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Subharmonics and homoclinics for a class of Hamiltonian-like equations

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SUBHARMONICS AND HOMOCLINICS FOR A CLASS OF HAMILTONIAN-LIKE EQUATIONS

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Abstract. We study the existence of periodic and homoclinic solutions for a class of non-autonomous second order advanced-delayed differential equations of the type

$$\ddot{u}(t) + f_0(t, u(t)) = \sum_{i=1}^{N} \left[f_i(t, u(t+\tau_i) - u(t)) - f_i(t-\tau_i, u(t) - u(t-\tau_i)) \right].$$

We prove, under some growth conditions on the non-linearities, the existence of non-constant periodic solutions with period any given positive integer. Using very simple arguments, the existence of a non-trivial homoclinic solution is also established. This homoclinic is obtained as the limit of subharmonics as the period goes to infinity. An application to the existence of periodic and homoclinic travelling waves in an infinite lattice of partciles with N -nearest-neighbour interaction and on-site potential is given.

1. Statement of the results

This note is concerned with the study of second order advanced-delayed ordinary differential equations of the form:

$$\ddot{u} + f_0(t, u) = \sum_{i=1}^{N} \left[f_i(t, A_{\tau_i} u) - f_i(t-\tau_i, A^*_{\tau_i} u) \right], \ t \in \mathbb{R}, \tag{1.1}$$

where $f_0, f_1, \dots, f_N \in C(\mathbb{R}^2)$ and $\tau_1, \dots, \tau_N > 0$, with $N \geq 1$. For $\tau > 0$, the forward and backward-difference operators A_τ and A^*_τ are defined by

$$A_\tau u(t) = u(t+\tau) - u(t) = A^*_\tau u(t+\tau). \tag{1.2}$$

The study of this class of ordinary differential equations is motivated both by [8] and [3]. In the former paper, D. Smets proved the existence of multibump type solutions for travelling waves in *non-autonomous* infinite lattices with nearest-neighbour interaction, without on-site potential. In [3], the author studied travelling waves in autonomous infinite lattices with nearest- neighbour interaction and on-site potential. Those travelling waves are solutions of an equation of the type (1.1), with $N = 1 = \tau_1$ and autonomous non-linearities. Roughly speaking the results of [3] say that there exist periodic solutions of any given period with relatively high speed provided the non-linearities satisfy some growth condition at infinity, and that, when the non-linearities satisfy some global growth condition, there exist non-trivial homoclinic solutions emanating from the origin of any given speed.

The aim of this note is to give some generalizations of the results obtained in [3]. A solution u of (1.1) is said to be periodic, say with period $k > 0$, if $u(\cdot + k) = u(\cdot)$. Suppose (1.1) possesses a stationary solution u_0. A solution $u \not\equiv u_0$ is said to be

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homoclinic to u_0 if $u(\pm\infty) = u_0$ and $\dot{u}(\pm\infty) = 0$, where the notation $u(\infty)$ stands for the limit of u at infinity, etc.

Statements of the results. Given an $f \in \{f_0, f_1, \dots, f_N\}$, we set

$$\mathcal{P}f(t,x) = \int_0^x f(t,z)dz.$$

We shall consider non-linearities $f_0, \dots, f_N$ of the type

(A.0) $f_i(x) = \alpha_i(t)x + g_i(t,x)$ for $i = 0, 1, \dots, N$,

where the g_i's satisfy either

(A.1) $g_i(t,x) = o(x)$ as $x \to 0$, uniformly in t, and

(A.2) $\mathcal{P}g_i \geq 0$ and there are constants $\beta > 2, r_0 > 0$ such that

$$0 < \beta\mathcal{P}g_i(\cdot,x) \leq xg_i(\cdot,x) \text{ for } |x| \geq r_0,$$

or

(A.3) there is a constant $\beta > 2$ such that

$$0 < \beta\mathcal{P}g_i(\cdot,x) \leq xg_i(\cdot,x) \text{ for } x \neq 0.$$

Clearly, (A.3) and the periodicity imply (A.1), and (A.2) implies that

(A.4) there are constants $a_0, a_1 > 0$ such that

$$g_i(\cdot,x) \geq a_0|x|^\beta - a_1 \text{ for all } x.$$

We shall denote by $\mathbb{T}, C(\mathbb{T})$ and $C(\mathbb{T}\times\mathbb{R})$ the unit circle, the space of 1-periodic continuous functions and the space of continuous functions in two variables which are 1-periodic in the first variable, respectively. Given a real-valued function α, we denote by α^+ and α^- its positive and negative part, respectively, i.e. $\alpha^+ = \max(0,\alpha)$ and $\alpha^- = -\min(0,\alpha)$.

Let $c_0 \geq 0$ be given by

$$c_0^2 = \begin{cases} 0 & \text{if } \quad 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} < \min\omega < \infty \\ \sum_{i=1}^N \tau_i\|\alpha_i^+\|_{L^\infty} & \text{if } \quad 0 \leq \min\omega \leq 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \end{cases}. \tag{1.3}$$

The main results of this note are the following

Theorem 1.1. *Let $f_0, f_1, \dots, f_N \in C(\mathbb{T}\times\mathbb{R})$ be given by (A.0), with $\alpha_0 \equiv 0$, and such that the g_i's satisfy (A.1),(A.2). Suppose in addition that $c_0 < 1$. Then, for any positive interger k, and every $\tau_1, \dots, \tau_N > 0$,* (1.1) *possesses a non-constant k-periodic solution.*

Theorem 1.2. *Let $f_0, f_1, \dots, f_N \in C(\mathbb{T}\times\mathbb{R})$ be given by (A.0). Suppose $\omega \equiv -\alpha_0 > 0$, and the g_i's satisfy (A.1),(A.3). If $c_0 < 1$, then, for every $\tau_1, \dots, \tau_N > 0$,* (1.1) *possesses a non-trivial homoclinic solution emanating from the origin.*

Remark. Theorem 1.1 and 1.2 hold also true when some of the α_i's or g_i's are time-independent, or when some of the g_i's are identically equal to zero. When all the f_i's are time independent, the period k in Theorem 1.1 is allowed to take any positive real value. □

Equation (1.1) can be interpreted as the Euler-Lagrange equation of the functional

$$\Phi(u) = \int_{\mathcal{T}} \left[\frac{1}{2}\dot{u}^2 - \mathcal{P}f_0(t,u) - \sum_{i=1}^N \mathcal{P}f_i(t, A_{\tau_i}u)\right], \tag{1.4}$$

on some appropriate Hilbert space, where $\mathcal{T} \subseteq \mathbb{R}$. When dealing with periodic solutions, say with period $k \in \mathbb{N}$, we shall take as $\mathcal{T}$ a segment of length k, and shall denote the functional (1.4) by Φ_k. In that case the natural space to work on is the space H_k^1 of k-periodic functions $u : \mathbb{R} \to \mathbb{R}$ whose restriction to $[0,k]$ belong to

$H^1([0,k])$. When dealing with homoclinic solutions, $\mathcal{T} = \mathbb{R}$, and the natural space to work on is $E = H^1(\mathbb{R})$. The functional (1.4) shall then be denoted by Φ_∞.

Organization of the paper. In the next section we collect some prelimaries results that are needed in the sequel. The second and third sections are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. In the last we give some application to the problem of travelling waves in infinite lattices with N-nearest neighbour interaction and on-site potential.

2. Preliminary results

In this section we collect some of the results we need to prove the theorems stated above. We shall denote by L^p_k the space of k-periodic functions $u : \mathbb{R} \to \mathbb{R}$ whose restriction to $[0,k]$ belong $L^p([0,k])$. A similar meaning is attached to C^m_k.

Lemma 2.1. *Let $k \in \mathbb{N}$. Then, the finite difference-operator defined by* (1.3) *maps continuously H^1_k into $L^\infty_k \cap L^2_k$, with*

$$\|A_\tau u\|_{L^2_k} \le \sqrt{\tau}\|\dot u\|_{L^2_k}, \quad \|A_\tau u\|_{L^\infty_k} \le l(k,\tau)\|\dot u\|_{L^2_k}, \tag{2.1}$$

with

$$l(k,\tau) = \begin{cases} \sqrt{\tau([\tau/k]+1)} & \text{if} \quad k < \tau \\ \sqrt{\tau} & \text{if} \quad k \ge \tau \end{cases}, \tag{2.2}$$

where $[s]$ denotes the integer part of s. Furthermore, A_τ maps continuously E into $L^\infty \cap L^2$, with

$$\max(\|A_\tau u\|_{L^2}, \|A_\tau u\|_{L^\infty}) \le \sqrt{\tau}\|\dot u\|_{L^2}. \tag{2.3}$$

Proof. Let $u \in H^1_k$. Applying Jensen's inequality, the change of variable $s \leftrightarrow s+t$, and Fubini's Theorem, we have

$$\begin{aligned}\int_0^k |A_\tau u(t)|^2 dt = \int_0^k \left(\int_t^{t+\tau} \dot u(s)ds\right)^2 dt \quad &\le \quad \int_0^k \left(\int_0^\tau \dot u^2(s+t)ds\right) dt \\ &= \quad \int_0^\tau \left(\int_0^k \dot u^2(s+t)dt\right) ds \\ &= \quad \tau\|\dot u\|^2_{L^2_k}.\end{aligned}$$

For the second estimate we use the Cauchy-Schwarz inequality:

$$|A_\tau u(t)| \le \sqrt{\tau}\left(\int_t^{t+\tau} \dot u^2(s)ds\right)^{1/2}.$$

If $k \ge \tau$, then

$$|A_\tau u(t)| \le \sqrt{\tau}\left(\int_0^k \dot u^2(s)ds\right)^{1/2} = \sqrt{\tau}\|\dot u\|_{L^2_k}.$$

It $k < \tau$, we set $n = [\tau/k]$. Then $nk \le \tau < (n+1)k$, and

$$|A_\tau u(t)| \le \sqrt{\tau}\left(\int_t^{t+(n+1)k} \dot u^2(s)ds\right)^{1/2} = \sqrt{\tau(n+1)}\|\dot u\|_{L^2_k}.$$

Suppose now $u \in E$. Using the same arguments as above, we have

$$\begin{aligned}\int_{\mathbb{R}} |A_\tau u(t)|^2 dt = \int_{\mathbb{R}} \left(\int_t^{t+\tau} \dot u(s)ds\right)^2 dt \quad &\le \quad \int_{\mathbb{R}} \left(\int_0^\tau \dot u^2(s+t)ds\right) dt \\ &= \quad \int_0^\tau \left(\int_{\mathbb{R}} \dot u^2(s+t)dt\right) ds \\ &= \quad \tau\|\dot u\|^2_{L^2},\end{aligned}$$

and

$$|A_\tau u(t)| \leq \sqrt{\tau}\left(\int_t^{t+\tau} \dot{u}^2(s)ds\right)^{1/2} \leq \sqrt{\tau}\|\dot{u}\|_{L^2} \quad (\forall t).$$

□

Recall that if I is a compact interval, then the embeddings $H^1(I) \hookrightarrow C(I)$ and $H^1(I) \hookrightarrow L^2(I)$ are compact. In particular, there is a positive constant $C_s > 0$ such that

$$\|u\|_{H^1(I)} \leq C_s\|u\|_{L^\infty(I)} \quad (\forall u \in H^1(I)). \tag{2.4}$$

The constant C_s may actually be taken to be $\sqrt{2}$ when the length $|I|$ of I is greater than or equal to 1. Let us also recall that any member u of E satisfies $u(\pm\infty) = 0$.

Proposition 2.1. *Let I be a compact interval and $g \in C(I \times \mathbb{R})$. Then the functional $G_I : H^1(I) \to \mathbb{R}$ defined by*

$$G_I(u) = \int_I \mathcal{P}g(t, u)$$

is C^1, and its derivative is given by

$$G_I'(u)\xi = \langle g(\cdot, u), \xi\rangle_{L^2(I)}.$$

Proof. Since $g \in C(I \times \mathbb{R})$ and I is compact, we have

$$\sup_{\|\xi\|_{H^1(I)}=1} |\langle g(\cdot, u), \xi\rangle_{L^2(I)}| \leq \|g(\cdot, u)\|_{L^2(I)} \leq |I|^{1/2}\|g(\cdot, u)\|_{L^\infty(I)} < \infty,$$

i.e. the linear map $\xi \mapsto L_u\xi = \langle g(\cdot, u), \xi\rangle_{L^2(I)}$ is bounded.

On the other hand, if we set $P = I \times [0, 1]$, and for every $u, \xi \in H^1(I)$ denote by u_ξ the function defined on P by $u_\xi(t, s) = u(t) + s\xi(t)$, we have

$$\begin{aligned} |G_I(u + \xi) - G_I(u) - L_u\xi| &= \left|\int_P [\mathcal{P}g(t, u + \xi) - \mathcal{P}g(t, u) - g(t, u)\xi]\, dt\right| \\ &= \left|\int_P [g(t, u_\xi) - g(t, u)]\, \xi ds dt\right| \\ &\leq \|\xi\|_{L^2(I)}\|g(\cdot, u_\xi) - g(\cdot, u)\|_{L^2(P)} \\ &\leq |I|^{1/2}\|\xi\|_{H^1(I)}\|g(\cdot, u_\xi) - g(\cdot, u)\|_{L^\infty(P)}. \end{aligned}$$

Since $K = u_\xi(P)$ is a compact subset of $\mathbb{R}$, and g is continuous (therefore uniformly on compact subsets), for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in K$, with $|x-y| \leq \delta$ we have $|g(t, x)-g(t, y)| \leq \epsilon|I|^{-1/2}$ for all t. On the other hand, we have $\|\xi\|_{L^\infty(I)} = \|u_\xi - u\|_{L^\infty(P)}$, therefore, if we choose ξ such that $\|\xi\|_{H^1(I)} \leq \delta/C_s$, where C_s is given by (2.4), then we have

$$|G_I(u + \xi) - G_I(u) - L_u\xi| \leq \epsilon\|\xi\|_{H^1(I)},$$

i.e. G_I is (Fréchet) differentiable, with $G_I'(u) = L_u$ for all $u \in H^1(I)$.

Continuity of G_I'. Let u be a member of $H^1(I)$ and (u_m) a sequence in $H^1(I)$ that converges to u. We have

$$\begin{aligned} |G_I'(u)\xi - G_I'(u_m)\xi| &= |\langle g(\cdot, u) - g(\cdot, u_m), \xi\rangle_{L^2(I)}| \\ &\leq \|g(\cdot, u) - g(\cdot, u_m)\|_{L^2(I)}\|\xi\|_{L^2(I)} \\ &\leq |I|^{1/2}\|\xi\|_{H^1(I)}\|g(\cdot, u) - g(\cdot, u_m)\|_{L^\infty(I)}. \end{aligned}$$

Since $H^1(I)$ is continuously embedded into $L^\infty(I)$ (the embedding is actually compact, but it is not needed), and g is uniformly continuous on compact subsets of $\mathbb{R}^2$, it follows that that $g(\cdot, u_m) \to g(\cdot, u)$ uniformly on I. Thus, for any positive number ϵ we have

$$\|g(\cdot, u) - g(\cdot, u_m)\|_{L^\infty(I)} \leq \epsilon|I|^{-1/2}$$

for m sufficiently large. Hence, for m large enough we have

$$\sup_{\|\xi\|_{H^1(I)}=1} |G_I'(u)\xi - G_I'(u_m)\xi| \leq \epsilon$$

i.e. $G_I'(u_m) \to G_I'(u)$ as $m \to \infty$. □

Proposition 2.2. *Let $g \in C(\mathbb{T}\times\mathbb{R})$ satisfies (A.1). Then the functional $G_\infty : E \to \mathbb{R}$ defined by*

$$G_\infty(u) = \int_{\mathbb{R}} \mathcal{P}g(t,u)$$

is C^1, and its derivative is given by

$$G_\infty'(u)\xi = \langle g(\cdot,u),\xi\rangle_{L^2}.$$

Proof. Let us first make sure that G_∞ is well-defined, i.e. it takes only finite values.

Thanks to (A.1), there is a $\delta > 0$ such that

$$|\mathcal{P}g(\cdot,x)| \leq x^2 \text{ for } |x| \leq \delta.$$

If $u \in E$, then $u(\pm\infty) = 0$, and therefore there is a positive number r depending on δ such that if $|t| \geq r$, then

$$|u(t)| \leq \frac{\delta}{2}.$$

It follows that

$$\begin{aligned} |G_\infty(u)| &\leq |\int_{|t|\leq r} \mathcal{P}g(t,u)| + \int_{|t|\geq r} u^2 \\ &\leq |\int_{|t|\leq r} \mathcal{P}g(t,u)| + \|u\|_{L^2}^2 < \infty, \end{aligned}$$

i.e. $-\infty < G_\infty(u) < \infty$.

Differentiability of G_∞: Let $u \in E$ be fixed and let $\epsilon > 0$. Thanks to (A.1), there is a positive number ρ_0 such that if $|x| \leq \rho_0$, then

$$|g(\cdot,x)| \leq \frac{\epsilon|x|}{3\tau(1+\|u\|_E)}. \tag{2.5}$$

Since $u(\pm\infty) = 0$, there is an r such that $|u(t)| \leq \rho_0/2$ whenever $|t| \geq r$. We set $I_r = [-r,r]$, and $\bar{I}_r^c = \mathbb{R}\setminus(-r,r)$. Thanks to Propsoition 2.1, we have $G_{I_r} \in C^1(H^1(I_r),\mathbb{R})$. Therefore, there is a positive number $\delta = \delta(\epsilon,r,u)$ (there is of course no loss of generality in assuming that $\delta \leq \min(1,\rho_0/2)$) such that if $\|\xi\|_E \leq \delta$, then

$$|G_{I_r}(u+\xi) - G_{I_r}(u) - G_{I_r}'(u)\xi| \leq \frac{\epsilon}{3}\|\xi\|_E. \tag{2.6}$$

Thanks to the mean value theorem, (2.5), and (2.3), we get

$$|\mathcal{P}g(t,u+\xi) - \mathcal{P}g(t,u)| \leq \epsilon|\xi|\frac{|u|+|\xi|}{3\tau(1+\|u\|_E)} \quad (\forall t \in \bar{I}_r^c).$$

It follows that

$$\begin{aligned} \int_{\bar{I}_r^c} |\mathcal{P}g(t,u+\xi) - \mathcal{P}g(t,u)| &\leq \frac{\epsilon}{3\tau(1+\|u\|_E)}\int_{\bar{I}_r^c} |\xi|(|u|+|\xi|) \\ &\leq \frac{\epsilon}{3\tau(1+\|u\|_E)}\|\xi\|_{L^2}\||u|+|\xi|\|_{L^2} \\ &\leq \frac{\epsilon}{3(1+\|u\|_E)}\|\xi\|_E(\|u\|_E+\|\xi\|_E) \\ &\leq \frac{\epsilon}{3}\|\xi\|_E. \end{aligned} \tag{2.7}$$

Again, by (2.5) we have

$$\begin{aligned}
\int_{\bar{I}_r^c} |g(t,u)\xi| &\leq \frac{\epsilon}{3\tau(1+\|u\|_E)}\int_{\bar{I}_r^c}|\xi||u| \\
&\leq \frac{\epsilon}{3\tau(1+\|u\|_E)}\|\xi\|_{L^2}\|u\|_{L^2} \\
&\leq \frac{\epsilon}{3(1+\|u\|_E)}\|\xi\|_E\|u\|_E \\
&\leq \frac{\epsilon}{3}\|\xi\|_E. \qquad (2.8)
\end{aligned}$$

Combining (2.6),(2.7), and (2.8) we get

$$\begin{aligned}
|G_\infty(u+\xi) - G_\infty(u) - \langle g(\cdot,u),\xi\rangle_{L^2}| &\leq |G_{I_r}(u+\xi) - G_{I_r}(u) - G'_{I_r}(u)\xi| \\
&+ \int_{\bar{I}_r^c} |\mathcal{P}g(t,u+\xi) - \mathcal{P}g(t,u)| \\
&+ \int_{\bar{I}_r^c} |g(t,u)\xi| \\
&\leq \frac{\epsilon}{3}\|\xi\|_E + \frac{\epsilon}{3}\|\xi\|_E + \frac{\epsilon}{3}\|\xi\|_E \\
&= \epsilon\|\xi\|_E,
\end{aligned}$$

i.e. G_∞ is differentiable, and the derivative is given precisely by

$$G'_\infty(u)\xi = \langle g(\cdot,u),\xi\rangle_{L^2}.$$

Continuity of G'_∞. Let $u \in E$ and $(u_m) \subset E$ a sequence that converges to u. Then, for some constant $K_0 \geq 0$ we have $\|u_m\|_E \leq K_0$ for all m. By (A.1), given $\epsilon > 0$, there is a positive number r such that if $|t| \geq r$, then

$$|g(t,u)| \leq \frac{\epsilon|u|}{4(1+\|u\|_E)} \text{ and } |g(t,u_m)| \leq \frac{\epsilon|u_m|}{4(1+K_0)},$$

for m suffisciently large. Setting $I_r = [-r,r]$ and $I_r^c = \mathbb{R}\setminus I_r$, we have

$$\begin{aligned}
|G'_\infty(u)\xi - G'_\infty(u_m)\xi| &\leq |\langle g(\cdot,u) - g(\cdot,u_m),\xi\rangle_{L^2}| \\
&\leq \|g(t,u) - g(t,u_m)\|_{L^2(I_r)}\|\xi\|_{L^2(I_r)} \\
&+ \|g(\cdot,u) - g(\cdot,u_m)\|_{L^2(\bar{I}_r^c)}\|\xi\|_{L^2(\bar{I}_r^c)} \\
&\leq \sqrt{2r}\|g(\cdot,u) - g(\cdot,u_m)\|_{L^\infty(I_r)} \\
&+ \left(\|g(\cdot,u)\|_{L^2(\bar{I}_r^c)} + \|g(\cdot,u_m)\|_{L^2(\bar{I}_r^c)}\right)\|\xi\|_{L^2} \\
&\leq \sqrt{2r}\|g(\cdot,u) - g(\cdot,u_m)\|_{L^\infty(I_r)}\|\xi\|_{L^2} \\
&+ \frac{\epsilon}{4}\left(\frac{\|u\|_{L^2}}{1+\|u\|_E} + \frac{\|u_m\|_{L^2}}{1+C}\right)\|\xi\|_{L^2} \\
&\leq \left(\sqrt{2r}\|g(\cdot,u) - g(\cdot,u_m)\|_{L^\infty(I_r)} + \frac{\epsilon}{2}\right)\|\xi\|_E.
\end{aligned}$$

Thus we have

$$\sup_{\|\xi\|_E=1}|G'_\infty(u)\xi - G'_\infty(u_m)\xi| \leq \sqrt{2r}\|g(\cdot,u) - g(\cdot,u_m)\|_{L^\infty(I_r)} + \frac{\epsilon}{2}.$$

Since $u_m \to u$ in E implies $u_m \to u$ on compact subsets of $\mathbb{R}$, therefore the boundedness of (u_m) in $L^\infty(I_r)$ and the uniform continuity of g on compact subsets of $\mathbb{R}^2$ implies that $g(\cdot,u_m) \to g(\cdot,u)$ uniformly on I_r, so that

$$\sqrt{2r}\|g(\cdot,u) - g(\cdot,u_m)\|_{L^\infty(I_r)} \leq \frac{\epsilon}{2}$$

for m sufficiently large. It results that

$$\sup_{\|\xi\|_E=1}|G'_\infty(u)\xi - G'_\infty(u_m)\xi| \leq \epsilon$$

for m large enough, i.e. G'_∞ is continuous. □

3. Existence of subharmonics

In this section we are going to give a proof of Theorem 1.1. We shall use a linking theorem of Rabinowitz, which we state below

Theorem 3.1 (Rabinowitz [4]). *Let $X = X_0 \oplus \hat{X}$ with $\dim X_0 < \infty$ and $J \in C^1(X)$ be (PS). Suppose in addition the following conditions are satisfied*

(J.3) $J|_{X_0} \leq 0$,
(J.4) *there are constants $\omega_*, \rho > 0$ such that $J|_{\hat{X}\cap(B_\rho\setminus\{0\})} > 0$ and $J|_{\hat{X}\cap S_\rho} \geq \omega_*$,*
(J.5) *for each finite-dimensional subspace $Y \subset X$, there is an $R = R(Y)$ such that $J \leq 0$ on $Y \setminus B_R$.*

Then, J possesses a positive critical value b characterized by

$$b = \inf_{h\in\Gamma} \max_{u\in \bar{B}_{R(X_1)}\cap X_1} J(h(u))$$

where

$$\Gamma = \{h \in C(\bar{B}_{R(X_1)} \cap X_1, X) | h(u) = u \text{ if } J(u) \leq 0\}$$

and $X_1 = X_0 \oplus \operatorname{span}\{v\}$, for any non-zero $v \in \hat{X}$.

The notations $B_r, \bar{B}_r$ and S_r stand for the open ball, the closed ball and the sphere centered at 0 with radius r, respectively.

Recall that a sequence (u_m) is called a Palais-Smale sequence – (PS) sequence in short – for J if $J(u_m)$ is bounded and $J'(u_m) \to 0$. The functional $J \in C^1(X)$ is said to satisfy the Palais-Smale condition – in short we shall say J is (PS) – if every (PS) sequence is precompact.

Lemma 3.1. *Let $k \in \mathbb{N}$. Under the assumptions of Theorem 1.1, $\Phi_k \in C^1(H^1_k, \mathbb{R})$, and any critical point of Φ_k is a classical solution of* (1.1).

Proof. Write

$$\Phi_k(u) = \frac{1}{2}B_k(u,u) - G_{0,k}(u) - \sum_{i=1}^N G_{i,k}(A_{\tau_i}u),$$

with

$$\begin{aligned} B_k(u,v) &= \langle \dot{u}, \dot{v}\rangle_{L^2_k} + \langle \omega u, v\rangle_{L^2_k} - \sum_{i=1}^N \alpha_i A_{\tau_i}u, A_{\tau_i}v\rangle_{L^2_k}, \\ G_{i,k}(u) &= \int_0^k \mathcal{P}g_i(t,u), \quad i = 0,1,2,\dots,N. \end{aligned}$$

B_k is a bounded (symmetric) bilinear form on H^1_k. Indeed, we have

$$\begin{aligned} |B_k(u,v)| &\leq \|\dot{u}\|_{L^2_k}\|\dot{v}\|_{L^2_k} + \|\omega u\|_{L^2_k}\|v\|_{L^2_k} + \sum_{i=1}^N \|\alpha_i A_{\tau_i}u\|_{L^2_k}\|A_{\tau_i}v\|_{L^2_k} \\ &\leq \|\dot{u}\|_{L^2_k}\|\dot{v}\|_{L^2_k} + \|\omega\|_{L^\infty}\|u\|_{L^2_k}\|v\|_{L^2_k} + \sum_{i=1}^N \|\alpha_i\|_{L^\infty}\|A_{\tau_i}u\|_{L^2_k}\|A_{\tau_i}v\|_{L^2_k} \\ &\leq \|\dot{u}\|_{L^2_k}\|\dot{v}\|_{L^2_k} + \|\omega\|_{L^\infty}\|u\|_{L^2_k}\|v\|_{L^2_k} + \sum_{i=1}^N \tau_i\|\alpha_i\|_{L^\infty}\|\dot{u}\|_{L^2_k}\|\dot{v}\|_{L^2_k} \\ &= (1 + \sum_{i=1}^N \tau_i\|\alpha_i\|_{L^\infty})\|\dot{u}\|_{L^2_k}\|\dot{v}\|_{L^2_k} + \|\omega\|_{L^\infty}\|u\|_{L^2_k}\|v\|_{L^2_k} \\ &\leq \max(1 + \sum_{i=1}^N \tau_i\|\alpha_i\|_{L^\infty}, \|\omega\|_{L^\infty})\|u\|_{H^1_k}\|v\|_{H^1_k}. \end{aligned}$$

It follows that $u \mapsto B_k(u,u)$ is C^∞. Since each of the operators $A_{\tau_i}: H^1_k \to H^1_k$ is bounded, we deduce from Proposition 2.1 that $G_{0,k}, G_{1,k}\circ A_{\tau_1}, \dots, G_{N,k}\circ A_{\tau_N} \in C^1(H^1_k, \mathbb{R})$. The remaining part is standard and shall therefore be omitted. □

Proposition 3.1. *Let $k \in \mathbb{N}$. Under the assumptions of Theorem 1.1, Φ_k is (PS).*

Proof. We first prove the boundedness of (PS) sequences, and next we prove their precompactness.

Boundedness: Given $s > 0$ we define the quadratic form $\mathcal{N}_s^2$ by

$$\mathcal{N}_s^2(u) = \|\dot{u}\|^2_{L^2_k} + s\|u\|^2_{L^2_k} - \sum_{i=1}^N \langle \alpha_i A_{\tau_i} u, A_{\tau_i} u\rangle_{L^2_k} \quad (\forall u \in H^1_k).$$

Then $\mathcal{N}_s = (\mathcal{N}_s^2)^{1/2}$ is a norm on H^1_k which is equivalent to the standard one.
Let $(u_m) \subset H^1_k$ be a (PS) sequence for Φ_k, i.e. for some constant $M \geq 0$ we have

$$|\Phi_k(u_m)| \leq M \quad (\forall m), \text{ and } \lim_{m\to\infty} \Phi'_k(u_m) = 0.$$

Then, for some positive integer m_0 we have $\|\Phi'_k(u_m)\| \leq 1$ whenever $m \geq m_0$.
Fixing $m \geq m_0$, we have

$$|\Phi'_k(u_m)u_m| \leq \mathcal{N}_s(u_m),$$

which implies

$$\begin{aligned} G'_{0,k}(u_m)u_m + \sum_{i=1}^N G'_{i,k}(, A_{\tau_i}u_m)A_{\tau_i}u_m &= -\Phi_k(u_m)u_m + \mathcal{N}_s^2(u_m) - s\|u_m\|^2_{L^2_\tau} \\ &\leq \mathcal{N}_s(u_m) + \mathcal{N}_s^2(u_m). \end{aligned} \tag{3.1}$$

We set

$$I_0 = \{t \in [0,k] : |u_m(t)| \leq r_0\}, \quad I_i = \{t \in [0,k] : |A_{\tau_i}u_m(t)| \leq r_0\}, 1 \leq i \leq N$$

and $\bar{I}_i = [0,k] \setminus I_i$ for $i = 0, 1, \dots, N$. Then

$$\int_{I_0} \mathcal{P}g_0(u_m) + \sum_{i=1}^N \int_{I_i} \mathcal{P}g_i(t, A_{\tau_i}u_m) \quad \leq \quad M_0 :=: k\sum_{i=0}^N \max_{\Omega_0} g_i,$$

with $\Omega_0 = \mathbb{T} \times [-r_0, r_0]$, and thanks to (A.2) and (3.1), we get

$$\begin{aligned} G_{0,k}(u_m) \quad &+ \quad \sum_{i=1}^N G_{i,k}(A_{\tau_i}u_m) \\ &= \int_{I_0} \mathcal{P}g_0(u_m) + \sum_{i=1}^N \int_{I_i} \mathcal{P}g_i(t, A_{\tau_i}u_m) \\ &+ \int_{\bar{I}_0} \mathcal{P}g_0(t, u_m) + \sum_{i=1}^N \int_{\bar{I}_i} \mathcal{P}g_i(t, A_{\tau_i}u_m) \\ &\leq M_0 + \beta^{-1}\left[\int_{\bar{I}_0} g_0(t,u_m)u_m + \sum_{i=1}^N \int_{\bar{I}_i} g_i(t, A_{\tau_i}u_m)A_{\tau_i}u_m\right] \\ &\leq M_0 + \beta^{-1}\left[G'_{0,k}(u_m)u_m + \sum_{i=1}^N G'_{i,k}(A_{\tau_i}u_m)A_{\tau_i}u_m\right], \end{aligned}$$

i.e.

$$G_{0,k}(u_m) + \sum_{i=1}^N G_{i,k}(A_{\tau_i}u_m) \leq M_0 + \beta^{-1}(\mathcal{N}_s(u_m) + \mathcal{N}_s^2(u_m)), \tag{3.2}$$

Also, thanks to (A.2), there is a constant $r \geq r_0$ such that the condition $|x| \geq r$ implies

$$(*) \qquad x^2 \leq x g_0(t,x) \quad (\forall t \in [0,1]).$$

Setting

$$I = \{t \in [0,k] : |u_m(t)| \leq r\}, \quad \bar{I} = [0,k] \setminus I,$$

we then deduce from $(*)$ and (3.1) that

$$\|u_m\|^2_{L^2_\tau} = \int_I u_m^2 + \int_{\bar{I}} u_m^2 \leq kr^2 + \int_{\bar{I}} g_0(t,u_m)u_m$$

that is,

$$\|u_m\|^2_{L^2_\tau} \leq kr^2 + \mathcal{N}_s(u_m) + \mathcal{N}_s^2(u_m). \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), it comes

$$\begin{aligned}
M' &= M_0 + skr^2/2 + M \\
&\geq M_0 + skr^2/2 + \Phi_k(u_m) \\
&= M_0 + skr^2/2 + \frac{1}{2}\mathcal{N}_s^2(u_m) - \frac{s}{2}\|u_m\|^2_{L^2_k} - G_{0,k}(u_m) + \sum_{i=1}^N G_{i,k}(A_{\tau_i}u_m) \\
&\geq M_0 + skr^2/2 + \frac{1}{2}\mathcal{N}_s^2(u_m) - \frac{s}{2}(kr^2 + \mathcal{N}_s(u_m) + \mathcal{N}_s^2(u_m)) \\
&- M_0 - \beta^{-1}(\mathcal{N}_s(u_m) + \mathcal{N}_s^2(u_m)),
\end{aligned}$$

i.e.

$$\frac{1}{2}(1 - \frac{2}{\beta} - s)\mathcal{N}_s^2(u_m) - (\frac{1}{s} + \frac{1}{\beta})\mathcal{N}_s(u_m) \leq M'.$$

Since $\beta > 2$, if we choose s such that $s < 1 - 2/\beta$, then the above inequality yields an upper bound for $\mathcal{N}_s(u_m)$ which is independent of m, i.e. (u_m) is bounded in $(H^1_k, \mathcal{N}_s)$ and therefore in $(H^1_k, \|\cdot\|_{H^1_k})$.

Precompactness: The boundedness of (u_m) in H^1_k allows us to extract a weakly convergent subsequence, which for simplicity we still denote by (u_m). Let $u \in H^1_k$ be its (weak) limit. Then u_m converges to u strongly in C^0_k as well as in L^2_k. Thanks to the continuity of the f_i's we get

$$\begin{aligned}
\lim_m \int_0^k [f_0(t,u_m)u_m - f_0(t,u)u] &= 0, \\
\lim_m \int_0^k [f_i(t,A_{\tau_i}u_m)A_{\tau_i}u_m - f_i(t,A_{\tau_i}u)A_{\tau_i}u] &= 0
\end{aligned}$$

for all $i = 1, 2, \ldots, N$. On the one hand, (u_m) being a bounded and (PS) sequence for Φ_k, we deduce that $\Phi'_k(u_m)u_m \to 0$ as $m \to \infty$. Note also that $\Phi'_k(u)u = 0$. Indeed, we can write $\Phi'_k(u)u$ in the following form

$$\Phi'_k(u)u = \Phi'_k(u_m)u + \underbrace{[\Phi'_k(u)u_m - \Phi'_k(u_m)u]}_{R_m} + [\Phi'_k(u)u - \Phi'_k(u)u_m],$$

where the first term goes to zero because (u_m) is a (PS) sequence, while the last one goes to zero because $u_m \rightharpoonup u$ weakly in H^1_k. Therefore we only have to show that the second term, R_m, goes to zero as well. For this, we write R_m as

$$R_m = \underbrace{G'_{0,k}(u_m)u - G'_{0,k}(u)u_m}_{R^0_m} + \sum_{i=1}^N \underbrace{\left[G'_{i,k}(A_{\tau_i}u_m)A_{\tau_i}u - G'_{i,k}(A_{\tau_i}u)A_{\tau_i}u_m\right]}_{R^i_m}.$$

Then we have

$$|R^0_m| = \left|\int_0^k (g_0(t,u_m)u - g_0(t,u)u_m)\right|$$

$$\begin{aligned} &\leq \int_0^k |g_0(t,u_m)u - g_0(t,u)u| + \int_0^k |g_0(t,u)u - g_0(t,u)u_m| \\ &\leq k^{1/2}\left(\|u\|_{L_k^2}\|g_0(\cdot,u)-g_0(\cdot,u_m)\|_{L_k^\infty} + \|g_0(\cdot,u)\|_{L_k^2}\|u-u_m\|_{L_k^\infty}\right), \end{aligned}$$

which, since $u_m \to u$ strongly in C_k^0, and g_0 is continuous, shows that $R_m^0 \to 0$. Replacing g_0, u, and u_m by $g_i, A_{\tau_i}u$ and $A_{\tau_i}u_m$, respectively, we get, for each $i = 1,2,\ldots,N$

$$\begin{aligned} |R_m^i| &\leq k^{1/2}\|A_{\tau_i}u\|_{L_k^2}\|g_0(\cdot,A_{\tau_i}u)-g_0(\cdot,A_{\tau_i}u_m)\|_{L_k^\infty} \\ &+ 2k^{1/2}\|g_0(\cdot,A_{\tau_i}u)\|_{L_k^2}\|u-u_m\|_{L_k^\infty} \end{aligned}$$

which shows that $R_m^i \to 0$ for each $i = 1,2,\ldots,N$. Consequently,

$$\begin{aligned} \lim_m \|\dot u_m\|_{L_k^2}^2 &= \lim_m \left[\Phi_k'(u_m)u_m + \int_0^k f_0(t,u_m)u_m + \sum_{i=1}^N \int_0^k f_i(t,A_{\tau_i}u_m)A_{\tau_i}u_m\right] \\ &= \int_0^k f_0(t,u)u + \sum_{i=1}^N \int_0^k f_i(t,A_{\tau_i}u)A_{\tau_i}u \\ &= -\Phi_k'(u)u + \|\dot u\|_{L_k^2}^2 = \|\dot u\|_{L_k^2}^2. \qquad (3.4) \end{aligned}$$

On the other hand the boundedness of (u_m) implies the one of $(\dot u_m)$ in L_k^2. It follows from (3.4) that $\dot u_m \to \dot u$ strongly in L_k^2. Hence $u_m \to u$ strongly in H_k^1. □

Set

$$E_0 = \{u \in H_k^1 | u(t) = u(0) \text{ for all } t\} \cong \mathbb{R}.$$

Then,

$$H_k^1 = E_0 \oplus E_0^\perp,$$

where the orthogonal complement $E_0^\perp$ of E_0 in H_k^1 is formed by functions with zero mean value.

We can now prove the following

Lemma 3.2. *Let $k \in \mathbb{N}$. Under the assumptions of Theorem 1.1, Φ_k satisfies the conditions (J.3)-(J.5) of Theorem 3.1, with $X = H_k^1, X_0 = E_0$, and $\hat X = E_0^\perp$.*

Proof. Condition (J.3) follows from the facts that $\mathcal{P}g_0 \geq 0$ and $\mathcal{P}g_i(\cdot,0) = 0$ for all $i = 1,2,\ldots,N$.

Condition (J.4): Let

$$0 < \epsilon < \frac{1-c_0^2}{k+2}.$$

There is a $\delta > 0$ such that if $|x| \leq \delta$, then

$$\mathcal{P}g_0(t,x) \leq \frac{\epsilon}{2}x^2, \quad \mathcal{P}g_i(t,x) \leq \frac{\epsilon}{2N\tau_i}x^2$$

for all t and every $i = 1,2,\ldots,N$. Let

$$\rho = \frac{\delta}{\max(C_s, \bar l(k))},$$

where

$$\bar l(k) = \max_{1\leq i\leq N} l(k,\tau_i),$$

and $l(\cdot,\cdot)$ is defined by (2.2). Choose a $u \in E_0^\perp$ with

$$0 < \|u\|_{H_k^1} \leq \rho.$$

Then by (2.1) and (2.4) we have,

$$\max(\|u\|_{L_k^\infty}, \|A_{\tau_1}u\|_{L_k^\infty}, \ldots, \|A_{\tau_N}u\|_{L_k^\infty}) \leq \delta.$$

On the other hand, for every u in $E_0^\perp$ we have $u(t_0) = 0$ for some $t_0 \in [0, k]$. Therefore,

$$|u(t)|^2 = \left| \int_{t_0}^t \dot u \right|^2 \le \int_0^k \dot u^2 = \|\dot u\|^2_{L^2_k},$$

i.e.

$$\|u\|_{L^\infty} \le \|\dot u\|_{L^2_k};$$

and we deduce that

$$\|u\|_{H^1_k} \le \sqrt{1+k}\|\dot u\|_{L^2_k} \quad (\forall u \in E_0^\perp).$$

Therefore

$$\begin{aligned}
\Phi_k(u) &\ge \int_0^k \left[\frac{1}{2}\dot u^2 - \frac{\epsilon}{2}u^2 - \frac{1}{2}\sum_{i=1}^N (\frac{\epsilon}{N\tau_i} + \alpha_i^+)(A_{\tau_i}u)^2 \right] \\
&\ge \frac{1}{2}(1 - c_0^2 - \epsilon)\|\dot u\|^2_{L^2_k} - \frac{\epsilon}{2}\|u\|^2_{L^2_k} \\
&\ge \frac{1 - c_0^2 - \epsilon}{2(k+1)}\|u\|^2_{H^1_k} - \frac{\epsilon}{2}\|u\|^2_{H^1_k} \\
&= \frac{k+2}{2(k+1)}\left(\frac{1-c_0^2}{k+2} - \epsilon \right)\|u\|^2_{H^1_k} > 0.
\end{aligned}$$

In particular

$$\Phi_k(u) \ge \frac{k+2}{2(k+1)}\left(\frac{1-c_0^2}{k+2} - \epsilon \right)\rho^2 \quad (\forall u; \|u\|_{H^1_k} = \rho).$$

Condition (J.5): Let Y be a finite-dimensional subspace of H^1_k. Then any two norms on Y are equivalent, in particular there is a positive constant λ depending only on Y such that

$$\|u\|_{L^\beta_k} \ge \lambda \|u\|_{H^1_k} \quad (\forall u \in Y).$$

If we denote by $S(Y)$ the unit sphere of Y in the Sobolev norm, then we have

$$\inf_{v \in S(Y)} \left\{ \|v\|^\beta_{L^\beta_k} + \sum_{i=1}^N \|A_{\tau_i}v\|^\beta_{L^\beta_k} \right\} \ge \lambda^\beta.$$

Given a non-zero $u \in Y$, we set

$$u = r\tilde u, \quad r = \|u\|_{H^1_k}.$$

Then $\tilde u \in S(Y)$. Thanks to (A.2) and (A.4), we get

$$\begin{aligned}
\Phi_k(u) &= \frac{r^2}{2}B_k(\tilde u, \tilde u) - \int_0^k \left[\mathcal{P}g_0(t, \tilde u) - \sum_{i=1}^N \mathcal{P}g_i(t, \tilde u) \right] \\
&\le \frac{1}{2}(1 + \sum_{i=1}^N \|\alpha_i^-\|_{L^\infty})r^2 - a_0 r^\beta \int_0^k (|\tilde u|^\beta + \sum_{i=1}^N |A_{\tau_i}\tilde u|^\beta) + a_1(N+1)k \\
&\le \frac{1}{2}(1 + \sum_{i=1}^N \|\alpha_i^-\|_{L^\infty})r^2 - a_0 \inf_{v \in S(Y)}(\|v\|^\beta_{L^\beta_k} + \sum_{i=1}^N \|A_{\tau_i}v\|^\beta_{L^\beta_k})r^\beta \\
&+ a_1(N+1)k \\
&\le \frac{1}{2}(1 + \sum_{i=1}^N \|\alpha_i^-\|_{L^\infty})r^2 - a_0\lambda^\beta r^\beta + a_1(N+1)k.
\end{aligned}$$

Since $\beta > 2$, there is an $R > 0$ depending on $\lambda(Y)$, and therefore on Y, such that if $u \in Y$, with $\|u\|_{H^1_k} > R$, then $\Phi_k(u) \le 0$, and (J.5) is satisfied. $\square$

Thanks to Theorem 3.1, Φ_k possesses a critical point u in H^1_k which corresponds to a k-periodic solution of (1.3). The characterization of the critical value $\Phi_k(u)$ shows that u is non-constant.

4. Existence of a homoclinic solution

The idea is to construct a sequence (u_k) such that for k large enough, each u_k is a non-constant k-periodic solution of (1.1). We shall then show the existence of a convergent subsequence whose limit is a non-trivial critical point of Φ_∞.

Lemma 4.1. *Under the assumptions of Theorem 1.2, Φ_∞ is a well defined, and C^1 functional on E. Furthermore, any critical point of Φ_∞ is a classical solution of* (1.1).

Proof. Write

$$\Phi_\infty(u) = \frac{1}{2}B_\infty(u,u) - G_{0,\infty}(u) - \sum_{i=1}^{N} G_{i,\infty}(A_{\tau_i}u),$$

where

$$\begin{aligned} B_\infty(u,v) &= \int_{\mathbb{R}} \left[\dot{u}\dot{v} + \omega uv - \sum_{i=1}^{N} \alpha_i (A_{\tau_i}u)(A_{\tau_i}v \right], \\ G_{i,\infty}(u) &= \int_{\mathbb{R}} \mathcal{P} g_i(t,u), \quad i = 0,1,\ldots,N. \end{aligned}$$

One easily shows that B_∞ is bounded, and since each the operators $A_{\tau_i} : E \to E$ is bounded, it follows from Proposition 2.2, that $G_{0,\infty}, G_{1,\infty} \circ A_{\tau_1}, \ldots, G_{N,\infty} \circ A_{\tau_N} \in C^1(E,\mathbb{R})$.
It is an easy exercise to compute $\Phi'_\infty(u)\xi$ for any $u, \xi \in E$. By standard bootstrap arguments one shows that if $u \in E$ is a critical point of Φ_∞, then it is a weak solution of (1.1). The continuity of u and the f_i's then imply $u \in C^2$, i.e. u is a classical solution. □

Proposition 4.1. *Under the assumptions of Theorem 1.2, any critical point $u \in E$ of Φ_∞ satisfies $\dot{u}(\pm\infty) = 0$, i.e. it is a homoclinic solution of* (1.1) *emanating from the origin.*

Proof. We shall prove that if $u \in E$ is a critical point of Φ_∞, then $\ddot{u} \in L^2$, which, obviously implies that $\dot{u} \in E$ and therefore $\dot{u}(\pm\infty) = 0$.
Thanks to (A.1), there is a $\delta > 0$ such that

$$\max_{0\le i\le N} |g_i(t,x)| \le |x| \text{ for } |x| \le \delta \text{ and } t \in [0,1].$$

Because $u \in E$, we have $u(t) \to 0$ as $t \to \infty$, and therefore, there is an $r_1 > 0$ such that

$$|u(t)| \le \delta/2 \text{ for all } |t| \ge r_1.$$

Setting

$$r_2 = r_1 + \max_{1\le i\le N} \tau_i,$$

it then follows that

$$\max_{1\le i\le N} |A_{\tau_i}u(t)| \le \delta \text{ for } t \in (-\infty, -r_2] \cup [r_1, \infty),$$

and

$$\max_{1\le i\le N} |A^*_{\tau_i}u(t)| \le \delta \text{ for } t \in (-\infty, -r_1] \cup [r_2, \infty).$$

Thanks to (1.1), for $|t| \geq r_2$, we have

$$\begin{aligned} \ddot{u}^2 &\leq (4N+2)\Bigg\{\omega^2(t)u^2 + g_0^2(t,u) + \sum_{i=1}^N \left[\alpha_i^2(t)|A_{\tau_i}u|^2 + \alpha_i^2(t-\tau_i)|A_{\tau_i}^* u|^2\right] \\ &\quad + \sum_{i=1}^N \left[|g_i(t, A_{\tau_i}u)|^2 + |g_i(t-\tau_i, A_{\tau_i}^* u)|^2\right]\Bigg\} \\ &\leq (4N+2)\left\{(1+\|\omega\|_{L^\infty}^2)u^2 + \sum_{i=1}^N (1+\|\alpha_i\|_{L^\infty}^2)\left[|A_{\tau_i}u|^2 + |A_{\tau_i}^* u|^2\right]\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{|t|\geq r_2} \ddot{u}^2 &\leq (4N+2)\left\{(1+\|\omega\|_{L^\infty}^2)\|u\|_{L^2}^2 + 2\sum_{i=1}^N \tau_i(1+\|\alpha_i\|_{L^\infty}^2)\|\dot{u}\|_{L^2}^2\right\} \\ &\leq (4N+2)\max\left\{1+\|\omega\|_{L^\infty}^2, 2\sum_{i=1}^N \tau_i(1+\|\alpha_i\|_{L^2}^2)\right\}\|u\|_E^2 < \infty, \end{aligned}$$

Hence

$$\int_{\mathbb{R}} \ddot{u}^2 = \int_{|t|\leq r_2} \ddot{u}^2 + \int_{|t|\geq r_2} \ddot{u}^2 < \infty.$$

This shows that $\ddot{u} \in L^2(\mathbb{R})$, and the proof is complete. □

For convenience Φ_k shall, from now on, be defined as an integral over $[-k/2, k/2]$ instead of $[0, k]$.

Lemma 4.2. *Under the hypotheses of Theorem 1.2, for every $k \in \mathbb{N}$ (1.1) possesses a non-trivial k-periodic solution.*

The proof of the above lemma shall follow from the standard version of the mountain pass theorem:

Theorem 4.1 (Ambrosetti-Rabinowitz [1])**.** *Let $J \in C^1(X, \mathbb{R})$ be (PS) and $J(0) = 0$. Suppose the following conditions are satisfied*

(J.1) *there are constants $\omega_*, \rho > 0$ such that $J|_{S_\rho} \geq \omega_*$,*
(J.2) *there is an $e \in X \backslash \bar{B}_\rho$ such that $J(e) \leq 0$.*

Then, J possesses a critical value $b \geq \omega_*$ characterized by

(4.1)
$$b = \inf_{\gamma\in\Gamma} \max_{0\leq s\leq 1} J(\gamma(s)),$$

where

(4.2)
$$\Gamma = \{\gamma \in C([0,1], X) | \gamma(0) = 0 \text{ and } \gamma(1) = e\}.$$

Observe that

(4.3)
$$B_k(v,v) \geq \epsilon_0 \|v\|_{H_k^1}^2 \quad (\forall v \in H_k^1),$$

where

(4.4)
$$\epsilon_0 = \begin{cases} \min\{\min\omega, 1 - \sum_{i=1}^N \tau_i\|\alpha_i^+\|_{L^\infty}\} & \text{if} \quad 0 < \min\omega < 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \\ \min\{1, \min\omega - 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty}\} & \text{if} \quad \min\omega > 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \end{cases}.$$

Indeed, if

$$0 < \min\omega < 4\sum_{i=1}^N \|\alpha_i^+\|_{L^\infty},$$

then for any $v \in H_k^1$ we have

$$\begin{aligned} B_k(v,v) &\geq \|\dot{v}\|_{L_k^2}^2 + \int_0^k \omega v^2 - \sum_{i=1}^N \int_0^k \alpha_i^+ (A_{\tau_i} v)^2 \\ &\geq \|\dot{v}\|_{L_k^2}^2 + \min \omega \|v\|_{L_k^2}^2 - \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \|A_{\tau_i} v\|_{L_k^2}^2 \\ &\geq \|\dot{v}\|_{L_k^2}^2 - \sum_{i=1}^N \tau_i \|\alpha_i^+\|_{L^\infty} \|\dot{v}\|_{L_k^2}^2 + \min \omega \|v\|_{L_k^2}^2 \\ &= (1 - \sum_{i=1}^N \tau_i \|\alpha_i^+\|_{L^\infty}) \|\dot{v}\|_{L_k^2}^2 + \min \omega \|v\|_{L_k^2}^2 \\ &\geq \min\{\min \omega, 1 - \sum_{i=1}^N \tau_i \|\alpha_i^+\|_{L^\infty}\} \|v\|_{H_k^1}^2, \end{aligned}$$

and if

$$\min \omega > 4 \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty},$$

then we have

$$\begin{aligned} B_k(v,v) &\geq \|\dot{v}\|_{L_k^2}^2 + \int_0^k \omega v^2 - \sum_{i=1}^N \int_0^k \alpha_i^+ (A_{\tau_i} v)^2 \\ &\geq \|\dot{v}\|_{L_k^2}^2 + \min \omega \|v\|_{L_k^2}^2 - \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \|A_{\tau_i} v\|_{L_k^2}^2 \\ &\geq \|\dot{v}\|_{L_k^2}^2 + \min \omega \|v\|_{L_k^2}^2 - 4 \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty} \|v\|_{L_k^2}^2 \\ &= \|\dot{v}\|_{L_k^2}^2 + (\min \omega - 4 \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty}) \|v\|_{L_k^2}^2 \\ &\geq \min\{1, \min \omega - 4 \sum_{i=1}^N \|\alpha_i^+\|_{L^\infty}\} \|v\|_{H_k^1}^2. \end{aligned}$$

Proof of Lemma 4.2. We only have to check that Theorem 4.1 is applicable, i.e. Φ_k is (PS), and satisfies the conditions (J.1) and (J.2).

Φ_k *is (PS).* We shall only prove the boundedness of (PS) sequences. The precompactness can be dealt with following the same line of arguments as in the proof of Proposition 3.1.

Let $(u_m) \subset H_k^1$ be a (PS) sequence, i.e. for some constant $M \geq 0$ we have

$$|\Phi_k(u_m)| \leq M \quad (\forall m) \text{ and } \lim_{m\to\infty} \Phi_k'(u_m) = 0.$$

Then, there is an integer m_0 such that

$$\|\Phi_k'(u_m)\| \leq 1 \quad (\forall m \geq m_0).$$

Fixing $m \geq m_0$, thanks to (A.3) and (4.3), we have

$$\begin{aligned} M + \frac{1}{\beta}\|u_m\|_{H_k^1} &\geq \Phi_k(u_m) - \frac{1}{\beta}\Phi_k'(u_m)u_m \\ &= (\frac{1}{2} - \frac{1}{\beta}) B_k(u_m, u_m) + \int_0^k \left[\frac{1}{\beta} g_0(t, u_m) u_m - \mathcal{P} g_0(t, u_m)\right] \\ &+ \sum_{i=1}^N \int_0^k \left[\frac{1}{\beta} g_i(t, A_{\tau_i} u_m) A_{\tau_i} u_m - \mathcal{P} g_i(t, A_{\tau_i} u_m)\right] \end{aligned}$$

$$\geq \quad (\frac{1}{2}-\frac{1}{\beta})\epsilon_0\|u_m\|^2_{H^1_k},$$

and thus

$$(\beta-2)\epsilon_0\|u_m\|^2_{H^1_k}-2\|u_m\|_{H^1_k}\leq 2\beta M.$$

Since $\beta>2$, the previous inequality shows that (u_m) is bounded in H^1_k.

Condition (J.1). We only have to show that

$$\Phi_k(u)=\frac{1}{2}B_k(u,u)+o(\|u\|^2_{H^1_k}).$$

Note that

$$\epsilon_0\|u\|^2_{H^1_k}\leq B_k(u,u)\leq \max(\|\omega\|_{L^\infty},1+\sum_{i=1}^N\tau_i\|\alpha_i^-\|_{L^\infty})\|u\|^2_{H^1_k}. \tag{4.5}$$

By (A.1), given $\epsilon>0$, there is a $\delta>0$ such that if $|x|\leq\delta$, then

$$\mathcal{P}g_0(\cdot,x)\leq\frac{\epsilon}{2}x^2,$$

and

$$\mathcal{P}g_i(\cdot,x)\leq\frac{\epsilon}{2N\tau_i}x^2\quad(\forall i=1,2,\dots,N).$$

Set

$$\rho=\frac{\delta}{\max(C_s,\bar{l}(k))},$$

where C_s is given by (2.4) and

$$\bar{l}(k)=\max_{1\leq i\leq N}l(k,\tau_i).$$

Let $u\in H^1_k$ with

$$\|u\|_{H^1_k}=\rho.$$

Then, thanks to (2.4) we have

$$\|u\|_{L^\infty_k}\leq\delta.$$

Also, thanks to (2.1), we have

$$\|A_{\tau_i}u\|_{L^\infty_k}\leq\delta\quad(\forall i=1,2,\dots,N).$$

It follows that,

$$\begin{aligned}0\leq\Phi_k(u)-\frac{1}{2}B_k(u,u) &\leq \frac{\epsilon}{2}\int_0^k(u^2+\sum_{i=1}^N\frac{1}{N\tau_i}(A_{\tau_i}u)^2)\\ &\leq \frac{\epsilon}{2}\|u\|^2_{H^1_k}.\end{aligned}$$

Since $\epsilon>0$ is arbitrary, we have

$$\Phi_k(u)-\frac{1}{2}B_k(u,u)=o(\|u\|^2_{H^1_k}).$$

In particular, if we choose ϵ such that

$$0<\epsilon<\epsilon_0,$$

where ϵ_0 is given by (4.4), then thanks to (4.5), we have

$$\Phi_k(u)\geq\frac{1}{2}(\epsilon_0-\epsilon)\rho^2>0.$$

Condition (J.2). Let u be a non-zero element of H^1_k and $r>0$. Thanks to (A.3), we have

$$\Phi_k(ru)\leq a_2k+\frac{r^2}{2}B_k(u,u)-a_0r^\beta\int_{-k/2}^{k/2}(|u|^\beta+\sum_{i=1}^N|A_{\tau_i}u|^\beta).$$

But $\beta > 2$, therefore $\Phi_k(ru) \to -\infty$ as $r \to \infty$ and there is an $r_u > 0$ such that $\Phi_k(ru) \le 0$ for $r \ge r_u$. □

From now, Φ_k is defined as an integral over $[-k/2, k/2]$. Let

$$k_0 = \max_{1\le i\le N}[\tau_i] + 1,$$

where $[*]$ denotes the integer part of $*$, and e_0 a non-zero member of $C^1([-k_0, k_0])$ whose support lies inside $(-k_0, 0)$. Denote by e_{2k_0} the $2k_0$-periodic extension of e_0 on the whole $\mathbb{R}$. Given $k > 2k_0$, we extend continuously e_0 on $[-k/2, k/2]$ and denote that extension by $\tilde{e}$, i.e. $\tilde{e}$ agrees with e_0 on $[-k_0, k_0]$, and equals 0 elsewhere. We now extend $\tilde{e}$ on the whole $\mathbb{R}$ in a k-periodic fashion, and denote that extension by $\tilde{e}_k$, i.e. $\tilde{e}_k$ is k-periodic and agrees with $\tilde{e}$ on $[-k/2, k/2]$.
Obviously, we have

$$\|\tilde{e}_k\|_{H^1_k} = \|e_0\|_{H^1(-k_0,0)} \quad (\forall k \ge 2k_0),$$

and for $k \ge 2k_0$ and $|t| \le k/2$ we have

$$A_{\tau_i}\tilde{e}_k(t) = \begin{cases} A_{\tau_i}\tilde{e}_{k_0}(t) & \text{if } t \in [-2k_0, 0] \\ 0 & \text{otherwise} \end{cases}.$$

Using the fact that $\tilde{e}_{2k_0}$ is $2k_0$-periodic, we have, for any $r > 0$, and any $i = 1, 2, \ldots, N$:

$$\begin{aligned} \int_{-k/2}^{k/2} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_k) &= \int_{-2k_0}^{0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}) \\ &= \int_{-2k_0}^{-k_0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}) + \int_{-k_0}^{0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}) \\ &= \int_{0}^{k_0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}) + \int_{-k_0}^{0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}) \\ &= \int_{-k_0}^{k_0} \mathcal{P}g_i(t, rA_{\tau_i}\tilde{e}_{2k_0}). \end{aligned}$$

Since $\mathcal{P}g_0(\cdot, 0) = 0$, and $\tilde{e}_{k_0}$ is a non-zero element of $H^1_{2k_0}$, we can always choose r in such a way that

$$\Phi_k(r\tilde{e}_k) = \Phi_{2k_0}(r\tilde{e}_{2k_0}) \le 0.$$

From now on, we assume that $k \ge 2k_0$, and set

$$e_k = r\tilde{e}_k,$$

where r is such that the above property holds. Denote by b_k the critical value of Φ_k given by (4.1), with

$$\Gamma = \Gamma_k := \{\gamma \in C([0, 1], H^1_k) | \gamma(0) = 0 \text{ and } \gamma(1) = e_k\}.$$

Let u_k be the corresponding critical point.

We have the following

Lemma 4.3. *The sequences (u_k), $(\dot{u}_k)$ and $(\ddot{u}_k)$ are uniformly bounded in $(C_b(\mathbb{R}), \|\cdot\|_{L^\infty})$. Furthermore, there is a positive integer k_* such that for $k \ge k_*$ the solution (u_k) is non-constant.*

Proof. Uniform upper bound for $\|u_k\|_{L^\infty}$: Let $\gamma_k \in \Gamma_k$ be given by $\gamma_k(s) = se_k$. Then

$$\Phi_k(\gamma_k(s)) = \Phi_{2k_0}(\gamma_{2k_0}(s)),$$

and we deduce that

$$b_k \le \max_{0\le s\le 1} \Phi_{2k_0}(\gamma_{2k_0}(s)) =: b_0. \tag{4.6}$$

Note that

$$b_k = \Phi_k(u_k) - \frac{1}{\beta}\Phi_k'(u_k)u_k \geq (\frac{1}{2} - \frac{1}{\beta})\epsilon_0 \|u_k\|^2_{H^1_k}.$$

Thus, by (4.6) we have

$$\|u_k\|_{H^1_k} \leq \left(\frac{2\beta b_0}{(\beta-2)\epsilon_0}\right)^{1/2}. \tag{4.7}$$

It then follows from (4.7) and (2.4) that

$$\|u_k\|_{L^\infty} \leq \sqrt{2}\|u_k\|_{H^1_k} \leq M_0 = 2\left(\frac{\beta b_0}{(\beta-2)\epsilon_0}\right)^{1/2}. \tag{4.8}$$

Uniform bound for $\|\ddot{u}_k\|_{L^\infty_k}$: It follows from the periodicity of u_k, the continuity of the f_i's, and from (1.1) that, for all t:

$$\begin{aligned}|\ddot{u}_k(t)| &\leq \max_{0\leq t\leq k}|f_0(t,u_k)| \\ &+ \sum_{i=1}^N \left[\max_{0\leq t\leq k}|f_i(t, A_{\tau_i}u_k(t))| + \max_{0\leq t\leq k}|f_i(t-\tau_i, A^*_{\tau_i}u_k(t))|\right] \\ &\leq \max_{z\in\Omega_0}|f_0(z)| + 2\sum_{i=1}^N \max_{z\in\Omega_1}|f_i(z)| =: M_2,\end{aligned}$$

where

$$\Omega_0 = [0,1]\times[-M_0, M_0], \quad \Omega_1 = [0,1]\times[-2M_0, 2M_0].$$

Thus

$$\|\ddot{u}_k\|_{L^\infty} = \|\ddot{u}_k\|_{L^\infty_k} \leq M_2 \quad (\forall k \geq 2k_0).$$

Uniform upper bound for $\|\dot{u}_k\|_{L^\infty}$: Let $|t| \leq k/2$. By the mean value theorem, we have

$$\dot{u}_k(t_k) = \int_{t-1}^t \dot{u}_k(s)ds = u_k(t) - u_k(t-1)$$

for some $t_k \in [t-1, t]$. It follows that

$$\begin{aligned}|\dot{u}_k(t)| = \left|\dot{u}_k(t_k) + \int_{t_k}^t \ddot{u}_k(s)ds\right| &\leq |u_k(t) + |u_k(t-1)| + \int_{t_k}^t |\ddot{u}_k(s)|ds \\ &\leq 2M_0 + \int_{t-1}^t M_2 ds \\ &= 2M_0 + M_2.\end{aligned}$$

Thus

$$\|\dot{u}_k\|_{L^\infty} = \|\dot{u}_k\|_{L^\infty_k} \leq M_2 := 2M_0 + M_2 \quad (\forall k \geq 2k_0).$$

Uniform lower bound for $\|u_k\|_{L^\infty}$. For each $i = 0, 1, \ldots, N$ we defined $Y_i : [0,\infty) \to \mathbb{R}$ by $Y_i(s) = 0$ if $s = 0$, and for $s > 0$

$$Y_i(s) = \max\{x^{-1}g_i(t,x) : t \in [0,1] \text{ and } 0 < |x| \leq s\}, 0 \leq i \leq N.$$

For each i, the map $s \mapsto Y_i(s)$ is continuous, non-decreasing, and non-negative.

Since

$$|u_k(t)| \leq s_k := \|u_k\|_{L^\infty_k},$$

for every t, it readily follows from the definition of Y_0 that

$$\frac{g_0(t, u_k(t))}{u_k(t)} \leq Y_0(s_k),$$

for every t for which the left hand side is well defined. Similarly, for each $i = 1, 2\ldots, N$, we have

$$\frac{g_i(t, Au_k(t))}{Au_k(t)} \leq Y_i(2s_k),$$

whenever the left hand side is well defined.

Using the definition of u_k, we infer that

$$\begin{aligned}
\epsilon_0\|u_k\|^2_{H^1_k} &\leq B_k(u_k,u_k)\\
&= \int_{-k/2}^{k/2}\left[\dot{u}_k^2+\omega u_k^2-\sum_{i=1}^N \alpha_i(A_{\tau_i}u_k)^2\right]\\
&= \int_{-k/2}^{k/2}\left[g_0(t,u_k)u_k+\sum_{i=1}^N g_i(t,A_{\tau_i}u_k)A_{\tau_i}u_k\right]\\
&\leq \int_{-k/2}^{k/2}\left[Y_0(s_k)u_k^2+\sum_{i=1}^N Y_i(2s_k)(A_{\tau_i}u_k)^2\right]\\
&\leq Y_0(s_k)\|u_k\|^2_{L^2_k}+\|\dot{u}_k\|^2_{L^2_k}\sum_{i=1}^N \tau_i Y_i(2s_k)\\
&\leq (Y_0(s_k)+\sum_{i=1}^N \tau_i Y_i(2s_k))\|u_k\|^2_{H^1_k}\\
&= Y(s_k)\|u_k\|^2_{H^1_k},
\end{aligned}$$

where

$$Y(s)=Y_0(s)+\sum_{i=1}^N \tau_i Y_i(2s)$$

It is clear that Y enjoys the same properties as the Y_i's, therefore, since $s_k\neq 0$, there is a positive number δ_0 which is independent of k such that

$$\|u_k\|_{L^\infty}\geq \delta_0. \tag{4.9}$$

Existence of k_*. We argue by contradiction. Suppose all the u_k are constants. Then we have

$$\delta_0\leq \|u_k\|_{L^\infty}=|u_k(0)|=\frac{\|u_k\|_{H^1_k}}{\sqrt{k}}\leq \frac{M_0}{\sqrt{2k}}$$

for k large enough. This contradicts the fact that δ_0 is non-zero. Therefore, there must be such a k_*. □

Lemma 4.4. *The sequence* (u_k) *possesses a convergent subsequence* $(\tilde{u}_k)$ *whose limit,* $\tilde{u}$*, belongs to* E *and is a non-zero critical point of* Φ_∞*.*

Proof. First, note that the sequences (u_k) and $(\dot{u}_k)$ are equicontinuous. Indeed, given $k\geq k_*$, and $t_1,t_2\in\mathbb{R}$, we have

$$\begin{aligned}
|u_k(t_2)-u_k(t_1)| &= \left|\int_{t_1}^{t_2}\dot{u}_k(s)ds\right|\leq M_1|t_2-t_1|\\
|\dot{u}_k(t_2)-\dot{u}_k(t_1)| &= \left|\int_{t_1}^{t_2}\ddot{u}_k(s)ds\right|\leq M_2|t_2-t_1|.
\end{aligned}$$

Hence, in view of Lemma 4.2 and thanks to Arzelà-Ascoli's Theorem, a subsequence $(\tilde{u}_k)$ converges in $C^1_{\text{loc}}(\mathbb{R})$, say to some $\tilde{u}$. Actually $\tilde{u}_k\to\tilde{u}$ in $C^2_{\text{loc}}(\mathbb{R})$ since each $\tilde{u}_k$ satisfies (1.1).

By (4.7), one infers hat

$$\int_{\mathbb{R}}(\dot{\tilde{u}}^2+\tilde{u}^2)\leq M_0^2/2,$$

i.e. $\tilde{u}\in E$. It only remains to show that

$$\Phi'_\infty(\tilde{u})=0.$$

Let ξ be a test function on $\mathbb{R}$. Denote by I_0, I_1, $\dots$, I_N the supports of $\xi(\cdot)$, $\xi(\cdot+\tau_1)$, $\dots$, $\xi(\cdot+\tau_N)$, respectively. Let $k\in\mathbb{N}$ be sufficiently large, so that

$$I:=\bigcup_{i=1}^{N} I_i\subset(-k/2,k/2).$$

Then, we have

$$\begin{aligned}\Phi'_\infty(\tilde u)\xi &= \Phi'_\infty(\tilde u)\xi-\Phi'_k(\tilde u_k)\xi\\ &= B_I(\tilde u-\tilde u_k,\xi)+(G'_{0,I}(\tilde u)-G'_{0,I}(\tilde u_k))\xi+(G'_I(\tilde u)-G'_I(\tilde u_k))\xi,\end{aligned}$$

with

$$\begin{aligned}B_I(w,\xi) &= \int_I\left[\dot w\dot\xi+\omega w\xi-\sum_{i=1}^{N}\alpha_i A_{\tau_i}wA_{\tau_i}\xi\right],\\ G'_{0,I}(w)\xi &= \int_I g_0(t,w)\xi,\\ G'_I(w)\xi &= \sum_{i=1}^{N}\int_I g_i(t,A_{\tau_i}w)A_{\tau_i}\xi.\end{aligned}$$

Note that

$$\begin{aligned}|B_I(\tilde u-\tilde u_k,\xi)| &\le \|\dot{\tilde u}-\dot{\tilde u}_k\|_{L^\infty(I)}\int_I|\dot\xi|+\|\tilde u-\tilde u_k\|_{L^\infty(I)}\int_I\omega|\xi|\\ &+ 2\|\tilde u-\tilde u_k\|_{L^\infty(I)}\sum_{i=1}^{N}\int_{I_i}|\alpha_i||A_{\tau_i}\xi|\\ &\le |I|^{1/2}\|\dot\xi\|_{L^2(I)}\|\dot{\tilde u}-\dot{\tilde u}_k\|_{L^\infty(I)}\\ &+ |I|^{1/2}\|\omega\|_{L^\infty}\|\xi\|_{L^2(I)}\|\tilde u-\tilde u_k\|_{L^\infty(I)}\\ &+ 2\|\tilde u-\tilde u_k\|_{L^\infty(I)}\sum_{i=1}^{N}\sqrt{\tau_i}\|\alpha_i\|_{L^\infty}\|\dot\xi\|_{L^2(I)}\\ &\le b_*\|\xi\|_E(\|\tilde u-\tilde u_k\|_{L^\infty(I)}+\|\dot{\tilde u}-\dot{\tilde u}_k\|_{L^\infty(I)})\\ &= b_*\|\xi\|_E\|\tilde u-\tilde u_k\|_{C^1(I)},\end{aligned}$$

where

$$b_*=\max\left(|I|^{1/2},\|\omega\|_{L^\infty}+2\sum_{i=1}^{N}\sqrt{\tau_i}\|\alpha_i\|_{L^\infty}\right),$$

Since $\tilde u_k\to\tilde u$ in $C^1_{\rm loc}$, it follows from the above estimate that $B_I(\tilde u-\tilde u_k,\xi)\to 0$ as $k\to\infty$.

Similarly, we have

$$\begin{aligned}|G'_{0,I}(\tilde u)\xi-G'_{0,I}(\tilde u_k)\xi| &\le |I|^{1/2}\|\xi\|_E\|g_0(\cdot,\tilde u)-g_0(\cdot,\tilde u_k)\|_{L^\infty(I)}\\ |G'_I(\tilde u)\xi-G'_I(\tilde u_k)\xi| &\le \sum_{i=1}^{N}\int_I|g_i(t,A_{\tau_i}\tilde u)-g_i(t,A_{\tau_i}\tilde u_k)||A_{\tau_i}\xi|\\ &\le |I|^{1/2}\|\xi\|_E\sum_{i=1}^{N}\sqrt{\tau_i}\|g_i(\cdot,A_{\tau_i}\tilde u)-g_i(\cdot,A_{\tau_i}\tilde u_k)\|_{L^\infty(I)}\end{aligned}$$

It then follows from the continuity of the g_i's, and the convergence $\tilde u_k\to\tilde u$ in $C^1_{\rm loc}$ that

$$\lim_k|G'_{0,I}(\tilde u)\xi-G'_{0,I}(\tilde u_k)\xi|=0\ \text{ and }\ \lim_k|G'_I(\tilde u)\xi-G'_I(\tilde u_k)\xi|=0.$$

Thus

$$|\Phi'_\infty(\tilde u)\xi|=\lim_k|\Phi'_\infty(\tilde u)\xi-\Phi'_k(\tilde u_k)\xi|=0.$$

Since ξ is arbitrary, we have $\Phi'_\infty(\tilde u)=0$.

From (4.9), one infers that

$$\|\tilde{u}\|_{L^\infty} \geq \delta_0 > 0,$$

that is, $\tilde{u} \not\equiv 0$. □

5. Applications: travelling waves in lattices with N-nearest-neighbour interaction

In this section we study periodic and homoclinic travelling waves in infinite lattices with N-nearest-neighbour interaction.

Consider an infinite lattice of particles subejcted to a potential V_0 and such that each particle interacts with its N (first) nearest neighbours[1], under the potentials $V_1, \ldots, V_N$. The equation of the motion of a single particle is described by Newton's law, i.e.

$$\ddot{q}_j + V_0'(q_j) = \sum_{i=1}^{N} \left[V_i'(q_{j+i} - q_i) - V_i'(q_i - q_{j-i})\right], \, j \in \mathbb{Z}. \tag{5.1}$$

A travelling wave is a solution of (5.1) of the form

$$q_j(t) = u(j - ct), \, j \in \mathbb{Z}, \tag{5.2}$$

where $c > 0$ is the wave speed and u the wave profile.
Inserting (5.2) into (5.1) yields the advanced-delayed ODE

$$c^2\ddot{u} + V_0'(u) = \sum_{i=1}^{N} \left[V_i'(A_i u) - V_i'(A_i^* u)\right]. \tag{5.3}$$

We shall consider potentials of the type

$$(V) \quad V_i(x) = \frac{\lambda_i}{2}x^2 + W_i(x), \quad i = 0, 1, \ldots, N.$$

Let $c_* \geq 0$ be given by

$$c_*^2 = \begin{cases} 0 & \text{if} \quad -\infty < \lambda_0 < -4\sum_{i=1}^{N} \max(0, \lambda_i) \\ \sum_{i=1}^{N} i \max(0, \lambda_i) & \text{if} \quad -4\sum_{i=1}^{N} \max(0, \lambda_i) \leq \lambda_0 \leq 0. \end{cases}.$$

Then we have the following

Corollary 5.1. *Let the $V_i \in C^1(\mathbb{R})$ be given by (V). Suppose $\lambda_0 = 0$, and each $W \in \{W_i : 0 \leq i \leq N\}$ satisfies the following conditions:*

- (W.1) *$W(x) = o(x^2)$ as $x \to 0$, and*
- (W.2) *$W \geq 0$ and there are constants $\beta > 2, r_0 > 0$ such that $\beta W(x) \leq xW'(x)$ for $|x| \geq r_0$.*

Then for any $\tau > 0$, and any $c > c_*$, (5.3) *possesses a non-constant τ-periodic solution.*

Corollary 5.2. *Let the $V_i \in C^1(\mathbb{R})$ be given by (V). Suppose $\lambda_0 < 0$ and each $W \in \{W_i : 0 \leq i \leq N\}$ satisfies the growth condition*

- (W.3) *there exists a constant $\beta > 2$ such that*

$$0 < \beta W(x) \leq xW'(x) \quad \forall x \neq 0.$$

Then for any $c > c_*$, (5.3) *possesses a non-trivial homoclinic solution emanating from 0.*

---

[1]This means that each particle interacts with $2N$ other particles.

The proofs shall be omitted. However, one can observe that (5.3) is of the type (1.1), with $\alpha_i(t) = c^{-2}\lambda_i, g_i(t,x) = c^{-2}W_i'(x)$, for $t, x \in \mathbb{R}$ and $i = 0, 1, 2, \ldots, N$, and $\tau_i = i$ for $i = 1, 2, \ldots, N$. The only difference is that (5.3) is non-autonomous while (1.1) is not. The proof of Corollary 5.1 (resp. Corollary 5.2) follows exactly the same line of arguments as the one of Theorem 1.1 (resp. Theorem 1.2). The only exception when dealing with periodic solutions for (5.3) is that the period is allowed to take any positive real value. The condition $c_0 < 1$ reads precisely $c > c_*$.

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