_s \times \mathbb{C}_x$ *with the projection to* $\mathbb{C}_x$*:*

- *for* x *in the regions I and II along* ℓ *so that* $-\delta < \mathrm{Im}\ s_1(\underline{x}) - s_2(\underline{x})| < 2\delta$*.*
- *for* x *in the regions I and IV, canonical distance* $< \delta/2$ *from the Stokes curve* ℓ'*;*

and the fiber over $x \in \mathbb{C}_x$ *as follows:*

- *unbounded in the* $-\infty$ *direction, bounded by* $\mathrm{Re}\ s = \mathrm{Re}\ s_2 + N$ *from the right;*
- *Upper boundary: (with slots cut out around singularities)*
  - *at* $\min\{\mathrm{Im}\ s_1 + \delta, \mathrm{Im}\ s_2 + 2\delta$ *for* $\mathrm{Re}\ s < \mathrm{Re}\ s_2$ *and at* $\min\{\mathrm{Im}\ s_1 + \delta, \mathrm{Im}\ s_2 + \delta\}$ *for* $\mathrm{Re}\ s > \mathrm{Re}\ s_2$*.*
  - *according to fig. 13 in I and IV canonical distance* $< \delta/2$ *from* ℓ'*.*
- *Lower boundary at* $\mathrm{Im}\ s_2 - B$*.*

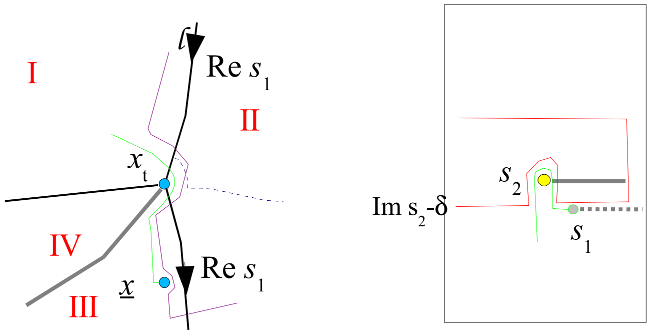

Figure 39**:** Lemma 5.21.

AND R_jG *is defined on all points of this chart that project to the region* IV*,*
THEN: R_jG *is also defined on the same set.*

Proof analogous to Lemma 5.19. □

Lemma 5.21 *Suppose* s_1 *is a moving and* s_2 *is a stationary singularity and* $s_1(x_t) = s_2(x_t)$*, and* Re s_1 *decreases along* ℓ *in the direction away from* x_t*. Let* $0 < \delta < C$*. We assume that the function* G *is constructed in region* $D \subset \mathbb{C}_s \times \mathbb{C}_x$ *with the projection to* $\mathbb{C}_x$*:*

- *for* x *in the region II, for all* x *such that* $\mathrm{Re}\ s_1 - s_2 > -\rho$, $\mathrm{Im}\ s_1 - s_2 < \delta$*;*
- *for* x *in the region I canonical distance* $< \delta/2$ *(i.e.* $\mathrm{Im}\ s_2 - s_1 < \delta$*) from the Stokes curve* ℓ*;*
- *for* x *in the region III canonical distance* $< \delta$ *(i.e.* $\mathrm{Im}\ s_2 - s_1 < 2\delta$*) from the Stokes curve* ℓ''*;*

and the fiber over $x \in \mathbb{C}_x$ *as follows:*

- *unbounded in the* $-\infty$ *direction, bounded by* $\mathrm{Re}\ s = N$ *from the right;*
- *Upper boundary at* $\mathrm{Im}\ s_2 + C$*;*
- *Lower boundary (with slots cut out around singularities):*
  - *for* $|\mathrm{Im}\ s_2 - s_1| < \delta$ *– according to fig.13;*
  - *for* $\delta \leq \mathrm{Im}\ s_2 - s_1 < 2\delta$ *– at* $\mathrm{Im}\ s_2 - \delta$

AND R_jG *is defined on all points of this chart that project to the region* I*,*
THEN: R_jG *is also defined on the same set.*

Proof is shown on fig. 39. □

5.4 Applications of lemmas 5.2-5.21 ...

We will now consider case by case situations when $\underline{x}$ is in each of the Stokes regions (except for region A which has already been considered) and along each of the internal Stokes curves or generalizes Stokes curves, and s is in one of the subsets of $\mathcal{S}$ described above, and point out which lemma needs to be applied in order to construct an integration path leading from $\underline{x}$ to an earlier Stokes regions. We will discuss cases of R_1G and R_2G separately.

5.4.1 ... towards constructing R_1G.

For $\underline{x}$ along the curve L_1:

- chart $(S+i0)$ – Lemma 5.15.
- chart $(S-i0,-s_1+i0)$ – Lemma 5.10; this uses the existence of R_1G in region B, strip between $S(\underline{x})$ and $-s_1(\underline{x})$, which will be shown in a minute.
- chart $(S-i0,-s_1-i0)$ – Lemma 5.19.

For $\underline{x}$ in the region B:

- above $S(x)$ – Lemma 5.15.
- between $S(\underline{x})$ and $-s_1(\underline{x})$ – Lemma 5.13, with $A=2\text{Im}\,[S(x_2)-S(x_1)]$. This lemma applies for all $\underline{x}$ in B except those for which Im $S(x_2)-S(\underline{x})<\delta$, and this case will be studied in the proof for $\underline{x}$ along L_2 and L_2'.
- between $-s_1$ and $-S$: from the flap along $(-s_1,\infty)$ to Im $[-S]+\delta$ – Lemma 5.18; from Im $[-s_1]-\delta$ to the flap along $(-s_1,\infty)$ – Lemma 5.2. [5]
- below $-S$ – Lemma 5.2, with a remark analogous to footnote [5] .

For $\underline{x}$ around the curve L_2:

- chart $(S+i0)$ – Lemma 5.15
- chart $(S-i0,-s_2+i0)$ – Lemma 5.10; application of this lemma uses the existence of R_1G in region B, in the strip between S and $-s_2$, which will be shown below.
- chart $(S-i0,-s_2-i0)$ – Lemma 5.19.

For $\underline{x}$ in the region C:

- above $S(x)$ – Lemma 5.15.
- between S and $-s_2$ – Lemma 5.13
- between $-s_2$ and $-s_1$: from the flap along $(-s_2,\infty)$ to Im $[-s_1]+\delta$ – Lemma 5.18. From Im $[-s_2]-\delta$ to the flap along $(-s_1,\infty)$ – Lemma 5.2, with a remark analogous to footnote [5] .
- between $-s_1$ and $-S$, below $-S$ – Lemma 5.2, with a remark analogous to footnote [5] .

For $\underline{x}$ along the curve L_2':

[5] To be more precise, Lemma 5.2 should be first applied to to the subset of region B where the flap along $(-s_1,\infty)$ is of size δ. Then, for every $0<\delta'<\delta$ consider the subset of point $\underline{x}$ in B distance $\geq\delta'$ from the Stokes curve L_2 and apply the same Lemma 5.2 with δ' instead of δ to construct the analytic continuation to the flap of this size for such $\underline{x}$.

- chart $(-s_2 + i0, S - i0)$ – Lemma 5.10;
- chart $(-s_2 + i0, S + i0)$ – Lemma 5.21;
- charts $(-s_1 + i0)$, $(-S + i0)$ – Lemma 5.2;
- charts $(-s_2 - i0, S + \underline{i0})$, $(-s_1 - i0, -s_{12'} + \underline{i0})$, $(-S - i0, -s_{2'} + \underline{i0})$ – Lemma 5.7 (which uses a construction in Zone D to be discussed below)
- charts $(-s_2 - i0, S - i0)$, $(-s_1 - i0, -s_{12'} - i0)$, $(-S - i0, -s_{2'} - i0)$ – proof follows from Lemma 5.4 and can be done together with the case of region D, strips between S and $-s_1$, between $-s_{12'}$ and $-s_{2'}$, and under $-s_2$, respectively.

For $\underline{x}$ in the region D:

- above $-s_2$ – by Lemma 5.3.
- between $-s_2$ and S, between $-s_1$ and $-s_{12'}$, between $-S$ and $-s_{2'}$ – an obvious modification of the Lemma 5.6.
- between S and $-s_1$, between $-s_{12'}$ and $-S$, under $-s_{2'}$ – are done by an obvious modification of Lemma 5.4, where the lower boundary can be made to contain a stationary singularity or go all the way down to $-i\infty$.

For $\underline{x}$ along the curve L_3:

- chart $(S + \underline{i0}, -s_1 - i0)$ – an obvious modification of Lemma 5.7
- chart $(S + \underline{i0}, -s_1 + i0)$ – the situation of $\underline{x}$ in region D reduces to that of $\underline{x}$ in region D′ by Lemma 5.16; for $\underline{x}$ in the region D′ apply Lemma 5.4;
- chart $(S - i0, -s_1 + i0)$ – use Lemma 5.17 to cross L_3 into the region D;
- chart $(S - i0, -s_1 - i0)$ – use Lemma 5.16 to cross L_3 into the region D.

For $\underline{x}$ along the curve L_3':

- chart $(-s_1 - i0, S + \underline{i0})$ – an obvious modification of Lemma 5.7;
- chart $(-s_1 + i0, S + \underline{i0})$ – an obvious modification of Lemma 5.5
- chart $(-s_1 - i0, S - i0)$ – Lemma 5.4
- chart $(-s_1 + i0, S - i0)$ – draw a piece of the integration path starting at $\underline{x}$ with Re $S(x) = const$ and Im $S(x)$ increasing, until reaching region D.

For $\underline{x}$ in the region D′:

- between $-s_1$ and S – Lemma 5.6;

- between S and $-s_{12'}$ – covering the region D$'$ by sufficiently small sets, can apply Lemmas 5.8 and 5.9 on each of them.

The rest of the first sheet is constructed analogously to the case of region D.

For $\underline{x}$ along the curve L_2'':

- charts $(S-i0)$, $(-s_1+i0, -s_{12'}-\underline{i0})$, $(-S+i0, -s_2-\underline{i0})$ – obvious modifications of Lemma 5.7;
- charts $(S+i0)$, $(-s_1+i0, -s_{12'}+i0)$, $(-S+i0, -s_2+i0)$ – Lemma 5.4;
- charts $(-s_{12'}-i0)$, $(-s_{2'}-i0)$ – Lemma 5.2

For $\underline{x}$ in the region E:

- above S – Lemma 5.4;
- between S and $-s_{12''}$ – The situaton of x canonical distance $< \delta/2$ from L_2'' has been treated above; the rest of the region F can be covered by projections of sets G as in Lemmas 5.8 and 5.9.
- between $-s_{12''}$ and $-s_1$ – Lemma 5.6.
- between $-s_1$ and $-s_{2''}$ – an obvious modification of Lemma 5.4, with a stationary singularity on the lower boundary of G_x.
- between $-s_{2''}$ and $-S$ – Lemma 5.6;
- below $-S$ – Lemma 5.2.

For $\underline{x}$ along the curve L_1'':

- chart $(-s_1+i0, S-i0)$ – Lemma 5.10;
- chart $(-S+i0)$ – Lemma 5.2;
- chart $(-s_1+i0, S+i0)$ – Lemma 5.21;
- charts $(-s_1-i0, S+\underline{i0})$, $(-S-i0, -s_{1''}+\underline{i0})$ – obvious modification of Lemma 5.7 using construction performed in the region F (see below);
- charts $(-s_1-i0, S-i0)$, $(-S-i0, -s_{1''}-i0)$ – Lemma 5.4;

For $\underline{x}$ in the region F:

- above $-s_1$ – Lemma 5.3
- between $-s_1$ and S, between $-S$ and $-s_{1''}$ – use an obvious modification of Lemma 5.6.

- between S and $-S$: without loss of generality, assume Im $S(\underline{x}) - [-S(\underline{x})] = 2\text{Im } S(\underline{x}) > 2\delta$ (otherwise reduce the situation to this one by drawing a piece of the integration path such that Im S increases and Re S stays constant along it). Now for the part of the strip from the flap along (S, ∞) to Im $(-S(\underline{x})) + \delta$ use Lemma 5.4, and for the part of the strip from Im $S(\underline{x})$ to Im $-S(\underline{x}) - \delta$ use Lemma 5.5.
- below $-s_{1''}$ – Lemma 5.4.

For $\underline{x}$ along the curve L_1'

- chart $(-S + i0, -s_{1'} - \underline{i0})$, resp., $(S - i0)$ – Lemma 5.7, resp., its obvious modification (uses construction in the region G that will be formulated later);
- chart $(-S + i0, -s_{1'} + i0)$, resp., chart $(S + i0)$ – Lemma 5.4, resp., its obvious modification
- chart $(-S - i0)$ – Lemma 5.2;

For $\underline{x}$ in the region G:

- above S – use an obvious modification of Lemma 5.4 for a red strip extending to $-i\infty$;
- between $-S$ and $-s_{1'}$ – covering the region G by appropriately small enough subsets, we can apply on each of them Lemmas 5.8 and 5.9;
- between $-s_{1'}$ and $-S$ – obvious modification of Lemma 5.6 for a red strip without flap;
- under $-S$ – Lemma 5.2, with a remark analogous to footnote [5] .

5.4.2 ... towards constructing R_2G.

For $\underline{x}$ along the curve L_1:

- chart $(S + i0)$ – Lemma 5.2;
- chart $(S - i0, -s_1 + i0)$, resp., $(-S + i0)$ – Lemma 5.7, resp., its obvious modification, using a construction in the region B (see below);
- chart $(S - i0, -s_1 - i0)$, resp., $(-S - i0)$ – Lemma 5.4, resp., its obvious modification where the stationary singularity is not visible on the first sheet.

For $\underline{x}$ in the region B:

- above S – Lemma 5.2
- between S and $-s_1$ – Lemma 5.6. (Note that for $\underline{x}$ canonical distance $< \delta/2$ from L_2 or L_2'', the analytic continuation to a flap of a correct size along the cut $(-s_1, \infty)$ will be obtained below)

- between $-s_1$ and $-S$ – covering the region B by appropriately small enough subsets, we can apply on each of them Lemmas 5.8 and 5.9. (Note that for $\underline{x}$ canonical distance $< \delta/2$ from L_2 or L_2'', the analytic continuation to a flap of a correct size along the cut $(-S, \infty)$ will be obtained below)
- below $-S$ – a modification of Lemma 5.4 where the red strip extends to $-i\infty$.

For $\underline{x}$ along the curve L_2:

- chart $(S + i0)$ – Lemma 5.2;
- chart $(S-i0, -s_2+i0)$, resp. $(-s_1+i0)$ and $(-S+i0)$ – Lemma 5.7, resp., its obvious modification (using a construction in the region C mentioned below);
- chart $(S-i0, -s_2-i0)$, resp. $(-s_1-i0)$ and $(-S-i0)$ – Lemma 5.4, resp., its obvious modification where the stationary singularity is not visible on the first sheet.

For $\underline{x}$ in the region C:

- above S – Lemma 5.5;
- between S and $-s_2$ – Lemma 5.6 with a remark similar to footnote [5] to account for the smaller flap sizes when Im $S(\underline{x})$ approaches A';
- between $-s_2$ and $-s_1$ – covering the region C by appropriately small enough subsets, we can apply on each of them Lemmas 5.8 and 5.9;
- between $-s_1$ adn $-S$ – a similar application of Lemmas 5.8 and 5.9, except that a singularity with which $-s_1$ should be confluent at x_2 is not visible on the first sheet;
- below $-S$ – an easy modification of Lemma 5.4 due to the fact that the singularity with which $-S$ would be confluent at x_2 is not visible on the first sheet.

For $\underline{x}$ along the curve L_2':

- chart $(-s_2 + i0, S - i0)$ – Lemma 5.17;
- chart $(-s_2 + i0, S + i0)$ – Lemma 5.16;
- charts $(-s_1 + i0)$, $(-S + i0)$ – use an obvious modification of Lemma 5.17 where the stationary singularity is not present on the same sheet as the region G.
- charts $(-s_2 - i0, S + \underline{i0})$, $(-s_1 - i0, -s_{12'} + \underline{i0})$, $(-S - i0, -s_{2'} + \underline{i0})$ – Lemma 5.10, using the construction in the region D to be performed below.
- charts $(-s_2 - i0, S + i0)$, $(-s_1 - i0, -s_{12'} + i0)$, $(-S - i0, -s_{2'} + i0)$ – Lemma 5.20.

For $\underline{x}$ in the region D:

- above $-s_2$ – use an obvious modification of Lemma 5.15 where the red region extends all the way up to $+i\infty$.
- between $-s_2$ and S, between $-s_1$ and $-s_{12}$, between $-S$ and $-s_{2'}$ (in both cases for $\underline{x}$ canonical distance $> \delta/2$ from L_2') – Lemma 5.13.
- Strip between $S(\underline{x})$ and $-s_1(\underline{x})$ – lemmas 5.14 and 5.15. (Note that for $\underline{x}$ canonical distance $< \delta/2$ from L_3 or L_3'', the analytic continuation to the flap of a correct sizewill be obtained below)
- Strip between $-s_{12'}(\underline{x})$ and $-S(\underline{x})$ – Lemmas 5.14 and 5.15 for the subset where Im $-s_{12'}+S \geq \delta$; when $0 <$ Im $-s_{12'} + S < \delta$ we can proceed as in the situation of two decoupled singularities, one diving under the cut starting at the other.
- under $-s_{2'}$ – Lemma 5.14.

For $\underline{x}$ along L_3 and L_3'. – The argument almost repeats that for R_1, once we interchange L_3 and L_3', reverse the roles of $-s_1$ and S, and reflect the charts in the s-plane with respect to a horizontal axis. The only asymmetry of the situation to keep in mind comes from the fact that we do not construct one of the flaps of the Riemann surface along the cut $[S(\underline{x}), +\infty)$.

For $\underline{x}$ in the region D$'$:

- between $-s_2$ and $-s_1$ – cover D$'$ by small enough subsets and apply Lemmas 5.8 and 5.9 on each of those subsets to cross the curve L_3 into the region D
- between $-s_1$ and S – an obvious modification of Lemma 5.6.
- between S and $-s_{12'}$ – from Im $S+\delta$ to Im $-s_{12'}$ – Lemma 5.5; from Im $S-\delta$ to Im $-s_{12'}$ – Lemma 5.2.

The construction of an analytic continuation to the rest of the first sheet can be done simultaneously for the regions D and D$'$; the arguments given for the region D apply.

For $\underline{x}$ along the curve L_2'':

- charts $(-s_1 + i0, -s_{12'} - \underline{i0})$, $(-S + i0, -s_2 - \underline{i0})$ – supplement to Lemma 5.11;
- charts $(-s_1 + i0, -s_{12'} + i0)$, $(-S + i0, -s_2 + i0)$ – Lemma 5.19;
- charts $(-s_1 - i0)$, $(-S - i0)$ – an obvious modification of Lemma 5.16 or 5.17.

For $\underline{x}$ in the region E:

- above $S(x)$ – Lemma 5.2;
- between S and $-s_{12''}$: from the flap along (S, ∞) down to Im $[-s_{12''}] + \delta$ – Lemma 5.2; from Im $S - \delta$ down to flap along $(-s_{12''}, \infty)$ – Lemma 5.18.
- between $-s_{12''}$ and $-s_1$ – Lemma 5.11.

- between $-s_1$ and $-s_{2''}$: without loss of generality, assume Im $-s_1(\underline{x})-[-s_{2''}(\underline{x})] > 2\delta$ (otherwise reduce the situation to this one by drawing a piece of the integration path such that Im S decreases and Re S stays constant along it). Now for the part of the strip from the flap along $(-s_1, \infty)$ to Im $(-s_{2''}(\underline{x})) + \delta$ use Lemma 5.15, and for the part of the strip from Im $-s_1(\underline{x})$ to the flap along $(-s_{2''}(\underline{x}), \infty)$ use Lemma 5.18.
- between $-s_{2''}$ and $-S$ – Lemma 5.11 and its supplement with $\varepsilon' = \min\{\delta, \text{Im } -s_1(\underline{x}) + s_2(\underline{x})\}$
- under use an obvious modification of Lemma 5.15 where the red region extends all the way up to $+i\infty$.

For $\underline{x}$ along the curve L_1''

- chart $(-s_1 + i0, S - i0)$ – Lemma 5.17;
- chart $(-s_1 + i0, S + i0)$ – Lemma 5.16;
- charts $(-s_1 - i0, S + i0)$, $(-S - i0, -s_{1'} + i0)$ – Lemma 5.20
- chart $(-s_1 - i0, S - i0)$, $(-S - i0, -s_{1'} - i0)$ – Lemma 5.17
- chart $(-S + i0)$ – use an obvious modification of Lemma 5.17 where the stationary singularity is not present on the same sheet as the region G.

For $\underline{x}$ in the region F:

- above $-s_1$ – obvious modification of Lemma 5.15 where the red region extends all the way up to $+i\infty$;
- between $-s_1(x)$ and $-S(x)$, between $-S(\underline{x})$ and $-s_{1'}(\underline{x})$ – Lemma 5.11.
- between S and $-S$: without loss of generality, assume Im $S(\underline{x}) - [-S(\underline{x})] = 2\text{Im } S(\underline{x}) > 2\delta$ (otherwise reduce the situation to this one by drawing a piece of the integration path such that Im S increases and Re S stays constant along it). Now for the part of the strip from the flap along (S, ∞) to Im $(-S(\underline{x})) + \delta$ use Lemma 5.14, and for the part of the strip from Im $S(\underline{x})$ to Im $-S(\underline{x}) - \delta$ use Lemma 5.15.
- below $-s_{1''}$ – Lemma 5.3.

For $\underline{x}$ along the curve L_1':

- chart $(-S - i0)$ – Lemma 5.15;
- chart $(-S + i0, -s_{1'} - \underline{i0})$ – an obvious modification of the supplement to Lemma 5.11;
- chart $(-S + i0, -s_{1'} + i0)$ – Lemma 5.19

For $\underline{x}$ in the region G:

- above S – Lemma 5.2, with a remark similar to footnote [5] ;
- between S and $-s_{1'}$: from the flap along (S, ∞) to Im $[-s_{1'}] + \delta$ use Lemma 5.2, with a remark analogous to footnote [5] , from Im $S - \delta$ to the flap along $(-s_{1'}, \infty)$ – Lemma 5.18.
- between $-s_{1'}$ and $-S$ – Lemma 5.11.
- below $-S$ – an obvous modification of Lemma 5.15 where the red strip is infinite in the vertical direction.

This finishes the proof of the theorem. □

6 Concluding remarks

It is immediate to see from the decription of $\mathcal{S}$ that once $\underline{x}$ goes one loop around the turning points x_1 or x_2, the locations of the singularities of the fiber $\mathcal{S}_{\underline{x}}$ remains the same up to a permutation, except for one singularity in each case: namely, the singularity $S(\underline{x})$ is present in the zones D and F, but the corresponding singularity is absent in the zones E and G.

Assuming that the series (8) converges and $\Phi(s, x) := \tilde{Y} f(s)$ indeed gives a solution to the equation (6), we can use the observation of [V83], page 243 and on, that the Laplace integral of $\Phi(s, x)$ gives a solution of (5) which is unramified at x_1 and x_2, and hence show that $\Phi(s, x)$ has only a removable singularity at $S(\underline{x})$ for $\underline{x}$ in the zones D and E, and also show the relations between other singularities in the zones D and E, F and G that would amount to asymptotic connection formulas. Also notice that we do not need to introduce a cut in $\tilde{\mathcal{D}}$ starting at a generalized turning point x_3, for the following reason. When we construct an integration path that leads from the region D to the region D′, we can circle around x_3 in either direction. Since x_3, unlike x_1 and x_2, is not a ramification point of $S(x)$, the result of the analytic continuation of $R_j F$ will not depend on the way we went around x_3.

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References

[CNP] B.Candelpergher, J.-C. Nosmas, F.Pham. *Approche de la résurgence.* Actualits Mathèmatiques. Hermann, Paris, 1993.

[DDP97] Delabaere, Dillinger, Pham. *Exact semiclassical expansions for one-dimensional quantum oscillators.* J.Math.Phys, 38 (1997)

[DP99] Delabaere, Pham. *Resurgent methods in semi-classical asymptotics.* Ann. Inst. Poincaré Phys. Théor. 77 (1999)

[E] J.Écalle, *Les fonctions résurgentes.* Publications Math. d'Orsay, preprint, 1981

[G] A.Getmanenko, *Resurgent analysis of the Witten Laplacian in one dimension.* `arXiv:0809.0441v2`.

[ShSt] V.E.Shatalov, B.Yu.Sternin, *Borel-Laplace transform and asymptotic theory. Introduction to resurgent analysis.* CRC Press, Boca Raton, FL, 1996.

[V83] A.Voros, *Return of the quatric oscillator. The complex WKB method.* Ann. Inst. H.Poincaré Phys. Théor. 39 (1983)

[Wh] H.Whitney, *Elementary Structure of Real Algebraic Varieties*, Annals of Math., 2nd Ser., Vol.66, No.3 (1957)