

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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(revised version: July 2009)

by

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Preprint no.: 38

2009



On Lightlike Geometry in Indefinite Kenmotsu Manifolds

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Abstract

We investigate some geometric aspects of lightlike hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field, by paying attention to the geometry of leaves of integrable distributions. Theorems on parallel vector fields, Killing distribution, geodesibility of those leaves are obtained. The geometrical configuration of such lightlike hypersurfaces and leaves of its screen integrable distributions are established. We show that any totally contact umbilical leaf of screen integrable distribution of a lightlike hypersurface cannot be an extrinsic sphere. We also prove the geometry any leaf of integrable distribution is closely related with the geometry of a normal bundle.

2000 Mathematics Subject Classification: 53C15, 53C25, 53C50.

Key words: Lightlike hypersurfaces, Screen distribution, Indefinite Kenmostu space form.

1 Introduction

In 1971, K. Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions. We call them Kenmotsu manifolds [9]. Several authors have studied some properties of Kenmotsu manifolds since then. In [8], for instance, the authors partially classified the Kenmotsu manifolds and considered manifolds admitting the transformation which keeps the Riemannian curvature tensor and Ricci tensor invariant. The contact geometry has significant use in differential equations, phase spaces of dynamical systems (see details in [12] and [21], for instance), and the literature about its lightlike case is very limited. Some specific discussions on this matter can be found in [18], [19] and references therein.

The present paper aims to investigate the geometry of lightlike hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field.

As it is well known, the geometry of lightlike submanifolds [3] are different because for the fact that their normal vector bundle intersects with the tangent bundle. Thus, the study becomes more difficult and strikingly different from the study of non-degenerate submanifolds. This means that one cannot use, in the usual way, the classical submanifold theory to define any induced object on a lightlike submanifold. To deal with this anomaly, the lightlike submanifolds

were introduced and presented in a book by Duggal and Bejancu [3]. They introduced a non-degenerate screen distribution to construct a nonintersecting lightlike transversal vector bundle of the tangent bundle. Several authors have studied a lightlike hypersurface of semi-Riemannian manifold (see [6] and many more references therein).

Physically, lightlike hypersurfaces are interesting in general relativity since they produce models of different types of horizons. On the latter, the relationship between Killing and geodesic notions is well specified. Lightlike hypersurfaces are also studied in the theory of electromagnetism (see, for instance [3], Chapter 8).

In a totally umbilical real lightlike hypersurface of an indefinite Kähler space form, Duggal and Bejancu proved that the nonzero mean curvature vector satisfies partial differential equations which imply that the nonzero mean curvature vector is not parallel. The usual terminology says that such an umbilical lightlike submanifold is not an *extrinsic sphere* (see [4] for more details). As the notion of totally umbilical submanifolds of Kählerian manifolds corresponds to that of totally contact umbilical submanifolds of Sasakian manifolds [11], the author in [15] showed that, in a totally contact umbilical lightlike hypersurface of an indefinite Sasakian space form, the nonzero mean curvature vector also is not parallel. But in [17] it is proved that any totally contact umbilical leaf of a screen integrable distribution of a lightlike hypersurface in an indefinite Sasakian space form is an extrinsic sphere.

In this paper, we focus on the similar mentioned notions above and those given in [7], [13], [14] and [16] on lightlike hypersurfaces of indefinite Sasakian manifolds. It is important to notice that Kenmotsu manifolds are different from Sasakian manifolds.

The paper is organized as follows. In section 2, we recall some basic definitions for indefinite Kenmotsu manifolds and lightlike hypersurfaces of semi-Riemannian manifolds. In section 3, we give the decomposition of almost contact metrics of lightlike hypersurfaces in indefinite Kenmotsu manifolds which are tangential to the structure vector field, supported by an example, as well as theorems on Lie derivatives and parallel second fundamental form. In section 4, we investigate the geometry of integrable distributions. Theorems on parallel vector fields, Killing distributions, geodesibility of lightlike hypersurfaces and of leaves of integrable distributions D , $D_0 \perp \langle \xi \rangle$ and D_0 are stated. By Theorems 5.4, 5.8 and 5.9 in section 5, we establish the geometrical configuration of such lightlike hypersurfaces, its screen distributions and leaves of its integrable screen distributions in Kenmotsu space forms. A characterization of totally contact umbilical lightlike hypersurfaces is given (Theorem 5.7). We show that any totally contact umbilical leaf of an integrable screen distribution of a lightlike hypersurface cannot be an extrinsic sphere (Theorem 5.10). By Theorem 5.12, we characterize the geometry of any leaf of integrable screen distribution.

2 Preliminaries

Let \overline{M} be a $(2n + 1)$ -dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \quad \text{and} \quad \overline{\phi}\xi = 0. \quad (2.1)$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on \overline{M} if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on \overline{M} and \overline{g} is a semi-Riemannian metric on \overline{M} such that, for any vector field $\overline{X}, \overline{Y}$ on \overline{M} [2]

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \quad \overline{g}(\overline{\phi}\overline{X}, \overline{\phi}\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}). \quad (2.2)$$

If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi}\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi}\overline{X}$ and $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric \overline{g} , we call \overline{M} an indefinite Kenmotsu manifold [9].

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$ -section if it is spanned by \overline{X} and $\overline{\phi}\overline{X}$, where \overline{X} is a unit tangent vector field orthogonal to ξ . The sectional curvature of a $\overline{\phi}$ -section σ is called a $\overline{\phi}$ -sectional curvature. If a Kenmotsu manifold \overline{M} has constant $\overline{\phi}$ -sectional curvature c , then, by virtue of the Proposition 12 in [9], the curvature tensor \overline{R} of \overline{M} is given by

$$\begin{aligned} \overline{R}(\overline{X}, \overline{Y})\overline{Z} &= \frac{c-3}{4} \{ \overline{g}(\overline{Y}, \overline{Z})\overline{X} - \overline{g}(\overline{X}, \overline{Z})\overline{Y} \} + \frac{c+1}{4} \{ \eta(\overline{X})\eta(\overline{Z})\overline{Y} \\ &\quad - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X}, \overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y}, \overline{Z})\eta(\overline{X})\xi + \overline{g}(\overline{\phi}\overline{Y}, \overline{Z})\overline{\phi}\overline{X} \\ &\quad - \overline{g}(\overline{\phi}\overline{X}, \overline{Z})\overline{\phi}\overline{Y} - 2\overline{g}(\overline{\phi}\overline{X}, \overline{Y})\overline{\phi}\overline{Z} \}, \quad \overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M}). \end{aligned} \quad (2.3)$$

A Kenmotsu manifold \overline{M} of constant $\overline{\phi}$ -sectional curvature c will be called *Kenmotsu space form* and denoted $\overline{M}(c)$.

Let $(\overline{M}, \overline{g})$ be a $(2n + 1)$ -dimensional semi-Riemannian manifold with index s , $0 < s < 2n + 1$ and let (M, g) be a hypersurface of \overline{M} , with $g = \overline{g}|_M$. M is a lightlike hypersurface of \overline{M} if g is of constant rank $2n - 1$ and the normal bundle TM^\perp is a distribution of rank 1 on M [3]. A complementary bundle of TM^\perp in TM is a rank $2n - 1$ non-degenerate distribution over M . It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As TM^\perp lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 2.1 [3] *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of TM^\perp on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on \mathcal{U} satisfying*

$$\overline{g}(N, E) = 1 \quad \text{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall W \in \Gamma(S(TM)|_{\mathcal{U}}). \quad (2.4)$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote $\Gamma(E)$ the smooth sections of the vector bundle E . Also by \perp and \oplus we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decomposition

$$\begin{aligned} TM &= S(TM) \perp TM^\perp, \\ \overline{TM} &= TM \oplus N(TM) = S(TM) \perp (TM^\perp \oplus N(TM)). \end{aligned} \quad (2.5)$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M}, \overline{g})$, then by using the second decomposition of (2.5) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.1, we have Gauss and Weingarten formulae in the form, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.6)$$

where $\nabla_X Y, A_N X \in \Gamma(TM)$. ∇ is an induced a symmetric linear connection on M , ∇^\perp is a linear connection on the vector bundle $N(TM)$, B is a symmetric bilinear form and A_N is the shape operator of M concerning N .

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.6) takes the form, for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \quad (2.7)$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.7), we obtain

$$B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \quad (2.8)$$

$$\tau(X) = \overline{g}(\nabla_X^\perp N, E), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \quad (2.9)$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM . We have

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)E, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \quad (2.10)$$

$$\text{and } \nabla_X E = -A_E^* X - \tau(X)E, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \quad (2.11)$$

where $\nabla_X^* PY$ and $A_E^* X$ belong to $\Gamma(S(TM))$. C , A_E^* and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \quad (2.12)$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$.

Also, we have, $g(A_E^* X, PY) = B(X, PY)$, $g(A_E^* X, N) = 0$, $B(X, E) = 0$.

Finally, using (2.7), \bar{R} and R are curvature tensor fields of \bar{M} and M are related as

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X \\ &+ \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ &- \tau(Y)B(X, Z)\} N,\end{aligned}\tag{2.13}$$

$$\text{where } (\nabla_X B)(Y, Z) = X.B(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

3 Lightlike hypersurfaces of indefinite Kenmotsu manifolds

Let $(\bar{M}, \bar{\phi}, \xi, \eta, \bar{g})$ be an indefinite Kenmotsu manifold and (M, g) be its lightlike hypersurface, tangent to the structure vector field ξ ($\xi \in TM$). If E is a local section of TM^\perp , then $\bar{g}(\bar{\phi}E, E) = 0$, and $\bar{\phi}E$ is tangent to M . Thus $\bar{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\bar{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\bar{\phi}(TM^\perp)$ as vector subbundle. If we consider a local section N of $N(TM)$, since $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$, we deduce that $\bar{\phi}E$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\bar{g}(\bar{\phi}N, N) = 0$, we see that the component of $\bar{\phi}N$ with respect to E vanishes. Thus $\bar{\phi}N \in \Gamma(S(TM))$. From (2.1), we have $\bar{g}(\bar{\phi}N, \bar{\phi}E) = 1$. Therefore, $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2. If M is tangent to the structure vector field ξ , then, we may choose $S(TM)$ so that ξ belongs to $S(TM)$. Using this, and since $\bar{g}(\bar{\phi}E, \xi) = \bar{g}(\bar{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n - 4$ on M such that

$$S(TM) = \left\{ \bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM)) \right\} \perp D_0 \perp \langle \xi \rangle,\tag{3.1}$$

where $\langle \xi \rangle$ is the distribution spanned by ξ , that is, $\langle \xi \rangle = \text{Span}\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\bar{\phi}$, i.e. $\bar{\phi}(D_0) = D_0$.

Example 3.1 We consider the 7-dimensional manifold $\bar{M} = \{(x_1, x_2, \dots, x_7) \in \mathbb{R}^7\}$, where $x = (x_1, x_2, \dots, x_7)$ are the standard coordinates in \mathbb{R}^7 . The vector fields

$$\begin{aligned}e_1 &= x_7 \frac{\partial}{\partial x_1}, \quad e_2 = x_7 \frac{\partial}{\partial x_2}, \quad e_3 = x_7 \frac{\partial}{\partial x_3}, \quad e_4 = x_7 \frac{\partial}{\partial x_4}, \quad e_5 = -x_7 \frac{\partial}{\partial x_5}, \\ e_6 &= -x_7 \frac{\partial}{\partial x_6}, \quad e_7 = -x_7 \frac{\partial}{\partial x_7}\end{aligned}\tag{3.2}$$

are linearly independent at each point of \bar{M} . Let \bar{g} be the Riemannian metric defined by $\bar{g}(e_i, e_j) = 0, \quad \forall i \neq j, \quad i, j = 1, 2, \dots, 7$ and $\bar{g}(e_k, e_k) = 1, \quad \forall k = 1, 2, 3, 4, 7, \quad \bar{g}(e_m, e_m) = -1, \quad \forall m = 5, 6$. Let η be the 1-form defined by $\eta(\bar{X}) = \bar{g}(\bar{X}, e_7)$, for any $\bar{X} \in \Gamma(T\bar{M})$.

Let $\bar{\phi}$ be the $(1, 1)$ tensor field defined by

$$\bar{\phi}e_1 = -e_2, \quad \bar{\phi}e_2 = e_1, \quad \bar{\phi}e_3 = -e_4, \quad \bar{\phi}e_4 = e_3, \quad \bar{\phi}e_5 = -e_6, \quad \bar{\phi}e_6 = e_5, \quad \bar{\phi}e_7 = 0.$$

Then using the linearity of $\bar{\phi}$ and \bar{g} , we have $\eta(e_7) = 1, \quad \bar{\phi}^2 \bar{X} = -\bar{X} + \eta(\bar{X})e_7, \quad \bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y})$, for any $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$. Thus, for $e_7 = \xi, (\bar{\phi}, \xi, \eta, \bar{g})$ defines an almost

contact metric structure on \overline{M} . Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the metric \overline{g} . Then, we have $[e_i, e_7] = e_i, \forall i = 1, 2, \dots, 6$ and $[e_i, e_j] = 0, \forall i \neq j, i, j = 1, 2, \dots, 6$. The metric connection $\overline{\nabla}$ of the metric \overline{g} is given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}) &= \overline{X}.\overline{g}(\overline{Y}, \overline{Z}) + \overline{Y}.\overline{g}(\overline{Z}, \overline{X}) - \overline{Z}.\overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]) \\ &\quad - \overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]), \end{aligned}$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by $\overline{\nabla}_{e_1}e_1 = -e_7, \overline{\nabla}_{e_2}e_2 = -e_7, \overline{\nabla}_{e_3}e_3 = -e_7, \overline{\nabla}_{e_4}e_4 = -e_7, \overline{\nabla}_{e_5}e_5 = e_7, \overline{\nabla}_{e_6}e_6 = e_7, \overline{\nabla}_{e_1}e_7 = e_1, \overline{\nabla}_{e_2}e_7 = e_2, \overline{\nabla}_{e_3}e_7 = e_3, \overline{\nabla}_{e_4}e_7 = e_4, \overline{\nabla}_{e_5}e_7 = e_5, \overline{\nabla}_{e_6}e_7 = e_6$. From these relations, it follows that the manifold \overline{M} satisfies $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$. Hence, \overline{M} is indefinite Kenmotsu manifold.

We now define a hypersurface M of $(\mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$ as $M = \{x \in \mathbb{R}^7 : x_5 = x_2\}$. Thus, the tangent space TM is spanned by $\{U_i\}_{1 \leq i \leq 6}$, where $U_1 = e_1, U_2 = e_2 - e_5, U_3 = e_3, U_4 = e_4, U_5 = e_6, U_6 = \xi$ and the 1-dimensional distribution TM^\perp of rank 1 is spanned by E , where $E = e_2 - e_5$. It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of \mathbb{R}^7 . Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}(e_2 + e_5)$. On the other hand, by using the almost contact structure of \mathbb{R}^7 and also by taking into account of the decomposition (3.1), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_3, \overline{\phi}F = -U_4$ and the distributions $\langle \xi \rangle, \overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by $\xi, \overline{\phi}E = U_1 + U_5$ and $\overline{\phi}N = \frac{1}{2}(U_1 - U_5)$. Hence, M is a lightlike hypersurface of \mathbb{R}^7 .

Moreover, from (2.5) and (3.1) we obtain the decomposition

$$TM = \left\{ \overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM)) \right\} \perp D_0 \perp \langle \xi \rangle \perp TM^\perp, \quad (3.3)$$

$$T\overline{M} = \left\{ \overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM)) \right\} \perp D_0 \perp \langle \xi \rangle \perp (TM^\perp \oplus N(TM)). \quad (3.4)$$

Now, we consider the distributions on $M, D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0, D' := \overline{\phi}(N(TM))$. Then D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle \xi \rangle. \quad (3.5)$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}N, V := -\overline{\phi}E$. Then, from (3.5), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi, QX = u(X)U$, where R and Q are the projection morphisms of TM into D and D' , respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to X and (2.1), one obtains $\overline{\phi}X = \phi X + u(X)N$, where ϕ is a tensor field of type $(1, 1)$ defined on M by $\phi X := \overline{\phi}RX$. Also, we obtain, for any $X \in \Gamma(TM)$,

$$B(X, \xi) = 0, \quad (3.6)$$

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \quad (3.7)$$

$$\text{and } \nabla_X \xi = X - \eta(X)\xi. \quad (3.8)$$

By using (2.1) we derive $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where v is 1-form locally defined on M by $v(\cdot) = g(U, \cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \quad (3.9)$$

For the sake of future use, we have the following identities: for any $X, Y \in \Gamma(TM)$,

$$C(X, \xi) = \theta(X), \quad (3.10)$$

$$B(X, U) = C(X, V) \quad (3.11)$$

$$(\nabla_X u)Y = -B(X, \phi Y) - u(Y)\tau(X) - \eta(Y)u(X), \quad (3.12)$$

$$(\nabla_X \phi)Y = \bar{g}(\bar{\phi}X, Y)\xi - \eta(Y)\phi X - B(X, Y)U + u(Y)A_N X. \quad (3.13)$$

Proposition 3.2 *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold \bar{M} with $\xi \in TM$. The Lie derivative of g with respect to the vector field V is given by,*

$$(L_V g)(X, Y) = X.u(Y) + Y.u(X) + u([X, Y]) - 2u(\nabla_X Y), \quad \forall X, Y \in \Gamma(TM). \quad (3.14)$$

Proof: The proof follows by direct calculation. \square

The relation (3.14) can be written in terms of B using the following relation,

$$u(\nabla_X Y) = B(X, \phi Y) + u(X)\eta(Y), \quad \forall X, Y \in \Gamma(TM). \quad (3.15)$$

As the geometry of a lightlike hypersurface depends on the chosen screen distribution, it is important to investigate the relationship between geometrical objects induced by two screen distributions. Suppose a screen $S(TM)$ changes to another screen $S(TM)'$. Following are the transformation equations due to this change (see details in [3], pages 164-165).

$$\begin{aligned} K'_i &= \sum_{j=1}^{2n-1} K_i^j (K_j - \epsilon_j c_j E), \\ N' &= N - \frac{1}{2} \left\{ \sum_{i=1}^{2n-1} \epsilon_i (c_i)^2 \right\} E + K, \\ \tau'(X) &= \tau(X) + B(X, K), \\ \nabla'_X Y &= \nabla_X Y + B(X, Y) \left\{ \frac{1}{2} \left(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2 \right) E - K \right\}, \end{aligned} \quad (3.16)$$

where $K = \sum_{i=1}^{2n-1} c_i K_i$, $\{K_i\}$ and $\{K'_i\}$ are the local orthonormal basis of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E . Here c_i and K_i^j are smooth functions on \mathcal{U} and $\{\epsilon_1, \dots, \epsilon_{2n-1}\}$ is the signature of the base $\{K_1, \dots, K_{2n-1}\}$. The Lie derivatives L_V and L'_V of the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related through the relation (see [14]):

$$(L'_V g)(X, Y) = (L_V g)(X, Y) - u(X)B(Y, W) - u(Y)B(X, W).$$

It is easy to check that the Lie derivative L_V is unique, that is, L_V is independent of $S(TM)$, if and only if, the second fundamental form h (or equivalently B) of M vanishes identically on M .

If a $(2n + 1)$ -dimensional Kenmotsu manifold \overline{M} has a constant $\overline{\phi}$ -sectional curvature c , then the Ricci tensor \overline{Ric} and the scalar curvature \overline{r} are given by [9]

$$\overline{Ric} = \frac{1}{2}(n(c - 3) + c + 1)\overline{g} - \frac{1}{2}(n + 1)(c + 1)\eta \otimes \eta, \quad (3.17)$$

$$\overline{r} = \frac{1}{2}(n(2n + 1)(c - 3) - n(c + 1)). \quad (3.18)$$

This means that \overline{M} is η -Einstein. Since \overline{M} is Kenmotsu and η -Einstein, by Corollary 9 in [9], \overline{M} is an Einstein one and consequently, $c + 1 = 0$, that is, $c = -1$. So, the Ricci tensor (3.17) becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n + 1)$.

Thus, if a Kenmotsu manifold \overline{M} is a space form, then it is η -Einstein and $c = -1$.

Let $\overline{M}(c)$ be an indefinite Kenmotsu space form and M be a lightlike hypersurface of $\overline{M}(c)$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (see Theorem 2.1) and by using (2.13), we obtain

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \quad (3.19)$$

Theorem 3.3 *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ of constant curvature c , with $\xi \in TM$. Then, the Lie derivative of the second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X, Y) = (1 - \tau(\xi))B(X, Y), \quad \forall X, Y \in \Gamma(TM). \quad (3.20)$$

Moreover, if $\tau(\xi) \neq 1$, then ξ is a Killing vector field with respect to the second fundamental form B if and only if M is totally geodesic.

Proof: Using (2.14), we obtain

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) - 2B(X, Y). \quad (3.21)$$

Likewise, Using again (2.14), we have

$$(\nabla_X B)(\xi, Y) = -B(X, Y). \quad (3.22)$$

Subtracting (3.21) and (3.22), we obtain

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = (L_\xi B)(X, Y) - B(X, Y). \quad (3.23)$$

From (5.9) and after calculations, the left hand side of (5.10) becomes

$$(\nabla_\xi B)(X, Y) - (\nabla_X B)(\xi, Y) = -\tau(\xi)B(X, Y). \quad (3.24)$$

The expressions (5.10) and (5.11) implies $(L_\xi B)(X, Y) = (1 - \tau(\xi))B(X, Y)$. The last assertion is obvious by definitions of Killing distribution and totally geodesic submanifold. \square

As an example to the last part of the Theorem 3.3, we have a lightlike hypersurface of an indefinite Kenmotsu space form, tangent to the structure vector field ξ , with parallel vector field U or V . In fact, when the vector field U or V is parallel, the differential 1-form τ vanishes on M and consequently, the equivalence of the Theorem 3.3 holds.

Next, we give characterization on parallel lightlike hypersurface of an indefinite Kenmotsu manifold. In fact, it shows that there do not exist non-totally geodesic totally umbilical lightlike hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field ξ .

The second fundamental form h of M is said to be parallel if $(\nabla_X h)(Y, Z) = 0, \forall X, Y, Z \in \Gamma(TM)$. That is,

$$(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z). \quad (3.25)$$

Theorem 3.4 *Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ of constant curvature c with $\xi \in TM$. If the second fundamental form h of M is parallel, then M is totally geodesic.*

Proof: Suppose that the second fundamental form h of M is parallel. Then (3.25) is satisfied. Using (3.25), we obtain

$$(\nabla_\xi B)(X, Y) = -\tau(\xi)B(X, Y). \quad (3.26)$$

From (2.14) and using (3.20), the left hand side of (3.26) becomes

$$(\nabla_\xi B)(X, Y) = (L_\xi B)(X, Y) - 2B(X, Y) = -(1 + \tau(\xi))B(X, Y). \quad (3.27)$$

From the expressions (3.26) and (3.27) we complete the proof. \square

This means that any parallel lightlike hypersurface M of an indefinite Kenmotsu manifold \overline{M} admits a *metric connection*.

The covariant derivative of the second fundamental form h depends on ∇, N and τ which depend on the choice of the screen vector bundle. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any $X, Y, Z \in \Gamma(TM)$,

$$\overline{g}((\nabla'_X h')(Y, Z), E) = \overline{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}_{(X, Y)}Z,$$

with $\mathcal{L}_{(X, Y)}Z = B(X, Y)B(Z, K) + B(X, Z)B(Y, K) + B(Y, Z)B(X, K)$. It is easy to check that the parallelism of h is independent of the screen distribution $S(TM)$ ($\nabla' h' \equiv \nabla h$) if and only the second fundamental form B of M vanishes identically on M .

From (2.3) and (2.13), a direct calculation shows that

$$\begin{aligned} (\nabla_X C)(Y, PZ) &= (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ) \\ &= \overline{g}(X, PZ)\theta(Y) - \overline{g}(Y, PZ)\theta(X). \end{aligned} \quad (3.28)$$

Lemma 3.5 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the covariant derivative of v and the Lie derivative of g with respect to the vector field U are given, respectively, by, for any $X, Y \in \Gamma(TM)$,*

$$(\nabla_X v)Y = -C(X, \phi Y) - v(X)\eta(Y) + \tau(X)v(Y), \quad (3.29)$$

$$(L_U g)(X, Y) = X.v(Y) + Y.v(X) + v([X, Y]) - 2v(\nabla_X Y). \quad (3.30)$$

Proof: The proof of (3.29) and (3.30) follows from direct calculations. \square

The Lie derivative (3.30) can be written in terms of the second fundamental form C of $S(TM)$ using the relation

$$v(\nabla_X Y) = C(X, \phi Y) + \eta(Y)v(X), \quad \forall X, Y \in \Gamma(TM). \quad (3.31)$$

The Lie derivative (3.30) depends on C and v which are not unique and their change can be seen as follows. Denote by κ the dual 1-form of $K = \sum_{i=1}^{2n-1} c_i K_i$ (characteristic vector field of the screen change defined in (3.16)) with respect to the induced metric g of M , that is $\kappa(X) = g(X, K)$, $\forall X \in \Gamma(TM)$. Let P and P' be projections of TM on $S(TM)$ and $S(TM)'$, respectively with respect to the orthogonal decomposition of TM . So, any vector field X on M can be written as $X = PX + \theta(X)E = P'X + \theta'(X)E$, where $\theta(X) = \overline{g}(X, N)$ and $\theta'(X) = \overline{g}(X, N')$. Then, using one of the relation in (3.16) we have $P'X = PX - \kappa(X)E$ and $C'(X, P'Y) = C'(X, PY)$, $\forall X, Y \in \Gamma(TM)$. The relationship between the second fundamental forms C and C' of the screen distribution $S(TM)$ and $S(TM)'$, respectively, is given by

$$C'(X, PY) = C(X, PY) - \frac{1}{2}\kappa(\nabla_X PY + B(X, Y)K). \quad (3.32)$$

So, (3.30) is independent of the screen distribution $S(TM)$ if and only if $\kappa(\nabla_X PY + B(X, Y)K) = 0$, $\forall X, Y \in \Gamma(TM)$.

Example 3.6 Let M be a hypersurface of \mathbb{R}^7 defined in the example 3.1. The tangent space TM is spanned by $\{U_i\}_{1 \leq i \leq 6}$, where $U_1 = e_1$, $U_2 = e_2 - e_5$, $U_3 = e_3$, $U_4 = e_4$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution TM^\perp of rank 1 is spanned by E , where $E = e_2 - e_5$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}(e_2 + e_5)$. It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of \mathbb{R}^7 having a local quasi-orthogonal field of frames $\{U_1, U_2 = E, U_3, U_4, U_5, U_6 = \xi, N\}$ along M . Denote by $\overline{\nabla}$ the Levi-Civita connection on \mathbb{R}^7 . Then, by straightforward calculations, we obtain

$$\overline{\nabla}_X N = 0, \quad \forall X \in \Gamma(TM).$$

Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae we have $A_N X = 0$, $A_E^* X = 0$ and $\nabla_X E = 0$, $\forall X \in \Gamma(TM)$. Therefore, by Theorem ?? and Proposition 2.7 in [3] page

89, the lightlike hypersurface M of \mathbb{R}^7 is totally geodesic and its distribution is parallel. The non-vanishing components of the Lie derivatives (3.14) and (3.30) are given by

$$\begin{aligned} L_V g(U_1, \xi) &= L_V g(\xi, U_1) = 1, & L_V g(U_5, \xi) &= L_V g(\xi, U_5) = -1, \\ L_V g(U, \xi) &= L_V g(\xi, U) = -1, & L_U g(V, \xi) &= L_U g(\xi, V) = -1, \\ L_U g(U_1, \xi) &= L_U g(\xi, U_1) = \frac{1}{2}, & L_U g(U_5, \xi) &= L_U g(\xi, U_5) = -\frac{1}{2}. \end{aligned}$$

4 Screen Integrable Lightlike Hypersurfaces of Indefinite Kenmotsu Manifolds

Let M be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. From the differential geometry of lightlike hypersurfaces, we recall the following desirable property for lightlike geometry. It is known that lightlike submanifolds whose screen distribution is integrable have interesting properties. Now, we study the geometry of integrable distributions with specific attention to the screen distribution $S(TM)$, the distributions D , D_0 and $D_0 \perp \langle \xi \rangle$. By Theorem 2.3 in [3] page 89, the screen distribution $S(TM)$ of M is integrable if and only if the second fundamental form of $S(TM)$ is symmetric on $\Gamma(S(TM))$. However, for any $X, Y \in \Gamma(D \perp \langle \xi \rangle)$, $u([X, Y]) = B(X, \phi Y) - B(\phi X, Y)$. So, it is very easy to see that the distribution $D \perp \langle \xi \rangle$ is integrable if and only if $B(X, \phi Y) = B(\phi X, Y)$.

Theorem 4.1 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$ such that the distribution $D \perp \langle \xi \rangle$ is integrable. Then, M is $D \perp \langle \xi \rangle$ -totally geodesic if and only if $\overline{\phi}(TM^\perp)$ is a $D \perp \langle \xi \rangle$ -Killing distribution.*

Proof: Since $D \perp \langle \xi \rangle$ is integrable, using (3.14) and (3.15), one obtains,

$$(L_V g)(X, Y) = -B(X, \phi Y) - B(\phi X, Y) = -2B(X, \phi Y), \quad X, Y \in \Gamma(D \perp \langle \xi \rangle).$$

Using (3.6) and the fact that $\overline{\phi}(D \perp \langle \xi \rangle) = D$, we complete the proof. \square

Note that the Theorem 4.1 also holds when the distribution $D \perp \langle \xi \rangle$ is replaced by D .

Example 4.2 Consider the lightlike hypersurface M of \mathbb{R}^7 defined in the example 3.6. Since M is totally geodesic, so it is obviously $D \perp \langle \xi \rangle$ -totally geodesic. Since the only nonvanishing brackets on the distribution $D \perp \langle \xi \rangle$ are $[V, \xi] = V$, $[E, \xi] = E$, $[F, \xi] = F$ and $[\overline{\phi}F, \xi] = \overline{\phi}F$, it is easy to check that the distribution $D \perp \langle \xi \rangle$ is integrable and $(L_V g)(X, Y) = -2B(X, \phi Y) = 0$, $X, Y \in \Gamma(D \perp \langle \xi \rangle)$, that is, $\overline{\phi}(TM^\perp)$ is a $D \perp \langle \xi \rangle$ -Killing distribution.

Proposition 4.3 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. If the screen distribution $S(TM)$ is integrable, then,*

$$(L_\xi C)(X, PY) = \tau(\xi)C(X, PY), \quad X, Y \in \Gamma(TM). \quad (4.1)$$

Proof: If the screen distribution $S(TM)$ of a lightlike hypersurface M is integrable, then, from (3.28) and using (3.10), we have, for any $X, Y \in \Gamma(TM)$,

$$(\nabla_\xi C)(X, PY) - (\nabla_X C)(\xi, PY) = \eta(PY)\theta(X) + \tau(\xi)C(X, PY). \quad (4.2)$$

On the other hand, using (3.10), we have

$$\begin{aligned} (\nabla_\xi C)(X, PY) &= \xi.C(X, PY) - C(\nabla_\xi X, PY) - C(X, \nabla_\xi(PY)) \\ &= (L_\xi C)(X, PY) - 2C(X, PY) + \eta(PY)\theta(X), \end{aligned} \quad (4.3)$$

$$\begin{aligned} \text{and } (\nabla_X C)(\xi, PY) &= X.C(\xi, PY) - C(\nabla_X \xi, PY) - C(\xi, \nabla_X PY) \\ &= -2C(X, PY). \end{aligned} \quad (4.4)$$

Putting (4.3) and (4.4) together in (4.2), we obtain (4.1). \square

Let us assume that the screen distribution $S(TM)$ of M is integrable and let M' be a leaf of $S(TM)$. Then, using (2.7) and (2.10), we obtain, for any $X, Y \in \Gamma(TM')$,

$$\bar{\nabla}_X Y = \nabla_X^* Y + C(X, Y)E + B(X, Y)N = \nabla'_X Y + h'(X, Y), \quad (4.5)$$

where ∇' and h' are the Levi-Civita connection and second fundamental form of M' in \bar{M} . Thus

$$h'(X, Y) = C(X, Y)E + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM'). \quad (4.6)$$

In the sequel, we need the following lemma.

Lemma 4.4 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmotsu manifold (\bar{M}, \bar{g}) with $\xi \in TM$ and M' be a leaf of $S(TM)$. Then, for any $X \in \Gamma(TM')$,*

$$\nabla'_X \xi = X - \eta(X)\xi, \quad (4.7)$$

$$\nabla'_X U = -v(X)\xi - v(A_N X)E - v(A_E^* X)N + \bar{\phi}(A_N X) + \tau(X)U, \quad (4.8)$$

$$\nabla'_X V = -u(X)\xi - u(A_N X)E - u(A_E^* X)N + \bar{\phi}(A_E^* X) - \tau(X)V. \quad (4.9)$$

Proof: From a straightforward calculation we complete the proof. \square

It is well known that the second fundamental form and the shape operators of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.10) and (2.11), in the case of lightlike hypersurfaces, the second fundamental forms on M and their screen distribution $S(TM)$ are related to their respective shape operators A_N and A_E^* . As the shape operator is an information tool in studying the geometry of submanifolds, their studying turns out very important. For instance, in [6] a class of lightlike hypersurfaces whose shape operators are the same as the one of their screen distribution up to a conformal non zero smooth factor in $\mathcal{F}(M)$ was considered. That work gave a way to generate, under some geometric conditions, an integrable canonical screen (see [6] for more details).

Next, we study these operators and give their implications in lightlike hypersurface of indefinite Kenmotsu manifolds with $\xi \in TM$.

Proposition 4.5 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and M' be a leaf of $S(TM)$. Then we have*

- (i) *The vector field U is parallel with respect to the Levi-Civita connection ∇' on M' if and only if*

$$A_N X = u(A_N X)U, \quad \forall X \in \Gamma(TM'),$$

v and τ vanish on M' .

- (ii) *The vector field V is parallel with respect to the Levi-Civita connection ∇' on M' if and only if*

$$A_E^* X = v(A_E^* X)V, \quad \forall X \in \Gamma(TM'),$$

u and τ vanishes on M' .

Proof: (i) Suppose U is parallel with respect to the Levi-Civita connection ∇' on M' . Then, by using (4.8), we have, for any $X \in \Gamma(TM')$,

$$\overline{\phi}(A_N X) = v(X)\xi + v(A_N X)E + v(A_E^* X)N - \tau(X)U. \quad (4.10)$$

Since $\overline{\phi}(A_N X) = \phi(A_N X) + u(A_N X)N$, by using (3.11), we obtain

$$\phi(A_N X) = v(X)\xi + v(A_N X)E - \tau(X)U. \quad (4.11)$$

Apply ϕ to (4.11) and by using (3.7) and the fact that $\phi U = 0$, we obtain

$$A_N X = \eta(A_N X)\xi + u(A_N X)U + v(A_N X)V \quad (4.12)$$

$$= \theta(X)\xi + u(A_N X)U + v(A_N X)V \quad (4.13)$$

$$= u(A_N X)U + v(A_N X)V, \quad (4.14)$$

since $\theta(X) = 0$, for any $X \in \Gamma(TM')$. Putting (4.12) in (4.8) and using (3.11), one obtains $v(X)\xi - \tau(X)U = 0$ which is equivalent to $v(X) = 0$ and $\tau(X) = 0$. Since $A_N X \in \Gamma(TM')$, then (4.12) is reduced to $A_N X = u(A_N X)U$. The converse is obvious. In the similar way, by using (4.9) the assertion (ii) follows. \square

Corollary 4.6 (to Proposition 4.5) *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and M' be a leaf of $S(TM)$ such U and V are parallel with respect to the Levi-Civita connection ∇' on M' . Then, the type number $t'(x)$ of M' (with $x \in M'$) satisfies $t'(x) \leq 1$.*

Proof: The proof follows from Proposition 4.5. \square

Let W be an element of $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ which is a non-degenerate vector subbundle of $S(TM)$ of rank 2. Then there exist non-zero functions a and b such that

$$W = aV + bU. \quad (4.15)$$

It is easy to check that $a = v(W)$ and $b = u(W)$. Let ω be a 1-form locally defined by $\omega(\cdot) = g(W, \cdot)$.

Lemma 4.7 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the covariant derivative of ω and the Lie derivative of g with respect to the vector field W are given, respectively, by*

$$(\nabla_X \omega)Y = -v(W)B(X, \phi Y) - u(W)C(X, \phi Y) - \omega(X)\eta(Y), \quad (4.16)$$

$$(L_W g)(X, Y) = X.\omega(Y) + Y.\omega(X) + \omega([X, Y]) - 2\omega(\nabla_X Y), \quad (4.17)$$

for any $X, Y \in \Gamma(TM)$.

Proof: Using (3.12) and (3.29), we obtain, for any $X, Y \in \Gamma(TM)$,

$$(\nabla_X \omega)Y = -v(W)B(X, \phi Y) - u(W)C(X, \phi Y) - \omega(X)\eta(Y) \quad (4.18)$$

which proves (4.16) and (4.17) follows from a direct calculation. \square

From (3.15) and (3.31), one obtains, for any $X, Y \in \Gamma(TM)$,

$$\omega(\nabla_X Y) = v(W)B(X, \phi Y) + u(W)C(X, \phi Y) + \omega(X)\eta(Y). \quad (4.19)$$

Lemma 4.8 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ and M' be a leaf of $S(TM)$. Then, for any, $X, Y \in \Gamma(TM')$,*

$$\omega(\nabla'_X Y) = -\omega(\overline{\phi}h'(X, \phi Y)), \quad (4.20)$$

$$\omega([X, Y]) = \omega(\overline{\phi}h'(\phi X, Y) - \overline{\phi}h'(X, \phi Y)). \quad (4.21)$$

Proof: Using (4.5) and (4.6), we obtain, for any $X, Y \in \Gamma(TM')$,

$$\begin{aligned} \omega(\nabla'_X Y) &= g(W, \nabla'_X Y) = v(W)u(\overline{\nabla}_X Y) + u(W)v(\overline{\nabla}_X Y) \\ &= v(W)B(X, \phi Y) + u(W)C(X, \phi Y) = -\omega(\overline{\phi}h'(X, \phi Y)) \end{aligned}$$

$$\text{and } \omega([X, Y]) = \omega(\nabla'_X Y) - \omega(\nabla'_Y X) = -\omega(\overline{\phi}h'(X, \phi Y) - \overline{\phi}h'(Y, \phi X))$$

which completes the proof. \square

Theorem 4.9 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the distribution $D_0 \perp \langle \xi \rangle$ is integrable if and only if*

$$C(\phi X, Y) = C(X, \phi Y), \quad B(\phi X, Y) = B(X, \phi Y), \quad (4.22)$$

$$\text{and } C(X, Y) = C(Y, X), \quad \forall X, Y \in \Gamma(D_0 \perp \langle \xi \rangle). \quad (4.23)$$

Proof: The proof follows from a direct calculation. \square

Note that when the distribution D_0 is integrable, the relations (4.22) and (4.23) are satisfied and vice versa.

Theorem 4.10 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold (\bar{M}, \bar{g}) with $\xi \in TM$. Suppose the distribution $D_0 \perp \langle \xi \rangle$ is integrable. Let M' be a leaf of $D_0 \perp \langle \xi \rangle$. Then*

- (i) *If M' is totally geodesic in M , then M' is auto-parallel with respect to the Levi-Civita connection ∇' in M and $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ is a Killing distribution on M' .*
- (ii) *If M' is parallel with respect to the Levi-Civita connection ∇' in M , then M' is totally geodesic.*

Proof (i) Writing $Y \in \Gamma(D_0 \perp \langle \xi \rangle)$ as $Y = \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} F_i + \eta(Y)\xi$, where $g(F_i, F_i) \neq 0$ and $\{F_i\}_{1 \leq i \leq 2n-4}$ an orthogonal basis of D_0 . So, it is easy to check that, for any $X, Y \in \Gamma(TM')$, $h'(X, \phi Y) = \sum_{i=1}^{2n-4} \frac{g(Y, F_i)}{g(F_i, F_i)} h'(X, \phi F_i)$. If M' is totally geodesic, then, for any $X, Y \in \Gamma(D_0 \perp \langle \xi \rangle)$, $h'(X, Y) = 0$. In particular $h'(X, \phi Y) = \sum_i \frac{g(Y, F_i)}{g(F_i, F_i)} h'(X, \phi F_i) = 0$. The auto-parallelism of M' follows from (4.20). Using (4.17), (4.20), (4.21) and the fact that $\omega(X) = 0, \forall X \in \Gamma(D_0 \perp \langle \xi \rangle)$, we obtain $(L_W g)(X, Y) = 0$. So $\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ is a Killing distribution on M' . (ii) If M' is parallel with respect to the connection in M , then, for any $X, Y, Z \in \Gamma(TM')$, $(\nabla'_X h')(Y, Z) = 0$. So, $(\nabla'_X C)(Y, Z) - C(Y, Z)\tau(X) = 0$ and $(\nabla'_X B)(Y, Z) + B(Y, Z)\tau(X) = 0$. Using (3.6) and (3.20), since $D_0 \perp \langle \xi \rangle$ integrable, for $Z = \xi$, we have,

$$0 = (\nabla'_\xi B)(X, Y) + \tau(\xi)B(X, Y) = -B(X, Y). \quad (4.24)$$

Also, using (4.1), we obtain, for any $X, Y \in \Gamma(TM')$,

$$0 = (\nabla'_\xi C)(X, Y) - \tau(\xi)C(X, Y) = -2C(X, Y). \quad (4.25)$$

From (4.24) and (4.25), we get $h'(X, Y) = 0$ which completes the proof. \square

Note that, the Lie derivative (4.17) can be expressed in functions of Lie derivatives (3.14) and (3.30) as, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned} (L_W g)(X, Y) &= X.v(W)u(Y) + Y.v(W)u(X) + X.u(W)v(Y) + Y.u(W)v(X) \\ &+ v(W)(L_V g)(X, Y) + u(W)(L_U g)(X, Y). \end{aligned} \quad (4.26)$$

Theorem 4.11 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold (\bar{M}, \bar{g}) with $\xi \in TM$. Suppose the distribution D_0 is integrable. Let M' be a leaf of D_0 . Then, the following assertions are equivalent:*

- (i) *M' is totally geodesic in M ,*
- (ii) *$A_E^* X$ and $A_N X \in \Gamma(\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))), \forall X \in \Gamma(TM')$,*
- (iii) *$\bar{\phi}(TM^\perp) \oplus \bar{\phi}(N(TM))$ is a Killing distribution on M' ,*
- (iv) *$\bar{\phi}(TM^\perp)$ and $\bar{\phi}(N(TM))$ are Killing distribution on M' .*

Proof: The equivalence of (i) and (ii) follows from direct calculations. Using the relation (4.26), we obtain the equivalence of (iii) and (iv). Next we prove the equivalence of (i) and (iii). Using the fact that ω vanishes on M' and the relation (4.19), and since D_0 is integrable, (4.17) becomes, for any $X, Y \in \Gamma(TM')$,

$$\begin{aligned} (L_W g)(X, Y) &= -v(W) \{B(X, \phi Y) + B(\phi X, Y)\} - u(W) \{C(X, \phi Y) + C(\phi X, Y)\} \\ &= -2\omega(\overline{\phi}h'(X, \phi Y)) \end{aligned} \quad (4.27)$$

Suppose M' is totally geodesic in M . Then, $h'(X, Y) = 0, \forall X, Y \in \Gamma(D_0)$. In particular $h'(X, \phi Y) = 0$, since $D_0 = \overline{\phi}(D_0)$. Therefore $(L_W g)(X, Y) = 0$ and $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ is a Killing distribution on M' . The converse is obvious by using (4.27). \square

A submanifold M is said to be $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic if for any $X \in \Gamma(D \perp \langle \xi \rangle)$ and $Y \in \Gamma(D')$, $B(X, Y) = 0$.

Let M be a $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic of an indefinite Kenmotsu manifold \overline{M} with $\xi \in TM$. Then, for any $X \in \Gamma(D \perp \langle \xi \rangle)$, $B(X, U) = 0$. Using (3.11), we have $u(A_N X) = \overline{g}(A_N X, V) = C(X, V) = B(U, U) = 0$ i.e $A_N X \in \Gamma(D \perp \langle \xi \rangle)$. Since $g(A_N X, N) = 0$, that is, $A_N X$ has no component in $\Gamma(TM^\perp)$, so we have

$$A_N X \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0 \perp \langle \xi \rangle) \quad (4.28)$$

The converse is obvious i.e if the relation (4.28) is satisfied, M is $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic. We have:

Proposition 4.12 *Let $(M, g, S(TM))$ be a $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Suppose the distribution D_0 is integrable, and vector fields U and V are parallel with respect to the Levi-Civita connection ∇' on M' . Let M' be a leaf of D_0 . Then, M' is totally geodesic in M if and only if the shape operators A_E^* and A_N vanish identically on M' .*

Proof: Suppose that M' is totally geodesic in M . Since $(M, g, S(TM))$ be a $(D \perp \langle \xi \rangle, D')$ -mixed totally geodesic and using the relation (4.28) and the Theorem 4.11, then, for any $X \in \Gamma(TM')$, $A_E^* X = u(A_E^* X)U$ and $A_N X = v(A_N X)V$. By Proposition 4.5, it is easy to check that the shape operators A_E and A_N vanish identically on M' . The converse is obvious. \square

5 Totally Contact Umbilical Leaf of Integrable Screen Distributions

In this section, we deal with the geometry of the mean curvature vector of a leaf of an integrable screen distribution of a lightlike hypersurface M of an indefinite Kenmotsu space form $\overline{M}(c)$ by introducing a new concept. First of all, a submanifold M is said to be totally umbilical lightlike hypersurface of the a semi-Riemannian manifold \overline{M} if the local second fundamental form B of M satisfies

$$B(X, Y) = \rho g(X, Y), \quad \forall X, Y \in \Gamma(TM) \quad (5.1)$$

where ρ is a smooth function on $\mathcal{U} \subset M$. If we assume that M is totally umbilical lightlike hypersurface of the a semi-Riemannian manifold \overline{M} , then we have $B(X, Y) = \rho g(X, Y)$, for any $X, Y \in \Gamma(TM)$, which implies, by using (3.6), that $0 = B(\xi, \xi) = \rho$. Hence M is totally geodesic. Therefore we have

Proposition 5.1 *Let (M, g) be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. If M is totally umbilical, then M is totally geodesic.*

It follows from the Proposition 5.1 that a Kenmotsu $\overline{M}(c)$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [1] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [11].

It is now important to investigate the parallelism of the nonzero mean curvature vector by regarding the effect of the totally contact umbilical condition on the geometry of lightlike submanifolds in Kenmotsu manifolds case. As it was done in case of lightlike hypersurfaces of indefinite Sasakian manifolds [17], the terminology of *extrinsic sphere* [4] is also going to be used in case of totally contact geodesic submanifolds. We say that a totally contact umbilical submanifold is an *extrinsic sphere* when it has parallel non zero mean curvature vector [4].

A submanifold M is said to be totally contact umbilical if its second fundamental form h of M satisfies [1]

$$h(X, Y) = \{g(X, Y) - \eta(X)\eta(Y)\} H + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \quad (5.2)$$

for any $X, Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$). Using (3.6), it is easy to check that a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold is η -totally umbilical.

Proposition 5.2 *Let (M, g) be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, $\overline{\phi}(TM^\perp)$ is a $D \perp \langle \xi \rangle$ -Killing distribution.*

Proof: Using (3.9) and (3.14), one obtains, for any $X, Y \in \Gamma(D \perp \langle \xi \rangle)$,

$$\begin{aligned} (L_V g)(X, Y) &= -B(X, \phi Y) - B(\phi X, Y) = -g(X, \phi Y) - g(\phi X, Y) \\ &= u(X)\theta(Y) + u(Y)\theta(X) = 0, \end{aligned}$$

which completes the proof. □

In the sequel, we need the following identities and lemma. For any $X, Y \in \Gamma(TM)$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \quad (\nabla_X \theta)Y = -C(X, Y) + \tau(X)\theta(Y). \quad (5.3)$$

Lemma 5.3 *Let (M, g) be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, $\forall X, Y, Z \in \Gamma(TM)$,*

$$\begin{aligned} (\nabla_X B)(Y, Z) &= \lambda \{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\} - \lambda \{\eta(Z)\overline{g}(\overline{\phi}X, \overline{\phi}Y) \\ &+ \eta(Y)\overline{g}(\overline{\phi}X, \overline{\phi}Z)\} + \{g(Y, Z) - \eta(Y)\eta(Z)\} (X.\lambda). \end{aligned} \quad (5.4)$$

Proof: The proof follows from direct computing using the identities (2.12), (3.6) and (5.3). \square

Theorem 5.4 *Let $\overline{M}(c)$ be an indefinite Kenmotsu space form and M be a totally contact umbilical lightlike hypersurface of $\overline{M}(c)$ with $\xi \in TM$. Then λ satisfies the partial differential equations*

$$E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0, \quad (5.5)$$

$$\xi \cdot \lambda + \lambda(\tau(\xi) + 1) = 0, \quad (5.6)$$

$$\text{and } PX \cdot \lambda + \lambda\tau(PX) = 0, \quad PX \neq \xi, \quad \forall X \in \Gamma(TM). \quad (5.7)$$

Proof: Let M be a totally contact umbilical lightlike hypersurface. Since $c = -1$, the direct calculation of the right hand side in (2.13) shows that, for any $X, Y \in \Gamma(TM)$,

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \quad (5.8)$$

Using (5.4), the equation (5.8) becomes

$$\begin{aligned} & \lambda \{B(X, Y)\theta(Z) + B(X, Z)\theta(Y)\} - \lambda\eta(Z) \{g(X, Y) - \eta(X)\eta(Y)\} \\ & - \lambda\eta(Y) \{g(X, Z) - \eta(X)\eta(Z)\} + \{g(Y, Z) - \eta(Y)\eta(Z)\} (X \cdot \lambda) \\ & - \lambda \{B(X, Y)\theta(Z) + B(Y, Z)\theta(X)\} + \lambda\eta(Z) \{g(X, Y) - \eta(X)\eta(Y)\} \\ & + \lambda\eta(X) \{g(Y, Z) - \eta(Y)\eta(Z)\} - \{g(X, Z) - \eta(X)\eta(Z)\} (Y \cdot \lambda) \\ & = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \end{aligned} \quad (5.9)$$

Regrouping like terms in (5.9) and using (3.9), we deduce

$$\begin{aligned} & \lambda \{B(X, Z)\theta(Y) - B(Y, Z)\theta(X)\} + \lambda \{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\} \\ & + \{g(Y, Z) - \eta(Y)\eta(Z)\} (X \cdot \lambda) - \{g(X, Z) - \eta(X)\eta(Z)\} (Y \cdot \lambda) \\ & = \tau(Y)B(X, Z) - \tau(X)B(Y, Z). \end{aligned} \quad (5.10)$$

Putting $X = E$ in (5.10), we find

$$-\lambda B(Y, Z) + \{g(Y, Z) - \eta(Y)\eta(Z)\} (E \cdot \lambda) = -\tau(E)B(Y, Z). \quad (5.11)$$

Take $Y = V$ and $Z = U$ in (5.11), we have $(B(V, U) = \lambda)$, $E \cdot \lambda + \lambda\tau(E) - \lambda^2 = 0$.

Finally, substituting $X = PX, Y = PY$ and $Z = PZ$ into (5.10) and taking into account that $S(TM)$ is nondegenerate, we obtain

$$\begin{aligned} & \{PX \cdot \lambda + \lambda\tau(PX)\} (PY - \eta(PY)\xi) + \lambda\eta(PX)PY \\ & = \{PY \cdot \lambda + \lambda\tau(PY)\} (PX - \eta(PX)\xi) + \lambda\eta(PY)PX. \end{aligned} \quad (5.12)$$

Putting $PX = \xi$ in (5.12), we have $\{\xi \cdot \lambda + \lambda(\tau(\xi) + 1)\} (PY - \eta(PY)\xi) = 0$ which leads, by taking $Y = V$, to $\xi \cdot \lambda + \lambda(\tau(\xi) + 1) = 0$.

If $PX, PY, PZ \in \Gamma(S(TM) - \langle \xi \rangle)$, then (5.12) becomes

$$\{PX \cdot \lambda + \lambda\tau(PX)\} PY = \{PY \cdot \lambda + \lambda\tau(PY)\} PX. \quad (5.13)$$

Now suppose that there exists a vector field X_0 on some neighborhood of M such that $PX_0 \cdot \lambda + \lambda\tau(PX_0) \neq 0$ at some point p in the neighborhood. Then, from (5.13) it follows that all vectors of the fibre $(S(TM) - \langle \xi \rangle)_p := (\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM)) \perp D_0)_p \subset S(TM)_p$ are collinear with $(PX_0)_p$. This contradicts $\dim(S(TM) - \langle \xi \rangle)_p > 1$. This implies (5.7). \square

A part of the Theorem 5.4 is similar to that of the generic submanifold of indefinite Sasakian manifolds case given in [15]. From the equations (5.5) and (5.7), the geometry of the mean curvature vector H of M is discussed. some equations are similar to those of the indefinite Kählerian case (see [3] for details). From (5.5) and (5.7), we have $\nabla_E^\perp H = \overline{g}(H, E)^2 N$, $\nabla_\xi^\perp H = -\overline{g}(H, E)N$ and $\nabla_{PX}^\perp H = 0$, $PX \neq \xi$, $\forall X \in \Gamma(TM)$. This means that H is not parallel on M .

Lemma 5.5 *Let M be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Then, the mean curvature vector H of M is $(S(TM) - \langle \xi \rangle)$ -parallel, that is, $\nabla_{PX}^\perp H = 0$, $PX \neq \xi$, $\forall X \in \Gamma(TM)$.*

Lemma 5.6 *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold \overline{M} with $\xi \in TM$. Then, M is $D \perp \langle \xi \rangle$ -totally geodesic if and only if, for any $X \in \Gamma(D \perp \langle \xi \rangle)$, $A_E^* X = u(A_N X)V$.*

Proof: The proof follows from straightforward calculation. \square

It is well known that if the lightlike hypersurface (M, g) is totally geodesic, the induced connection ∇ on M is torsion-free and g -metric. Also, the shape operator A_E^* vanishes identically on M (see Theorem 2.2 in [3] page 88). This vanishing property failed when the lightlike hypersurface M , with $\xi \in TM$, is $D \perp \langle \xi \rangle$ -totally geodesic. That is, only some privileged conditions on M and its the screen distribution may enable to get the $D \perp \langle \xi \rangle$ -version of the Theorem ???. Now, say that the screen distribution $S(TM)$ is totally umbilical if on any coordinates neighborhood $\mathcal{U} \subset M$, there exists a smooth function φ such that

$$C(X, PY) = \varphi g(X, PY), \forall X, Y \in \Gamma(TM|_{\mathcal{U}}). \quad (5.14)$$

If we assume that the screen distribution $S(TM)$ of the lightlike hypersurface M with $\xi \in TM$ is totally umbilical, then it follows that C is symmetric on $\Gamma(S(TM)|_{\mathcal{U}})$ and hence according to Theorem 2.3 in [3], the distribution $S(TM)$ is integrable. Also, we have $A_N X = \varphi PX$ and $C(E, PX) = 0$. Since $\overline{\phi}\xi = 0$ and by using $\eta(A_N X) = -\theta(X)$, we have $\eta(A_N \xi) = \varphi \overline{g}(\xi, \xi) = -\theta(\xi) = 0$ which implies that $\varphi = 0$, so the screen distribution $S(TM)$ is totally geodesic. Therefore, we have the following result.

Theorem 5.7 *Let $(M, g, S(TM))$ be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$, with $\xi \in TM$, such that $S(TM)$ is totally umbilical. Then, the following assertions are equivalent:*

- (i) M is $D \perp \langle \xi \rangle$ -totally geodesic,

(ii) $A_E^*X = 0, \forall X \in \Gamma(D \perp \langle \xi \rangle)$,

(iii) $\bar{\phi}(TM^\perp)$ is $D \perp \langle \xi \rangle$ -parallel.

Proof: Since the screen distribution $S(TM)$ is totally umbilical, $S(TM)$ is totally geodesic, that is, for any $X, Y \in \Gamma(S(TM))$, $C(X, Y) = 0$. In particular, for any $X \in \Gamma(\bar{\phi}(TM^\perp) \perp D_0 \perp \langle \xi \rangle)$, $C(X, V) = u(A_N X) = 0$. Since $C(E, V) = 0$, for any $X_0 \in \Gamma(D \perp \langle \xi \rangle)$, $u(A_N X_0) = 0$ and using the Lemma 5.6, the equivalence of (i) and (ii) follows.

Now, we want to show the equivalence of (ii) and (iii). First of all, we have

$$\bar{\nabla}_{X_0} V = -(\bar{\nabla}_{X_0} \bar{\phi})E - \bar{\phi}(\nabla_{X_0} E) = -u(X_0)\xi + \bar{\phi}(A_E^* X_0) - \tau(X_0)V. \quad (5.15)$$

Writing the left hand side of (5.15) as $\bar{\nabla}_{X_0} V = \nabla_{X_0} V + u(A_E^* X_0)N$, we deduce

$$\nabla_{X_0} V = \bar{\phi}(A_E^* X_0) - u(X_0)\xi - u(A_E^* X_0)N - \tau(X_0)V. \quad (5.16)$$

Suppose $A_E^* X_0 = 0, \forall X_0 \in \Gamma(D \perp \langle \xi \rangle|_{\mathcal{U}})$. Then, the relation (5.16) becomes, $\nabla_{X_0} V = -\tau(X_0)V$. Since the normal bundle $\bar{\phi}(TM^\perp)$ is a distribution on M of rank 1 and spanned by V , then, for any $Y_0 \in \Gamma(\bar{\phi}(TM^\perp))$, $\nabla_{X_0} Y_0 = (X_0.v(Y_0) - v(Y_0)\tau(X_0))V \in \Gamma(\bar{\phi}(TM^\perp))$, since $Y_0 = v(Y_0)V$. So, the distribution $\bar{\phi}(TM^\perp)$ is $D \perp \langle \xi \rangle$ -parallel. Conversely, suppose the distribution $\bar{\phi}(TM^\perp)$ is $D \perp \langle \xi \rangle$ -parallel. Then, for any $X_0 \in \Gamma(D \perp \langle \xi \rangle)$ and $Y_0 = v(Y_0)V \in \Gamma(\bar{\phi}(TM^\perp)|_{\mathcal{U}})$, $\nabla_{X_0} Y_0 \in \Gamma(\bar{\phi}(TM^\perp)|_{\mathcal{U}})$. In particular, by taking $Y_0 = V$, we have $\nabla_{X_0} V \in \Gamma(\bar{\phi}(TM^\perp)|_{\mathcal{U}})$. Since $\bar{\phi}(TM^\perp)$ is spanned by V , there exist a smooth function $\varepsilon \neq 0$ on M such that $\nabla_{X_0} V = \varepsilon V$. Using (5.16), we have $\varepsilon = \bar{g}(\bar{\phi}(A_E^* X_0), U) - \tau(X_0) = -\bar{g}(A_E^* X_0, N) - \tau(X_0) = -\tau(X_0)$. Since $\nabla_{X_0} V = -\tau(X_0)V$ and u vanishes on $D \perp \langle \xi \rangle$, from (5.16), we obtain $\bar{\phi}(A_E^* X_0) = -u(A_E^* X_0)N$. Applying $\bar{\phi}$ to this equation, using (3.6) and the fact that M is totally contact umbilical lightlike hypersurface, one obtains, for any $X_0 \in \Gamma(D \perp \langle \xi \rangle)$, $A_E^* X_0 = -\lambda u(X_0)U = 0$. This completes the proof. \square

The Theorem 5.7 can be extended by using Theorem 2.2 in [3] (page 88) in order to get more information about the geometry of lightlike hypersurface M .

We say that the screen distribution $S(TM)$ is totally contact umbilical if the local second fundamental form C of $S(TM)$ satisfies

$$C(X, Y) = \alpha(g(X, Y) - \eta(X)\eta(Y)) + \eta(X)C(Y, \xi) + \eta(Y)C(X, \xi), \quad (5.17)$$

where α is a smooth function on $\mathcal{U} \subset M$. If we assume that the screen distribution of the lightlike hypersurface M of an indefinite Kenmotsu manifold, with $\xi \in TM$, is totally contact umbilical, then it follows that C is symmetric on $\Gamma(S(TM))$ and hence, by Theorem 2.3 page 89, the distribution $S(TM)$ is integrable.

Theorem 5.8 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu space form $\bar{M}(c)$ with $\xi \in TM$ such that $S(TM)$ is totally contact umbilical. Then $S(TM)$ is totally contact geodesic.*

Proof: By a direct calculation of the right hand side in (3.28) and using (5.17), we get

$$\begin{aligned}
& (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) + \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ) \\
&= \{g(Y, PZ) - \eta(Y)\eta(PZ)\} (X.\alpha) - \{g(X, Z) - \eta(X)\eta(PZ)\} (Y.\alpha) \\
&+ \alpha \{B(X, PZ)\theta(Y) - B(Y, PZ)\theta(X)\} + 2\alpha \{\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ)\} \\
&+ \eta(PZ) \{\tau(X)\theta(Y) - \tau(Y)\theta(X)\} + \theta(Y)g(X, PZ) - \theta(X)g(Y, PZ) \\
&+ \tau(Y)C(X, PZ) - \tau(X)C(Y, PZ). \tag{5.18}
\end{aligned}$$

Putting $X = E$ in (5.18) and in the right hand side of (3.28), we obtain

$$\begin{aligned}
& \{g(Y, PZ) - \eta(Y)\eta(PZ)\} (E.\alpha) - \alpha B(Y, PZ) + \eta(PZ) \{\tau(E)\theta(Y) - \tau(Y)\} \\
& - g(Y, PZ) + \tau(Y)C(E, PZ) - \tau(E)C(Y, PZ) \\
&= -\bar{g}(Y, PZ). \tag{5.19}
\end{aligned}$$

Replacing $Y = PZ = U$ in (5.19), we have $-\alpha B(U, U) = -\alpha C(U, U) = -\alpha^2 = 0$. \square

It is easy to check that, when the screen distribution $S(TM)$ of a lightlike hypersurface M with $\xi \in TM$ is η -totally umbilical, its second fundamental form $h^* = C \otimes E$ vanishes identically, that is, $S(TM)$ is totally geodesic. This allows us to say that the Theorem 5.7 also holds when totally umbilical condition is replaced by η -totally umbilical condition on $S(TM)$.

Theorem 5.9 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmotsu manifold (\bar{M}, \bar{g}) with $\xi \in TM$. Suppose any leaf M' of $S(TM)$ is totally contact umbilical immersed in \bar{M} as non-degenerate submanifold. Then, the mean curvature vector H' of M' satisfies the partial differential equations*

$$\bar{g}(\nabla_\xi^\perp H', E) + \bar{g}(H', E) = 0, \tag{5.20}$$

$$\bar{g}(\nabla_\xi^\perp H', N) + \bar{g}(H', N) = 0, \tag{5.21}$$

$$\bar{g}(\nabla_X^\perp H', E) = 0, \quad \bar{g}(\nabla_X^\perp H', N) = 0, \quad \forall X \in \Gamma(TM' - \langle \xi \rangle), \tag{5.22}$$

where ∇^\perp is a linear connection on $N(TM) \oplus TM^\perp$ defined by $\nabla_X^\perp E = \nabla_X^* E = -\tau(X)E$ and $\nabla_X^\perp N = \nabla_X N = \tau(X)N$.

Proof: By combining the first equations of (2.7) and (2.10), we obtain

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X^* Y + C(X, Y)E + B(X, Y)N \\
&= \nabla'_X Y + h'(X, Y), \quad \forall X, Y \in \Gamma(TM'). \tag{5.23}
\end{aligned}$$

Denote by H' the mean curvature vector of M' . As $N(TM) \oplus TM^\perp$ is the normal bundle of M' , there exist smooth functions λ and ρ such that $H' = \lambda E + \rho N$. Since M' is totally contact umbilical immersed in \bar{M} we have

$$h'(X, Y) = (g(X, Y) - \eta(X)\eta(Y)) H' + \eta(X)h'(Y, \xi) + \eta(Y)h'(X, \xi). \tag{5.24}$$

Since $h'(X, \xi) = 0$, for any $X \in \Gamma(TM')$, from (5.23) we obtain

$$\bar{\nabla}_X Y = \nabla'_X Y + (g(X, Y) - \eta(X)\eta(Y)) H' \quad (5.25)$$

which implies

$$\begin{aligned} \bar{\nabla}_X \bar{\nabla}_Y Z &= \nabla'_X \nabla'_Y Z + \{g(X, \nabla'_Y Z) - \eta(X)\eta(\nabla'_Y Z)\} H' \\ &+ \{g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z) - \eta(Z)g(X, Y) + 2\eta(X)\eta(Y)\eta(Z) \\ &- \eta(Z)\eta(\nabla'_X Y) - \eta(Y)g(X, Z) - \eta(Y)\eta(\nabla'_X Z)\} H' \\ &+ \{g(Y, Z) - \eta(Y)\eta(Z)\} \bar{\nabla}_X H'. \end{aligned} \quad (5.26)$$

Since $S(TM')$ is integrable, $\theta([X, Y]) = 0$, for any $X, Y \in \Gamma(TM')$ and we have

$$\bar{\nabla}_{[X, Y]} Z = \nabla'_{[X, Y]} Z + \{g([X, Y], Z) - \eta([X, Y])\eta(Z)\} H'. \quad (5.27)$$

From (5.26), (5.27) and (4.7)-(4.9), after calculations, we obtain

$$\begin{aligned} \bar{R}(X, Y)Z &= R'(X, Y)Z + \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} H' \\ &+ \{g(Y, Z) - \eta(Y)\eta(Z)\} \bar{\nabla}_X H' - \{g(X, Z) - \eta(X)\eta(Z)\} \bar{\nabla}_Y H'. \end{aligned} \quad (5.28)$$

Consequently,

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, E) &= \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \bar{g}(H', E) + \{g(Y, Z) \\ &- \eta(Y)\eta(Z)\} \bar{g}(\bar{\nabla}_X H', E) - \{g(X, Z) - \eta(X)\eta(Z)\} \bar{g}(\bar{\nabla}_Y H', E), \end{aligned} \quad (5.29)$$

$$\begin{aligned} \bar{g}(\bar{R}(X, Y)Z, N) &= \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \bar{g}(H', N) + \{g(Y, Z) \\ &- \eta(Y)\eta(Z)\} \bar{g}(\bar{\nabla}_X H', N) - \{g(X, Z) - \eta(X)\eta(Z)\} \bar{g}(\bar{\nabla}_Y H', N). \end{aligned} \quad (5.30)$$

From (5.29) and using (2.3), we obtain

$$\begin{aligned} 0 &= \{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\} \bar{g}(H', E) + \{g(Y, Z) \\ &- \eta(Y)\eta(Z)\} \bar{g}(\bar{\nabla}_X H', E) - \{g(X, Z) - \eta(X)\eta(Z)\} \bar{g}(\bar{\nabla}_Y H', E). \end{aligned} \quad (5.31)$$

Taking $X = \xi$ in this equation, we have, for $Y = U$ and $Z = V$, $\bar{g}(\bar{\nabla}_\xi H', E) + \bar{g}(H', E) = 0$.

Now, if $X, Y, Z \in \Gamma(TM' - \xi)$, from (5.31), we have

$$\bar{g}(\bar{\nabla}_X H', E)Y = \bar{g}(\bar{\nabla}_Y H', E)X. \quad (5.32)$$

Likewise, from (5.30) and (2.3), we have

$$\bar{g}(\bar{\nabla}_\xi H', N) + \bar{g}(H', N) = 0 \quad \text{and} \quad \bar{g}(\bar{\nabla}_X H', N)Y = \bar{g}(\bar{\nabla}_Y H', N)X. \quad (5.33)$$

Now suppose that there exists a vector field X_0 on some neighborhood of M' such that $\bar{g}(\bar{\nabla}_{X_0} H', E) \neq 0$ and $\bar{g}(\bar{\nabla}_{X_0} H', N) \neq 0$ at some point p in the neighborhood. From (5.32) and (5.33) it follows that all vectors of the fibre $TM' - \langle \xi \rangle$ are collinear with $X_0|_p$. This contradicts $\dim(TM' - \langle \xi \rangle) > 1$. This implies $\bar{g}(\bar{\nabla}_X H', E) = 0$ and $\bar{g}(\bar{\nabla}_X H', N) = 0$,

$\forall X \in \Gamma(TM' - \langle \xi \rangle)$. These lead, respectively, to $g(\nabla_X^\perp H', E) = 0$ and $g(\nabla_X^\perp H', N) = 0$ which completes the proof. \square

From (5.20) and (5.22), we have $\nabla_\xi^\perp H' = H'$ and $\nabla_X^\perp H' = 0, \forall X \in \Gamma(TM' - \langle \xi \rangle)$. So, $\nabla_X^\perp H' = H'$, for any $X \in \Gamma(TM')$. This means that the mean curvature vector H' is not parallel on M' and consequently, we have

Theorem 5.10 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmostu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Suppose any leaf M' of $S(TM)$ is totally contact umbilical immersed in \overline{M} as non-degenerate submanifold. Then M' cannot be an extrinsic sphere.*

Next we deal with the geometry of the normal bundle $TM^\perp \oplus N(TM)$ and we show there exists a close relationship between its geometry with the geometry of a leaf of an integrable screen distribution of a lightlike hypersurface M of an indefinite Kenmostu space form $\overline{M}(c)$. Let \widehat{W} be an element of $TM^\perp \oplus N(TM)$ which is a non-degenerate of rank 2. Then there exist non zero functions α and β such that

$$\widehat{W} = \alpha E + \beta N, \quad (5.34)$$

where α and β are defined as $\alpha = \overline{g}(\widehat{W}, N)$ and $\beta = \overline{g}(\widehat{W}, E)$. Let $\mathcal{A}_{\widehat{W}}$ be a tensor field of type $(1, 1)$ locally defined by the combination of the shape operators A_E^* and A_N , that is,

$$\mathcal{A}_{\widehat{W}}X = \alpha A_E^*X + \beta A_NX, \quad \forall X \in \Gamma(TM). \quad (5.35)$$

Lemma 5.11 *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmostu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then $\mathcal{A}_{\widehat{W}}X = 0, \forall X \in \Gamma(TM)$ if and only if $A_E^*X = 0$ and $A_NX = 0, \forall X \in \Gamma(TM)$.*

Proof: Suppose that $\mathcal{A}_{\widehat{W}}X = 0, \forall X \in \Gamma(TM)$. Then, $\alpha A_E^*X + \beta A_NX = 0$. So, for any $Y \in \Gamma(TM)$, $\alpha g(A_E^*X, Y) + \beta g(A_NX, Y) = 0$, i.e. $\overline{g}(\widehat{W}, C(X, Y)E + B(X, Y)N) = 0$ which implies that $B(X, Y) = 0$ and $C(X, Y) = 0$, since $TM^\perp \oplus N(TM)$ is a nondegenerate distribution of rank 2. By Theorem 2.2 and Proposition 2.7 in [3] (pp. 88 and 89, respectively), A_E^* and A_N vanish identically on M . The converse is obvious. \square

Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmostu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. The Lie derivative $L_{\widehat{W}}$ of \overline{g} with respect to the vector field \widehat{W} is given by

$$\begin{aligned} (L_{\widehat{W}}\overline{g})(X, Y) &= -2\alpha B(X, Y) - 2\beta C(X, Y) + \beta \{\tau(X)\theta(Y) + \tau(Y)\theta(X)\} \\ &+ (X.\beta)\theta(Y) + (Y.\beta)\theta(X), \quad \forall X, Y \in \Gamma(TM). \end{aligned} \quad (5.36)$$

Let M' be a leaf of $S(TM)$. Then, on M' , the relation (5.36) becomes

$$(L_{\widehat{W}}\overline{g})(X, Y) = -2\alpha B(X, Y) - 2\beta C(X, Y), \quad \forall X, Y \in \Gamma(TM'). \quad (5.37)$$

The action of the Levi-Civita connection ∇' (defined in (4.5)) on the normal bundle $TM^\perp \oplus N(TM)$ is given by

$$\nabla'_X \widehat{W} = -\mathcal{A}_{\widehat{W}} X + \nabla'^\perp_X \widehat{W}, \quad \forall X, Y \in \Gamma(TM'), \quad (5.38)$$

where $\nabla'^\perp_X \widehat{W} = \{X.\alpha - \alpha\tau(X)\} E + \{X.\beta + \beta\tau(X)\} N$.

Theorem 5.12 *Let $(M, g, S(TM))$ be a screen integrable lightlike hypersurface of an indefinite Kenmostu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Let M' be a leaf of $S(TM)$. Then, the following assertions are equivalent:*

- (i) M' is totally geodesic in M ,
- (ii) $\mathcal{A}_{\widehat{W}} X = 0, \forall X \in \Gamma(TM')$,
- (iii) $TM^\perp \oplus N(TM)$ is parallel distribution on M' ,
- (iv) $TM^\perp \oplus N(TM)$ is Killing distribution on M' .

Proof: The equivalence of (i) and (ii) follows from Lemma 5.11. Now we prove the equivalence of (i) and (iv). Suppose that M' is totally geodesic in M . Then $h'(X, Y) = 0, \forall X, Y \in \Gamma(TM')$. So, with the aid of (4.6), we have $C(X, Y) = 0$ and $B(X, Y) = 0$. Using (5.37), we obtain $(L_{\widehat{W}} \overline{g})(X, Y) = 0, \forall X, Y \in \Gamma(TM')$. Suppose that $TM^\perp \oplus N(TM)$ is a Killing distribution on M' . Then, using (5.37), we have $\alpha B(X, Y) + \beta C(X, Y) = 0$, that is, for any $X, Y \in \Gamma(TM')$, $\overline{g}(\widehat{W}, C(X, Y)E + B(X, Y)N) = 0$ which implies that $C(X, Y) = 0$ and $B(X, Y) = 0$, since $TM^\perp \oplus N(TM)$ is a nondegenerate distribution of rank 2. Consequently, $h'(X, Y) = 0, \forall X, Y \in \Gamma(TM')$. Next we prove the equivalence of (ii) and (iii). Suppose that $TM^\perp \oplus N(TM)$ is a parallel distribution on M' . Then exist functions φ and ε such that $\nabla'_X \widehat{W} = \varphi E + \varepsilon N$. Using (5.38), it is easy to check that $\varphi = \overline{g}(\nabla'_X \widehat{W}, N) = X.\alpha - \alpha\tau(X)$ and $\varepsilon = \overline{g}(\nabla'_X \widehat{W}, E) = X.\beta + \beta\tau(X)$. This implies that $\nabla'_X \widehat{W} = \nabla'^\perp_X \widehat{W}$, that is $\mathcal{A}_{\widehat{W}} X = 0$. The converse is obvious. \square

Acknowledgements.

The author would like to thank the Max-Planck Institute for Mathematics in the Sciences in Leipzig (Germany) for the support during this work.

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