Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Euler-Poincaré Flows on the Loop Bott-Virasoro Group and Space of Tensor Densities and $2+1$ Dimensional Integrable Systems

by

Partha Guha

Preprint no.: 40 2009

Euler-Poincaré Flows on the Loop Bott-Virasoro Group and Space of Tensor Densities and $2 + 1$ Dimensional Integrable Systems

Partha Guha
Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103 Leipzig
GERMANY
and
S.N. Bose National Centre for Basic Sciences
JD Block, Sector-3, Salt Lake
Calcutta-700098, INDIA.

Abstract

Following the work of Ovsienko and Roger (Comm. Math. Phys. 273 (2007) 357-378) we study a new kind of deformation of loop Virasoro algebra. Using this new algebra we formulate the Euler-Poincaré flows on the coadjoint orbit of loop Virasoro algebra. We show that the Calogero-Bogoyavlenskii-Schiff equation and various other $(2 + 1)$-dimensional Korteweg–deVries (KdV) type systems follow from this construction. Using the right invariant H^1 inner product on the Lie algebra of loop Bott-Virasoro group we formulate Euler-Poincaré framework of the $2 + 1$-dimensional of the Camassa-Holm equation. This equation appears to be the Camassa-Holm analogue of the Calogero-Bogoyavlenskii-Schiff type $2 + 1$-dimensional KdV equation. We also derive the $(2 + 1)$-dimensional generalization of the Hunter-Saxton equation. Finally, we give an Euler-Poincaré formulation of one-parameter family of $1 + 1$-dimensional partial differential equations, known as the *b-field equations*. Later we extend our construction to algebra of loop tensor densities to study the Euler-Poincaré framework of the $2 + 1$-dimensional extension of b-field equations.

Mathematics Subject Classifications (2000): 53A07, 53B50.

Key Words : diffeomorphism, loop Virasoro algebra, tensor densities, Calogero-Bogoyavlenskii-Schiff equation, $2 + 1$-dimensional Camassa equation, b-field equation.

1 Introduction

The study of higher dimensional integrable systems is one of the most challenging areas in integrable systems. Early in the study of integrable systems, the main thrusts were restricted to the $(1+1)$-dimensional systems because of the difficulty of finding the physically significant high-dimensional solutions which are localized in all directions. Recently much progress has been achieved in understanding the properties and solutions for two-dimensional integrable models such as Kadomtsev-Petvashvili (KP), Davey-Stewartson (DS) equations [1]. One of the most striking feature of $(2+1)$-dimensional system is the exponentially localized structures, called dromions, which are driven by two perpendicular line ghost solitons in the case of DS equation or two non-perpendicular line ghost solitons in the case of KP equation. It should be notice that the name dromions as well as their spectral meaning were introduced by Fokas and Santini [21]. Recently the rich dromion structures were found in $(2+1)$ dimensional KdV equations also [39, 40, 48, 52].

After the discovery of dromions, the question arises whether there exist exponentially localized structures in $(2+1)$-dimensional breaking soliton equations as well. In such systems the spectral parameter becomes a multivalued function, in other words, spectral parameter possesses so-called breaking behaviour. The solutions of these equations may become multivalued. There is an equation exibiting breaking solitons, formulated by O. Bogoyavlenskii, as one of the $2+1$ dimensional reductions of the self dual Yang-Mills equations. In a series of papers Bogoyavlenskii [4, 5] studied such breaking soliton equations. He extended the well-known Lax representation to the generalized form

$$L_t = P(L) + \sum_{k=1}^{n} R_k(L, L_{y_k}) + [L, A].$$

Here $P(L)$ and $R_k(L, L_{y_k})$ are certain meromorphic functions of the operator L.

Bogoyavlenskii constructed several hydrodynamic-type systems which are connected to the Toda lattice and the Volterra model. It has been shown that these systems possess the breaking behaviour, the Hamiltonian forms and conservation laws. The continuous limits of these systems include the equation

$$v_t = 4vv_y + 2v_x\partial_x^{-1}v_y - v_{xxy} + \beta_0(6vv_x - v_{xxx}) \tag{1}$$

which, after the substitution $v = u_x$, is reduced to potential form

$$u_{tx} = 4u_x u_{xy} + 2u_y u_{xx} - u_{xxxy}, \tag{2}$$

where we set $\beta_0 = 0$.

J. Schiff [53] obtained above equation in a different route. He obtained from reduction of the self-dual Yang-Mills equations from four to three dimensions. There has been considerable interest to show that the self-dual Yang-Mills equations as a *master integrable equation*, from which many integrable systems can be obtained by suitable reductions. It has been shown in [55] that the generalized SDYM equations contain (as dimensional reductions) various $(2+1)$-dimensional integrable soliton hierarchies which generalize the nonlinear Schrödinger and KdV hierarchies.

One can also derive $(2+1)$-dimensional KdV type systems from a different method. In geodesic coordinates the Gauss equation is reduced to the Schrödinger equation where the Gaussian curvature plays the role of a potential. It can be shown that a special case is governed by the KdV equation for the Gaussian curvature. In this framework Konopelchenko [37] studied the integrable dynamics of curvature via the KdV equation, higher KdV equations and other $(2+1)$-dimensional integrable equations with breaking solitons. The bihamiltonian operators for 2+1 dimensional integrable systems were introduced in [22, 23, 24].

In an interesting paper Fokas et al [20] proposed an algorithmic construction of $(2+1)$ dimensional integrable system

$$q_{xt} - \nu q_{xxxt} + a q_{xy} + b q_{xxxy} + c\Big(q_{xx}q_y + 2q_x q_{xy}\Big) - c\nu\Big(q_{xxxx}q_y + 2q_{xxx}q_{xy}\Big). \quad (3)$$

which yield peakon/dromion type solutions. This equation can be identified with the potential form of the Camassa-Holm analogue of the Calogero-Bogoyavlenskii-Schiff equation. For $\nu = 0$, this reduces to Camassa-Holm equation.

Recently the one–parameter family of shallow water equations of the following form

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}, \quad (4)$$

where b is a real parameter, has drawn some attention. This equation is known as the *b-field* equation. It was introduced by Degasperis, Holm and Hone [13, 14], who showed the existence of multi-peakon solutions for any value of b, although only the special cases $b = 2, 3$ are integrable, having bihamiltonian formulations. The $b = 2$ case is the well-known Camassa-Holm (CH) equation [7] and $b = 3$ is the integrable system discovered by Degasperis and Procesi [15]. One must note that for $b = 2, 3$ the equation (4) is hydrodynamically relevant [34, 11]. Incidentally $b = 2$ case was later recognized as being included in a class of integrable equations derived from hereditary symmetries in Fokas and Fuchssteiner [19]

Using the Helmholz field $m := u - u_{xx}$, the b-field equation or the DHH equation (4) allows reformulation in the compact form

$$m_t + um_x + bu_x m = 0 \ , \quad (5)$$

where the three terms correspond respectively to evolution, convection and stretching of the one-dimensional flow. In this paper we study an Euler-Poincaré formulation of $(2+1)$-dimensional b-field equation.

1.1 Motivation, result and organization

In a very recent work, an Euler-Poincaré framework of the Degasperis and Procesi (DP) equation has been formulated. It turns out that the DP equation is the Euler-Poincaré flow on the combined space of Hill's (second order) and first order differential operators on circle [29]. It has been generalized to the two component generalization of the DP equation. It has been shown also that the Hamiltonian structure obtained from the EP framework exactly coincides with the Hamiltonian structues of the DP equation obtained by Degasperis, Holm and Hone. In this paper *we give a much shorter derivation* of the DP and the b-field equation using the deformation of vector field structure on S^1. Following the work of Ovsienko and Roger [44] we study a new kind of deformation of loop Virasoro algebra. Using this new algebra we able to derive the $2+1$-dimensional b field equation.

The aim of this paper is to contribute towards a theory of integrable type geodesic flows on infinite-dimensional Lie groups which has attracted tremendous attention since Arnold's seminal paper [2] on Euler equation in hydrodynamics. Later, Ebin and Marsden [17] established a proper geometric setting of this problem. They showed that the geodesic spray was smooth. This led to very nice existence proofs; the limit of zero viscosity for manifolds with no boundary was shown to exist for the first time. It would be worth to mention that the in recent years equations like the Camassa-Holm equation model for shallow water waves or the Hunter-Saxton [33] model for nemetic liquid crystals, the geometric structures was used to to study qualitative properties of the solutions.

The KdV equation is an Euler-Poincaré equation on the Virasoro-Bott group (see [35, 47, 54]). This group is defined as the unique (up to isomorphism) non-trivial central extension of the group $\mathit{Diff}(S^1)$ of all diffeomorphisms of S^1. The inertia operator is given by the standard L^2-metric on S^1. It is known that the two-component KdV and Camassa-Holm equations are also geodesic flows on the extended Virasoro-Bott group [26, 27, 28].

The infinite-dimensional groups also play important role for the construction of (2+1)-dimensional integrable systems. The $(2+1)$-dimensional KdV and nonlinear Schrödinger equation can be derived from the toroidal Lie algebra. Here the variable x is associated to the action of the usual affine part of the toroidal Lie algebra [3, 50], while evolutions in y and t are indiced by the action of the genuine toroidal part. The weight of v and the relative and the relative weight of y and t are balanced with that of x, thus it allows us two freedoms to determine the weights for all the variables.

In this paper we study the (formal) Euler-Poincaré framework [42] of 2+1 dimensional KdV type systems. Till now there is no systematic construction of $2+1$-dimensional integrable systems from the study of geodesic flows. In particular, we show that the Calogero–Bogoyavlenskii–Schiff equation arises as a geodesic flow on loop Bott-Virasoro group. This equation is an eminent member of the $(2+1)$-dimensional family of KdV equations [32]. In the second half of the paper we construct higher dimensional Camassa-Holm equation. We show that the $(2+1)$-dimensional Camassa–Holm equation arises as geodesic flow with respect to the right invariant H^1 metric on the cotangent loop Virasoro group. We also compute the $(2+1)$-dimensional Hunter-Saxton equation. In fact the result of this paper was annonced [30] in the Oberwolfach meeting on geometrical mechanics. We hope in our forthcoming work we will consider the singular solutions of all these equations.

The paper is **organized** as follows. In Section 2 we present the Euler-Poincaré formalism and frozen Poisson structures. Loop Virasoro algebra is introduced in Section 3. We also derive 2+1 KdV equation in this section. Section 4 is devoted to the derivation of the 2+1 dimensional Camassa-Holm equation and the Hunter-Saxton equation. In Section 5 we present the Euler-Poincaré framework of the b-field equation. The formulation of 2+1 dimensional b- field equation is given Section 6.

2 The Euler-Poincaré formalism

The Euler-Poincaré equations were born in 1901 (see [42]) when Poincaré made a extensive generalization of the classical Euler equations for the rigid body and ideal fluids. He did this by formulating the equations on a general Lie algebra, with the rigid body being associated with the rotation Lie algebra and fluids with the Lie algebra of divergence free vector fields.

Let G be a Lie group and $\mathfrak{g}$ be its corresponding Lie algebra and its dual is denoted by $\mathfrak{g}^*$. The dual space $\mathfrak{g}^*$ to any Lie algebra $\mathfrak{g}$ carries a natural Lie-Poisson structure:

$$\{f,g\}_{LP}(\mu) :=< [df,dg], \mu >$$

for any $\mu \in \mathfrak{g}^*$ and $f,g \in C^\infty(\mathfrak{g}^*)$.

The Hamiltonian vector field on $\mathfrak{g}^*$ corresponding to a Hamiltonian function f, computed with respect to the Lie-Poisson structure is given by

$$\frac{d\mu}{dt} = ad^*_{df}\mu, \qquad \mu \in \mathfrak{g}^*. \tag{6}$$

We denote $E(u) = \frac{1}{2} < u, Iu >$ for the energy quadratic form on $\mathfrak{g}$. It is used to define the Riemannian metric. We identify the Lie algebra and its dual with this quadratic form. This identification is done via *inertia operator*. Let I be an inertia operator

$$I : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and then $u \in \mathfrak{g}^*$ evolve by

$$\frac{du}{dt} = (I^{-1}u) \cdot u, \tag{7}$$

where right hand side denote the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. This equation is called the Euler-Poincaré equation.

Definition 2.1 *The Euler-Poincaré equation on $\mathfrak{g}^*$ corresponding to the Hamiltonian $H(\mu) = \frac{1}{2} < I^{-1}\mu, \mu >$ is given by*

$$\frac{du}{dt} = ad^*_{I^{-1}u}u, \qquad I^{-1}u \in \mathfrak{g}.$$

It characterizes an evolution of a point $u \in \mathfrak{g}^*$.

2.1 Frozen Lie-Poisson structure

Consider the dual of the Lie algebra of $\mathfrak{g}^*$ with a Poisson structure given by the "frozen" Lie-Poisson structure. In otherwords, we fix some point $\mu_0 \in \mathfrak{g}^*$ and define a Poisson structure given by

$$\{f,g\}_{LP}(\mu) :=< [df(\mu), dg(\mu)], \mu_0 >,$$

which satisfies Jacobi identity. It plays an important role in integrable systems. We can give another interpretation [9, 10] of frozen structure from the definition of cocycle. Given an inertia operator $I : \mathfrak{g} \to \mathfrak{g}^*$ one can define a constant Poisson structure

$$\{f, g\}_0(\mu) = \langle df, I\, dg \rangle \qquad \text{where} \qquad \mu \in \mathfrak{g}^*.$$

A two-cocycle ω is called a coboundary if there is a point $\mu_0 \in \mathfrak{g}^*$ such that

$$\omega(\mathfrak{p}, \mathfrak{q}) = \langle [\mathfrak{p}, \mathfrak{q}], \mu_0 \rangle,$$

where $\mathfrak{p}, \mathfrak{q} \in \mathfrak{g}$.

Since the Poisson structure is generated by a coboundary of ω, we obtain

$$\{f, g\}_0(\mu) = \langle [df(\mu), dg(\mu)], \mu_0 \rangle.$$

This behaves like a Lie-Poisson structure frozen at the point $\mu_0 \in \mathfrak{g}^*$ and this coincides with the previous definition of frozen structure.

It is easy to check that the above Poisson structures are compatible, i.e., their linear combination or pencil of Poisson structures

$$\{\, ,\, \}_l = \{\, ,\, \}_0 + \lambda \{\, ,\, \}_{LP} \tag{8}$$

is again a Poisson structure for all $\lambda \in \mathbb{R}$.

It was shown by Khesin and Misiolek [36] that

Proposition 2.2 *The brackets $\{\cdot, \cdot\}_{LP}$ and $\{\cdot, \cdot\}_0$ are compatible for every "freezing" point $u_0 \in \mathfrak{g}^*$.*

At this point we can introduce the bihamiltonian structure. The notion of integrability can be understood from this structure. The standard way to understand bihamiltonian vector fields on the dual of the Lie algebra is associated to Lie-Poisson structures.

Definition 2.3 *A vector field X on $\mathfrak{g}^*$ is called bi-Hamiltonian if there are two functions, H_1 and H_2 such that X is a Hamiltonian vector field of H_1 with respect to the Poisson structure $\{\, ,\, \}_{LP}$ and is a Hamiltonian vector field of H_2 with respect to $\{\, ,\, \}_0$.*

3 Loop Virasoro algebra and $2+1$-dimensional KdV flows

We wish to extend the Virasoro algebra to the case of two space variables. A natural way to do this is to consider the loops on it. One defines the loop group on $\mathit{Diff}(S^1)$ as follows

$$L\left(\mathit{Diff}(S^1)\right) = \left\{\phi : S^1 \to \mathit{Diff}(S^1) \mid \quad \phi \quad \text{is differentiable}\right\},$$

the group law being given by

$$(\phi \circ \psi)(y) = \phi(y) \circ \psi(y), \qquad y \in S^1.$$

In the similar way, we construct the Lie algebra $L(Vect(S^1))$ consisting of vector fields on S^1 depending on one more independent variable $y \in S^1$. The loop variable is thus denoted by y and the variable on the "target" copy of S^1 by x. The elements of $L(Vect(S^1))$ are of the form: $f(x,y)\frac{\partial}{\partial x}$ where $f \in C^\infty(S^1 \times S^1)$ and the Lie bracket reads as follows [44]

$$\left[f(x,y)\frac{\partial}{\partial x}, g(x,y)\frac{\partial}{\partial x}\right] = \left(f(x,y)\, g_x(x,y) - f_x(x,y)\, g(x,y)\right)\frac{\partial}{\partial x}.$$

It is easy to convince oneself that $L\left(Vect(S^1)\right)$ is the Lie algebra of $L\left(Diff(S^1)\right)$ in the usual weak sense for the infinite-dimensional case; a one-parameter group argumentation gives an identification between the tangent space to $L\left(Diff(S^1)\right)$ at the identity and $L\left(Vect(S^1)\right)$, equipped with its Lie bracket. In future we will denote $L\left(Vect(S^1)\right)$ by $\tilde{\mathfrak{g}}$. The natural pairing between the loop Virasoro algebra and its dual is given by

$$\langle f(x,y)\,\frac{\partial}{\partial x}, v(x,y)\, dx^2\rangle = \int_{S^1\times S^1} f v \, dxdy. \tag{9}$$

3.1 Cocycle and extension of loop Virasoro algebra

Consider the following "modified" Gelfand-Fuchs cocycle on $Vect(S^1)$:

$$\omega_{mGF}(f(x)\frac{d}{dx}, g(x)\frac{d}{dx}) = \int_{S^1}(af'g'' + bf'g)dx, \tag{10}$$

where the first term is the original Gelfand-Fuchs cocycle.

This cocycle is cohomologues to the Gelfand-Fuchs cocycle, hence, the corresponding central-extension is isomorphic to the Virasoro algebra. The additional term is a coboundary term. It is easy to check that the functional

$$\int_{S^1} f'g \; dx = \frac{1}{2}\int_{S^1}(f'g - fg')dx$$

depends on the commutator of $f\frac{d}{dx}$ and $g\frac{d}{dx}$.

Let us give the explicit formulæ of non-trivial 2-cocycles [25] on $\widetilde{\mathfrak{g}}$. A distribution $\lambda \in C^\infty(S^1)'$ corresponds to a 2-cocycle of the first class given by [56]

$$\mu_\lambda\left((f,u),(g,v)\right) = \lambda\Big(\int_{S^1} f\, g_{xxx}\, dx\Big),$$

these are the Virasoro type extensions. For the particular case where $\lambda(a(y)) = \int_{S^1} a(y)dy$, such a 2-cocycle will be denoted by μ_1 so that one has

$$\mu_1\left((f,u),(g,v)\right) = \int_{S^1\times S^1} f\, g_{xxx}\, dxdy. \tag{11}$$

We define the Lie algebra $\widehat{\mathfrak{g}}$ as the one-dimensional central extension of $\widetilde{\mathfrak{g}}$ given by the cocycles μ_1. As a vector space,

$$\widehat{\mathfrak{g}} = \widetilde{\mathfrak{g}} \oplus \mathbb{R},$$

where the summand $\mathbb{R}$ is the center of $\widehat{\mathfrak{g}}$. The commutator in $\widehat{\mathfrak{g}}$ is given by the following explicit expression which readily follows from the above formulæ.

$$\left[(f\,\frac{\partial}{\partial x}\,,a),(g\,\frac{\partial}{\partial x}\,,b)\right] = (f\,g_x - f_x\,g)\,\frac{\partial}{\partial x} + \int_{S^1\times S^1} f\,g_{xxx}\,dxdy\,. \tag{12}$$

where the last term is an element of the center of $\widehat{\mathfrak{g}}$.

3.2 Calogero-Bogoyavlenskii-Schiff type $2+1$-dimensional KdV equation

We recall the Kirillov-Segal result and generalize it to the case of the Lie algebra $\widehat{\mathfrak{g}}$.

Proposition 3.1 *The coadjoint action of the Lie algebra $\widehat{\mathfrak{g}}$ is given by*

$$\widehat{ad}^*_{f(x,y)\frac{\partial}{\partial x}}\,(v(x,y)\,dx^2) = (f\,v_x + 2\,f_x\,v + c_1\,f_{xxx} + c_2\,f_x)\,dx^2, \tag{13}$$

while the center acts trivially.

Corollary 3.2 *The Hamiltonian operator $\mathcal{O}_{LV}$ corresponding to the coadjoint action of the loop Virasoro algebra is given by*

$$\mathcal{O}_{LV} = \partial_x v + v\partial_x + c_1\partial_x^3 + c_2\partial_x. \tag{14}$$

Given a functional H on $\widehat{\mathfrak{g}}^*$ which is a (pseudo)differential polynomial:

$$H(g,\,v) = \int_{S^1\times S^1} h\left(g,\,v,\,g_x,\,v_x,\,g_y,\,v_y,\,\partial_x^{-1}g,\,\partial_x^{-1}v,\,\partial_y^{-1}g,\,\partial_y^{-1}v,\,g_{xy},\,v_{xy},\ldots\right)\,dxdy,$$

where h is a polynomial in an infinite set of variables.

For instance,

$$\begin{aligned}\frac{\delta H}{\delta v} \;&=\; h_v - \frac{\partial}{\partial x}\,(h_{v_x}) - \frac{\partial}{\partial y}\left(h_{v_y}\right) - \partial_x^{-1}\left(h_{\partial_x^{-1}v}\right) - \partial_y^{-1}\left(h_{\partial_y^{-1}v}\right)\\ &\quad + \frac{\partial^2}{\partial x^2}\,(h_{v_{xx}}) + \frac{\partial^2}{\partial x\partial y}\left(h_{v_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(h_{v_{yy}}\right) \pm \cdots\end{aligned}$$

where, as usual, h_v means the partial derivative $\frac{\partial h}{\partial v}$, similarly $h_{v_x} = \frac{\partial h}{\partial v_x}$.

3.2.1 The Bogoyavlenskii-Konopelchenko equation

The Euler-Poincaré formalism of the Bogoyavlenskii-Konopelchenko equation

$$v_t + \beta v_{xxy} + 3\alpha + v_{xxx} + vv_x + 2\beta vv_y + \beta v_x\partial_x^{-1}v_y = 0, \qquad \partial_x^{-1}v = \int^x v\,dx \tag{15}$$

is closely related to the Calogero-Bogoyavlenskii-Schiff equation. In fact, this is a combination of KdV and Calogero-Bogoyavlenskii-Schiff flows. Eqn. (15) models the $(2+1)$-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis.

Corollary 3.3 *The Euler-Poincaré flow restricted to hyperplane $c_1 = 1, c_2 = 0$ at yields the Bogoyavlenskii-Konopelchenko equation for the Hamiltonian*

$$H = \frac{1}{2}\int_{S^1\times S^1} (v^2 + v\partial_x^{-1}v_y)\, dxdy.$$

Another class of $(2+1)$-dimensional KdV equation was proposed by Lou and his collaborators ([38], [39], [40]) to study the rich dromion structures, defined as

$$u_t + u_{xxx} = 3(u\partial_y^{-1}u_x)_x. \tag{16}$$

This equation reduces to the usual $(1+1)$-dimensional KdV equation.

We use "frozen " Lie-Poisson structure to compute the Hamiltonian operator at $(v(x)dx^2, c_1, c_2) = (0, 0, 1)$. We also assume that the only cocycle term is $\int_S^1 f'g$. The Hamiltonian operator computed at the freezing at the point $(0, 0, 1)$ yields a truncated Hamiltonian operator

$$\tilde{\mathcal{O}}_1 = \partial_x.$$

We also compute the Hamiltonian operator at $(v(x)dx^2, c_1, c_2) = (0, 1, 0)$, given by

$$\tilde{\mathcal{O}}_2 = \partial_x^3.$$

Proposition 3.4 *The second class $(2+1)$-dimensional KdV follows from the following combination of flows on $\widehat{\mathfrak{g}}^*$*

$$v_t = \mu\tilde{\mathcal{O}}_1\frac{\delta H_1}{\delta v} + \lambda\tilde{\mathcal{O}}_2\frac{\delta H_2}{\delta v},$$

where $\frac{\delta H_1}{\delta v} = (v\partial_y^{-1}v_x)$ and $\frac{\delta H_2}{\delta v} = v$ respectively.

Proof: By direct computation.
□

4 H^1 metric and $2+1$-dimensional Camassa-Holm and Hunter-Saxton systems

In this section we study the Camassa-Holm analogue of the $(2+1)$-dimensional KdV equations. Let us start with the explicit expression for the coadjoint action of $\mathfrak{g}$ with respect to right invariant H^1-metric.

Let us introduce H^1 norm on the algebra $\tilde{\mathfrak{g}}$.

Definition 4.1 *The H^1- Sobolev norm on the loop Virasoro algebra is defined as*

$$< f(x,y)\frac{\partial}{\partial x}, u(x,y)\, dx^2 >_{H^1} = \int_{S^1} fu dx + \nu\int_{S^1} \partial_x f\partial_x u dx, \tag{17}$$

Now we compute the coadjoint action.

Proposition 4.2 *The coadjoint action with respect to H^1 metric of the loop Virasoro algebra $\widehat{\mathfrak{g}}$ is given by*

$$\widehat{ad}^*_{f(x,y)\frac{\partial}{\partial x}}\, v\, dx^2 = (f\, \tilde{v}_x + 2\, f_x\, \tilde{v} + c_1\, f_{xxx} + c_2\, f_x)\, dx^2,$$

where $\tilde{v} = (1 - \nu\, \partial^2)v$.

Proof: We know that

$$< \widehat{ad}^*_{f\frac{\partial}{\partial x}}\, v\, dx^2, h\, \frac{\partial}{\partial x} >_{H^1} = < v\, dx^2, [f\, \frac{\partial}{\partial x}, h\, \frac{\partial}{\partial x}] >_{H^1}$$

$$= < g\, \frac{\partial}{\partial x}, [\Big(f\, h_x - f_x\, h\Big)\frac{\partial}{\partial x} + \Big(\int_{S^1\times S^1} f\, h_{xxx}\, dx\, dy, \int_{S^1\times S^1} f\, h_x\, dx\, dy\Big) >_{H^1} .$$

Thus from the R.H.S. we obtain the matrix expression.

We compute now the L.H.S. of the above equation Let us denote

$$\hat{f} = (f\frac{\partial}{\partial x}, \mathbf{c}), \qquad \hat{g} = (g\frac{\partial}{\partial x}, \mathbf{d}), \qquad \hat{h} = (h\frac{\partial}{\partial x}, \mathbf{e}),$$

where $\mathbf{c} = (c_1, c_2)$, $\mathbf{d} = (d_1, d_2)$ and $\mathbf{e} = (e_1, e_2)$.

Now we compute the L.H.S.

$$L.H.S. = \int_{S^1\times S^1} (\widehat{ad}^*_{\hat{f}}\hat{g})\hat{h}\, dx\, dy + \nu \int_{S^1\times S^1} (\widehat{ad}^*_{\hat{f}}\hat{g})'\hat{h}'\, dx\, dy$$
$$= \int_{S^1\times S^1} [(1 - \nu{\partial_x}^2)\widehat{ad}^*_{\hat{f}}\hat{g}\hat{h}\, dx\, dy.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.
□

Lemma 4.3 *The Hamiltonian operator corresponding to the coadjoint action of the loop Virasoro algebra with respect to H^1 metric is given by*

$$\mathcal{O}_{H^1} \;=\; (1 - \nu\partial_x^2)^{-1}\left(\partial_x\tilde{v} + \tilde{v}\partial_x + c_1\partial_x^3 + c_2\partial_x\right), \tag{18}$$

where $\tilde{v} = (1 - \nu\partial^2)v$.

Let us study the Euler-Poincaré flow associated to H^1 metric on the coadjoint orbit of the cotangent loop Virasoro algebra $\widehat{\mathfrak{g}}$.

Proposition 4.4 *The Euler-Poincaré flow with respect to H^1- norm on dual space of loop Virasoro algebra becomes*

$$v_t = \mathcal{O}_{H^1}\frac{\delta H}{\delta v}, \tag{19}$$

where $\mathcal{O}_{H^1}$ is defined by (18). Suppose the quadratic Hamiltonian on $\tilde{\mathfrak{g}}^*$ is defined as

$$H = \frac{1}{2}\int_{S^1\times S^1} v\partial_x^{-1}v_y\, dxdy,$$

then the Euler-Poincaré flow yields

$$\tilde{v}_t = \tilde{v}_x\partial_x^{-1}v_y + 2\tilde{v}v_y + c_1 v_{xxy} + c_2 v_y. \tag{20}$$

Corollary 4.5 *The Euler-Poincaré flow restricted to hyperplane* $c_2 = 0$ *yields the Camassa-Holm analogue of the Calogero-Bogoyavlenskii-Schiff equation*

$$v_t - \nu v_{xxt} + c_1 v_{xxy} + (v_x - \nu v_{xxx})\partial_x^{-1} v_y + 2(v - \nu v_{xx})v_y = 0 \tag{21}$$

for the Hamiltonian $H = \int_{S^1 \times S^1} g \partial_x^{-1} v_y \, dxdy$.

Corollary 4.6 *The potential form of the* $2+1$*-dimensional Camassa-Holm equation takes the form*

$$u_{xt} - \nu\, u_{xxxt} + c_1\, u_{xxxy} \;+\; (u_{xx}u_y + 2u_x u_{xy}) - \nu(u_{xxxx}u_y + 2u_{xxx}u_{xy} = 0 \tag{22}$$

for all $v = u_x$.

Remark In a special $(1+1)$-dimensional case $(y = x)$, Eqn. (22) reduces to potential Camassa-Holm equation. If we further assume $\nu = 0$, then Eqn. (22) reduces to potential KdV equation.

Corollary 4.7 *The Euler-Poincaré flow restricted to hyperplane* c_1 *and* c_2 *yields the modified Calogero-Bogoyavlenskii-Schiff equation*

$$v_t - \nu v_{xxt} + c_3 v_y + c_1 v_{xxy} + (v_x - \nu v_{xxx})\partial_x^{-1} v_y + 2(v - \nu v_{xx})v_y, = 0$$

and potential form of takes the form

$$u_{xt} - \nu\, u_{xxxt} + c_3\, u_{xy} \;+\; c_1\, u_{xxxy} \;+\; (u_{xx}u_y + 2u_x u_{xy}) - \nu(u_{xxxx}u_y + 2u_{xxx}u_{xy}) = 0.$$

Corollary 4.8 *If we assume* $\frac{1}{\nu} \to 0$, *then Eqn. (40) takes the form*

$$v_{xxt} + v_{xxx}\partial_x^{-1} v_y + 2v_{xx})v_y = 0, \tag{23}$$

it is known as $2+1$ *dimensional Hunter-Saxton equation. For* $(1+1)$*-dimensional case* $(y = x)$, *this reduces to the Hunter-Saxton equation*

$$v_{xxt} + v_{xxx}v + 2v_{xx}v_x = 0. \tag{24}$$

The potential form of takes the form

$$u_{xxxt} + u_{xxxx}u_y + 2u_{xxx}u_{xy} = 0. \tag{25}$$

5 Euler-Poincaré framework of $1+1$-dimensional b-field equation

Denote $\mathcal{F}_\mu(S^1)$ the space of tensor-densities of degree μ on S^1

$$\mathcal{F}_\mu = \{a(x)dx^\mu \mid a(x) \in C^\infty(S^1)\},$$

where μ is the degree, x is a local coordinate on S^1. As a vector space, $\mathcal{F}_\mu(S^1)$ is isomorphic to $C^\infty(S^1)$ [43].

Geometrically we say

$$\mathcal{F}_\lambda \in \Gamma(\Omega^{\otimes\lambda}), \qquad \text{where} \quad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

$\Omega = T^*S^1$ is the cotangent bundle of S^1. Here $\mathcal{F}_0(M) = C^\infty(M)$, the space $\mathcal{F}_1(M)$ and $\mathcal{F}_{-1}(M)$ coincide with the spaces of differential forms and vector fields respectively.

Definition 5.1 *The b- bracket between $v(x)\frac{d}{dx}$ and $w(x)\frac{d}{dx}$ is defined as*

$$[v, w]_b = vw_x - (b-1)v_x w \tag{26}$$

This b-bracket can also be expressed as

$$[v, w]_b = \frac{b}{2}[v, w] - \frac{b-2}{2}[v, w]^{sym}, \tag{27}$$

where $[v, w] = vw_x - v_x w$ and $[v, w]^{sym} = vw_x + v_x w$.

Remark The b-bracket can be interpretred as an action of $Vect(S^1)$ on $\mathcal{F}_{-(b-1)}(S^1)$, a tensor densities on S^1 of degree $-(b-1)$. For $b = 2$ this is just a vector field action corresponding to a Lie algebra. Moreover because of $[v, w]^{sym}$ term b-bracket is not a skew-symmetric bracket, it is a deformation of the bracket of vector fields.

There is a pairing

$$\langle , \rangle : \ \mathcal{F}_\mu \otimes \mathcal{F}_{1-\mu} \to \mathbb{R}$$

given by

$$\langle a(x)(dx)^\mu, b(x)(dx)^{1-\mu}\rangle = \int_{S^1} a(x)b(x)\, dx \tag{28}$$

which is $Diff(S^1)$-invariant. A vector field $f(x)\frac{d}{dx}$ acts on the space of tensor densities $\mathcal{F}_\mu$ by the Lie derivative

$$L^\mu_{f(x)\frac{d}{dx}}(a(x)) = \Big(f(x)a'(x) + \mu f'(x)a(x)\Big)(dx)^\mu. \tag{29}$$

We denote b-algebra by $\mathcal{F}_{-(b-1)}$ and its dual by $\mathcal{F}_b$. Thus we can define a pairing according to (9)

$$\langle a(x)(dx)^{-(b-1)}, b(x)(dx)^b\rangle = \int_{S^1} a(x)b(x)\, dx.$$

Let us compute the coadjoint action with respect to the b–field equation.

Lemma 5.2

$$(ad^{H^1})^*_f(u) = (1-\nu\partial^2)^{-1}[f(1-\nu\partial^2)u_x + bf_x(1-k\partial^2)u]. \tag{30}$$

Proof: We know

$$< ad^*_f(u), g >_{H^1} = - < u, [f,g]_b >_{H^1}$$
$$\equiv - < udx^b, (fg' - (b-1)f'g)(dx)^{1-b} >_{H^1},$$

hence the pairing is well-defined. Let us compute

$$\text{R.H.S.} \quad = \quad \int_{S^1}(ufg' - (b-1)uf'g)dx + \nu\int_{S^1} u'(fg' - (b-1)f'g)'dx$$
$$= \int_{S^1}[f(1-\nu\partial^2)u' + bf'(1-\nu\partial^2)u$$
$$L.H.S. = \int_{S^1}(ad^{H^1})^*_f u)gdx + \nu\int_{S^1}(ad^{H^1})^*_f u'g'dx$$
$$= \int_{S^1}[(1-\nu\partial^2)ad^{H^1})^*_f u]gdx.$$

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.
□

Using the Helmholtz operator we express $m = (1-\nu\partial^2)u$. Thus, we express the Hamiltonian operator corresponding to (30) as

$$\mathcal{O}_1 = -(1-\nu\partial^2)^{-1}(m_x + bm\partial). \tag{31}$$

The Euler-Poincaré equation

$$u_t = \mathcal{O}_1\frac{\delta H}{\delta u} \qquad \text{for} \quad H = \int_{S^1} u^2\, dx,$$

can be rewritten as

$$m_t = \mathcal{O}\frac{\delta H}{\delta u}, \tag{32}$$

where $\mathcal{O} = (m_x + bm\partial)$.

Using the EP formula (32) we construct b-field equation.

Proposition 5.3 *The Euler-Poincaré flow on the dual space of b-algebra yields the b-field equation*

$$m_t + m_x u + bmu_x = 0.$$

This is a new derivation of the b-field equation.

5.1 Hamiltonian structure of the Degasperis-Procesi equation and EP framework

Degasperis et al studied Hamiltonian structures for $b = 3$ case of the b-field equation or the DHH equation, in other words, they exhibits bihamiltonian features of the Degasperis-Procesi system. They expressed the Degasperis-Procesi equation as

$$m_t = B_i \frac{\delta H_i}{\delta m} \qquad i = 0, 1, \tag{33}$$

where $m = u - u_{xx}$. Thus they studied the flow of Helmholtz function. They showed that there is only one local Hamiltonian structure

$$B_0 = \partial_x(1 - \partial_x^2)(4 - \partial_x^2), \tag{34}$$

and the second Hamiltonian structure is given by

$$B_1 = m^{2/3}\partial_x m^{1/3}(\partial_x - \partial_x^3)^{-1} m^{1/3}\partial_x m^{2/3}, \tag{35}$$

which can be simplified to

$$B_1 \equiv \hat{B} = \frac{2}{9}(3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x).$$

Proposition 5.4 *The Degasperis-Procesi equation*

$$m_t = \hat{B}\frac{\delta H_1}{\delta m}, \qquad \hat{B} = (3m\partial + m_x)(\partial - \partial^3)^{-1}(3m\partial + 2m_x) \tag{36}$$

is equivalent to

$$m_t = \mathcal{O}\frac{\delta H}{\delta u} \qquad \text{for} \;\; H = \int_{S^1} u^2 \, dx,$$

where $\mathcal{O} = (m_x + bm\partial)$.

Proof: Our goal is to show

$$\frac{2}{9}(\partial - \partial^3)^{-1}(3m\partial + 2m_x)\frac{\delta H_1}{\delta m} = \frac{\delta H}{\delta u},$$

where $H_1 = \frac{9}{4}\int_{S^1} m \, dx$. If we insert $\frac{\delta H_1}{\delta m} = \frac{9}{4}$ to left hand side of above equation we obtain

$$(\partial - \partial^3)^{-1} m_x = u,$$

where we use $u = (1 - \partial^2)^{-1} m$. Thus we obtain

$$m_t = (3m\partial + m_x)\frac{\delta H}{\delta u}$$

where $H = \frac{1}{2}\int_{S^1} u^2 \, dx$.

Therefore the Degasperis-Holm-Hone form of Hamiltonian structure coincides with our Hamiltonian structure.

□

5.1.1 First Hamiltonian structure of b-field equation

Let us compute the Hamiltonian operator at a frozen point $m(x) = m_0$. Since m_0 is constant so the Hamiltonian operator at the frozen point becomes

$$\mathcal{O}_0 = 3m_0\partial. \tag{37}$$

Actually freezing at the point m_0 yields a Poisson structure induced by a coboundary, which is always a trivial Poisson structure. For all practical purposes we can normalize this $\mathcal{O}_0$ operator or taking $m_0 = \frac{1}{3}$. We show that this leads us to the first Hamiltonian operator of the Degasperis-Procesi equation.

Proposition 5.5 *The Degasperis-Procesi equation with respect to first Hamiltonian structure of Degasperis-Holm-Hone exactly coincides with $\hat{\mathcal{O}}_0 = \partial$, where the corresponding Hamiltonian $\hat{H}$ satisfies*

$$\frac{\delta\hat{H}}{\delta u} = (2u^2 - u_x^2 - uu_{xx}). \tag{38}$$

Proof: It is easy to check that

$$\frac{\delta\hat{H}}{\delta u} = (4 - \partial^2)\frac{\delta H_0}{\delta u},$$

where the first DHH hamiltonian is given by $H_0 = \frac{1}{6}\int_{S^1} u^3\, dx$.

Thus we obtain

$$m_t = \partial(4 - \partial^2)\frac{\delta H_0}{\delta u}.$$

Using the chain rule formula for variational derivatives

$$\frac{\delta H_0}{\delta u} = (1 - \partial^2)\frac{\delta H_0}{\delta m}$$

we obtain

$$m_t = \partial(4 - \partial^2)(1 - \partial^2)\frac{\delta H_0}{\delta m}.$$

Hence we obtain the first Hamiltonian structure B_0 of Degasperis, Holm and Hone from our method.

□

6 EP formalism for $2+1$-dimensional b-field equation

Consider $\tilde{G}_1 = LG_1$ be the associated loop group corresponding to G_1 whose Lie algebra is given by

$$\tilde{\mathfrak{g}}_1 = L(\mathcal{F}_{-(b-1)}).$$

Consider an action of $L(Vect(S^1))$ on $L(\mathcal{F}_{-(b-1)})$

$$L_{f\,\frac{\partial}{\partial x}}(g\,(dx)^{-(b-1)}) == (f\,g_x - (b-1)f_x\,g)(dx)^{-(b-1)}, \tag{39}$$

this yields a new bracket.

Let us introduce H^1 norm on the algebra $\tilde{\mathfrak{g}}_1$.

Definition 6.1 *The H^1- Sobolev norm on the loop tensor density algebra is defined as*

$$< f(x,y)(dx)^{-(b-1)}, u(x,y)(dx)^b >_{H^1} = \int_{S^1} f u dx + \nu \int_{S^1} \partial_x f \partial_x u dx, \tag{40}$$

Proposition 6.2 *The coadjoint action with respect to H^1 metric of the Lie algebra $\widehat{\mathfrak{g}}$ is given by*

$$\widehat{ad}^*_{(f\frac{d}{dx})}\left(v(x,y)(dx)^b\right) = +\left(f\,\tilde{v}_x + b\,f_x\,\tilde{v}\right)dx^b,$$

where $\tilde{v} = (1 - \nu\,\partial^2)v$.

Corollary 6.3 *The Hamiltonian operator corresponding to the coadjoint action of the cotangent loop Virasoro algebra with respect to H^1 metric is given by*

$$\widehat{\mathcal{O}} \; = \; -(1-\nu\partial_x^2)^{-1}(\partial_x\tilde{v} + (b-1)\tilde{v}\partial_x) \tag{41}$$

Proposition 6.4 *The Euler-Poincaré flow on the $\tilde{\mathfrak{g}}_1$ orbit yields the $2+1$-dimensional b-field equation*

$$v_t - \nu v_{xxt} + \lambda\partial_x^{-1}v_{yy} + (v_x - \nu v_{xxx})\partial_x^{-1}v_y + b(v - \nu v_{xx})v_y = 0 \tag{42}$$

where the Hamiltonian is given by

$$H = \frac{1}{2}\int_{S^1\times S^1} v\partial_x^{-1}v_y\,dxdy.$$

The potential form of (42) yields

$$u_{xt} - \nu u_{xxxt} + \lambda u_{yy} + (u_{xx}u_y + bu_x u_{xy}) - \nu(u_{xxxxy}u_y + bu_{xxx}u_{xy}) = 0. \tag{43}$$

Introducing the quantity

$$m = v - \nu v_{xx},$$

which is just the Helmholtz operator action on v. Therefore the one-parameter family of $2+1$-dimensional peakon-type pde's (or $2+1$-dimensional b-field equations) may be written in the following form

$$m_t + m_x\partial_x^{-1}v_y + bv_y m + \lambda\partial_x^{-1}v_{yy} = 0, \tag{44}$$

which reduces to $1+1$-dimensional b–field equation for $y = x$ and $\lambda = 0$. It is clear that equation (44) becomes $2+1$-dimensional Camassa-Holm and $2+1$ dimensional Degasperis-Procesi equation for $b = 2$ and $b = 3$ respectively.

7 Conclusion and Outlook

We have examined various extensions of $2+1$-dimensional KdV equations and $2+1$-dimensional generalized Camassa-Holm type systems. In particular, we have shown that all these equations constitute geodesic flows on the loop Bott-Virasoro group. In fact three famous $(2+1)$-dimensional partial differential equations: Calogero -Bogoyavlenskii-Schiff (CBS), $2+1$-dimensional Camassa-Holm (CH_2) and $2+1$-dimensional Hunter-Saxton (HS_2) can be described as Euler-Poincaré flows on the dual space of loop Virasoro orbit.

After that we have given the Euler-Poincaré formalism of the new $(1+1)$-dimensional b-field equation, proposed by Degasperis et al, on the space of tensor algebra. We also extend the EP framework to $(2+1)$-dimensional b-field equation, which includes the $(2+1)$-dimensional Degasperis-Procesi equation too. Therefore, this paper has further strengthened the programme of Euler-Poincaré and integrable geodesic flows on extended group of diffeomorphisms.

Acknowledgement

The author is profoundly grateful to Professors Jerry Marsden, Tudor Ratiu and Valentin Ovsienko for stimulating discussions and various constructive suggestions. He is also grateful to Professor Thanasis Fokas for his interest and encouragement. He expresses grateful thanks to Professor Jürgen Jost for gracious hospitality at the Max Planck Insitute for Mathematics in the Sciences.

References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, nonlinear evolution equations and inverse scattering, London Mathematical Society Lecture Note Series, **149**. Cambridge University Press, 1991.

[2] V.I. Arnold, Sur la géométrie differentielle des groupes de Lie de dimenson infinie et ses applications à l'hydrodynamique des fluids parfaits, Ann. Inst. Fourier, Grenoble, vol. 16, 319-361, 1966.

[3] Yu. Billig, An extension of the KdV hierarchy arising from a representation of a toroidal Lie algebra, J. Algebra 217, 40-64 (1999).

[4] O.I. Bogoyavlensky, Breaking solitons in 2 + 1-dimensional integrable equations, Russian Math. Surveys 45,4 1-86 (1990).

[5] O.I. Bogoyavlensky, Breaking solitons III, Izv. Akad. Nauk SSSR Ser. Matem. 54, 123-131 (1990), English transl. in Math. USSR Izv. 36, 129-137 (1991).

[6] F. Calogero, *A method to generate solvable nonlinear evolution equations.* Lett. Nuovo Cimento 14, 443-447 (1975).

[7] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons.* Phys. Rev. Lett. 71 (1993), no. 11, 1661–1664.

[8] R. Camassa, D. Holm and J.M. Hyman, *A New Integrable Shallow Water Equation.* Advances in Applied Mechanics 31 (1994), 1-33.

[9] A. Constantin and B. Kolev, *Integrability of invariant metrics on the Virasoro group.* Phys. Lett. A 350 (2006), no. 1-2, 75–80.

[10] A. Constantin and B. Kolev, *Integrability of invariant metrics on the diffeomorphism group of the circle.* J. Nonlinear Sci. 16 (2006), no. 2, 109–122.

[11] A. Constantin and D. Lannes, *The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations*, Arch. Ration. Mech. Anal. 192 (2009) 165-186.

[12] Ming Chen, Si-Qi Liu and Y. Zhang, *A 2-component generalization of the Camassa–Holm equation and its solutions.* nlin.SI/0501028.

[13] A. Degasperis, D.D. Holm, A.N.W. Hone, *A New Integrable Equation with Peakon Solutions.* NEEDS 2001 Proceedings, Theoret. and Math. Phys. 133 (2002) 170-183.

[14] A. Degasperis, D.D. Holm, A.N.W. Hone, *Integrable and non-integrable equations with peakons*, Nonlinear physics: theory and experiment, II (Gallipoli, 2002), 37–43, World Sci. Publishing, River Edge, NJ, 2003.

[15] A. Degasperis and M. Procesi, *Asymptotic integrability. Symmetry and perturbation theory*, (Rome, 1998), 23–37, World Sci. Publishing, River Edge, NJ, 1999.

[16] C. Devchand and J. Schiff, *The supersymmetric Camassa–Holm equation and geodesic flow on the superconformal group.* J. Math. Phys. 42 (2001), no. 1, 260–273.

[17] D. Ebin and J. Marsden, Groups of diffeomorphisms and themotion of an incompressible fluid. Ann. Math. 92, 102-163, 1970.

[18] G. Falqui, *On a Camassa-Holm type equation with two dependent variables.* nlin.SI/0505059.

[19] A.S. Fokas and B. Fuchssteiner, *Bäcklund transformations for hereditary symmetries*, Nonlinear Anal. 5 (1981), no. 4, 423–432.

[20] A.S. Fokas, P.J. Olver, P. Rosenau, *A plethora of integrable bi-Hamiltonian equations.* Algebraic aspects of integrable systems, 93–101, Progr. Nonlinear Differential Equations Appl., 26, Birkhäuser Boston, Boston, MA, 1997.

[21] A.S. Fokas and P.M. Santini, *Dromions and a Boundary Value Problem for the Davey-Stewartson I Equation*, Physica D 44, 99-130 (1990).

[22] A.S. Fokas and P.M. Santini, *The Recursion Operator of the Kadomtsev-Petviashvili Equation and the Squared Eigenfunction of the Schrödinger Operators*, Stud. Appl. Math. 75, 179-186 (1986)

[23] A.S. Fokas and P.M. Santini, *Recursion Operators and Bi-Hamiltonian Structures in Multidimensions I*, Comm. Math. Phys. 115, 375-419 (1988).

[24] A.S. Fokas and P.M. Santini, *Recursion Operators and Bi-Hamiltonian Structures in Multidimensions II*, Commun. in Math. Phys. 116, 449-474(1988).

[25] I.M. Gelfand, D.B Fuks, *Cohomologies of the Lie algebra of vector fields on the circle*, Func. Anal. Appl. **2**:4 (1968) 92–93.

[26] P. Guha, *Integrable geodesic flows on the (super)extension of the Bott-Virasoro group.* Lett. Math. Phys. 52 (2000), no. 4, 311–328.

[27] P. Guha, *Geodesic flows, bi-Hamiltonian structure and coupled KdV type systems.* J. Math. Anal. Appl. 310 (2005) 45-56.

[28] P. Guha and P. Olver, *Geodesic Flow and Two (Super) Component Analog of the Camassa–Holm Equation*, SIGMA 2 (2006), 054, 9 pages.

[29] P. Guha, *Euler-Poincaré formalism of (two component) Degasperis-Procesi and Holm-Staley type systems*. J. Nonlinear Math. Phys. 14 (2007), no. 3, 390–421.

[30] P. Guha, *Euler-Poincaré Flows on the Space of Tensor Densities and Integrable Systems*, Oberwolfach Report 5, No. 3 (2008) 1875-1880.

[31] I.M. Gelfand, I.M. Graev and A.M. Vershik, *Models of representations of current groups.* Representations of Lie groups and Lie algebras (Budapest, 1971), 121–179, Akad. Kiad, Budapest, 1985.

[32] A. N. W. Hone, *Reciprocal link for* $2+1$*-dimensional extensions of shallow water equations*. Appl. Math. Lett. 13 (2000), no. 3, 37–42.

[33] J.K. Hunter and R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. 51 (1991) 1498-1521.

[34] R.S. Johnson, *Camassa-Holm, Korteweg-de Vries and related models for water waves*, J. Fluid Mech. 455 (2002) 63-82.

[35] A. Kirillov, *Infinite-dimensional Lie groups: their orbits, invariants and representations. The geometry of moments*, Lecture Notes in Math., **970**, 101–123, Springer, Berlin, 1982.

[36] B. Khesin, G. Misiolek, *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math. **176**:1 (2003) 116–144.

[37] B. G. Konopelchenko, *Solitons in Multidimensions.* (Singapore: World Scientific) (1993).

[38] S.-Y. Lou, *Searching for Higher Dimensional Integrable Models from Lower Ones via Painlevé Analysis*, Phys. Rev. Lett. 80 (1998) 5027.

[39] J. Lin,, S.-Y. Lou and K. Wang, *High-dimensional Virasoro integrable models and exact solutions*. Phys. Lett. A 287 (2001), no. 3-4, 257–267.

[40] Y. -S. Li and Y. -J. Zhang, *Symmetries of a* $(2+1)$*-dimensional breaking soliton equation*. J. Phys. A 26 (1993), no. 24, 7487–7494.

[41] G. Misiolek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group.* J. Geom. Phys. 24 (1998) 203-208.

[42] J.E. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*. Springer-Verlag, New York, 1994.

[43] V. Ovsienko and C. Roger, *Generalizations of Virasoro group and Virasoro algebra through extensions by modules of tensor-densities on* S^1 Indag. Math. (N.S.) **9**:2 (1998) 277–288.

[44] V. Ovsienko and C. Roger, *Looped cotangent Virasoro algebra and non-linear integrable systems in dimension* $2+1$ Comm. Math. Phys., 273 (2007) 357-378. math-ph/0602043.

[45] V. Ovsienko, *Coadjoint representation of Virasoro-type Lie algebras and differential operators on tensor-densities*, DMV Sem., 31, Birkhäuser, Basel, 2001, 231–255.

[46] P. Olver and P. Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support.* Phys. Rev. E (3) 53 (1996), no. 2, 1900–1906.

[47] V. Yu. Ovsienko and B.A. Khesin, *KdV super equation as an Euler equation.* Funct. Anal. Appl. 21, 329-331, 1987.

[48] R. Radha and M. Lakshmanan, *Dromion-like structures in the* $(2+1)$*-dimensional breaking soliton equation.* Phys. Lett. A 197 (1995), no. 1, 7–12.

[49] P. Rosenau, *Nonlinear dispersion and compact structures.* Phys. Rev. Lett. 73 (1994), no. 13, 1737–1741.

[50] E. Ramos, C.-H. Sah, R Shrock, *Algebras of diffeomorphisms of the* N*-torus*, J. Math. Phys. **31**:8 (1990), 1805–1816.

[51] A. Reiman, M. Semenov-Tyan-Shanskii, *Hamiltonian structure of equations of Kadomtsev-Petviashvili type*, Differential geometry, Lie groups and mechanics, VI. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 133 (1984), 212–227.

[52] H.-Y. Ruan and Y.-X. Chen, *Dromion interactions of* $(2+1)$*-dimensional KdV-type equations.* J. Phys. Soc. Japan 72 (2003), no. 3, 491–495.

[53] J. Schiff, *Integrability of Chern-Simons-Higgs vortex equations and a reduction of the self-dual Yang-Mills equations to three dimensions*, in Painlevé Transcendents, D. Levi and P. Winternitz (eds.), NATO ASI Series B, vol. 278, Plenum Press, New York (1992)

[54] G. Segal, *Unitary representations of some infinite-dimensional groups*, Comm. Math. Phys. **80**:3 (1981) 301–342.

[55] I.A.B. Strachan, *Some integrable hierarchies in* $(2+1)$ *dimensions and their twistor description*, J. Math. Phys. 34 (1993), no. 1, 243–259.

[56] P. Zusmanovich, *The second homology group of current Lie algebras*, in: K-theory (Strasbourg, 1992), Astérisque No. 226 (1994), 11, 435–452.