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infinite lattices

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PERIODIC AND HOMOCLINIC TRAVELLING WAVES IN INFINITE LATTICES

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ABSTRACT. Consider an infinite lattice of particles in one dimension subjected to a potential f and such that each site interacts (only) with its nearest neighbours under an interaction potential V . The dynamics of the system is described by the infinite system of second order differential equations

$$\ddot{q}_j + f'(q_j) = V'(q_{j+1} - q_j) - V'(q_j - q_{j-1}), \quad j \in \mathbb{Z}.$$

We investigate the existence of travelling wave solutions. Two kinds of such solutions are studied: periodic and homoclinic ones. On the one hand, we prove under some growth conditions on f and V , the existence of non-constant periodic solutions of any given period $\tau > 0$, and any given speed $c > c_0$. On the other hand, under very similar conditions, we establish the existence of non-trivial homoclinic solutions, of any given speed $c > c_0$, emanating from the origin. These homoclinics are obtained as limits of periodic solutions by letting the period go to infinity.

1. INTRODUCTION

We consider an infinite lattice of particles in one dimension subjected to a potential f . In addition, each particle interacts with its nearest neighbours under a (non-linear) interaction potential V . The dynamics of the system is described by the infinite system of second order ordinary differential equations:

$$(1.1) \quad \ddot{q}_j(t) + f'(q_j) = V'(q_{j+1}(t) - q_j(t)) - V'(q_j(t) - q_{j-1}(t)), \quad t \in \mathbb{R}, j \in \mathbb{Z},$$

where $f, V \in C^1(\mathbb{R})$.

For $f \equiv 0$, (1.1) is the usual lattice equation, sometimes called the Fermi-Pasta-Ulam (FPU) lattice. For $f(x) = K(1 - \cos x)$, with $K > 0$, (1.1) is sometimes called the discrete sine-Gordon (DSG) equation, even if V is not harmonic, i.e. $V(x) \neq \frac{\alpha}{2}x^2$.

As it is well known, (1.1) is an infinite-dimensional Hamiltonian system whose Hamiltonian is ‘formally’ given by

$$H = \sum_{j \in \mathbb{Z}} \left[\frac{1}{2} p_j^2 + f(q_j) + V(q_{j+1} - q_j) \right].$$

Under the summation, the first term represent the kinetic energy of the j -th particle and the remaining terms (containing q_j) represent its ‘potential energy’.

The aim of this paper is to investigate the existence of periodic and homoclinic travelling waves. A travelling wave, say with speed $c > 0$ and profile u , is a solution of (1.1) of the form

$$(1.2) \quad q_j(t) = u(j - ct), \quad j \in \mathbb{Z}.$$

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Plugging the Ansatz (1.2) into (1.1) yields the second order backward-forward differential equation for the wave profile:

$$(1.3) \quad c^2 \ddot{u} + f'(u) = V'(Au) - V'(A^*u),$$

where the difference operators A and A^* are defined by

$$Au(t) = u(t+1) - u(t) = A^*u(t+1), \quad t \in \mathbb{R}.^1$$

The study of lattice dynamical systems goes back to Fermi, Pasta and Ulam [3] who studied by numerical methods the dynamics of a finite lattice of particles with nearest-neighbour interaction. Since then, the study of infinite lattices has become a mathematical subject on its own right. Many authors have studied the FPU lattice. However, the most significant results we can think of are certainly due to Friesecke and Wattis [4]. They proved for the first time a global existence result for travelling waves in the FPU lattice. Their approach was to minimize the ‘kinetic energy’ over the set of states with a given ‘potential energy’. They solved the variational problem, under some superquadratic growth condition on the interaction potential, using Lions’ concentration compactness principle. The speed of travelling waves was then given as some unknown Lagrange multiplier. Also, they showed the optimality of the growth condition satisfied by V by proving the non-existence of travelling waves for quadratic potentials. Smets and Willem [12] also proved the existence of travelling waves, with prescribed speed instead. They used a completely different approach from the one in [4]. More precisely, using a variant of the mountain pass theorem, a travelling wave was obtained as a critical point of ‘the action functional’ defined on some Hilbert space on which the Palais-Smale compactness condition ² is not satisfied.

The DSG equation have also drawn the attention of many authors. For the sake of brevity, we will just mention the recent paper by Kreiner and Zimmer [5] in which they considered the DSG equation in a generalized set up. There, they proved the existence of non-constant periodic travelling waves with period bigger than some unknown constant. For this, they considered an interaction potential V of type (A.0) with $\alpha > 0$ and non-quadratic part W satisfying the global growth condition (A.3). Furthermore, they also proved the existence of homoclinic travelling waves with prescribed speed, but the non-quadratic part of the coupling has a special form, namely $W(x) = \epsilon_0|x|^\beta$ with $\epsilon_0 > 0$ and $\beta \geq 3$.

Roughly speaking, two main results (Theorem 1.1 and Theorem 1.2) are established in the present note. In Theorem 1.1, we prove, under some superquadratic growth condition on both V and f , the existence of a periodic travelling wave of any given period and whose speed is bounded below by ‘ $\max(0, \alpha = V''(0))$ ’ (same lower bound as in [12]). The quadratic part of the coupling is not necessarily positive, i.e. α ranges over \mathbb{R} and its non-quadratic part, W , belongs to a larger class than the one considered in [4, 12, 5]. The non-quadratic parts of both f and V satisfy the growth condition (A.2), which is a growth condition at infinity. The assumptions on f are precisely those on the potential (or Hamiltonian) in [8]. Therefore, periodic travelling waves with period 1 always exist since they are solutions of the Hamiltonian equation

$$(HS) \quad \ddot{u} + c^{-2}f'(u) = 0,$$

which, by a result of Rabinowitz [8] possesses indeed a 1-periodic solution.

In Theorem 1.2 we prove the existence of a travelling wave which is homoclinic to 0. Specifically, this is obtained as limit of periodic solutions by letting the period go to infinity. This approach is borrowed from [11] where a homoclinic solution of

¹The variable t here does not represent the time but the displacement

²In the future to be referred to as the (PS) condition.

(HS) is constructed as limit of a sequence of subharmonics. It is, of course, not difficult to show that Theorem 1.2 holds true for a linear (or weak) coupling, i.e. $V(x) = \frac{\alpha}{2}x^2$. Such a case was studied in [6], with techniques other than those we are using and completely different assumptions on f .

Note that if $V'(0) = 0$ and x_0 is a critical point of f , then the constant function $t \mapsto x_0$ is a solution of (1.3), in particular it is 1-periodic. Conversely, if u is a 1-periodic solution of (1.3), as aforementioned, u solves (HS). For this reason, we shall only deal with periodic solutions with periods in $(0, 1) \cup (1, \infty)$.

As it is well known in the Calculus of Variations, solutions of (1.3) can be obtained as critical points of the the action functional

$$(1.4) \quad \Phi(u) = \int_{\mathcal{T}} \left[\frac{c^2}{2} \dot{u}^2 - f(u) - V(Au) \right] dt,$$

defined on some appropriate Hilbert space E , where $\mathcal{T} \subseteq \mathbb{R}$.

When the Palais-Smale compactness condition is satisfied, the critical points of Φ can be detected with the aid of the mountain pass theorem, or some of its variants. It will be shown that under some conditions on f and V , Φ satisfies the (PS) condition when \mathcal{T} is a segment. However, if $\mathcal{T} = \mathbb{R}$, this condition is never fulfilled. This is due to the invariance of the functional under the action $\mathbb{R} \times E \rightarrow E, (s, u) \mapsto u(\cdot + s)$.

Organization of the paper. The main results are stated in the end of the present section. Preliminary results are collected in the next section. The third section is devoted to the proof of the first result (Theorem 1.1), while the second result (Theorem 1.2) is proven in the last section.

Main results. Throughout this paper, we shall consider potentials V and f of the type

$$(A.0) \quad V(x) = \frac{\alpha}{2}x^2 + W(x), \quad f(x) = -\frac{\omega_0}{2}x^2 + g(x), \text{ where } \alpha \text{ and } \omega_0 \text{ are real constants.}$$

The non-quadratic part $h \in \{g, W\}$ shall satisfy either

$$(A.1) \quad h(x) = o(x^2) \text{ as } x \rightarrow 0,$$

$$(A.2) \quad h \geq 0 \text{ and there are constants } \beta > 2, r_0 > 0 \text{ such that}$$

$$\beta h(x) \leq xh'(x) \text{ for } |x| \geq r_0,$$

or

$$(A.3) \quad \text{there is a constant } \beta > 0 \text{ such that}$$

$$= 0 < \beta h(x) \leq xh'(x) \text{ for } x \neq 0.$$

It is clear that if h satisfies (A.3), it also satisfies (A.1), and if it satisfies either (A.3) or (A.2), then there exists some constants $a_0, a_1 > 0$ such that the following holds true:

$$(A.4) \quad h(x) \geq a_0|x|^\beta - a_1 \text{ for all } x.$$

Let $c_0 \geq 0$ be given by

$$(1.5) \quad c_0 := \begin{cases} 0 & \text{if } 4\alpha^+ < \omega_0 < \infty \\ \sqrt{\alpha^+} & \text{if } 0 \leq \omega_0 \leq 4\alpha \end{cases},$$

where the notation α^+ stands for $\max(0, \alpha)$.

The main results are the following:

Theorem 1.1. *Let V and f be C^1 functions given by (A.0), with $\omega_0 = 0$. Suppose W and f satisfy (A.1),(A.2). Then, for every $\tau > 0$, and every $c > c_0$, (1.3) possesses a non-constant τ -periodic solution.*

Theorem 1.2. *Let V and f be C^1 functions given by (A.0), where $\omega_0 > 0$, and the non-linearities W and g satisfy (A.3). Then, for every $c > c_0$, (1.3) possesses a non-trivial homoclinic solution emanating from the origin.*

Remark 1.1. *Theorem 1.2 remains true in the particular case of a linear coupling, i.e. when $V(x) = \alpha x^2/2$, where α is a real constant. The case $V \equiv 0$ corresponds to the one-dimensional Hamiltonian system (HS), and will not therefore be considered in this note. For $\alpha > 0 \equiv W$, one can think of (1.3) as describing the dynamics of an infinite chain of particles connected with springs of common constant α . In the literature such a coupling is often referred to as harmonic. In the case of quadratic potential and coupling, it is easy to show with a direct method (Fourier transform) that there are no travelling waves (see, e.g., [4]). Note that, under the assumptions of Theorem 1.1 (resp. of Theorem 1.2), (1.1) admits 0 as a solution which will be referred to as the trivial one.*

2. PRELIMINARY RESULTS

In this section we give some preliminaries that are needed in the sequel. Let us first fix some notations. Given $\tau > 0$, we denote by H_τ^1 the space of τ -periodic functions whose restriction to $[0, \tau]$ belong to $H^1([0, \tau])$. Similarly, we introduce the notations L_τ^p and C_τ^k .

Lemma 2.1. *The difference operator $A : H_\tau^1 \rightarrow L_\tau^\infty \cap L_\tau^2$ is bounded and*

$$(2.1) \quad \|Au\|_{L_\tau^2} \leq \|\dot{u}\|_{L_\tau^2}, \quad \|Au\|_{L_\tau^\infty} \leq l(\tau)\|\dot{u}\|_{L_\tau^2},$$

with

$$(2.2) \quad l(\tau) = \begin{cases} \sqrt{[1/\tau] + 1} & \text{if } 0 < \tau < 1 \\ 1 & \text{if } \tau \geq 1 \end{cases},$$

where $[s]$ denotes the integer part of s .

Proof. Let $u \in H_\tau^1$ and $t \in \mathbb{R}$. Using Jensen's inequality successively, the change of variable $s \leftrightarrow s + t$, and Fubini's Theorem, we get

$$\begin{aligned} \int_0^\tau \left(\int_t^{t+1} \dot{u}(s) ds \right)^2 dt &\leq \int_0^\tau \left(\int_t^{t+1} \dot{u}^2(s) ds \right) dt \\ &= \int_0^\tau \left(\int_0^1 \dot{u}^2(s+t) ds \right) dt \\ &= \int_0^1 \left(\int_0^\tau \dot{u}^2(s+t) dt \right) ds \\ &= \int_0^1 \|\dot{u}\|_{L_\tau^2}^2 ds. \end{aligned}$$

Therefore

$$\|Au\|_{L_\tau^2}^2 = \int_0^\tau (Au(t))^2 dt \leq \|\dot{u}\|_{L_\tau^2}^2.$$

For the second estimate we use the Cauchy-Schwarz inequality:

$$|Au(t)| \leq \int_t^{t+1} |\dot{u}(s)| ds \leq \left[\int_t^{t+1} \dot{u}^2(s) ds \right]^{1/2}.$$

Therefore, if $\tau \geq 1$, we get

$$|Au(t)| \leq \left[\int_t^{t+\tau} \dot{u}^2(s) ds \right]^{1/2} = \|\dot{u}\|_{L_\tau^2}.$$

If $0 < \tau < 1$, we set

$$n := [1/\tau].$$

Then $n\tau \leq 1 < (n+1)\tau$. Again, by Cauchy-Schwarz inequality, it follows that

$$|Au(t)| \leq \left[\int_t^{t+(n+1)\tau} \dot{u}^2(s) ds \right]^{1/2} = \left[(n+1) \int_0^\tau \dot{u}^2(s) ds \right]^{1/2} \leq \sqrt{n+1} \|\dot{u}\|_{L^2_\tau},$$

and the proof is complete. \square

Similarly one can prove that

Lemma 2.2. *The difference operator $A : E \rightarrow L^\infty \cap L^2$ is bounded and*

$$(2.3) \quad \|Au\|_{L^2} \leq \|\dot{u}\|_{L^2}, \quad \|Au\|_{L^\infty} \leq \|\dot{u}\|_{L^2}.$$

Proof. See [12, Proposition 1] \square

Finally, let us recall the following

Proposition 2.1. *Let I be a compact interval. Then, there is a positive constant C_s such*

$$\|u\|_{L^\infty(I)} \leq C_s \|u\|_{H^1(I)} \quad (\forall u \in H^1(I)).$$

Moreover, the embeddings $H^1(I) \hookrightarrow C(I)$ and $H^1(I) \hookrightarrow L^2(I)$ are compact.

One can choose C_s to be $\sqrt{2}$ if $|I| \in [1, \infty)$, and $\sqrt{|I|} + 1/\sqrt{|I|}$ if $|I| \in (0, 1)$, where $|I|$ stands for the length of I .

Proposition 2.2. *Let $h \in C^1(\mathbb{R})$ and I a compact interval. Then, the functional $\bar{h}_I : H^1(I) \rightarrow \mathbb{R}$ defined by*

$$\bar{h}_I(u) = \int_I h(u)$$

is C^1 and its derivative is given by

$$\bar{h}'_I(u)\xi = \langle h'(u), \xi \rangle_{L^2(I)}$$

for every $u, \xi \in H^1(I)$.

Proof. We set

$$L_u \xi = \langle h'(u), \xi \rangle_{L^2(I)}.$$

It is clear that for every $u \in H^1(I)$, the map L_u is linear. On the other hand, h' and $u \in H^1(I)$ are continuous, therefore $\|h'(u)\|_{L^\infty(I)} < \infty$. Thus

$$|L_u \xi| = |\langle h'(u), \xi \rangle_{L^2(I)}| \leq \|h'(u)\|_{L^2(I)} \|\xi\|_{L^2(I)} \leq \sqrt{|I|} \|h'(u)\|_{L^\infty(I)} \|\xi\|_{H^1(I)},$$

i.e.

$$\sup_{\|\xi\|_{H^1(I)}=1} |L_u \xi| \leq \sqrt{|I|} \|h'(u)\|_{L^\infty(I)} < \infty,$$

meaning that L_u is bounded.

We shall now prove that $\bar{h}'_I(u) = L_u$ for every $u \in H^1(I)$.

Set

$$P = I \times [0, 1].$$

Given $u, \xi \in H^1(I)$, we set

$$u_\xi(t, s) = u(t) + s\xi(t) \quad (\forall (t, s) \in P)$$

Then

$$\begin{aligned} |\bar{h}_I(u + \xi) - \bar{h}_I(u) - L_u \xi| &= \left| \int_I [h(u + \xi) - h(u) - h'(u)\xi] \right| \\ &= \left| \int_P [h'(u_\xi) - h'(u)] \xi ds dt \right| \\ &= |\langle h'(u_\xi) - h'(u), \xi \rangle_{L^2(P)}| \\ &\leq \|\xi\|_{L^2(I)} \|h'(u_\xi) - h'(u)\|_{L^2(P)} \end{aligned}$$

$$\leq \sqrt{|I|} \|\xi\|_{H^1(I)} \|h'(u_\xi) - h'(u)\|_{L^\infty(P)},$$

Since u_ξ is continuous and P compact, so is its image $K = u_\xi(P)$. On the other hand, h' being continuous on \mathbb{R} is certainly uniformly continuous on compact subsets, therefore, for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $y, z \in K$, if $|y - z| \leq \delta$, then

$$|h'(y) - h'(z)| \leq \epsilon / \sqrt{|I|}.$$

Hence, if

$$\|\xi\|_{L^\infty(I)} = \|u_\xi - u\|_{L^\infty(P)} \leq \delta,$$

then we have

$$|h'(u_\xi(t, s)) - h'(u(t))| \leq \epsilon \quad (\forall (t, s) \in P),$$

i.e.

$$\|h'(u_\xi) - h'(u)\|_{L^\infty(P)} \leq \epsilon.$$

It follows that

$$|\bar{h}_I(u + \xi) - \bar{h}_I(u) - L_u \xi| \leq \epsilon \|\xi\|_{H^1(I)},$$

i.e. $\bar{h}'_I(u) = L_u$.

It only remains to prove the continuity of \bar{h}'_I . So, let $u \in H^1(I)$ and let (u_m) be a sequence in $H^1(I)$ that converges to u . We want to prove that $\bar{h}'_I(u_m)$ converges to $\bar{h}'_I(u)$.

The boundedness of (u_m) in $L^\infty(I)$, resulting from its convergence in $H^1(I)$ and the continuous embedding of $H^1(I)$ into $L^\infty(I)$, the continuity of u and the uniform continuity of h' on compact subsets imply that $h'(u_m) \rightarrow h'(u)$ in $L^\infty(I)$. Hence, given $\epsilon > 0$ there is a positive integer m_0 (depending on ϵ) such that if $m \geq m_0$ then

$$\|h'(u) - h'(u_m)\|_{L^\infty(I)} \leq \epsilon / \sqrt{|I|}.$$

For any $\xi \in H^1(I)$ we have

$$\begin{aligned} |\bar{h}'_I(u)\xi - \bar{h}'_I(u_m)\xi| &= |\langle h'(u) - h'(u_m), \xi \rangle_{L^2(I)}| \\ &\leq \|\xi\|_{L^2(I)} \|h'(u) - h'(u_m)\|_{L^2(I)} \\ &\leq \sqrt{|I|} \|\xi\|_{H^1(I)} \|h'(u) - h'(u_m)\|_{L^\infty(I)}, \end{aligned}$$

so that

$$\sup_{\|\xi\|_{H^1(I)}=1} |\bar{h}'_I(u)\xi - \bar{h}'_I(u_m)\xi| \leq \sqrt{|I|} \|h'(u) - h'(u_m)\|_{L^\infty(I)}.$$

Thus, if $m \geq m_0$, then

$$\sup_{\|\xi\|_{H^1(I)}=1} |\bar{h}'_I(u)\xi - \bar{h}'_I(u_m)\xi| \leq \epsilon,$$

which proves the continuity of \bar{h}'_I . \square

To end this section, let us prove the following

Proposition 2.3. *Let $h \in C^1(\mathbb{R})$ satisfies (A.1). Then, the functional $\bar{h}_\infty : E = H^1(\mathbb{R}) \rightarrow \mathbb{R}$ defined by*

$$\bar{h}_\infty(u) = \int_{\mathbb{R}} h(u)$$

is C^1 and its derivative is given by

$$\bar{h}'_\infty(u)\xi = \langle h'(u), \xi \rangle_{L^2}$$

for every $u, \xi \in E$.

Proof. Step 1: Well-definedness. Let us first make sure that \bar{h}_∞ takes only finite values. Thanks to (A.1), there is a $\delta > 0$ such that if $|x| \leq \delta$, then

$$h(x) \leq x^2.$$

On the other hand, if u is a member of E , then $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$, and therefore there is an $r > 0$, depending on δ , such that if $|t| \geq r$, then

$$|u(t)| \leq \frac{\delta}{2}.$$

It follows that

$$\begin{aligned} 0 \leq |\bar{h}_\infty(u)| &= \left| \int_{\mathbb{R}} h(u) \right| \leq \int_{|t| \leq r} |h(u)| + \int_{|t| \geq r} u^2 \\ &\leq \int_{|t| \leq r} |h(u)| + \|u\|_{L^2}^2 < \infty, \end{aligned}$$

Step 2: Differentiability of \bar{h}_∞ . We claim that for every $u \in E$, the linear map L_u defined by

$$L_u \xi = \langle h'(u), \xi \rangle_{L^2} \quad (\forall \xi \in E)$$

is bounded.

Let us fix a $u \in E$. Since $h'(x) = o(x)$ as $x \rightarrow 0$, given $\epsilon > 0$, there is a $\rho_0 > 0$ such that if $|x| \leq \rho_0$ then

$$(2.4) \quad |h'(x)| \leq \frac{\epsilon|x|}{3(1 + \|u\|_E)}.$$

On the other $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$, therefore there is an $r > 0$ depending on ρ_0 such that if $|t| \geq r$ then

$$|u(t)| \leq \rho_0.$$

Thus

$$\begin{aligned} \int_{|t| \geq r} |h'(u)|^2 &\leq \left(\frac{\epsilon}{3(1 + \|u\|_E)} \right)^2 \int_{|t| \geq r} u^2 \\ &\leq \left(\frac{\epsilon}{3(1 + \|u\|_E)} \right)^2 \|u\|_{L^2}^2 \\ (2.5) \quad &\leq \left(\frac{\epsilon}{3} \right)^2 < \infty, \end{aligned}$$

and

$$\int_{\mathbb{R}} |h'(u)|^2 \leq \int_{-r}^r |h'(u)|^2 + \left(\frac{\epsilon}{3} \right)^2 < \infty,$$

i.e. $h'(u) \in L^2$. It follows from Cauchy-Schwarz inequality that

$$|\langle h'(u), \xi \rangle_{L^2}| \leq \|h'(u)\|_{L^2} \|\xi\|_{L^2} \leq \|h'(u)\|_{L^2} \|\xi\|_E$$

so that

$$\sup_{\|\xi\|_E=1} |\langle h'(u), \xi \rangle_{L^2}| \leq \|h'(u)\|_{L^2} < \infty.$$

We can now prove that $\bar{h}'_\infty(u) = L_u$ for every $u \in E$. We set $I = [-r, r]$. Then, by Proposition 2.1 we have $\bar{h}_I \in C^1(E, \mathbb{R})$. Therefore, there is a positive number $\delta = \delta(\epsilon, r, u)$ (which one can assume to be less than or equal to $\min(1, \rho_0/2)$) such that if $\|\xi\|_E \leq \delta$ then

$$(2.6) \quad |\bar{h}_I(u + \xi) - \bar{h}_I(u) - \bar{h}'_I(u)\xi| \leq \frac{\epsilon}{3} \|\xi\|_E.$$

For the rest of this step we assume that $\|\xi\|_E \leq \delta$. Thanks to the mean value theorem, and (2.4), we get

$$|h(u + \xi) - h(u)| \leq \epsilon |\xi| \frac{|u| + |\xi|}{3(1 + \|u\|_E)}$$

whenever $|t| \geq r$. It follows that

$$\begin{aligned}
\int_{|t| \geq r} |h(u + \xi) - h(u)| &\leq \frac{\epsilon}{3(1 + \|u\|_E)} \int_{|t| \geq r} |\xi|(|u| + |\xi|) \\
&\leq \frac{\epsilon}{3(1 + \|u\|_E)} \|\xi\|_{L^2} (\|u\|_{L^2} + \|\xi\|_{L^2}) \\
&\leq \frac{\epsilon}{3(1 + \|u\|_E)} \|\xi\|_E (\|u\|_E + \|\xi\|_E) \\
&\leq \frac{\epsilon}{3(1 + \|u\|_E)} \|\xi\|_E (1 + \|u\|_E) \\
(2.7) \qquad \qquad \qquad &\leq \frac{\epsilon}{3} \|\xi\|_E.
\end{aligned}$$

Combining (2.5), (2.6), and (2.7) we get

$$\begin{aligned}
|\bar{h}_\infty(u + \xi) - \bar{h}_\infty(u) - L_u \xi| &\leq |\bar{h}_I(u + \xi) - \bar{h}_I(u) - \bar{h}'_I(u) \xi| \\
&\quad + \left| \int_{|t| \geq r} (h(u + \xi) - h(u)) \right| + \left| \int_{|t| \geq r} h'(u) \xi \right| \\
&\leq \frac{\epsilon}{3} \|\xi\|_E + \frac{\epsilon}{3} \|\xi\|_E + \int_{|t| \geq r} |h'(u)| |\xi| \\
&\leq \frac{2\epsilon}{3} \|\xi\|_E + \left(\int_{|t| \geq r} |h'(u)|^2 \right)^{1/2} \|\xi\|_{L^2} \\
&\leq \frac{2\epsilon}{3} \|\xi\|_E + \frac{\epsilon}{3} \|\xi\|_E \\
&= \epsilon \|\xi\|_E,
\end{aligned}$$

i.e. \bar{h}_∞ is differentiable, and the derivative is given precisely by

$$\bar{h}'_\infty(u) \xi = \langle h'(u), \xi \rangle_{L^2}.$$

Continuity of \bar{h}'_∞ : Let $u \in E$ and $(u_m) \subset E$ a sequence which converges to u in E . Then, (u_m) is bounded in E , i.e. there is a constant $K \geq 0$ independent of m such that $\|u_m\|_E \leq K$ for all m .

Given $\epsilon > 0$, there is a $r > 0$ such that for $|t| \geq r$ and m sufficiently large we have

$$|h'(u)| \leq \frac{\epsilon |u|}{4(1 + \|u\|_E)} \quad |h'(u_m)| \leq \frac{\epsilon |u_m|}{4(1 + K)},$$

Thus we have

$$\begin{aligned}
&\sup_{\|\xi\|_E=1} |\bar{h}'_\infty(u) \xi - \bar{h}'_\infty(u_m) \xi| \\
&\leq \left(\int_{\mathbb{R}} |h'(u) - h'(u_m)|^2 \right)^{1/2} \\
&\leq \left(\int_{-r}^r |h'(u) - h'(u_m)|^2 \right)^{1/2} + \left(\int_{|t| \geq r} |h'(u) - h'(u_m)|^2 \right)^{1/2} \\
&\leq \left(\int_{-r}^r |h'(u) - h'(u_m)|^2 \right)^{1/2} + \frac{\epsilon}{4} \left[\frac{\|u\|_{L^2}}{1 + \|u\|_E} + \frac{\|u_m\|_{L^2}}{1 + K} \right] \\
&\leq \sqrt{2r} \|h'(u) - h'(u_m)\|_{L^\infty([-r, r])} + \frac{\epsilon}{2}.
\end{aligned}$$

Since $u_m \rightarrow u$ in E implies $u_m \rightarrow u$ on compact subsets of \mathbb{R} . Therefore the boundedness of (u_m) in $L^\infty([-r, r])$ and the uniform continuity of h' on compact subsets implies that $h'(u_m) \rightarrow h'(u)$ uniformly on $[-r, r]$. Hence, for m sufficiently large we have

$$\|h'(u) - h'(u_m)\|_{L^\infty([-r, r])} \leq \epsilon/2\sqrt{2r}.$$

We conclude that, for m sufficiently large we have

$$\sup_{\|\xi\|_E=1} |\bar{h}'_\infty(u)\xi - \bar{h}'_\infty(u_m)\xi| \leq \epsilon,$$

i.e. \bar{h}'_∞ is continuous. \square

3. EXISTENCE OF PERIODIC TRAVELLING WAVES

In this section we are going to give a detailed proof of Theorem 1.1. The main tool to achieve this goal is a version of the mountain pass theorem, which we shall state below.

Before doing so, we fix some terminology. Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. We say that a sequence $\{u_m\} \subset X$ is a Palais-Smale for J if the sequence $\{J(u_m)\} \subset \mathbb{R}$ is bounded and $J'(u_m) \rightarrow 0$ as $m \rightarrow \infty$. The functional J is said to satisfy the Palais-Smale compactness condition (we will often say J is (PS)), if every Palais-Smale sequence is precompact.

Theorem 3.1 (Rabinowitz [9]). *Let $X = X_0 \oplus \hat{X}$ with $\dim X_0 < \infty$ and $J \in C^1(X)$ be (PS). Suppose in addition the following conditions are satisfied*

(J.3) $J|_{X_0} \leq 0$,

(J.4) *there are constants $\omega, \rho > 0$ such that $J > 0$ in $\hat{X} \cap (B_\rho \setminus \{0\})$ and $J \geq \omega$ on $\hat{X} \cap S_\rho$,*

(J.5) *for each finite-dimensional subspace $Y \subset X$, there is an $R = R(Y)$ such that $J \leq 0$ on $Y \setminus B_R$.*

Then, Φ possesses a positive critical value b characterized by

$$b = \inf_{h \in \Gamma} \max_{u \in \bar{B}_{R(X_1)} \cap X_1} J(h(u)),$$

where

$$\Gamma = \{h \in C(\bar{B}_{R(X_1)} \cap X_1, X) \mid h(u) = u \text{ if } J(u) \leq 0\}$$

and $X_1 = X_0 \oplus \text{span}\{v\}$, for any non-zero $v \in \hat{X}$.

The notations B_r, \bar{B}_r and S_r stand for the open ball, the closed ball and the sphere centered at 0 with radius r , respectively.

Proposition 3.1. *Let $V, f \in C^1(\mathbb{R})$ satisfy the assumptions of Theorem 1.1. Then $\Phi_\tau \in C^1(H_\tau^1, \mathbb{R})$, with*

$$\Phi'_\tau(u)\xi = \int_0^\tau [c^2 \dot{u}\dot{\xi} - f'(u)\xi - V'(Au)A\xi],$$

for all $u, \xi \in H_\tau^1$. Furthermore, any critical point of Φ_τ is a classical solution of (1.3).

Proof. Write

$$\Phi_\tau(u) = \frac{1}{2}B_\tau(u, u) - G_\tau(u) - \tilde{W}_\tau(u),$$

with

$$\begin{aligned} B_\tau(u, v) &= \int_0^\tau [c^2 \dot{u}\dot{v} + \omega_0 uv - \alpha AuAv] = \int_0^\tau [c^2 \dot{u}\dot{v} - \alpha AuAv], \\ G_\tau(u) &= \int_0^\tau g(u), \\ \tilde{W}_\tau(u) &= \int_0^\tau W(Au). \end{aligned}$$

One can easily check that B_τ is a bounded (symmetric) bilinear form on H_τ^1 , therefore the functional

$$u \mapsto B_\tau(u, u)$$

is C^1 (it is actually C^∞). Note that $A : H_\tau^1 \rightarrow H_\tau^1$ is a bounded linear operator and $\tilde{W}_\tau = W_\tau \circ A$ where W_τ is defined similarly to G_τ . Hence, by Proposition 2.2 we have $G_\tau, \tilde{W}_\tau \in C^1(H_\tau^1, \mathbb{R})$.

If $u \in H_\tau^1$ is critical point of Φ_τ . Then, for every test function ξ we have

$$\begin{aligned} 0 &= \int_0^\tau [c^2 \dot{u} \dot{\xi} - f'(u)\xi - V'(Au)A\xi] \\ &= \int_0^\tau [c^2 \dot{u} \dot{\xi} - f'(u)\xi + (V'(Au) - V'(A^*u))\xi] \end{aligned}$$

i.e. u is a weak solution of (1.3), and u is continuous because H_τ^1 continuously embeds into C_τ^0 . Since $V', f' \in C(\mathbb{R})$, thanks to (1.3), we have $\ddot{u} \in C^0$, and therefore $u \in C^2$, i.e. is a classical solution. \square

Proposition 3.2. *Under the assumptions of Theorem 1.1, Φ_τ is (PS).*

Proof. We first prove that (PS) sequences are bounded and next, that they are precompact.

Boundedness: Given $s > 0$ we set

$$N_s(u) = \left\{ \int_0^\tau [c^2 \dot{u}^2 - \alpha(Au)^2 + su^2] \right\}^{1/2} \quad \text{for } u \in H_\tau^1.$$

Then N_s defines a norm on H_τ^1 which is equivalent to the standard one.

Let $(u_m) \subset H_\tau^1$ be a (PS) sequence, i.e. for some constant $M \geq 0$ we have

$$|\Phi_\tau(u_m)| \leq M \quad (\forall m), \quad \text{and} \quad \lim_{m \rightarrow \infty} \Phi'_\tau(u_m) = 0.$$

Then, for some positive integer m_0 we have $\|\Phi'_\tau(u_m)\| \leq 1$ whenever $m \geq m_0$.

Fixing $m \geq m_0$, we have

$$|\Phi'_\tau(u_m)u_m| \leq N_s(u_m),$$

which implies

$$\begin{aligned} \int_0^\tau [f'(u_m)u_m + W'(Au_m)Au_m] &= -\Phi_\tau(u_m)u_m + N_s^2(u_m) - s\|u_m\|_{L^2}^2 \\ (3.1) \qquad \qquad \qquad &\leq N_s(u_m) + N_s^2(u_m). \end{aligned}$$

We set

$$I_1 = \{t \in [0, \tau] : |u_m(t)| \leq r_0\}, \quad I_2 = \{t \in [0, \tau] : |Au_m(t)| \leq r_0\},$$

and $\bar{I}_i = [0, \tau] \setminus I_i$ for $j = 1, 2$. Then,

$$\begin{aligned} \int_{I_1} g(u_m) + \int_{I_2} W(Au_m) &\leq \tau \left[\max_{|x| \leq r_0} g(x) + \max_{|x| \leq r_0} W(x) \right] \\ &=: K_0, \end{aligned}$$

and thanks to (A.2) and (3.3), we get

$$\begin{aligned} \int_0^\tau [g(u_m) + W(Au_m)] &= \int_{I_1} g(u_m) + \int_{I_2} W(Au_m) \\ &+ \int_{\bar{I}_1} g(u_m) + \int_{\bar{I}_2} W(Au_m) \\ &\leq K_0 + \beta^{-1} \left[\int_{\bar{I}_1} g'(u_m)u_m + \int_{\bar{I}_2} W'(Au_m)Au_m \right] \\ &\leq K_0 + \beta^{-1} \int_0^\tau [g'(u_m)u_m + W'(Au_m)Au_m] \\ &\leq K_0 + \beta^{-1}(N_s(u_m) + N_s^2(u_m)), \end{aligned}$$

i.e.

$$(3.2) \quad \int_0^\tau [g(u_m) + W(Au_m)] \leq K_0 + \beta^{-1}(N_s(u_m) + N_s^2(u_m)),$$

Also, thanks to (A.2), there is a constant $r \geq r_0$ such that the condition $|x| \geq r$ implies

$$(*) \quad x^2 \leq xg'(x).$$

Setting

$$I = \{t \in [0, \tau] : |u_m(t)| \leq r\}, \quad \bar{I} = [0, \tau] \setminus I,$$

we then deduce from (*) and (3.1) that

$$\begin{aligned} \|u_m\|_{L_\tau^2}^2 &= \int_I u_m^2 + \int_{\bar{I}} u_m^2 \leq r^2\tau + \int_{\bar{I}} g'(u_m)u_m \\ &\leq r^2\tau + \int_0^\tau g'(u_m)u_m, \end{aligned}$$

that is,

$$(3.3) \quad \|u_m\|_{L_\tau^2}^2 \leq r^2\tau + N_s(u_m) + N_s^2(u_m).$$

Combining (3.1), (3.2) and (3.3), it results

$$\begin{aligned} K_0 + sr^2\tau/2 + M &\geq K_0 + sr^2\tau/2 + \Phi_\tau(u_m) \\ &= \frac{1}{2}N_s^2(u_m) - \frac{s}{2}\|u_m\|_{L_\tau^2}^2 - \int_0^\tau [g(u_m) + W(Au_m)] \\ &+ K_0 + sr^2\tau/2 \\ &\geq K_0 + sr^2\tau/2 + \frac{1}{2}N_s^2(u_m) - \frac{s}{2}(r^2\tau + N_s(u_m) + N_s^2(u_m)) \\ &- K_0 - \beta^{-1}(N_s(u_m) + N_s^2(u_m)), \end{aligned}$$

i.e.

$$(1/2 - 1/\beta - s/2)N_s^2(u_m) - (1/s + 1/\beta)N_s(u_m) \leq K_0 + sr^2\tau/2 + M.$$

Choosing s such that $s < 1 - 2/\beta$, we deduce from the above inequality that (u_m) is bounded in (H_τ^1, N_s) and therefore in $(H_\tau^1, \|\cdot\|_{H_\tau^1})$.

Precompactness: The boundedness of (u_m) in H_τ^1 allows us to extract a weakly convergent subsequence, which for simplicity we still denote by (u_m) . Let $u \in H_\tau^1$ be its (weak) limit. Then u_m converges to u strongly in C_τ^0 as well as in L_τ^2 .

Note that $\Phi'_\tau(u)u = 0$. Indeed,

$$\Phi'_\tau(u)u = \Phi'_\tau(u_m)u + \underbrace{[\Phi'_\tau(u)u_m - \Phi'_\tau(u_m)u]}_{\Sigma_m} + [\Phi'_\tau(u)u - \Phi'_\tau(u)u_m].$$

The first term goes to zero because (u_m) is a (PS) sequence and the last one goes to zero too because $u_m \rightharpoonup u$ weakly in H_τ^1 . The second term can be written as

$$\Sigma_m = \underbrace{\int_0^\tau [g'(u_m)u - g'(u)u_m]}_{\Sigma_m^1} + \underbrace{\int_0^\tau [W'(Au_m)Au - W'(Au)Au_m]}_{\Sigma_m^2},$$

and

$$\begin{aligned} |\Sigma_m^1| &\leq \int_0^\tau |g'(u_m)u - g'(u)u| + \int_0^\tau |g'(u)u - g'(u)u_m| \\ &\leq \sqrt{\tau} (\|u\|_{L_\tau^2} \|g'(u) - g'(u_m)\|_{L_\tau^\infty} + \|g'(u)\|_{L_\tau^2} \|u - u_m\|_{L_\tau^\infty}). \end{aligned}$$

Since $u_m \rightarrow u$ and g' is uniformly continuous on compact sets, it follows from the above estimate that $\Sigma_m^1 \rightarrow 0$. Replacing g, u , and u_m by W, Au and Au_m , respectively, one shows that $\Sigma_m^2 \rightarrow 0$. Hence $\Sigma_m \rightarrow 0$ and we conclude that $\Phi'_\tau(u)u = 0$.

Thanks to the continuity of V' and f' we get

$$\begin{aligned} \lim_m \int_0^\tau [f'(u_m)u_m + V'(Au_m)Au_m] &= \int_0^\tau [f'(u)u + V'(Au)Au] \\ &= -\Phi'_\tau(u)u + c^2 \|\dot{u}\|_{L^2_\tau}^2 \\ &= c^2 \|\dot{u}\|_{L^2_\tau}^2. \end{aligned}$$

On one hand, the boundedness of (u_m) and its definition imply that $\Phi'_\tau(u_m)u_m \rightarrow 0$ as $m \rightarrow \infty$. Consequently,

$$\begin{aligned} \lim_m \|\dot{u}_m\|_{L^2_\tau}^2 &= c^{-2} \lim_m \left[\Phi'_\tau(u_m)u_m + \int_0^\tau (f'(u_m)u_m + V'(Au_m)Au_m) \right] \\ (3.4) \quad &= \|\dot{u}\|_{L^2_\tau}^2. \end{aligned}$$

On the other hand the boundedness of (u_m) implies the one of (\dot{u}_m) in L^2_τ . It follows from (3.4) that $\dot{u}_m \rightarrow \dot{u}$ in L^2_τ . Hence $u_m \rightarrow u$ in H^1_τ . \square

Set

$$E_0 = \{u \in H^1_\tau | u(t) = u(0) \text{ for all } t\}.$$

Then E_0 can be identified with \mathbb{R} and

$$H^1_\tau = E_0 \oplus E_0^\perp,$$

where E_0^\perp denotes the orthogonal complement of E_0 in H^1_τ . Obviously, u belongs to E_0^\perp if and only if its mean value over a period equals zero. Furthermore, for every $u \in E_0^\perp$ we have

$$(3.5) \quad \|u\|_{L^\infty} \leq \|\dot{u}\|_{L^2_\tau}, \quad \|u\|_{L^2_\tau} \leq \sqrt{\tau} \|\dot{u}\|_{L^2_\tau}.$$

We have

Lemma 3.1. *Under the assumptions of Theorem 1.1, Φ_τ satisfies the conditions (J.3)-(J.5) of Theorem 3.1, with $X = H^1_\tau$, $X_0 = E_0$, and $\hat{X} = E_1$.*

Proof. Condition (J.3) follows from the facts that $f \geq 0$ and $V(0) = 0$.

Condition (J.4): Let ϵ be such that

$$0 < \epsilon < \frac{c^2 - c_0^2}{1 + \tau}.$$

By (A.1), there is a $\delta > 0$ such that

$$\max(f(x), W(x)) \leq \frac{\epsilon}{2} x^2 \text{ whenever } |x| \leq \delta.$$

Set

$$\rho = \frac{\delta}{\max(l(\tau), C_s)}.$$

where $l(\tau)$ is given by (2.2), and C_s by Proposition 2.1. Choose a $u \in E_0^\perp$ such that

$$0 < \|u\|_{H^1_\tau} \leq \rho.$$

Then, by Proposition 2.1, and (2.1), we have

$$\max(\|Au\|_{L^\infty_\tau}, \|u\|_{L^\infty_\tau}) \leq \delta.$$

Also, (3.5) implies that

$$\|u\|_{H^1_\tau}^2 \leq (1 + \tau) \|\dot{u}\|_{L^2_\tau}^2 \quad (\forall u \in E_0^\perp).$$

Therefore

$$\begin{aligned} \Phi_\tau(u) &\geq \int_0^\tau \left[\frac{c^2}{2} \dot{u}^2 - \frac{\epsilon}{2} u^2 - \frac{\alpha}{2} (Au)^2 - \frac{\epsilon}{2} (Au)^2 \right] \\ &\geq \frac{c^2}{2} \|\dot{u}\|_{L^2_\tau}^2 - \frac{\epsilon}{2} \|u\|_{L^2_\tau}^2 - \frac{c_0^2}{2} \|\dot{u}\|_{L^2_\tau}^2 - \frac{\epsilon}{2} \|\dot{u}\|_{L^2_\tau}^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{c^2 - c_0^2}{2} \|\dot{u}\|_{L_\tau^2}^2 - \frac{\epsilon}{2} \|u\|_{H_\tau^1}^2 \\
&\geq \frac{c^2 - c_0^2}{2(1 + \tau)} \|u\|_{H_\tau^1}^2 - \frac{\epsilon}{2} \|u\|_{H_\tau^1}^2 \\
&= \frac{1}{2} \left(\frac{c^2 - c_0^2}{1 + \tau} - \epsilon \right) \|u\|_{H_\tau^1}^2 > 0.
\end{aligned}$$

In particular we have

$$\Phi_\tau(u) \geq \left(\frac{c^2 - c_0^2}{1 + \tau} - \epsilon \right) \rho^2 / 2 > 0$$

for $\|u\|_{H_\tau^1} = \rho$.

Condition (J.5): Let Y be a finite-dimensional subspace of H_τ^1 . Then, any two norms on Y are equivalent, therefore there is a positive constant λ depending only on Y such that

$$\|u\|_{L_\tau^\beta} \geq \lambda \|u\|_{H_\tau^1} \quad (\forall u \in Y).$$

Let $S(Y)$ be the unit sphere of Y (with respect to the Sobolev norm). Then,

$$\inf_{u \in S(Y)} \left\{ \|u\|_{L_\tau^\beta}^\beta + \|Au\|_{L_\tau^\beta}^\beta \right\} \geq \lambda^\beta.$$

Given a non-zero $u \in Y$, we set

$$\tilde{u} = \frac{u}{r}, \quad r = \|u\|_{H_\tau^1}.$$

Then $\tilde{u} \in S(Y)$. Thanks to (A.2) and (A.4), we get

$$\begin{aligned}
\Phi_\tau(u) &= \frac{r^2}{2} B_\tau(\tilde{u}, \tilde{u}) - \int_0^\tau [g(u) + W(u)] \\
&\leq \frac{1}{2} (1 + \alpha^-) r^2 - a_0 r^\beta \int_0^\tau (|\tilde{u}|^\beta + |A\tilde{u}|^\beta) + 2a_1 \tau \\
&\leq \frac{1}{2} (1 + \alpha^-) r^2 - a_0 \inf_{v \in S(Y)} (\|v\|_{L_\tau^\beta}^\beta + \|Av\|_{L_\tau^\beta}^\beta) r^\beta + 2a_1 \tau \\
&\leq \frac{1}{2} (1 + \alpha^-) r^2 - a_0 \lambda^\beta r^\beta + 2a_1 \tau.
\end{aligned}$$

Since $\beta > 2$, there is an $R > 0$ depending on λ , and therefore on Y , such that if $u \in Y$, with $\|u\|_{H_\tau^1} > R$, then $\Phi_\tau(u) \leq 0$. Hence Φ_τ satisfies (J.5). \square

Thanks to Theorem 3.1, Φ_τ possesses a critical point u in H_τ^1 which is, of course, a τ -periodic solution of (1.3). If u were constant, the corresponding critical value, $\Phi_\tau(u)$, would be non-positive, contrary to Theorem 3.1. Therefore u is necessarily non-constant.

We shall end this section with a ‘particular case’ of Theorem 1.1.

Theorem 3.2. *Let $f \equiv 0$ and $V \in C^1(\mathbb{R})$ be as in Theorem 1.1. Then, for every $\tau > 0$ and every $c > c_0$, (1.3) possesses a non-constant τ -periodic solution.*

This theorem is more general than the one prove in [7], in the sense that we only require a growth condition on W at infinity. However, to carry out the construction of a travelling wave solution u for which $\dot{u} \in L^2$, as it is done in [7], one definitely needs a global growth condition on W . The proof shall follow from the standard mountain pass theorem, whose statement is given below:

Theorem 3.3 (Ambrosetti-Rabinowitz [1]). *Let $J \in C^1(X, \mathbb{R})$ be (PS) and $J(0) = 0$. Suppose the following conditions are satisfied*

- (J.1) *there are constants $\omega, \rho > 0$ such that $J|_{S_\rho} \geq \omega$,*
- (J.2) *there is an $e \in X \setminus \bar{B}_\rho$ such that $J(e) \leq 0$.*

Then, J possesses a critical value $b \geq \omega$ characterized by

$$(3.6) \quad b = \inf_{\gamma \in \Gamma} \max_{0 \leq s \leq 1} J(\gamma(s)),$$

where

$$(3.7) \quad \Gamma = \{\gamma \in C([0, 1], E) \mid \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Note that under the assumptions of Theorem 3.2, if λ is a real constant, and u a solution of (1.3), so is $u + \lambda$. Therefore, we can search τ -periodic solutions of (1.3) in the space

$$E_\tau = \{u \in H_\tau^1 : u(0) = 0\}$$

which is a Hilbert space for the (equivalent) norm

$$\|u\|_{E_\tau} = \|\dot{u}\|_{L_\tau^2}.$$

Proof of Theorem 3.2. We shall only check the (PS) condition and (J.1) since condition (J.2) can be easily checked as (J.5) in the proof of Lemma 3.1.

(PS) condition. Let $(u_m) \subset E_\tau$ be a (PS) sequence for Φ_τ . Then, for some non-negative constant M we have $|\Phi_\tau(u_m)| \leq M$ for all n , and for m sufficiently large we have $\|\Phi'_\tau(u_m)\| \leq 1$. If we set $I_1 = \{t \in [0, \tau] : |u_m(t)| \leq r_0\}$, $I_2 = \{t \in [0, \tau] : |Au_m(t)| \leq r_0\}$, and $\bar{I}_i = [0, \tau] \setminus I_i$, $i = 1, 2$, then we have

$$\int_{I_1} g(u_m) + \int_{I_2} W(Au_m) \leq K := \tau \left[\max_{|x| \leq r_0} g(x) + \max_{|x| \leq r_0} W(x) \right].$$

Hence

$$\begin{aligned} M &\geq \Phi_\tau(u_m) \\ &= \frac{1}{2} B_\tau(u_m, u_m) - \int_0^\tau [g(u_m) + W(Au_m)] \\ &\geq \frac{1}{2} B_\tau(u_m, u_m) - \int_{\bar{I}_1} g(u_m) - \int_{\bar{I}_2} W(Au_m) - K \\ &\geq \frac{1}{2} B_\tau(u_m, u_m) - \frac{1}{\beta} \left[\int_{\bar{I}_1} g'(u_m) u_m + \int_{\bar{I}_2} W'(Au_m) Au_m \right] - K \\ &\geq \frac{1}{2} B_\tau(u_m, u_m) - \frac{1}{\beta} \int_0^\tau [g'(u_m) u_m + W'(Au_m) Au_m] - K \\ &= \frac{1}{2} B_\tau(u_m, u_m) + \frac{1}{\beta} [\Phi'_\tau(u_m) u_m - B_\tau(u_m, u_m)] - K \\ &\geq \left(\frac{1}{2} - \frac{1}{\beta} \right) B_\tau(u_m, u_m) - \frac{1}{\beta} \|u_m\|_{E_\tau} - K. \end{aligned}$$

Since

$$B_\tau(u, u) = \int_0^\tau [c^2 \dot{u}^2 - \alpha (Au)^2] \geq (c^2 - c_0^2) \|\dot{u}\|_{L_\tau^2}^2 \quad (\forall u \in E_\tau),$$

we deduce that

$$(\beta - 2)(c^2 - c_0^2) \|u_m\|_{E_\tau}^2 - 2\beta \|u_m\|_{E_\tau} \leq 2\beta(M + K),$$

which shows that (u_m) is bounded in $E_\tau \subset H_\tau^1$. Hence a subsequence, still denoted u_m , converges weakly in E_τ and strongly in L_τ^∞ , say to some $u \in E_\tau$. Since V' is continuous, and $u_m \rightarrow u$ strongly in L_τ^∞ , we have

$$\lim_{m \rightarrow \infty} \int_0^\tau V'(Au_m) Au = \int_0^\tau V'(Au) Au.$$

Note that

$$\Phi'_\tau(u_m) u = c^2 \langle u_m, \dot{u} \rangle_{E_\tau} - \int_0^\tau V'(Au_m) Au,$$

therefore, using the fact that (u_m) is a (PS) sequence and $u_m \rightharpoonup u$ weakly in E_τ , we get

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \Phi'_\tau(u_m)u \\ &= \lim_{m \rightarrow \infty} \left[c^2 \langle u_m, u \rangle_{E_\tau} - \int_0^\tau V'(Au_m)Au \right] \\ &= c^2 \langle u, u \rangle_{E_\tau} - \int_0^\tau V'(Au)Au = \Phi'_\tau(u)u. \end{aligned}$$

On the other hand the boundedness of (u_m) and the fact that it is a (PS) sequence imply that $\Phi'_\tau(u_m)u_m \rightarrow 0$. Thus, using once more the fact that V' is continuous and $u_m \rightarrow u$ strongly in L^∞_τ , we have

$$\begin{aligned} \lim_{m \rightarrow \infty} c^2 \|u_m\|_{E_\tau}^2 &= \lim_{m \rightarrow \infty} \left[\Phi'_\tau(u_m)u_m + \int_0^\tau V'(Au_m)Au_m \right] \\ &= \int_0^\tau V'(Au)Au \\ &= \Phi'_\tau(u)u + \int_0^\tau V'(Au)Au \\ &= c^2 \|u\|_{E_\tau}^2, \end{aligned}$$

and the convergence $u_m \rightarrow u$ is strong in E_τ .

Condition (J.1). Let ϵ be a such that

$$0 < \epsilon < c^2 - c_0^2.$$

Thanks to (A.1), there is a positive number δ (depending on ϵ) such that if $|x| \leq \delta$, then

$$W(x) \leq \frac{\epsilon}{2} x^2.$$

Let $u \in E_\tau$ with $\|u\|_{E_\tau} = \delta$. Then we have

$$\|Au\|_{L^\infty_\tau} \leq \|u\|_{E_\tau} = \delta,$$

so that

$$\begin{aligned} \Phi_\tau(u) &\geq \frac{1}{2} \int_0^\tau [c^2 \dot{u}^2 - \alpha(Au)^2 - \epsilon(Au)^2] \\ &\geq \frac{1}{2} (c^2 - c_0^2 - \epsilon) \|\dot{u}\|_{L^2_\tau}^2 \\ &= \frac{1}{2} (c^2 - c_0^2 - \epsilon) \delta^2 \end{aligned}$$

□

4. EXISTENCE OF HOMOCLINIC TRAVELLING WAVES

In this section we are going to construct homoclinic travelling waves as limits of periodic ones. We need the the standard version of the mountain pass theorem in order to prove the existence of periodic solutions for (1.3) under the assumptions of Theorem 1.2.

Note that

$$(4.1) \quad B_\tau(u, u) \geq \epsilon_0 \|u\|_{H^1_\tau}^2 \quad (\forall u \in H^1_\tau),$$

where

$$(4.2) \quad \epsilon_0 = \begin{cases} \min(c^2 - \alpha, \omega_0) & \text{if } 0 < \omega_0 \leq 4\alpha \\ \min(\omega_0 - 4\alpha^+, c^2) & \text{if } \omega_0 > 4\alpha^+ \end{cases}.$$

Indeed, if $0 < \omega_0 \leq 4\alpha$, then for every $u \in H^1_\tau$, we have

$$B_\tau(u, u) \geq c^2 \|\dot{u}\|_{L^2_\tau}^2 - \alpha^+ \|\dot{u}\|_{L^2_\tau}^2 + \omega_0 \|u\|_{L^2_\tau}^2$$

$$\geq \min(c^2 - \alpha^+, \omega_0) \|u\|_{H_\tau^1}^2,$$

and if $\omega_0 > 4\alpha^+$, then for every $u \in H_\tau^1$, we have

$$\begin{aligned} B_\tau(u, u) &\geq c^2 \|\dot{u}\|_{L_\tau^2}^2 + (\omega_0 - 4\alpha^+) \|u\|_{L_\tau^2}^2 \\ &\geq \min(c^2, \omega_0 - 4\alpha^+) \|u\|_{H_\tau^1}^2. \end{aligned}$$

We have the following

Lemma 4.1. *Under the assumptions of Theorem 1.2, (1.3) possesses non-trivial periodic solutions of any given period $\tau > 0$ and any given speed $c > c_0$.*

Proof. We only have to check that Φ_τ is (PS), and satisfies the conditions (J.1) and (J.2) of Theorem 3.3.

Φ_τ is (PS). We will only prove the boundedness of (PS) sequences. The precompactness can be dealt with following the same line of arguments as in the proof of Proposition 2.2.

Let $(u_m) \subset H_\tau^1$ be a (PS) sequence, i.e. for some constant $M \geq 0$ we have $|\Phi_\tau(u_m)| \leq M$ for all m and $\Phi'_\tau(u_m) \rightarrow 0$ as $m \rightarrow \infty$. Then, there is an integer m_0 such that $\|\Phi'_\tau(u_m)\| \leq 1$ for all $m \geq m_0$.

Fixing $m \geq m_0$, thanks to (A.3) and (4.1) we have

$$\begin{aligned} M + \frac{1}{\beta} \|u_m\|_{H_\tau^1} &\geq \Phi(u_m) - \frac{1}{\beta} \Phi'(u_m)u_m \\ &= \left(\frac{1}{2} - \frac{1}{\beta}\right) B_\tau(u_m, u_m) + \int_0^\tau \left[\frac{1}{\beta} g'(u_m)u_m - g(u_m) \right] \\ &\quad + \int_0^\tau \left[\frac{1}{\beta} W'(Au_m)Au_m - W(Au_m) \right] \\ &\geq \epsilon_0 \left(\frac{1}{2} - \frac{1}{\beta}\right) \|u_m\|_{H_\tau^1}^2, \end{aligned}$$

Thus

$$\epsilon_0(\beta - 2) \|u_m\|_{H_\tau^1}^2 - 2 \|u_m\|_{H_\tau^1} \leq 2\beta M,$$

showing that (u_m) is bounded in H_τ^1 for $\beta > 2$.

Condition (J.1). Thanks to (4.1), we have

$$\begin{aligned} \Phi_\tau(u) + G_\tau(u) + \tilde{W}_\tau(u) &= \frac{1}{2} B_\tau(u, u) \\ &\geq \frac{1}{2} \epsilon_0 \|u\|_{H_\tau^1}^2. \end{aligned}$$

Now we only have to show that

$$G_\tau(u) + \tilde{W}_\tau(u) = o(\|u\|_{H_\tau^1}^2).$$

Given, $\epsilon > 0$, thanks to (A.1), there is a $\delta > 0$ such that, if $|x| \leq \delta$, then

$$\max(g(x), W(x)) \leq \epsilon x^2/2.$$

Set

$$\rho = \frac{\delta}{\max(l(\tau), C_s)},$$

where the constant C_s is given by Proposition 2.1, and $l(\tau)$ by (2.2). Let u be a member of H_τ^1 with

$$\|u\|_{H_\tau^1} = \rho.$$

Then for every t we have

$$|u(t)| \leq \|u\|_{L^\infty} \leq C_s \|u\|_{H_\tau^1} \leq \delta.$$

Also, thanks to (2.1), we have, for every t

$$|Au(t)| \leq \|Au\|_{L^\infty} \leq l(\tau) \|\dot{u}\|_{L_\tau^2} \leq \delta.$$

Thus,

$$G_\tau(u) + \tilde{W}_\tau(u) \leq \frac{\epsilon}{2} \int_0^\tau (u^2 + (Au)^2) \leq \frac{\epsilon}{2} \|u\|_{H_\tau^1}^2.$$

Since $\epsilon > 0$ is arbitrary, this shows that

$$G_\tau(u) + \tilde{W}_\tau(u) = o(\|u\|_{H_\tau^1}^2).$$

In particular, if we choose ϵ such that

$$0 < \epsilon < \epsilon_0,$$

then, thanks to (4.1), we have

$$\Phi_\tau(u) \geq (\epsilon_0 - \epsilon)\rho^2/2 > 0.$$

Condition (J.2). Thanks to (A.4), there are constants $a_0, a_1 > 0$ such that

$$\min(g(x), W(x)) \geq a_0|x|^\beta - a_1 \text{ for all } x.$$

Let u be a non-zero element of H_τ^1 and $r > 0$. Then we have

$$\Phi_\tau(ru) \leq 2a_1\tau + \frac{r^2}{2} \int_0^\tau (c^2\dot{u}^2 + \omega_0u^2 - \alpha(Au)^2) - a_0r^\beta \int_0^\tau (|u|^\beta + |Au|^\beta).$$

But $\beta > 2$, therefore $\Phi_\tau(ru) \rightarrow -\infty$ as $r \rightarrow \infty$ and there is an $r_u > 0$ such that $\Phi_\tau(ru) \leq 0$ for $r \geq r_u$. \square

Remark 4.1. *In order to construct a homoclinic solution of (1.3) we need non-constant periodic solutions. At this point we only have non-trivial periodic solutions. However, we will see that for a suitable choice of e (see condition (J.2) of Theorem 3.3) the periodic solutions whose existence is guaranteed by Lemma 4.1 are actually non-constant.*

We denote by Φ_∞ the functional Φ defined on $E = H^1(\mathbb{R})$ by (1.4), i.e. with $\mathcal{T} = \mathbb{R}$. It is worth mentioning that members u of E satisfy $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$ (see, e.g., [11]). We are going to construct a family $U^{[0]}$ such that each of its members is a periodic solution of (1.3). Next, we shall prove the existence of a convergent sequence $(\bar{u}_k)_{k \in \mathbb{N}}$ in $U^{[0]}$ whose limit \bar{u} belongs to E , and is a critical point of Φ_∞ .

Proposition 4.1. *Under the assumptions of Theorem 1.2, $\Phi_\infty \in C^1(E, \mathbb{R})$ and its derivative is given by*

$$\Phi'_\infty(u)\xi = \int_{\mathbb{R}} [c^2\dot{u}\dot{\xi} - f'(u)\xi - V'(Au)A\xi].$$

Proof. Write

$$\Phi_\infty(u) = \frac{1}{2}B_\infty(u, u) - G_\infty(u) - \tilde{W}_\infty(u).$$

with

$$\begin{aligned} B_\infty(u, v) &= \int_{\mathbb{R}} [c^2\dot{u}\dot{v} + \omega_0uv - \alpha(Au)(Av)] \\ G_\infty(u) &= \int_{\mathbb{R}} g(u) \\ \tilde{W}_\infty(u) &= \int_{\mathbb{R}} W(Au). \end{aligned}$$

B_∞ is a bounded (symmetric) bilinear form therefore $u \mapsto B_\infty(u, u)$ is C^∞ . Since $A : E \rightarrow E$ is bounded and $\tilde{W}_\infty = W_\infty \circ A$, where W_∞ is defined similarly to G_∞ , thanks to Proposition 2.3 we have $G_\infty, \tilde{W}_\infty \in C^1(E, \mathbb{R})$. \square

Lemma 4.2. *Under the assumptions of Theorem 1.2 the critical points of Φ_∞ are classical solutions of (1.3). Moreover, any critical point u of Φ_∞ satisfies $\dot{u}(t) \rightarrow 0$ as $|t| \rightarrow \infty$, i.e. it is a homoclinic solution of (1.3) emanating from the origin.*

Proof. The first part can be dealt with like in [12]. Therefore we will only focus on the second statement.

We shall prove that if $u \in E$ is a critical point of Φ_∞ , then $\ddot{u} \in L^2$, which, obviously implies that $\dot{u} \in E$, and therefore $\dot{u}(t) \rightarrow 0$ as $|t| \rightarrow \infty$.

Since $g'(x) = o(x)$ and $W'(x) = o(x)$ as $x \rightarrow 0$, there is a $\delta > 0$ such that

$$\max(|g'(x)|, |W'(x)|) \leq |x| \text{ for } |x| \leq \delta.$$

Because $u \in E$, we have $u(t) \rightarrow 0$ as $t \rightarrow \infty$, and therefore, there is an $r_1 > 0$ such that

$$|u(t)| \leq \delta/2 \text{ for all } |t| \geq r_1.$$

It then follows that

$$|Au(t)| \leq \delta \text{ for } t \in (-\infty, -r_1 - 1] \cup [r_1, \infty),$$

and

$$|A^*u(t)| \leq \delta \text{ for } t \in (-\infty, -r_1] \cup [r_1 + 1, \infty).$$

Thanks to (1.3), we have, for $|t| \geq r_1 + 1$

$$\begin{aligned} \ddot{u}^2 &\leq 6c^{-2} [\omega_0^2 u^2 + \alpha^2 (|Au|^2 + |A^*u|^2) + |g'(u)|^2 + |W'(Au)|^2 + |W'(A^*u)|^2] \\ &\leq 6c^{-2} [\omega_0^2 u^2 + \alpha^2 (|Au|^2 + |A^*u|^2) + (u^2 + |Au|^2 + |A^*u|^2)] \\ &\leq 6c^{-2} (1 + \max(\alpha^2, \omega_0^2)) (u^2 + |Au|^2 + |A^*u|^2). \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} \ddot{u}^2 &\leq c^{-2} \int_{-r_1-1}^{r_1+1} \ddot{u}^2 + 6c^{-2} (1 + \max(\alpha^2, \omega_0^2)) \int_{|t| \geq r_1+1} (u^2 + |Au|^2 + |A^*u|^2) \\ &\leq c^{-2} \int_{-r_1-1}^{r_1+1} \ddot{u}^2 + 6c^{-2} (1 + \max(\alpha^2, \omega_0^2)) \int_{\mathbb{R}} (u^2 + |Au|^2 + |A^*u|^2) \\ &\leq c^{-2} \int_{-r_1-1}^{r_1+1} \ddot{u}^2 + 6c^{-2} (1 + \max(\alpha^2, \omega_0^2)) (\|u\|_{L^2}^2 + 2\|\dot{u}\|_{L^2}^2) \\ &\leq c^{-2} \int_{-r_1-1}^{r_1+1} \ddot{u}^2 + 12c^{-2} (1 + \max(\alpha^2, \omega_0^2)) \|u\|_E^2. \end{aligned}$$

This shows that $\ddot{u} \in L^2$, and the proof is complete. \square

4.1. Construction of the family $U^{[0]}$. For convenience, we shall now define Φ_τ as an integral over $[-\tau/2, \tau/2]$.

Let e_0 be a non-zero member of $C^1([-1, 1])$ whose support lies inside $(-1, 0)$. Denote by e_2 the 2-periodic extension of e_0 on \mathbb{R} . Given $\tau > 4$, we denote by e the continuous extension of e_0 on $[-\tau/2, \tau/2]$, i.e. e agrees with e_0 on $[-1, 1]$ and vanishes elsewhere. Let us now extend e on the whole \mathbb{R} in a τ -periodic fashion and denote that extension by e_τ , i.e. e_τ is τ -periodic and agrees with e on $[-\tau/2, \tau/2]$.

Obviously, for every $\tau \geq 4$ we have

$$\|e_\tau\|_{E_\tau} = \|e_0\|_{H^1(-1,0)},$$

and for $|t| \leq \tau/2$

$$Ae_\tau(t) = \begin{cases} Ae_2(t) & \text{if } t \in [-2, 0] \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, for any $r > 0$, we have

$$\int_{-\tau/2}^{\tau/2} V(rAe_\tau) = \int_{-2}^0 V(rAe_2)$$

$$\begin{aligned}
&= \int_{-2}^{-1} V(rAe_2) + \int_{-1}^0 V(rAe_2) \\
&= \int_0^1 V(rAe_2) + \int_{-1}^0 V(rAe_2) \\
&= \int_{-1}^1 V(rAe_2).
\end{aligned}$$

The third line follows from the fact that e_2 is 2-periodic.

Since $f(0) = 0$, it follows that

$$\Phi_\tau(re_\tau) = \Phi_2(re_0).$$

e_0 being a non-zero element of H_2^1 , for $r > 0$ sufficiently large we have $\Phi_2(re_0) \leq 0$.

Now, set

$$\bar{e}_\tau = re_\tau,$$

where $r > 0$ is large enough so that $\Phi_2(re_0) \leq 0$. Denote by b_τ the critical value of Φ_τ given by (3.6), with

$$\Gamma = \Gamma_\tau := \{\gamma \in C([0, 1], H_\tau^1) \mid \gamma(0) = 0 \text{ and } \gamma(1) = \bar{e}_\tau\}.$$

Let u_τ be the corresponding critical point.

Lemma 4.3. *Under the assumptions of Theorem 1.2, there are positive constants b_0, δ_0 and M_0 independent of τ such that $b_\tau \leq b_0$ and $\delta_0 \leq \|u_\tau\|_{L_\tau^\infty} \leq M_0$ for all $\tau \geq 4$. Furthermore, there is $\tau_0 > 0$ such that for any $\tau \geq \tau_0$, u_τ is a non-constant τ -periodic solution of (1.3).*

Proof. *Uniform upper bound for b_τ :* Let $\gamma_\tau \in \Gamma_\tau$ be given by $\gamma_\tau(s) = s\bar{e}_\tau$. Then

$$\Phi_\tau(\gamma_\tau(s)) = \Phi_2(\gamma_2(s)),$$

and we deduce that

$$(4.3) \quad b_\tau \leq \max_{0 \leq s \leq 1} \Phi_2(\gamma_2(s)) =: b_0.$$

Uniform upper bound for $\|u_\tau\|_{E_\tau}$: Note that

$$b_\tau = \Phi_\tau(u_\tau) - \frac{1}{\beta} \Phi'_\tau(u_\tau)u_\tau \geq \left(\frac{1}{2} - \frac{1}{\beta}\right)\epsilon_0 \|u_\tau\|_{E_\tau}^2,$$

where the inequality follows from (A.3). Thus, by (2.7) we have

$$(4.4) \quad \|u_\tau\|_{E_\tau} \leq \left(\frac{2\beta b_0}{(\beta - 2)\epsilon_0}\right)^{1/2}.$$

Thanks to Proposition 2.1 and (4.4) we get

$$(4.5) \quad \|u_\tau\|_{L_\tau^\infty} \leq \sqrt{2}\|u_\tau\|_{E_\tau} \leq M_0 := 2 \left(\frac{\beta b_0}{(\beta - 2)\epsilon_0}\right)^{1/2}.$$

Let $Y_1, Y_2 : [0, \infty) \rightarrow [0, \infty)$ be defined by $Y_1(s) = 0 = Y_2(s)$ if $s = 0$ and for $s > 0$

$$Y_1(s) = \max_{0 < |x| \leq s} x^{-1}g'(x), \quad Y_2(s) = \max_{0 < |x| \leq s} x^{-1}W'(x).$$

Clearly the maps $s \mapsto Y_1(s)$ and $s \mapsto Y_2(s)$ are continuous, non-decreasing, and non-negative, and so is $s \mapsto Y(s) = Y_1(s) + Y_2(2s)$.

For every t we have

$$|u_\tau(t)| \leq s_\tau := \|u_\tau\|_{L_\tau^\infty},$$

for every t , it readily follows from the definition of Y_1 that

$$\frac{g'(u_\tau(t))}{u_\tau(t)} \leq Y_1(s_\tau),$$

for every t for which the left hand side is well defined. Similarly, we have

$$\frac{W'(Au_\tau(t))}{Au_\tau(t)} \leq Y_2(2s_\tau),$$

whenever the left hand side is well defined.

Using the fact that u_τ is a critical point for Φ_τ , we get

$$\begin{aligned} \epsilon_0 \|u_\tau\|_{H_\tau^1}^2 &\leq B_\tau(u_\tau, u_\tau) \\ &= \int_{-\tau/2}^{\tau/2} [g'(u_\tau)u_\tau + W'(Au_\tau)Au_\tau] \\ &\leq \int_{-\tau/2}^{\tau/2} [Y_1(s_\tau)u_\tau^2 + Y_2(2s_\tau)(Au_\tau)^2] \\ &\leq Y_1(s_\tau)\|u_\tau\|_{L_\tau^2}^2 + Y_2(2s_\tau)\|\dot{u}_\tau\|_{L_\tau^2}^2 \\ &\leq Y(s_\tau)\|u_\tau\|_{H_\tau^1}^2. \end{aligned}$$

Since u_τ is non-trivial, we have

$$Y(s_\tau) \geq \epsilon_0.$$

Hence, thanks to the aforementioned properties of Y , it results that

$$(4.6) \quad \|u_\tau\|_{L_\tau^\infty} \geq \delta_0,$$

where δ_0 is a positive constant which is independent of τ .

Existence of τ_0 . Suppose all the u_τ 's are constants. Then we have

$$\delta_0 \leq \|u_\tau\|_{L_\tau^\infty} = |u_\tau(0)| = \frac{\|u_\tau\|_{H_\tau^1}}{\sqrt{\tau}} \leq \frac{M_0}{\sqrt{2\tau}},$$

for all $\tau \geq 4$, contradicting the fact that $\delta_0 \neq 0$. \square

We set

$$U^{[0]} = \{u_\tau\}_{\tau \geq \tau_0}, \quad U^{[1]} = \{\dot{u}_\tau\}_{\tau \geq \tau_0}, \quad U^{[2]} = \{\ddot{u}_\tau\}_{\tau \geq \tau_0}.$$

Lemma 4.4. *For each $i = 0, 1, 2$, the family $U^{[i]}$ is uniformly bounded in the space $C_b(\mathbb{R})$ equipped with the sup-norm. Furthermore the families $U^{[0]}$ and $U^{[1]}$ are equicontinuous.*

Proof. The boundedness of $U^{[0]}$ readily follows from (4.5).

Boundedness of $U^{[2]}$. Given $w \in U^{[2]}$, there is a $\tau \geq \tau_0$ such that $w = \ddot{u}_\tau$, and u_τ satisfies (1.3). Since V' and f' are continuous, and the image of u_τ is contained in $[-M_0, M_0]$, it follows from (1.3) that, for all t ,

$$\begin{aligned} |w(t)| &\leq c^{-2} \max_{t \in \mathbb{R}} [|f'(u_\tau(t))| + |V'(Au_\tau(t))| + |V'(A^*u_\tau(t))|] \\ &\leq c^{-2} \left[\max_{|x| \leq M_0} |f'(x)| + 2 \max_{|x| \leq 2M_0} |V'(x)| \right] \\ &= M_2. \end{aligned}$$

Thus

$$\|w\|_{L^\infty} \leq M_2 \quad (\forall w \in U^{[2]}).$$

Boundedness of $U^{[1]}$. Given $v \in U^{[1]}$, there exists $\tau \geq \tau_0$ such that $v = \dot{u}_\tau$. By the mean value theorem, we have

$$v(t_\tau) = \int_{t-1}^t v = u_\tau(t) - u_\tau(t-1)$$

for some $t_\tau \in [t-1, t]$. It follows that

$$|v(t)| = \left| v(t_\tau) + \int_{t_\tau}^t \dot{v}(s) ds \right| \leq |u_\tau(t) - u_\tau(t-1)| + \int_{t_\tau}^t |\dot{v}| ds$$

$$\begin{aligned} &\leq 2M_0 + \int_{t-1}^t M_2 ds \\ &= 2M_0 + M_2. \end{aligned}$$

Thus

$$\|v\|_{L^\infty} \leq M_1 := 2M_0 + M_2.$$

Equicontinuity: Given $u \in U^{[0]}$, and $t_1, t_2 \in \mathbb{R}$, we have

$$\begin{aligned} |u(t_2) - u(t_1)| &= \left| \int_{t_1}^{t_2} \dot{u} \right| \leq M_1 |t_2 - t_1| \\ |\dot{u}(t_2) - \dot{u}(t_1)| &= \left| \int_{t_1}^{t_2} \ddot{u} \right| \leq M_2 |t_2 - t_1|. \end{aligned}$$

□

4.2. Existence of homoclinics.

Lemma 4.5. *There is a sequence $(\bar{u}_k) \subset U^{[0]}$ which converges to a non-trivial critical point $\bar{u} \in E$ of Φ_∞ .*

Proof. In view of Lemma 4.4 and thanks to Arzelà-Ascoli's Theorem, there exists a sequence (\bar{u}_k) , with $\bar{u}_k = u_{\tau_k}$, such that $\bar{u}_k \rightarrow \bar{u}$ in $C_{\text{loc}}^1(\mathbb{R})$. Since each member of $U^{[0]}$ satisfies (1.3), the convergence is actually in $C_{\text{loc}}^2(\mathbb{R})$.

By (4.5), one infers hat

$$\int_{\mathbb{R}} (\dot{\bar{u}}^2 + \bar{u}^2) \leq M_0^2/2,$$

i.e. $\bar{u} \in E$.

Let ξ be a test function on \mathbb{R} . Denote by I_0 and I_1 the supports of $\xi(\cdot)$ and $\xi(\cdot+1)$ respectively. Let $k \in \mathbb{N}$ be sufficiently large, so that $I := I_0 \cup I_1 \subset (-\tau_k/2, \tau_k/2)$. Then, we have

$$\begin{aligned} \Phi'_\infty(\bar{u})\xi &= \Phi'_\infty(\bar{u})\xi - \Phi'_{\tau_k}(\bar{u}_k)\xi \\ &= (B_I(\bar{u} - \bar{u}_k, \xi)) + (G'_I(\bar{u})\xi - G'_I(\bar{u}_k)\xi) + (\tilde{W}'_I(\bar{u})\xi - \tilde{W}'_I(\bar{u}_k)\xi), \end{aligned}$$

with

$$\begin{aligned} B_I(v, \xi) &= \int_I [c^2 \dot{v} \dot{\xi} - \omega_0 v \xi + \alpha A v A \xi], \\ G'_I(v) \xi &= \int_I g'(v) \xi, \\ \tilde{W}'_I(v) \xi &= \int_I W'(A v) \xi. \end{aligned}$$

Note that

$$\begin{aligned} |B_I(\bar{u} - \bar{u}_k, \xi)| &= \left| \int_I [c^2 (\dot{\bar{u}} - \dot{\bar{u}}_k) \dot{\xi} - \omega_0 (\bar{u} - \bar{u}_k) \xi + \alpha A (\bar{u} - \bar{u}_k) A \xi] \right| \\ &\leq |I|^{1/2} \|\xi\|_E (c^2 \|\dot{\bar{u}} - \dot{\bar{u}}_k\|_{L^\infty(I)} + (\omega_0 + 2|\alpha|) \|\bar{u} - \bar{u}_k\|_{L^\infty(I)}), \end{aligned}$$

where $|I|$ is the length of I . Since $(\bar{u}_k, \dot{\bar{u}}_k) \rightarrow (\bar{u}, \dot{\bar{u}})$ uniformly on compact subsets of \mathbb{R} , it follows from the above estimate that $B_I(\bar{u} - \bar{u}_k, \xi) \rightarrow 0$ as $k \rightarrow \infty$.

Similarly, we have

$$\begin{aligned} |G'_I(\bar{u})\xi - G'_I(\bar{u}_k)\xi| &\leq |I|^{1/2} \|\xi\|_E \|g'(\bar{u}) - g'(\bar{u}_k)\|_{L^\infty(I)} \\ |\tilde{W}'_I(\bar{u})\xi - \tilde{W}'_I(\bar{u}_k)\xi| &\leq |I|^{1/2} \|\xi\|_E \|W'(A\bar{u}) - W'(A\bar{u}_k)\|_{L^\infty(I)} \end{aligned}$$

It then follows from the continuity of g' , and W' , and the uniform convergence of $(\bar{u}_k, \dot{\bar{u}}_k)$ on compact subsets of \mathbb{R} that

$$\lim_{k \rightarrow \infty} |G'_I(\bar{u})\xi - G'_I(\bar{u}_k)\xi| = 0,$$

$$\lim_{k \rightarrow \infty} |\tilde{W}'_I(\bar{u})\xi - \tilde{W}'_I(\bar{u}_k)\xi| = 0.$$

Thus

$$|\Phi'_\infty(\bar{u})\xi| = \lim_{k \rightarrow \infty} |\Phi'_\infty(\bar{u})\xi - \Phi'_{\tau_k}(\bar{u}_k)\xi| = 0.$$

Since ξ is arbitrary, we conclude that $\Phi'_\infty(\bar{u}) = 0$, i.e. \bar{u} is a critical point of Φ_∞ .

To prove that $\bar{u} \neq 0$, take the limit in (4.6), with u_τ replaced by \bar{u}_k . We get

$$\sup_{t \in \mathbb{R}} |\bar{u}(t)| \geq \delta_0 > 0,$$

which shows that $\bar{u} \neq 0$. □

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