Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Higher-order Abel equations: Lagrangian formalism, Darboux polynomials and constants of the motion

by

José Cariñena, Partha Guha, and Manuel Rañada

Preprint no.: 39 2009

Higher-order Abel equations: Lagrangian formalism, Darboux polynomials and constants of the motion

José Cariñena[†], Partha Guha[‡§] and Manuel F. Rañada[†]

[†] *Departamento de Física Teórica and IUMA, Facultad de Ciencias Universidad de Zaragoza, 50009 Zaragoza, Spain*

[‡] *Max Planck Institute for Mathematics in the Sciences Inselstrasse 22, D-04103 Leipzig, Germany*

[§] *S.N. Bose National Centre for Basic Sciences, JD Block Sector-3, Salt Lake, Calcutta-700098, India*

Abstract

A geometric approach is used to study a family of higher-order nonlinear Abel equations. The inverse problem of the Lagrangian dynamics is studied in the particular case of the second-order Abel equation and it is proved the existence of two alternative Lagrangian formulations, both Lagrangians being of a non-natural class (neither potential nor kinetic term). These higher-order Abel equations are studied by means of their Darboux polynomials and Jacobi multipliers. In all the cases a family of constants of the motion is explicitly obtained. The general n-dimensional case is also studied.

Keywords: Higher-order Riccati equations. Higher-order Abel equations. Lagrangian formalism. Constants of the motion. Darboux polynomials. Jacobi multipliers.

Running title: Higher-order Abel equations.

PACS numbers: 02.30.Hq ; 45.20.Jj

AMS classification: 34A26 ; 34A34 ; 34C14 ; 37J05 ; 70H03 ; 70H33

[a)]*E-mail address:* jfc@unizar.es
[b)]*E-mail address:* partha@bose.res.in
[c)]*E-mail address:* mfran@unizar.es

1 Introduction

The first-order Riccati equation

$$y' = P(x)y^2 + Q(x)y + R(x)$$

is important mainly because it is a nonlinear one but directly related to the general linear differential equation of second-order via a Cole-Hopf transformation. It is usually considered as the first instance in the study of nonlinear equations [1] and is endowed with many interesting properties. For example, it is a Lie system admitting a nonlinear superposition principle and it is the only nonlinear equation of the form $y' = f(x,y)$, where $f(x,y)$ is a rational function of the variable y with coefficients analytic in x, that possesses the Painlevé property (nevertheless the Lie-Scheffers theory or the Painlevé approach will not be considered in this paper).

The (first-order) Riccati equation is therefore a nonlinear equation that has been intensively studied by many authors. The important point is that it has been proved that it admits higher-order generalisations which are also studied by making use of several different approaches [2, 3, 4] (according to Davis these higher-order equations were first considered by Vessiot in 1895). All the higher-order Riccati equations can be linearised via a Cole-Hopf transformation to linear differential equations. It is known, that the higher-order Riccati equations play the role of Bäcklund transformations for integrable partial differential equations of higher-order than the KdV equation. The Riccati chain without potential is naturally associated to Faá di Bruno polynomials. The Faá di Bruno polynomials appear in several branches of mathematics and physics and can be introduced in several ways.

In fact higher-order Riccati equations are related to the existence of symmetries [5, 6], Darboux polynomials [7, 8, 9] and Jacobi multipliers [10, 11, 12]. We also mention that the second-order Riccati equation has been studied in [13] from a geometric perspective and it has been proved to admit two alternative Lagrangian formulations, both Lagrangians being of a non-natural class (neither potential nor kinetic term). An analysis of the higher-order Riccati equations and all these properties (Lagrangians, symmetries, Darboux polynomials and Jacobi multipliers) is presented in [14].

The Abel differential equation can be considered as the simplest nonlinear extension of the Riccati equation [15, 16, 17]. The Abel equation of the first kind [18, 19, 20, 21, 22] is given by

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3\,.$$

There is also another related equation, called Abel equation of second kind, given by

$$[g_0(x) + g_1(x)y]y' = f_0(x) + f_1(x)y + f_2(x)y^2 + f_3(x)y^3\,,$$

which is reducible to the previous one [23, 24] and it is not going to be considered in this paper. On one hand it has striking similarities with the Riccati equation but on the other side, as the non-linearity is of higher degree, the properties are different (and in fact more difficult to be studied). The objective of this paper is to study a chain of higher-order Abel equations using as an approach the analysis of the differential geometric properties, the Lagrangian formalism and the theory of Darboux polynomials and Jacobi multipliers.

The plan of the article is as follows: In Section 2 we review the hierarchy of higher-order Riccati equations and then introduce in a similar way the hierarchy of higher-order Abel equations. Section 3 is devoted to a particular case of second-order Abel equation. We study the existence of a Lagrangian formulation, obtain some constants of the motion and establish the relationship of this equation with the theory of Darboux polynomials and Jacobi multipliers. Section 4 is devoted to the third and fourth-order equations and in Section 5 we consider the general n-dimensional case. Finally in Section 6 we make some brief comments.

2 Riccati and Abel equations

Let us start with the definition of higher-order Riccati equations. It is known that these equations can be obtained by reduction from the Matrix Riccati equation. The matrix Riccati equation plays an important part in the theory of linear Hamiltonian systems, the calculus of variations, and other related topics.

2.1 Hierarchy of higher-order Riccati equations

Let us denote by $\mathbb{D}_R$ the following differential operator, depending on a real parameter $k \in \mathbb{R}$, that will be called 'differential operator of Riccati'

$$\mathbb{D}_R = \frac{d}{dt} + k\,x(t)\,,$$

in such a way that the action of $\mathbb{D}_R$ leads to the following family of differential expressions

$$\begin{aligned}
\mathbb{D}_R x &= \Big(\frac{d}{dt} + kx\Big)\,x = \dot{x} + kx^2 \\
\mathbb{D}_R^2 x &= \Big(\frac{d}{dt} + kx\Big)^2 x = \ddot{x} + 3kx\dot{x} + k^2x^3 \\
\mathbb{D}_R^3 x &= \Big(\frac{d}{dt} + kx\Big)^3 x = \dddot{x} + 4kx\ddot{x} + 6k^2x^2\dot{x} + 3k\dot{x}^2 + k^3x^4 \\
\mathbb{D}_R^4 x &= \Big(\frac{d}{dt} + kx\Big)^4 x = x^{iv)} + 5kx\,\dddot{x} + 10k\dot{x}\ddot{x} + 15k^2x\dot{x}^2 + 10k^2x^2\ddot{x} + 10k^3x^3\dot{x} + k^4x^5\,.
\end{aligned}$$

The Riccati equation of order m of the higher-order Riccati hierarchy (o chain), is given by

$$\mathbb{D}_R^m x = 0\,, \quad m = 1, 2, \ldots$$

In fact, the most general form of a Riccati equation of order m is just a superposition of all the previous equations (linear combination the different members of the hierarchy)

$$(p_0\mathbb{D}_R^n + p_1\mathbb{D}_R^{n-1} + \ldots + p_{n-1}\mathbb{D}_R + p_n)x + p_{n+1} = 0\,,$$

where each p_i is a function of t.

These equations have certain properties that make them interesting from both physical and mathematical points of view. Next we point out some of them.

(1) The higher-order Riccati equation of order m, member of the Riccati hierarchy, admits the maximal number of Lie point symmetries that can admit an equation of order m.

(2) The higher-order Riccati equation of order m can be linearised and presented as a linear equation of order $m+1$.

(3) The dimensional reduction of a linear equation of order $m+1$ leads to the Riccati equation of order m.

2.2 Hierarchy of higher-order Abel equations

The most natural generalisation of the Riccati equation is

$$\dot{x} = f(t, x),$$

where $f(t, x)$ is a polynomial in the variable x (with coefficients depending on t). The particular case of $f(t, x)$ being a cubic polynomial

$$f(t, x) = A_0(t) + A_1(t)x + A_2(t)x^2 + A_3(t)x^3\,,$$

is called Abel equation. Such an equation can be considered as the simplest nonlinear extension of the Riccati equation.

Let us denote by $\mathbb{D}_A$ the following differential operator, depending on a real parameter $k \in \mathbb{R}$, to be called 'Abel differential operator',

$$\mathbb{D}_A = \frac{d}{dt} + k\,x^2(t)\,,$$

in such a way that the action of $\mathbb{D}_A$ leads to a family of k-dependent differential equations whose first members are given by

$$\begin{aligned}
\mathbb{D}_A^0 x &= x \\
\mathbb{D}_A x &= \left(\frac{d}{dt} + kx^2\right) x = \dot{x} + kx^3 \\
\mathbb{D}_A^2 x &= \left(\frac{d}{dt} + kx^2\right)^2 x = \ddot{x} + 4kx^2\dot{x} + k^2x^5 \\
\mathbb{D}_A^3 x &= \left(\frac{d}{dt} + kx^2\right)^3 x = \dddot{x} + 5kx^2\ddot{x} + 8kx\dot{x}^2 + 9k^2x^4\dot{x} + k^3x^7 \\
\mathbb{D}_A^4 x &= \left(\frac{d}{dt} + kx^2\right)^4 x = x^{iv)} + 2k(4\dot{x}^3 + 13x\dot{x}\ddot{x} + 3x^2\,\dddot{x}) + 2k^2x^3(22\dot{x}^2 + 7x\dot{x}) + 16k^3x^6\dot{x} + k^4x^9
\end{aligned}$$

We call to this family the hierarchy of higher-order Abel equations. The Abel equation of order m, written in the so-called simplified form, is given by

$$\mathbb{D}_A^m x = 0\,, \quad m = 1, 2, \ldots$$

Actually, the most general form of the Abel equation of order m is just a superposition of all the previous equations (linear combination of the different members of the hierarchy with functions $p_i(t)$ as coefficients)

$$(p_0\mathbb{D}_A^n + p_1\mathbb{D}_A^{n-1} + \ldots + p_{n-1}\mathbb{D}_A + p_n)x + p_{n+1} = 0\,.$$

3 Abel equation of second-order

In this section we will analyse the particular case of the second-order Abel equation. In particular, we describe the Lagrangian formulation of the second-order Abel equation.

3.1 Lagrangian formalism

The action of $\mathbb{D}_A^2$ on the function $x(t)$ leads to the nonlinear equation

$$\frac{d^2x}{dt^2} + 4kx^2\Big(\frac{dx}{dt}\Big) + k^2x^5 = 0\,, \qquad (1)$$

that represents the Abel equation of second-order. It can be presented as a system of two first-order equations

$$\begin{aligned} \frac{d}{dt}x &= v \\ \frac{d}{dt}v &= -4kx^2v - k^2x^5 \end{aligned}$$

that determines a dynamical system that, in differential geometric terms, is represented by the following vector field

$$\Gamma^{(2)} = v\,\frac{\partial}{\partial x} + F_{A2}\,\frac{\partial}{\partial v}\,, \qquad F_{A2} = -4kx^2v - k^2x^5\,. \qquad (2)$$

defined on the phase space $\mathbb{R}^2$ with coordinates (x, v).

It has been proved in [13] that the second-order Riccati equation

$$\frac{d^2x}{dt^2} + 3kx\Big(\frac{dx}{dt}\Big) + k^2x^3 = 0$$

can be considered as the Lagrange equation determined by the following Lagrangian

$$L_R = \frac{1}{v + k\,x^2}\,.$$

Proposition 1 *The nonlinear Abel equation of second-order (1) admits a Lagrangian formulation with a non-polynomial Lagrangian.*

Proof: There are two different ways of obtaining a Lagrangian function for the nonlinear Abel equation; the Helmholtz approach and the generalisation of the method used for the corresponding Riccati case.

The Helmholtz conditions are a set of conditions that a multiplier matrix $g_{ij}(x,\dot{x},t)$ must satisfy in order for a given system of second-order equations

$$\ddot{x}_j = f_j(x,\dot{x},t)\,, \quad j = 1,2,\dots,n,$$

when written of the form

$$g_{ij}\ddot{x}_j = g_{ij}f_j(x,\dot{x},t)\,, \quad i,j = 1,2,\dots,n,$$

to be the set of Euler-Lagrange equations for a certain Lagrangian L [25, 26, 27, 28] (the summation convention on repeated indices is assumed). If a matrix solution g_{ij} is obtained then it can be identified with the Hessian matrix of L, that is $g_{ij} = \partial L/\partial v_i \partial v_j$, and a Lagrangian L can be obtained by direct integration of the g_{ij} functions. The two first conditions just impose regularity and symmetry of the matrix g_{ij}; the two other are equations introducing relations between the derivatives of g_{ij} and the derivatives of the functions f_i. Here we only write the fourth set of conditions that determine the time-evolution of the g_{ij}

$$\Gamma(g_{ij}) = g_{ik}A_{kj} + g_{jk}A_{ki}\,, \quad A_{ab} = -\frac{1}{2}\frac{\partial f_a}{\partial v_b}\,.$$

When the system is one-dimensional we have $i = j = k = 1$ and then the three first set of conditions become trivial and the fourth one reduces to one single first-order P.D.E.

$$\Gamma(g) + \Big(\frac{\partial f}{\partial v}\Big)g \equiv v\Big(\frac{\partial g}{\partial x}\Big) + f\Big(\frac{\partial g}{\partial v}\Big) + \Big(\frac{\partial f}{\partial v}\Big)g = 0 \tag{3}$$

that in the case of the Abel equation becomes

$$v\Big(\frac{\partial g}{\partial x}\Big) - (4kx^2v + k^2x^5)\Big(\frac{\partial g}{\partial v}\Big) - 4kx^2 g = 0\,. \tag{4}$$

So, the problem reduces to find the function g as a solution of this equation. Once a solution g is known a Lagrangian L is obtained by integrating two times the function g. The funtion L obtained from g is unique up to addition of a gauge term

Next we consider the second method that is specific for this particular nonlinear problem. The starting point is the idea that, since the Abel equation is very close related with the Riccati equation, it seems natural to assume that the Abel Lagrangian must be a non-polynomial function similar to that of the second-order Riccati equation.

Let us begin by considering the following one degree of freedom Lagrangian

$$L = \frac{1}{(v + k\,U(x,t))^m}\,. \tag{5}$$

From such a Lagrangian we arrive to the following second-order nonlinear equation

$$\ddot{x} + (\frac{2+m}{1+m})\,k\,U'_x\,\dot{x} + (\frac{1}{1+m})\,k^2UU'_x + kU'_t = 0\,. \tag{6}$$

Hence, in the particular case of U and m being given by

$$U(x,t) = x^3\,, \quad m = 2\,,$$

then the Lagrangian (5) leads to (1). Thus the second-order Abel equation (1) turns out to be the Euler-Lagrange equation of the Lagrangian function

$$L_A = \frac{1}{(v + k\,x^3)^2}\,. \tag{7}$$

Finally, as a byproduct of this approach, we have also obtained the Lagrangians for the whole family of nonlinear equations (6) depending of a function U. □

As a corollary of this proposition we can state that when the function U is time-independent the nonlinear equation (6) has a first-integral that can be interpreted as a preserved energy. That is, if we restrict the study to nonlinear equations arising from a time-independent Lagrangian of the form

$$L = \frac{1}{(v + k\,U(x))^m}$$

then we can define an associated Lagrangian energy E_L by the usual procedure

$$E_L = \Delta(L) - L\,, \quad \Delta = v\,\frac{\partial}{\partial v}\,,$$

and we arrive to

$$E_L = \frac{-\,((1+m)\,v + \,k\,U(x))}{(v + k\,U(x))^{m+1}}\,, \qquad \frac{d}{dt}E_L = 0.$$

In the particular case of the Abel Lagrangian L_A we have

$$E_{L_A} = -\,\frac{(3v + k\,x^3)}{(v + k\,x^3)^3}\,, \qquad \frac{d}{dt}\,E_{L_A} = 0\,. \tag{8}$$

Note that L_A is non-natural and, as there is neither kinetic term T nor potential function V, the energy cannot be of the standard form $E_L = T + V$. But, in spite of its rather peculiar form, E_{L_A} is a conserved function for the Abel equation.

An important property of the Lagrangian formalism is that for one degree of freedom systems if an equation admits a Lagrangian formulation then the Lagrangian is not unique [36, 37]. This property can be proved in two different ways. First, the Helmholtz equation (4) is a linear equation in partial derivatives and thus it admits many different particular solutions. Moreover it is clear from the form of the equation (3) that if g_1 is a particular solution then $g_2 = f g_1$ with $\Gamma(f) = 0$ is also a solution. A second method is related with the properties of the symplectic formalism. In a two-dimensional manifold all the symplectic forms must be proportional. Hence if ω_L is known then any other symplectic form ω_2 must be proportional to ω_L, that is $\omega_2 = f\omega_L$. Then

$$i(\Gamma_L)\,\omega_2 = f\,i(\Gamma_L)\,\omega_L = f\,dE_L\,.$$

The right-hand side is an exact one-form if, and only if, $df \wedge dE_L = 0$, which shows that f must be a function of E_L. In this case it can be proved that the new symplectic form ω_2 is derivable from an alternative Lagrangian $L_2 \neq L$ for Γ_L.

In the particular case of the Abel system $\Gamma^{(2)}$, several alternatives Lagrangians can be obtained that, in most of cases, are of non-algebraic character (with logarithm terms). Nevertheless, in the particular case of f given by $f = (-1/E_{L_A})^{4/3}$, we have obtained the following algebraic function

$$\widetilde{L}_A = (3v + k\,x^3)^{2/3} \tag{9}$$

as a new alternative Lagrangian for the Abel equation (1). This new Lagrangian is equivalent to L_A in the sense that both determine the same dynamics. It determines a new energy $\widetilde{E}_{L_A}$ that is a constant of the motion for the Abel equation; nevertheless it must not be considered as a new fundamental constant since it is a function of the original energy E_{L_A}.

3.2 Constants of the motion and geometric formalism

A function T that satisfies the following property

$$\frac{d}{dt}\,T \neq 0 \quad ,\dots, \quad \frac{d^m}{dt^m}\,T \neq 0\,, \quad \frac{d^{m+1}}{dt^{m+1}}\,T = 0\,,$$

is called a generator of integrals of motion of degree m. Notice that this means that the function T is a non-constant function generating a constant of motion by successive time derivations.

Let us denote by $T_1^{(2)}$ the following function

$$T_1^{(2)} = \frac{x}{v + k\,x^3}\,.$$

Then we have that under the evolution given by Abel's equation 1)

$$\frac{d}{dt}\,T_1^{(2)} = T_2^{(2)} = 1\,, \quad \frac{d}{dt}\,T_2^{(2)} = 0\,.$$

Thus, the function J_{t1} defined by

$$J_{t1} = T_1^{(2)} - t\,,$$

is a time-dependent constant of the motion for the Abel equation.

This means that we have obtained two constants of the motion (of quite different nature) for the Abel equation of second-order: the energy E_{L_A} and the time-dependent function J_{t1}.

In differential geometric terms a time-independent Lagrangian function L determines an exact two-form ω_L defined as

$$\theta_L = \Bigl(\frac{\partial L}{\partial v_x}\Bigr)\,dx\,, \qquad \omega_L = -\,d\theta_L\,,$$

and L is said to be regular when the 2-form ω_L is symplectic. In the particular case of L given by (7) ω_{L_A} is given by

$$\omega_{L_A} = \Bigl(\frac{6}{(v + k\,x^3)^4}\Bigr)\,dx \wedge dv\,,$$

and the dynamical vector field $\Gamma^{(2)}$ is the solution of the equation

$$i(\Gamma^{(2)})\,\omega_{L_A} = dE_{L_A}\,.$$

Next we consider two interesting class of symmetries: 'master symmetries' and 'non-Cartan symmetries'. The idea is that, in differential geometric terms, constants of motion that depend of the time but in a polynomial way are related with the existence of master symmetries [29, 30, 31, 32] and in some very particular cases with non-Cartan symmetries.

Given a dynamics represented by a certain vector field Γ, then a vector field Z satisfying

$$[Z,\Gamma] \,=\, \widetilde{Z} \neq 0\,, \qquad [\,\widetilde{Z}\,,\Gamma] = 0\,,$$

is called a 'master symmetry' of degree $m = 1$ for Γ. When Z is such that

$$[Z,\Gamma] = \widetilde{Z} \neq 0\,,\ [\,\widetilde{Z}\,,\Gamma] \neq 0 \quad \text{and} \quad [\,[\,\widetilde{Z}\,,\Gamma]\,,\Gamma] = 0\,,$$

then Z is called a 'master symmetry' of degree $m = 2$. The generalisation to higher values of m is straightforward:

$$(\mathrm{ad}(\Gamma))^{m+1}(Z) = 0, \quad \text{but} \quad (\mathrm{ad}(\Gamma))^{m}(Z) \neq 0\,.$$

It is well-known that symmetries are important because they give rise to constants of the motion and reduction procedures. Master symmetries, which are a rather peculiar class of symmetries, determine time-dependent constants of motion (the system is time-independent but the constant is however time-dependent). This can be seen as follows: if Z is a master symmetry of degree one, the time-dependent vector field Y_Z determined by Z as follows [32]

$$Y_Z = Z + t\,[Z,\Gamma] + (\frac{1}{2})\,t^2\,[\,[Z,\Gamma]\,,\Gamma]$$

is a time-dependent symmetry of $\Gamma_t = \partial/\partial t + \Gamma$, which is the suspension of the vector field Γ [33]. This symmetry determines a time-dependent constant of motion $J_t = T - t\,\Gamma(T)$ that depends linearly of t (for $m = 2$ the corresponding constant J_t will be quadratic in t and for $m = 3$ will be cubic).

Let Z_1 be the Hamiltonian vector field of the function $T_1^{(2)}$, that is, the unique solution of the equation

$$i(Z_1)\,\omega_L = dT_1^{(2)}\,,$$

which is given by

$$Z_1 = -\,(\frac{1}{6})P_{A1}^2\left(x\frac{\partial}{\partial x} + (v - 2kx^3)\,\frac{\partial}{\partial v}\right), \qquad P_{A1} = v + k\,x^3\,.$$

Then Z_1 is a symplectic symmetry (that is, $\mathcal{L}_{Z_1}\,\omega_L = 0$) because it is the Hamiltonian vector field of $T_1^{(2)}$, and moreover it is a dynamical symmetry because

$$i([Z_1\,,\,\Gamma^{(2)}]\omega_L = i(Z_1)(\mathcal{L}_{\Gamma^{(2)}}\omega_L) - \mathcal{L}_{\Gamma^{(2)}}(i(Z_1)\omega_L) = -\mathcal{L}_{\Gamma^{(2)}}(dT_1^{(2)}) = 0\,,$$

and therefore, as ω_L is non-degenerate, $[Z_1\,,\,\Gamma^{(2)}]=0$.

Note however that Z_1 is not a symmetry of the energy since $Z_1(E_{L_A})\neq 0$.

Thus, Z_1 is a dynamical but non-Cartan symmetry of the Lagrangian system [34, 35]. These symmetries are rather peculiar and only appear in some very particular cases. In particular it was proved in [35] that if the Hamiltonian vector field X_F with the function F as Hamiltonian in a symplectic manifold (M,ω) is a dynamical but non-Cartan symmetry, then $X_F(H)$ must be a numerical constant $X_F(H)=\alpha\neq 0$. In this case we are considering, $F=T_1^{(2)}$ and we have $Z_1(E_{L_A})=\alpha=-1$.

We close this section by recalling that the Riccati equation was endowed with similar properties but the function $T_1^{(2)}$ was the Lagrangian L_R itself [14].

3.3 Darboux polynomial and Jacobi multiplier approach

The existence of constants of the motion and the Lagrangian inverse problem for polynomial vector fields are two questions related with two important ideas: Jacobi multipliers and Darboux polynomials.

Let U be an open subset of $\mathbb{R}^n$. We say that a polynomial function $\mathcal{D}:U\to\mathbb{R}$ is a Darboux polynomial for a polynomial vector field X if there is a polynomial function f defined in U such that $X\mathcal{D}=f\mathcal{D}$ [7, 8, 9, 14]. The function f is said to be the cofactor corresponding to such Darboux polynomial and the pair $(f,\mathcal{D})$ a Darboux pair.

When $f=0$, then the Darboux polynomial is a first integral. We say that $\mathcal{D}$ is a proper Darboux polynomial if $f\neq 0$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are Darboux polynomials with the same cofactor, the quotient $\mathcal{D}_1/\mathcal{D}_2$ is a first integral.

On the other side given a vector field X in an oriented manifold (M,Ω), a function R such that $R\,i(X)\Omega$ is closed is said to be a Jacobi multiplier (JM) for X. Recall that the divergence of the vector field X (with respect to the volume form Ω) is defined by the relation

$$\mathcal{L}_X\Omega=(\operatorname{div}X)\,\Omega\,.$$

This means that R is a multiplier if and only if $R\,X$ is a divergence-less vector field and then

$$\mathcal{L}_{RX}\Omega=(\operatorname{div}RX)\,\Omega=[X(R)+R\operatorname{div}X]\,\Omega=0\,,$$

and therefore we see that R is a last multiplier for X if and only if

$$X(R)+R\operatorname{div}X=0\,. \tag{10}$$

Note that if R is a never vanishing Jacobi multiplier, then fR is a Jacobi multiplier too if and only if f is a constant of motion.

The remarkable point is that if $\mathcal{D}_1,\ldots,\mathcal{D}_k$, are Darboux polynomials with corresponding cofactors f_i, $i=1,\ldots,k$, one can look for multiplier factors of the form

$$R=\prod_{i=1}^{k}\mathcal{D}_i^{\nu_i} \tag{11}$$

and then

$$\frac{X(R)}{R}=\sum_{i=1}^{k}\nu_i\frac{X(\mathcal{D}_i)}{\mathcal{D}_i}=\sum_{i=1}^{k}\nu_i\, f_i\,,$$

and therefore, if the coefficients ν_i can be chosen such that

$$\sum_{i=1}^{k}\nu_i\, f_i=-\mathrm{div}\, X \tag{12}$$

holds, then we arrive to

$$\frac{X(R)}{R}=\sum_{i=1}^{k}\nu_i\, f_i=-\mathrm{div}\, X\,,$$

and consequently R is a Jacobi last multiplier for X.

Finally, if R is a Jacobi multiplier for a vector field which corresponds to a second-order differential equation, there is an essentially unique Lagrangian L (up to addition of a gauge term) such that $R=\partial^2L/\partial v^2$ [10, 11, 12].

From these general concepts we can return to the Abel equation. In this case the polynomial $\mathcal{D}_1$ defined by

$$\mathcal{D}_1(x,v)=v+kx^3$$

is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-kx^2$ since

$$\left(v\,\frac{\partial}{\partial x}+F_{A2}\,\frac{\partial}{\partial v}\right)(v+kx^3)=-kx^2(v+kx^3)\,.$$

The divergence of the vector field $\Gamma^{(2)}$ is $-4kx^2$, and then, according to (12), we see that there is a multiplier of the form

$$R=\mathcal{D}_1^{\nu_1}\,,$$

with $\nu_1=-4$. Consequently, the Abel equation admits a Lagrangian description by means of a function L_1 such that

$$\frac{\partial^2L_1}{\partial v^2}=(v+kx^3)^{-4}\,,$$

from where we obtain the Lagrangian $L_1=L_A$ given by (7).

But the polynomial $\mathcal{D}_2$ defined by

$$\mathcal{D}_2(x,v)=3v+kx^3$$

is a Darboux polynomial for $\Gamma^{(2)}$ with cofactor $-3kx^2$, because

$$\left(v\,\frac{\partial}{\partial x}+F_{A2}\,\frac{\partial}{\partial v}\right)(3v+kx^3)=3kx^2v-3(4kx^2v+k^2x^5)=-3kx^2(3v+kx^3)\,,$$

and then, using the equation (12) we can find another Jacobi multiplier of the form $\mathcal{D}_2^{\nu_2}$ with $\nu_2 = -4/3$. The Abel equation admits a Lagrangian description by means of a function L_2 such that

$$\frac{\partial^2 L_2}{\partial v^2} = (3v + kx^3)^{-4/3}\,,$$

from where we obtain the Lagrangian $L_2 = \widetilde{L}_A$ given by (9).

Remark that, as indicated above, if P and Q are two Darboux polynomials with the same cofactor then P/Q is a constant of the motion. This is just what happens with the energy E_{L_A} obtained in (8) which is given by $\mathcal{D}_2/\mathcal{D}_1^3$ (up to the sign).

4 Abel equations of third and fourth-order

4.1 Abel equation of third-order

The action of the operator $\mathbb{D}_A$ three times on the function $x(t)$ leads to the following nonlinear equation

$$\frac{d^3x}{dt^3} + 5kx^2\Big(\frac{d^2x}{dt^2}\Big) + 8kx\Big(\frac{dx}{dt}\Big)^2 + 9k^2x^4\Big(\frac{dx}{dt}\Big) + k^3x^7 = 0\,, \tag{13}$$

that represents the third-order element of the Abel equation chain. It can be presented as a system of three first-order equations

$$\left\{\begin{array}{rcl} \dfrac{dx}{dt} &=& v \\ \dfrac{dv}{dt} &=& a \\ \dfrac{da}{dt} &=& -5kx^2a - 8kxv^2 - 9k^2x^4v - k^3x^7 \end{array}\right. \tag{14}$$

that represents a dynamical system that, in differential geometric terms, is represented by the following vector field in the phase space $\mathbb{R}^3$, with coordinates (x, v, a)

$$\Gamma^{(3)} = v\,\frac{\partial}{\partial x} + a\,\frac{\partial}{\partial v} + F_{A3}\,\frac{\partial}{\partial a}\,, \qquad F_{A3} = -5kx^2a - 8kxv^2 - 9k^2x^4v - k^3x^7\,. \tag{15}$$

In what follows we make use of the following polynomials

$$P_{A0} = x\,, \qquad P_{A1} = v - F_{A1} = v + k\,x^3\,, \qquad P_{A2} = a - F_{A2} = a + 4kx^2v + k^2x^5\,,$$

defined on the phase space and obtained by making use of the substitution $\dot{x} \mapsto v$ and $\ddot{x} \mapsto a$. Then we have

$$\begin{array}{rcl} \Gamma^{(3)}(P_{A0}) + kx^2P_{A0} &=& v + k\,x^3, \\ \Gamma^{(3)}(P_{A1}) + kx^2P_{A1} &=& a + 4kx^2v + k^2x^5, \\ \Gamma^{(3)}(P_{A2}) + kx^2P_{A2} &=& F_{A3} + 5kx^2a + 8kxv^2 + 9k^2x^4v + k^3x^7. \end{array}$$

that can be rewritten as follows

$$\begin{aligned}\Gamma^{(3)}(P_{A0}) + kx^2 P_{A0} &= P_{A1},\\ \Gamma^{(3)}(P_{A1}) + kx^2 P_{A1} &= P_{A2},\\ \Gamma^{(3)}(P_{A2}) + kx^2 P_{A2} &= 0.\end{aligned}$$

Note that according to these properties P_{A2} is a Darboux polynomial with $f = -kx^2$ as cofactor. The divergence of the vector field $\Gamma^{(3)}$ is $-5kx^2$, and using relation (12) we see that $R = (P_{A2})^{\mu_2}$ with $\mu_2 = -5$ is a Jacobi multiplier.

Next let $T_1^{(3)}$ be the following function

$$T_1^{(3)} = \frac{x}{P_{A2}}\,,$$

and then we have

$$\Gamma^{(3)}(T_1^{(3)}) = T_2^{(3)} = \frac{v + kx^3}{P_{A2}}\,, \qquad \Gamma^{(3)}(T_2^{(3)}) = T_3^{(3)} = 1\,, \qquad \Gamma^{(3)}(T_3^{(3)}) = T_4^{(3)} = 0\,.$$

This means that $T_1^{(3)}$ and $T_2^{(3)}$ are generators of of constants of motion for the third-order element of the Abel equation chain represented by the dynamical vector field $\Gamma^{(3)}$. Thus we can state the following proposition.

Proposition 2 *The two functions J_{t1} and J_{t2} defined as*

$$J_{t1} = T_2^{(3)} - t\,, \qquad J_{t2} = T_1^{(3)} - t\,T_2^{(3)} + (\frac{1}{2})t^2\,,$$

are time-dependent constants of the motion for the Abel equation of third-order.

Note that J_{t1} is linear in the time t and J_{t2} is quadratic. So these expressions are similar to the constants of the motion determined by master symmetries; nevertheless in this third-order case we have not made use of any symplectic structure and we have obtained these functions without relating them with symmetries of a symplectic structure. This is a interesting situation deserving an additional analysis in the next sections.

Note also that both, J_{t1} and J_{t2}, can be written as quotients of polynomials; so if we consider the system as a time-dependent system then the dynamics is geometrically represented by the vector field $\Gamma_t^{(3)} = \Gamma^{(3)} + \partial/\partial t$ and the following polynomials

$$\mathcal{D}_2 = P_{A1} - tP_{A2}\,, \qquad \mathcal{D}_3 = P_{A0} - tP_{A1} + (\frac{1}{2})t^2 P_{A2}\,,$$

are two Darboux polynomials with the same cofactor as P_{A2}

$$\Gamma_t^{(3)}(\mathcal{D}_i) = \Big(\Gamma^{(3)} + \frac{\partial}{\partial t}\Big)(\mathcal{D}_i) = -\,kx^2\,\mathcal{D}_i\,, \quad i = 2,3.$$

4.2 Abel equation of fourth-order

The action of $\mathbb{D}_A$ four times on the function $x(t)$ leads to the following nonlinear equation

$$x^{iv)} + 2k(4\dot{x}^3 + 13x\dot{x}\ddot{x} + 3x^2\,\dddot{x}) + 2k^2x^3(22\dot{x}^2 + 7x\ddot{x}) + 16k^3x^6\dot{x} + k^4x^9 = 0\,, \tag{16}$$

that represents the fourth-order element of the Abel equation chain. This equation determines a dynamical system that, in geometric terms, can be represented by the following vector field on $\mathbb{R}^4$ as phase space, with coordinates (x, v, a, w):

$$\Gamma^{(4)} = v\,\frac{\partial}{\partial x} + a\,\frac{\partial}{\partial v} + w\,\frac{\partial}{\partial a} + F_{A4}\,\frac{\partial}{\partial w} \tag{17}$$

where

$$F_{A4} = -2k(4v^3 + 13xva + 3x^2w) - 2k^2x^3(22v^2 + 7xa) - 16k^3x^6v - k^4x^9.$$

Now we introduce the polynomial P_{A3}

$$P_{A3} = w - F_{A3} = w + 5kx^2a + 8kxv^2 + 9k^2x^4v + k^3x^7$$

obtained from the expression of $\mathbb{D}_A^3\,x$ with the substitution $\dot{x} \mapsto v$, $\ddot{x} \mapsto a$ y $\dddot{x} \mapsto w$. Then we have

$$\begin{aligned}
\Gamma^{(4)}(P_{A0}) + kx^2P_{A0} &= v + k\,x^3 \\
\Gamma^{(4)}(P_{A1}) + kx^2P_{A1} &= a + 4kx^2v + k^2x^5 \\
\Gamma^{(4)}(P_{A2}) + kx^2P_{A2} &= w + 5kx^2a + 8kxv^2 + 9k^2x^4v + k^3x^7 \\
\Gamma^{(4)}(P_{A3}) + kx^2P_{A3} &= F_{A4} + k(8v^3 + 26xva + 6x^2w) + k^2(44x^3v^2 + 14x^4a) \\
&\qquad + 16k^3x^6v + k^4x^9
\end{aligned}$$

that can be rewritten as follows

$$\begin{aligned}
\Gamma^{(4)}(P_{A0}) + kx^2P_{A0} &= P_{A1} \\
\Gamma^{(4)}(P_{A1}) + kx^2P_{A1} &= P_{A2} \\
\Gamma^{(4)}(P_{A2}) + kx^2P_{A2} &= P_{A3} \\
\Gamma^{(4)}(P_{A3}) + kx^2P_{A3} &= 0
\end{aligned}$$

Let now $T_1^{(4)}$ be the following function

$$T_1^{(4)} = \frac{x}{P_{A3}},$$

and then we have

$$\Gamma^{(4)}(T_1^{(4)}) = T_2^{(4)}\,, \qquad \Gamma^{(4)}(T_2^{(4)}) = T_3^{(4)}\,, \qquad \Gamma^{(4)}(T_3^{(4)}) = T_4^{(4)}\,, \qquad \Gamma^{(4)}(T_4^{(4)}) = 0\,,$$

with $T_2^{(4)}$, $T_3^{(4)}$, and $T_4^{(4)}$ given by

$$T_2^{(4)} = \frac{v + k\,x^3}{P_{A3}}\,, \quad T_3^{(4)} = \frac{a + 4kx^2v + k^2x^5}{P_{A3}}\,, \quad T_4^{(4)} = \frac{w + \ldots + k^3x^7}{P_{A3}} = 1$$

Proposition 3 *The three functions J_{t1}, J_{t2}, and J_{t3} defined as*

$$\begin{aligned} J_{t1} &= T_3 - t \\ J_{t2} &= T_2 - t\,T_3 + (\frac{1}{2})t^2 \\ J_{t3} &= T_1 - t\,T_2 + (\frac{1}{2})t^2 T_3 - (\frac{1}{6})t^3 \end{aligned}$$

are time-dependent constants of the motion for the fourth-order element of the Abel equation chain.

The situation is similar to the $n = 3$ case and the functions J_{tr}, $r = 1, 2, 3$, are polynomials of order r in the variable t.

5 Equation of Abel of order n

We have seen that the second-order element of the Abel equation chain is endowed with some specific properties (e.g., it admits a Lagrangian description) but the of third and fourth-order elements of the chain also enjoy very similar properties. Now in this section we study the equation of order n and prove that these properties characterise to all the equations of the family in an independent of the order way.

The equation of Abel of order n can be obtained as the equation arising from the action of the operator $\mathbb{D}_A$ on the equation of order $n-1$

$$\mathbb{D}_A(\mathbb{D}_A^{n-1}\,x) = \mathbb{D}_A^n x = 0\,.$$

This equation determines a dynamical system that, in geometric terms, can be represented by the following vector field defined on the phase space $\mathbb{R}^n$, with coordinates $(x = x_1, x_2, x_3, \ldots, x_n)$:

$$\Gamma^{(n)} = x_2\,\frac{\partial}{\partial x} + x_3\,\frac{\partial}{\partial x_2} + x_4\,\frac{\partial}{\partial x_3} + \ldots + F_{An}\,\frac{\partial}{\partial x_n}, \tag{18}$$

where F_{An} is obtained a from the expression for $\mathbb{D}_A^n\,x$ with the substitution $x \mapsto x_1$, $\dot{x} \to x_2$, $\ddot{x} \to x_3$, $\dddot{x} \to x_4$, ...

In the previous sections we have made use of the polynomials P_{A0}, P_{A1}, P_{A2}, and P_{A3} defined in the phase space and whose explicit expressions, when written in the notation of the coordinates $x_1, x_2, x_3, \ldots, x_n$, were given by

$$\begin{aligned} P_{A0} &= x_1\,, \\ P_{A1} &= x_2 - F_{A1} = x_2 + k\,x_1^3\,, \\ P_{A2} &= x_3 - F_{A2} = x_3 + 4kx_1^2 x_2 + k^2 x_1^5\,, \\ P_{A3} &= x_4 - F_{A3} = x_4 + 5kx_1^2 x_3 + 8kx_1 x_2^2 + 9k^2 x_1^4 x_2 + k^3 x_1^7\,. \end{aligned}$$

In the general case we have $P_{An-1} = x_n - F_{An-1}$ that leads to an expression of the form

$$P_{An-1} = x_n - F_{An-1} = x_n + (n+1)kx^2 x_{n-1} + \ldots + k^{n-1} x^{2n-1}\,.$$

Proposition 4 *The action of the dynamical vector field $\Gamma^{(n)}$ on the n polynomials P_{Ar}, $r = 0, 1, 2, \dots, n-1$, is given by*

$$
\begin{array}{ll}
(i) & \Gamma^{(n)}(P_{Ar}) + kx^2 P_{Ar} = P_{Ar+1}\,, \quad r = 0, 1, 2, \dots, n-2, \\
(ii) & \Gamma^{(n)}(P_{An-1}) + kx^2 P_{An-1} = 0.
\end{array}
$$

The property (i) can be proved by induction. The property (ii) follows by direct calculus. □

Proposition 5 *The time evolution of the functions P_{Ar}/P_{An-1} is given by*

$$
\Gamma^{(n)}\Big(\frac{P_{Ar}}{P_{An-1}}\Big) = \frac{P_{Ar+1}}{P_{An-1}}\,, \quad r = 0, 1, 2, \dots, n-2.
$$

By direct calculus we have

$$
\Gamma^{(n)}\Big(\frac{P_{Ar}}{P_{An-1}}\Big) = \frac{\Gamma^{(n)}(P_{Ar})P_{An-1} - P_{Ar}\Gamma^{(n)}(P_{An-1})}{(P_{An-1})^2}\,.
$$

Then, making use of the properties (i) and (ii), we arrive at

$$
\Gamma^{(n)}\Big(\frac{P_{Ar}}{P_{An-1}}\Big) = \frac{[P_{Ar+1} - kx^2 P_{Ar}]P_{An-1} - P_{Ar}[-kx^2 P_{An-1}]}{(P_{An-1})^2} = \frac{P_{Ar+1}}{P_{An-1}}. \qquad \square
$$

Notice that the first and the last derivatives in this series, corresponding to $r = 0$ and $r = n-2$, become

$$
\frac{d}{dt}\Big(\frac{x}{P_{An-1}}\Big) = \frac{P_{A1}}{P_{An-1}}\,, \qquad \frac{d}{dt}\Big(\frac{P_{An-2}}{P_{An-1}}\Big) = 1\,.
$$

Now let $T_1^{(n)}$ be the following function defined by

$$
T_1^{(n)} = \frac{x}{P_{An-1}},
$$

and then, making use of the two preceding Propositions we can obtain the values of the sequence of time derivatives of the functions $T_k^{(n)}$, which are given by

$$
\begin{array}{rcl}
\Gamma^{(n)}(T_1^{(n)}) & = & T_2^{(n)} = \dfrac{x_2 + kx^3}{P_{An-1}} = \dfrac{P_{A1}}{P_{An-1}} \\
\Gamma^{(n)}(T_2^{(n)}) & = & T_3^{(n)} = \dfrac{x_3 + 4kx^2 x_2 + k^2 x^5}{P_{An-1}} = \dfrac{P_{A2}}{P_{An-1}} \\
\Gamma^{(n)}(T_3^{(n)}) & = & T_4^{(n)} = \dfrac{x_4 + 5kx^2 x_3 + 8kx x_2^2 + 9k^2 x^4 x_2 + k^3 x^7}{P_{An-1}} = \dfrac{P_{A3}}{P_{Rn-1}} \\
\dots & & \dots\dots\dots\dots\dots\dots\dots\dots \\
\dots & & \dots\dots\dots\dots\dots\dots\dots\dots \\
\Gamma^{(n)}(T_{n-1}^{(n)}) & = & T_n^{(n)} = 1 \\
\Gamma^{(n)}(T_n^{(n)}) & = & 0\,.
\end{array}
$$

From here we can state the existence of a family of $n-1$ time-dependent constants of the motion.

Proposition 6 *The $(n-1)$ functions J_{tr}, $r = 1,2,3,\dots,n-1$, defined as the following polynomials of order r in the variable t*

$$
\begin{aligned}
J_{t1} &= T^{(n)}_{n-1} - t \\
J_{t2} &= T^{(n)}_{n-2} - t\,T^{(n)}_{n-1} + (\frac{1}{2})\,t^2 \\
J_{t3} &= T^{(n)}_{n-3} - t\,T^{(n)}_{n-2} + (\frac{1}{2})\,t^2\,T^{(n)}_{n-1} - (\frac{1}{6})\,t^3 \\
\dots & \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
J_{tn-1} &= T^{(n)}_{1} - t\,T^{(n)}_{2} + (\frac{1}{2})\,t^2\,T^{(n)}_{3} - \dots\dots + (-1)^n(\frac{1}{n!})\,t^n
\end{aligned}
$$

are $n-1$ functionally independent time-dependent constants of the motion for the Abel equation of order n.

An alternative form of proving the existence of all these constant of the motion is as follows. The n polynomials $\mathcal{D}_a$, $a = 1,2,\dots,n$, defined in the extended phase space $\mathbb{R}^n \times \mathbb{R}$ as

$$
\begin{aligned}
\mathcal{D}_1 &= P_{An-1}\,, \\
\mathcal{D}_2 &= P_{An-2} - tP_{An-1}\,, \\
\mathcal{D}_3 &= P_{An-3} - tP_{An-2} + (\frac{1}{2})t^2 P_{An-1}\,, \\
\mathcal{D}_4 &= P_{An-4} - tP_{An-3} + (\frac{1}{2})t^2 P_{An-2} - (\frac{1}{6})\,t^3 P_{An-1}\,, \\
\dots & \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\mathcal{D}_n &= P_{A0} - tP_{A1} + \dots + (-1)^n(\frac{1}{n!})\,t^n P_{An-1}\,,
\end{aligned}
$$

are n Darboux polynomials with the same cofactor

$$\Gamma^{(n)}_t(\mathcal{D}_a) = \Big(\Gamma^{(n)} + \frac{\partial}{\partial t}\Big)(\mathcal{D}_a) = -\,kx^2\,\mathcal{D}_a\,, \quad a = 1,2,\dots,n.$$

Hence the functions

$$J_{tab} = \frac{\mathcal{D}_a}{\mathcal{D}_b}\,, \quad a,b = 1,2,\dots,n,$$

are constants of the motion. In fact, we can arrange all these functions as the entries of an n-dimensional matrix $[J_{tab}]$ that becomes a matrix formed by constants of the motion (the diagonal elements are just ones) with the fundamental set of functions J_{tk} placed in the first row.

Finally, the divergence of the vector field $\Gamma^{(n)}$ is given by $\operatorname{div}\Gamma^{(n)} = -(n+2)kx^2$. Thus, using relation (12), we obtain the following Jacobi multipliers for the Abel equation of order n (or for the dynamical vector field $\Gamma^{(n)}$)

$$R_a = (\mathcal{D}_a)^{\mu_n}\,, \quad \mu_n = -(n+2)\,, \quad a = 1,2,\dots,n.$$

We note that these n Jacobi multipliers, although different $R_b \neq R_a$, $b \neq a$, they are however essentially the same since they are proportional by a constant of the motion.

6 Final comments

We have studied a chain of higher-order nonlinear Abel equations using, as starting point, the idea that they have many similarities with the higher-order nonlinear Riccati equations. We have made use of the Lagrangian formalism (inverse problem, non-polynomial Lagrangians, nonstandard symmetries) in the case of the second-order equation and of other mathematical tools (Darboux polynomials and Jacobi multipliers) in the case of higher-order nonlinearities. All these questions seems to be really interesting and we think they deserve a deeper study.

Finally, we mention that all these equations possess (for any order of the equation) a family of constants of the motion J_{tk} that depend of the time as a polynomial in t. In the symplectic case functions of such a class are associated to master symmetries of the (Lagrangian or Hamiltonian) system, but in the general Abel case we have proved the existence of such constants without any symplectic structure. This is in fact a very interesting fact that must be studied.

Acknowledgments

JFC and MFR acknowledge support from research projects MTM-2006-10531, FIS-2006-01225, and E24/1 (DGA). PG thanks the Departamento de Física Teórica de la Universidad de Zaragoza for its hospitality and acknowledges support from Max Planck Institute for Mathematics in the Sciences, Leipzig.

References

[1] H.T. Davis, "Introduction to Nonlinear Differential and Integral Equations" (Dover, New York, 1962).

[2] L. Erbe, "Existence of oscillatory solutions and asymptotic behavior for a class of third-order linear differential equations", Pacific J. Math. **64**, 369–385 (1976).

[3] L. Erbe, "Comparison theorems for second-order Riccati equations with applications", SIAM J. Math. Anal. **8**, 1032–1037 (1977).

[4] A.M. Grundland and D. Levi, "On higher-order Riccati equations as Bäcklund transformations", J. Phys. A **32**, 3931–3937 (1999).

[5] S. Moyo and P.G.L. Leach, "Exceptional properties of second and third-order ordinary differential equations of maximal symmetry", J. Math. Anal. Appl. **252**, 840–863 (2000).

[6] M. Euler, N. Euler and P.G.L. Leach, "The Riccati and Ermakov-Pinney hierarchies", J. Nonlinear Math. Phys. **14**, 290–310 (2007).

[7] G. Darboux, "Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré", Bull. Sci. Math. (2) **2**, 60–96, 123–144, 151–200 (1878).

[8] S. Labrunie, "On the polynomial first integrals of the (a, b, c) Lotka-Volterra system", J. Math. Phys. **37**, 5539–5550 (1996).

[9] A.J. Maciejewski and M. Przybylska, "Darboux polynomials and first integrals of natural polynomial Hamiltonian systems", Phys. Lett. A **326**, 219–226 (2004).

[10] M.C. Nucci and P.G.L. Leach, "Jacobi's last multiplier and symmetries for the Kepler problem plus a lineal story", J. Phys. A **37**, 7743–7753 (2004).

[11] M.C. Nucci, "Jacobi last multiplier and Lie symmetries: A novel application of an old relationship", J. Nonl. Math. Phys. **12**, 284–304 (2005).

[12] M.C. Nucci and P.G.L. Leach, "Jacobi's last multiplier and Lagrangians for multidimensional systems", J. Math. Phys. **49**, 073517 (2008).

[13] J.F. Cariñena, M.F. Rañada and M. Santander, "Lagrangian formalism for nonlinear second-order Riccati systems: one-dimensional integrability and two-dimensional superintegrability", J. Math. Phys. **46**, 062703 (2005).

[14] J.F. Cariñena, P. Guha and M.F. Rañada, "A geometric approach to higher-order Riccati chain: Darboux polynomials and constants of the motion", Proceedings of the "International Workshop on Higher Symmetries", Madrid November 2008 ; Journal of Physics Conference Series (to be published).

[15] G.M. Murphy, *Ordinary differential equations and their solutions* (Van Nostrand, Princeton, 1960).

[16] D. Zwillinger, *Handbook of differential equations*, 3rd ed. p. 120 (Academic Press, Boston, MA, 1997).

[17] A.D. Polyanin and V.F. Zaitsev, *Handbook of exact solutions for ordinary differential equations*, 2nd ed., p. 138 (Chapman & Hall/CRC, Boca Raton, FL, 2003)

[18] M. Briskin, J.P. Francoise and Y. Yomdin, "The Bautin ideal of the Abel equation", Nonlinearity 11, 431–443 (1998).

[19] V.M. Boyko, "Nonlocal symmetry and integrable classes of Abel equation", in Proceedings of the Fifth International Conference Symmetry in Nonlinear Mathematical Physics (Kiev, 2003), Editors A.G. Nikitin, V.M. Boyko, R.O. Popovych and I.A. Yehorchenko, Proceedings of the Institute of Mathematics, Kiev, vol. 50, Part 1, 47-51 (2004).

[20] E.S. Cheb-Terrab and A.D. Roche, "An Abel ordinary differential equation class generalizing known integrable classes", European J. Appl. Math. **14**, 217–229 (2003).

[21] D.E. Panayotounakos, "Exact analytic solutions of unsolvable classes of first and second-order nonlinear ODEs. I. Abel's equations", Appl. Math. Lett. **18**, 155–162 (2005).

[22] V.M. Boyko, "Symmetry, equivalence and integrable classes of Abel equations", in Symmetry and Integrability of Equations of Mathematical Physics, Institute of Mathematics, Kiev, 2006, vol. 3, no. 2, 39-48 ; nlin.SI/0404020.

[23] F. Schwarz, “Algorithmic solutions of Abel’s equations”, Computing **61**, 39–46 (1998).

[24] F. Schwarz, “Symmetry analysis of Abel’s equations”, Stud. Apppl. Math. **100**, 269–94 (1998).

[25] W. Sarlet, “The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics”, J. Phys. A **15**, 1503–1517 (1982).

[26] M. Crampin, G. Prince and G. Thompson, “A geometrical version of the Helmholtz conditions in time-dependent Lagrangian dynamics”, J. Phys. A **17**, 1437–1447 (1984).

[27] J. Lopuszanski, *The inverse variational problem in classical mechanics* (World Scientific Publishing, 1999).

[28] J.F. Cariñena and M.F. Rañada, “Helmholtz conditions and alternative Lagrangians: Study of an integrable Henon-Heiles system”, Intern. J. of Theor. Phys. **38**, 2049-2061 (1999).

[29] W. Sarlet and F. Cantrijn, “Higher-order Noether symmetries and constants of motion”, J. Phys. A **14**, 479–492 (1981).

[30] P.A. Damianou, “Symmetries of Toda equations”, J. Phys. A **26**, 3791–3796 (1993).

[31] R.L. Fernandes, “On the master symmetries and bi-Hamiltonian structure of the Toda Lattice”, J. Phys. A **26**, 3797–3803 (1993).

[32] M.F. Rañada, “Superintegrability of the Calogero-Moser system: constants of motion, master symmetries, and time-dependent symmetries”, J. Math. Phys. **40**, 236–247 (1999).

[33] R. Abraham and J.E. Marsden, *Foundations of mechanics*, 2nd edition (Benjamin/Cummings Publishing, Reading, Mass., 1978).

[34] M. Crampin, “Tangent bundle geometry for Lagrangian dynamics”, J. Phys. A **16**, 3755–3772 (1983).

[35] C. Lopez, E. Martinez, M.F. Rañada, “Dynamical symmetries, non-Cartan symmetries and superintegrability of the n-dimensional harmonic oscillator”, J. Phys. A **32**, 1241–1249 (1999).

[36] D.G. Currie and E.J. Saletan, “q-equivalent particle Hamiltonians. I. The classical one-dimensional case”, J. Math. Phys. **7**, 967–974 (1966).

[37] S. Hojman and H. Harleston, “Equivalent Lagrangians: multidimensional case”, J. Math. Phys. **22**, 1414–1419 (1981).