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by

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Abstract

We employ generalized Sundman transformation method to obtain certain new first integrals of autonomous second-order ordinary differential equations belonging to the Painlevé-Gambier classification scheme. In particular this method yields systematically both known and unknown first integrals of a large number of the autonomous Painlevé-Gambier equations.

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Contents

1 Introduction

The problem of constructing solutions of a given differential equation forms the cornerstone of their analysis. Not unrelated to this problem, is the issue of determining first integrals of the differential equation under consideration. This is because the existence of a sufficient number of first integrals often enables us to construct a solution by mere elimination of the derivatives of the dependent variable. Again the existence of one or more first integrals allows for a reduction of the order of the differential equations. It allows us, at least, to reduce the order of the differential equation.

In the case of linear ordinary differential equations, we have a number of well defined methods for their solution. The same however, cannot be said for nonlinear ordinary differential equation. In fact in the early days, the majority of the methods employed for solving differential equations were by and large *ad hoc* in nature. It was only through the efforts of S. Lie, towards the end of the nineteenth century that many such *ad hoc* methods were gradually systemized. Besides, it is generally acknowledged that whenever a differential equation is amenable to a solution, it is because of some sort of underlying symmetry of the equation. Much of Lie's work was concerned with point transformations of the form

$$(t, x) \mapsto (T, X) \;\text{ where }\; T = G(t, x), X = F(t, x),$$

with the transformation often involving one or more continuous real parameters.

Furthermore, towards the very end of the nineteenth century, the fact that a given differential equation could be transformed to a linear equation, that is, it could be *linearized* came to light. This provided a mechanism to work out the solutions of many nonlinear differential equations, by systematically transforming them to linear equations, whose solutions may be worked out relatively easily and then inverting the latter to arrive at a solution of the original nonlinear ordinary differential equation. There is no doubt that this is an attractive proposition. Indeed, Lie himself [16] solved the linearization problem for second-order ordinary differential equations, in the sense that he found the general form of all second-order ordinary differential equations, that could be reduce to a linear equation by changing the independent and dependent variables [16]. He showed that if a second-order ordinary differential equation is linearizable, then it should be at most cubic in the first-order derivatives and in addition provided a linearization test in terms of its coefficients. Tresse [24] also worked on the same problem and wrote the linearization criterion in terms of relative invariants of the equivalence group of point transformation. Cartan, on the other hand studied the same problem from a differential geometric point of view.

A point transformation is generally preferred for the simple reason that they are easier to work with. Moreover, under point transformation the Lie symmetries are preserved. It is well known that all second-order linear ordinary differential equations possess eight Lie point symmetries, which far exceed the number, namely two, required to arrive at a quadrature. Consequently it is of no harm to look beyond point transformations when dealing with second-order ODEs.

One such generalization consists in looking for nonlocal transformations, under which a given ordinary differential equation is linearizable. This problem was studied by Duarté *et al* [7] and considers transformations of the form

$$X(T) = F(t, x), \quad dT = G(t, x)dt. \tag{1.1}$$

Here F and G are arbitrary smooth function and assume that the Jacobian $J \equiv \frac{\partial(T,X)}{\partial(t,x)} \neq 0$. If one knows the functional form of $x(t)$ then the latter transformation ceases to be nonlocal. But knowledge of $x(t)$ is what we are interested in, in the first place. Consequently (1.1) constitutes a particular type of nonlocal transformation. It must be pointed out that term nonlocal is very general in nature and it is therefore better to refer to such a transformation as a generalized Sundman transformation (GST), in view of its similarity with the original transformation used in Sundman's analysis. Linearization method for the equation of motion play important roles in Celestial Mechanics. One of the pioneering contributor's to this field was K.F. Sundman [22] who introduced the transformation $dt = rd\tau$ in the study of the 3-body problem, where r is dependent variable (radial component). About a quarter century ago Sundman method was revitalized by Szebehely and Bond [23], who considered transformation of the dependent variable too $r = F(\rho)$. Nowadays, theoretical importance of the generalized Sundman transformations pop up in various places of mechanics and dynamical systems. In particular all these Sundman type transformations are especially effective for the solution of several nonlinear ODEs. Many authors (for example, [5]) have studied Sundman transformation under the name of non-point transformations.

It is interesting to note that, as a rule the GST preserve the time independent first integrals. However, for some differential equations the time dependent first integrals are also preserved.

In [7] the authors derived the most general condition under which a second-order ordinary differential equation is transformable to the linearized equation

$$X''(T) = 0,$$

(here $X' = \frac{dX}{dT}$) under a generalized Sundman transformation. Then using the fundamental invariants of this equation they obtained the first integrals of second-order ordinary differential equations, which could be linearized. The case of the general anharmonic oscillator was studied by Euler and Euler in [8]. In this interesting paper Euler and Euler investigated the Sundman symmetries of second-order and third-order nonlinear ODEs. These symmetries, which are in general nonlocal transformations can be calculated systematically and can be used to find first integrals of the equations. Euler *et al* [9], used the generalized Sundman transformation to obtain a relation between a generalized Emden-Fowler equation and the first Painlevé transcendent.

Of late there have been a number of papers concerned with linearization of third-order ODEs by using the generalized Sundman transformations as also by other methods [3, 20]. Unlike second order equation, the third order nonlinear ODEs can be linearized through a wide class of transformations, Viz., invertible point transformation [13], contact transformation [4, 13] and other methods [6]. In this paper, we will concentrate on generalized Sundman transformations. In a recent survey [10] the results from literature under what conditions a nonlinear ordinary differential of third order can be linearized into the form $d^3x/dt^3 = 0$ are summarized using invertible point transformations.

In this paper we shall confine our attention to the study of second-order ordinary differential equations of the Painlevé-Gambier classification. As is well known these equations arose in studies made by Painlevé and Gambier of second-order ODEs which did not have movable critical points. More specifically we shall examine whether there exists first integrals of the equations of the Painlevé-Gambier classification which are not mentioned in the classic text by Ince [14]. The method to be used for this purpose will be the generalized Sundman transformation.

1.1 Result and plan of the paper

In this paper we compute new first integrals of some of the autonomous Painlevé-Gambier equations. As is well known by a first integral we mean the following. Let $I(\mathbf{x}, t)$ be a C^1 function on an open set $U \in \mathbb{R}^n$. Then $I(\mathbf{x}, t)$ is a first integral of the vector field $\mathbf{f}.\partial_{\mathbf{x}}$ corresponding to the system of ODEs $\dot{\mathbf{x}} = f(\mathbf{x})$ if and only if it is constant along any solution of the equation. This means that given a time interval T, $I(\mathbf{x}(t), t)$ is independent of t for all $t \in T$.

In summary, we have obtained the known and eight new first integrals of Painlevé-Gambier equations bearing numbers 11, 17, 37, 41, 43, 18, 21 and 22 (see [14]). All the known integrals are time-independent and are given in Ince's book. The time-dependent integrals appear to be new to the literature. We have also computed the Sundman symmetries of these equations as well as their solutions. Our main results are displayed in the following table.

Painlevé-Gambier equation	time-dependent first integrals	time-independent
$\ddot{x} - \frac{1}{x}\dot{x}^2 = 0$	$\frac{t}{x}\dot{x} - ln\, x$	$\frac{\dot{x}}{x}$
$\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0$	$tx^{\frac{1-m}{m}}\dot{x} - mx^{\frac{1}{m}}$	$x^{\frac{1-m}{m}}\dot{x}$
$\ddot{x} - \left(\frac{1}{2x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t}{x^{1/2}(x-1)}\dot{x} - ln\,\frac{x^{1/2}-1}{x^{1/2}+1}$	$-\frac{1}{x^{1/2}(x-1)}\dot{x}$
$\ddot{x} - \frac{2}{3}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}} + 3(-1)^{1/3}x^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x)$	$\frac{\dot{x}}{x^{2/3}(x-1)^{2/3}}$
$\ddot{x} - \frac{3}{4}\left(\frac{1}{x} + \frac{1}{x-1}\right)\dot{x}^2 = 0$	$\frac{t\dot{x}}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}} + 4x^{1/4}(-1)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x)$	$\frac{\dot{x}}{x^{3/4}(x-1)^{3/4}}$
$\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0$	$\frac{1}{2i}\left(\frac{\dot{x}}{2x^{1/2}} + x\right)e^{-\int 2x^{1/2}dt}$	$\frac{\dot{x}^2}{4x} - x^2$
$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0$	$\frac{1}{2i}\left(\frac{\dot{x}}{x^{\frac{3}{4}}} + 2x^{\frac{3}{4}}\right)e^{-\frac{3}{4}\int x^{1/2}dt}$	$\frac{\dot{x}^2}{x^{3/2}} - 4x^{3/2}$
$\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0$	$\frac{1}{2i}\left(\frac{\dot{x}}{x^{\frac{3}{4}}} + 2x^{\frac{-1}{4}}\right)e^{\frac{1}{2}\int x^{-1/2}dt}$	$\frac{\dot{x}^2}{x^{3/2}} - \frac{4}{x^{1/2}}$

The organization of the paper is as follows. In section 2 we introduce the notion of a generalized Sundman transformation and the define the associated Sundman symmetry. Section 3 begins with a general discussion of the generalized Sundman transformation for the Jacobi equation and proceeds to outline the format for their explicit evaluation. It then examines, as a special case of the Jacobi equation, particular equations of the Painlevé-Gambier classification, notably the equations numbered 11, 17, 37, 41 and 43 of Ince's book, from the viewpoints of the generalized Sundman transformation, the associated Sundman symmetry including also their solution. In addition some new first integrals of these equations are also deduced. In section 4, three more equations of the Painlevé-Gambier classification (namely the equations numbered 18, 21 and 22) are similarly analyzed and these are also seen to arise as special cases of the Jacobi equation.

2 Generalized Sundman transformation

Let us consider an nth-order ordinary differential equation given by

$$x^{(n)} = w(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}) \tag{2.1}$$

where $x = x(t)$ and $x^{(k)} = \frac{d^k x}{dt^k}$. Formally we define a generalized Sundman transformation for (2.1) as follows.

Definition 2.1 (Sundman transformation) *A coordinate transformation of the form*

$$X(T) = F(t, x), \quad dT = G(t, x)dt, \quad \frac{\partial F}{\partial x} \neq 0, \quad G \neq 0 \tag{2.2}$$

is said to be a generalized Sundman transformation of the equation (2.1), if differentiable functions F and G are determined such that (2.1) is transformed to the autonomous equation

$$X^{(n)} = w_0(X, X', \ldots, X^{(n-1)}) \tag{2.3}$$

where $X' = \frac{dX}{dT}$ etc.

This notion of generalized Sundman transformation, as a kind of nonlocal extension of invertible point transformation was made by Duarte *et al.* Its nonlocal character is apparent from the fact that $T = \int G(t, x(t))\, dt$. However, as nonlocal transformations have a wide range of meanings, depending on the context in which they arise, and are certainly more general then point transformation [2, 21] we prefer to refer to the above type of transformation as a generalized Sundman transformation. The name being borrowed owing to the similarity they bear with the original transformation, due to Sundman [22] in respect of the nonlocal character of the independent (temporal) variable.

If (2.3) happens to be a linear ordinary differential equation, then we say that the original ordinary differential equation (2.1), is linearizable. In the event $w_0 = 0$ one says that the equation (2.1) has been mapped to the free particle equation.

Closely related to the concept of a generalized Sundman transformation is the notion of an associated Sundman symmetry. This is similar in spirit to the existence of a Lie symmetry under point transformations.
Let us suppose we have a generalized Sundman transformation (GST)

$$X(T) = F(\tilde{t}, \tilde{x}), \;\; dT = G(\tilde{t}, \tilde{x})d\tilde{t}$$

which maps the equation

$$\tilde{x}^{(n)} = w(\tilde{t}, \tilde{x}, \dot{\tilde{x}}, \ldots, \tilde{x}^{(n-1)}) \longmapsto X^{(n)} = w_0(X, X', \ldots X^{(n-1)})$$

If there exists a transformation of the differentiable function $F(\tilde{t}, \tilde{x})$ and $G(\tilde{t}, \tilde{x})$ considered as functions of $F(t, x)$ and $G(t, x)$, such that our original differential equation (2.1) remains invariant, under the transformation then the transformation defines a Sundman symmetry. Formally it may be defined as follows.

Definition 2.2 (Sundman symmetry) *A Sundman symmetry [8] for equation (2.1) is a transformation of the form*

$$F(\tilde{t},\tilde{x}) = M(F(t,x),G(x,t)), \qquad G(\tilde{t},\tilde{x})d\tilde{t} = N(F(t,x),G(t,x))dt \tag{2.4}$$

where M and N are some differentiable functions such that the transformation keeps (2.1) invariant. In other words (2.1) is transformed to

$$\tilde{x}^{(n)} = w(t;\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\ldots,\tilde{x}^{(n-1)}). \tag{2.5}$$

If $M(F,G) = F$ and $N(F,G) = G$ then of course, the symmetry is trivial. The set of conditions on the differentiable functions F and G when the differential equation (2.1) is mapped to the autonomous differential equation (2.3)are referred to as *the Sundman determining equations*, this will be illustrated later.

A Sundman symmetry (2.4) is obtained by choosing M and N in such a way that the Sundman determining equations remains invariant. If

$$X = F(\tilde{t},\tilde{x}), \quad dT = G(\tilde{t},\tilde{x})d\tilde{t}$$

transforms (2.5) to (2.3) and

$$X = M(F(t,x),G(t,x)), \quad dT = N(F(t,x),G(t,x))dt$$

also transforms (2.1) to (2.3), then the composition of these two GST leads to the Sundman symmetry (2.4) for (2.1).

3 GST for Jacobi's equation

We begin this section by considering the well known Jacobi equation. The reason for this is that, many of the second-order equations of the Painlevé-Gambier classification may then be regarded as special cases of this rather general equation, as we shall see below.

We start with the Jacobi equation, devised by Jacobi, given by [15, 17]

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + \phi_t\dot{x} + B(t,x) = 0. \tag{3.1}$$

This equation may be transformed to $X'' = 0$ under the transformation (2.4) when its coefficients satisfy the Sundman determining equation, given by the following relations:

$$\frac{1}{2}\phi_x(F,G;t,x) = \frac{F_{xx}}{F_x} - \frac{G_x}{G} \tag{3.2}$$

$$\phi_t(F,G;t,x) = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G} \tag{3.3}$$

$$B(F,G;t,x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x}. \tag{3.4}$$

Further it admits a Sundman symmetry of the form (2.4) if and only if M and N are given by

$$M(F,G) = M(F(t,x)) \;\text{ and }\; N(F,G) = G(t,x)\psi(F). \tag{3.5}$$

The Sundman symmetry of (3.1) is of the form

$$F(\tilde{x},\tilde{t}) = M(F(x,t)), \tag{3.6}$$

$$G(\tilde{t},\tilde{x}) = G(t,x)\frac{dM(F(t,x))}{dF}dt \tag{3.7}$$

with no further condition on the differentiable function M.
This follows from the following observation. Suppose for the sake of notational convenience we denote

$$F(\tilde{t},\tilde{x}) = \hat{F} \;\text{ and }\; G(\tilde{t},\tilde{x}) = \hat{G}.$$

The invariance of the Sundman determining equations requires each expression occurring in (3.2)-(3.4) to be invariant. From (3.4) we observe, making use of (3.5)

$$\frac{\hat{F}_{tt}}{\hat{F}_x} - \frac{\hat{G}_t}{\hat{G}}\frac{\hat{F}_t}{\hat{F}_x} = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + \left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right)\frac{F_t^2}{F_x}.$$

The left hand side is clearly an invariant, provided

$$\left(\frac{M''(F)}{M'(F)} - \frac{\psi'(F)}{\psi(F)}\right) = 0 \tag{3.8}$$

which in turn implies

$$\psi(F) = \frac{dM}{dF}, \tag{3.9}$$

where we have chosen the integration constant to be unity. It may be verified that (3.2) and (3.3) are also invariant under (3.5) provided the condition (3.9) holds, i.e.,

$$\frac{\hat{F}_{xx}}{\hat{F}_x} - \frac{\hat{G}_x}{\hat{G}} = \frac{F_{xx}}{F_x} - \frac{G_x}{G}$$

$$\frac{2\hat{F}_{xt}}{\hat{F}_x} - \frac{\hat{F}_t}{\hat{F}_x}\frac{\hat{G}_x}{\hat{G}} - \frac{\hat{G}_t}{\hat{G}} = \frac{2F_{xt}}{F_x} - \frac{F_t}{F_x}\frac{G_x}{G} - \frac{G_t}{G}.$$

It is interesting to note that many of the second-order ordinary differential equations of the Painlevé-Gambier classification appear as special cases of the Jacobi equation. In the following section we consider some of these ODEs.

3.1 Some special cases of the Jacobi equation

In this subsection we proceed to examine certain particular cases of the Jacobi equation and explicitly derive the forms of the functions F and G. We present this method algorithmically.

Step I: Writing G in term of F We assume $B(t,x)=0$ and ϕ_t=0. Since $B(t,x)=0$, from (3.4) we can set

$$G = a(x)F_t \tag{3.10}$$

where a is an arbitrary function of x.

Step II: Expressing F and its derivatives in terms of coefficients Again since ϕ_t=0, from (3.3)and using (3.10) we have

$$\frac{F_{xt}}{F_x} - \frac{a_x(x)}{a(x)}\frac{F_t}{F_x} - \frac{F_{tt}}{F_t} = 0$$

i.e.,

$$\frac{\partial}{\partial t}(\frac{F_x}{F_t}) = \frac{a_x(x)}{a(x)}.$$

Integrating this with respect to t we have

$$\frac{F_x}{F_t} = \frac{a_x}{a}t + b(x), \tag{3.11}$$

where b is an arbitrary function of x. Finally from (3.2) we get

$$\frac{F_x}{G} = c(t)e^{\frac{\phi}{2}} = c(t)K(x), \tag{3.12}$$

where c is an arbitrary function of t and

$$e^{\phi/2} = K(x). \tag{3.13}$$

Since $\phi_t = 0$, hence r.h.s. is independent of t.

Step III: Equations and solutions of coefficients Using (3.10)and (3.11) one can show that (3.12) can be reduced to

$$\frac{a_x}{a^2}t + \frac{b(x)}{a} = c(t)K(x). \tag{3.14}$$

There are two possibilities (a) $c(t)=c_0$ (constant), in this case a is also constant; (b) $c(t)=t$. The second case is more interesting. Equating coefficient of t from (3.14) leads to

$$\frac{a_x}{a^2} = K(x), \tag{3.15}$$

which implies

$$a(x) = -\frac{1}{K_1(x) + f} \tag{3.16}$$

where

$$K_1(x) = \int K(x)dx \tag{3.17}$$

and f is an arbitrary constant. Assuming $f = 0$ one finds

$$a(x) = -\frac{1}{K_1(x)}. \tag{3.18}$$

Step IV: Finding F and G using solutions of coefficients Using (3.18) in (3.11) and with $b(x) = 0$ we find that

$$\frac{F_x}{F_t} = -\frac{K(x)}{K_1(x)}t$$

or

$$\frac{K_1(x)}{K(x)}F_x + tF_t = 0. \tag{3.19}$$

By using the method of characteristics we obtain the general solution of $F(t, x)$ in the form

$$F(t, x) = J\left(\frac{K_1(x)}{t}\right), \tag{3.20}$$

where $J(\lambda)$ is any arbitrary function of the characteristic coordinate $\lambda = \frac{K_1(x)}{t}$. Hence from (3.10) using (3.18) and F as given by (3.20) we easily find that

$$G(t, x) = \frac{1}{t^2}J'(\lambda). \tag{3.21}$$

It is interesting to note that when $J(\lambda) = \lambda$ then the nonlocal character of the transformation vanishes, for we have

$$X = F(t, x) = \frac{K_1(x)}{t} \text{ and } G(t, x) = \frac{1}{t^2} \text{ so that } dT = \frac{1}{t^2}dt \text{ leading to } T = -\frac{1}{t}. \tag{3.22}$$

Step V: Finding first integrals from F and G As the standard first integrals of the linear ODE $X'' = 0$ are

$$I_1 = X' = \frac{dX}{dT} \text{ and } I_2 = X - TX'$$

respectively, we obtain as a result of the GST, these in the following form:

$$I_1 = \frac{F_x}{G}\dot{x} + \frac{F_t}{G} = tK(x)\dot{x} - K_1(x) \tag{3.23}$$

and

$$I_2 = X - TX' = F(t,x) - (tK(x)\dot{x} - K_1(x)) \int G(t,x)dt. \tag{3.24}$$

In particular when F and G are given by (3.22), then I_2 assumes the following simple form

$$I_2 = \dot{x}K(x). \tag{3.25}$$

It is important to note that, in the following examples we shall repeatedly use this expression, in order to compare the results of our calculations with the known time independent first integrals given in Ince's book [14].

3.2 Examples from Painlevé-Gambier class of equations

The Painlevé transcendents are defined by second-order ordinary differential equations whose singularities have the Painlevé property, i.e., the only movable singularities are poles. This property is shared by all linear ordinary differential equations but it is extremely rare in non-linear equations. One of the remarkable properties of Painlevé equations is that while they introduce new special functions, new 'transcendents', they also possess some solutions that can be expressed in terms of elementary functions [12]. These solutions fall into two classes – (a) solutions which are rational in the independent variable and (b) solutions which are expressed in terms of the classical special functions. Since the latter are the solutions of linear equations, this second kind of solutions is referred to as the 'linearizable' case, obviously these exist only for special values of parameters.

In this subsection we intend to focus on equations which do not belong to the six Painlevé transcendent equations. In particular we shall study equations related to the Painlevé-Gambier classification. Painlevé, Gambier and their pupils found *fifty* second-order ODEs possessing canonical form whose solutions do not have any movable critical singularities i.e., they possess the Painlevé property. In particular, using the generalized Sundman transformations we have obtained certain new first integrals related to the equations 11, 17, 37, 41 and 43 of the Painlevé-Gambier classification, as given in Ince's classic text [14]. The results are presented below.

3.2.1 Painlevé-Gambier equation XI

The first system we are going to examine is equation number 11 in the Painlevé-Gambier classification:

$$\ddot{x} - \frac{1}{x}\dot{x}^2 = 0 \tag{3.26}$$

Comparison with the Jacobi equation (3.1) reveals that

$$\frac{1}{2}\phi_x = -\frac{1}{x}, \quad \phi_t = 0 \;\text{ and }\; B(t,x) = 0. \tag{3.27}$$

Hence from (3.13) we have

$$K(x) = e^{\frac{\phi}{2}} = \frac{1}{x} \tag{3.28}$$

and from (3.17)

$$K_1(x) = \ln x \tag{3.29}$$

Therefore making use of (3.22) we have

$$F = \frac{\ln x}{t} \quad \text{and} \quad G(t,x) = \frac{1}{t^2} \tag{3.30}$$

while from (3.23) and (3.24) the first integrals for this equation are

$$I_1 = \frac{t}{x}\dot{x} - \ln x \tag{3.31}$$

and

$$I_2 = \frac{\dot{x}}{x}. \tag{3.32}$$

Notice that whereas the time independent first integral I_2 is mentioned in [14] the remaining first integral I_1 is time dependent and is not stated therein.

3.2.2 The Sundman symmetry for $\ddot{x} - \frac{1}{x}\dot{x}^2 = 0$

To deduce the Sundman symmetry for this equation, it is convenient to assume that $J(\lambda) = \lambda^2$,in the rest of this subsection, so that from (3.20) we have

$$F(t,x) = \left(\frac{K_1(x)}{t}\right)^2 = \left(\frac{\ln x}{t}\right)^2. \tag{3.33}$$

Now the Sundman symmetry of (3.26) being of the form (3.6), we assume that

$$\hat{F} = F(\tilde{t}, \tilde{x}) = M(F(t,x)).$$

Consequently with F given, as in (3.33) one finds that

$$\tilde{x} = \exp\left(\tilde{t}\sqrt{M(F)}\right). \tag{3.34}$$

On the other hand from (3.7), using (3.21) to calculate G which now is given by $G(t,x) = \frac{2\ln x}{t^3}$ we have

$$\hat{G}d\tilde{t} = G\frac{dM(F)}{dF}dt \;\; \text{implies} \;\; \frac{\ln\tilde{x}}{\tilde{t}^3}d\tilde{t} = \frac{\ln x}{t^3}\frac{dM}{dF}dt.$$

Upon using (3.34) to eliminate $\tilde{x}$ from the l.h.s, of the above expression, we obtain the following transformation for the time variable:

$$\tilde{t} = -\left[c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{dM(F)}{dF}dt\right]^{-1}. \tag{3.35}$$

Here c is a constant of integration. Substituting this expression in (3.34) we get the transformation for the spatial variable, *viz*

$$\tilde{x} = \exp\left(-\frac{\sqrt{M(F)}}{c + \int \frac{\ln x}{t^3\sqrt{M(F)}}\frac{dM(F)}{dF}dt}\right). \tag{3.36}$$

Here $M(F)$ is an arbitrary function of F and c is a constant of integration. Equations (3.35 and (3.36) constitute a Sundman symmetry for the Painlevé-Gambier XI equation.

3.2.3 Solving the Painlevé-Gambier XI equation

In this subsection we shall illustrate how, making use of Sundman symmetries, one can easily construct a solution of the differential equation under consideration. Since the nonlocal transformation maps the ODE to the free particle equation, namely $X'' = 0$, we make use of the fact that $X(T) = aT$, where a is a constant, is a solution of this differential equation. Next we use the fact that, by definition $X = F(t, x)$, consequently using the results of the previous subsection we may write

$$X = \left(\frac{\ln x}{t}\right)^2. \tag{3.37}$$

It follows that

$$x = \exp(t\sqrt{X}). \tag{3.38}$$

On the other hand using

$$G(t, x) = \frac{2\ln x}{t^3}$$

(from the previous section) and the definition $dT = Gdt$, one gets

$$dT = \frac{2\ln x}{t^3}dt = \frac{2t\sqrt{X}}{t^3}dt = \frac{2\sqrt{aT}}{t^2}dt$$

whence the variables may be separated (use having been made of the solution $X = aT$), to obtain

$$T = \frac{1}{4}\left[b - \frac{2\sqrt{a}}{t}\right]^2. \tag{3.39}$$

From $X = aT$ by using this expression for T and (3.37) we get finally the solution of the ODE in the form

$$x = \exp\left(\pm\frac{\sqrt{a}t}{2}\left(b - \frac{2\sqrt{a}}{t}\right)\right) = \exp\left(\pm(\alpha t + \beta)\right). \tag{3.40}$$

In other words $x(t) = Ae^{Bt}$ where A and B are arbitrary constants.
Of course, one could also have obtained this solution simply by eliminating $\dot{x}$ from the two first integrals I_1 and I_2 by mere inspection.
The procedure described in the preceding three subsections in some detail, may also be applied, with little difficulty to several other equations of the Painlevé-Gambier classification. Therefore we shall state only the final results for the subsequent examples.

3.2.4 Painlevé-Gambier equation XVII

The second equation we are going to examine is

$$\ddot{x} - \frac{m-1}{mx}\dot{x}^2 = 0. \tag{3.41}$$

Here

$$\frac{1}{2}\phi_x = -\frac{m-1}{mx} \quad \phi_t = 0 \text{ and } B(t,x) = 0. \tag{3.42}$$

As in the preceding case we find here

$$K(x) = e^{\frac{\phi}{2}} = x^{-\frac{m-1}{m}} \text{ and } K_1(x) = mx^{\frac{1}{m}}. \tag{3.43}$$

The first integrals for this equation are then found to be

$$I_1 = tx^{\frac{1-m}{m}}\dot{x} - mx^{\frac{1}{m}} \tag{3.44}$$

and

$$I_2 = x^{\frac{1-m}{m}}\dot{x}, \tag{3.45}$$

where I_2 is a time independent first integral cited in [14] but I_1 is a new time dependent first integral.

Regarding the Sundman symmetry of this equation, by making the same assumptions as in case of Painlevé-XI, we arrive at the following expression for the symmetry:

$$\tilde{x} = \left(\frac{\tilde{t}\sqrt{M(F)}}{m}\right)^m \text{ and } \tilde{t} = -\left[c + m\int \frac{x^{1/m}}{t^3\sqrt{M}}\frac{dM}{dF}dt\right]^{-1}. \tag{3.46}$$

3.2.5 Painlevé-Gambier equation XXXVII

The third equation in the Painlevé-Gambier classification we are going to study is

$$\ddot{x} - \left\{\frac{1}{2x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0. \tag{3.47}$$

Here

$$\frac{1}{2}\phi_x = -\left\{\frac{1}{2x} + \frac{1}{x-1}\right\} \tag{3.48}$$

So

$$K(x) = e^{\frac{\phi}{2}} = \frac{1}{x^{\frac{1}{2}}(x-1)}; \qquad K_1(x) = \ln\frac{x^{1/2}-1}{x^{1/2}+1} \tag{3.49}$$

As before we find, taking $J(\lambda) = \lambda$ for simplicity that

$$F(t,x) = \frac{\frac{x^{1/2}-1}{x^{1/2}+1}}{t}, \qquad G(t,x) = \frac{1}{t^2} \tag{3.50}$$

The associated first integrals for this equation are

$$I_1 = \frac{t}{x^{1/2}(x-1)}\dot{x} - \ln\frac{x^{1/2}-1}{x^{1/2}+1} \tag{3.51}$$

and

$$I_2 = -\frac{1}{x^{1/2}(x-1)}\dot{x} \tag{3.52}$$

While I_2 is a time independent first integrals and is given in [14], I_1 is a new time dependent first integral.
The Sundman symmetry for this case (assuming as before $J(\lambda) = \lambda^2$) for convenience)is given by the following:

$$\tilde{x} = \frac{1+e^{\tilde{t}\sqrt{M}}}{1-e^{\tilde{t}\sqrt{M}}} \quad \text{and} \quad \tilde{t} = -\left[c + \int \ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)\frac{1}{\sqrt{M}}\frac{dM}{dF}dt\right]^{-1}. \tag{3.53}$$

3.2.6 Painlevé-Gambier equation XLI

The equation is

$$\ddot{x} - \frac{2}{3}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}\dot{x}^2 = 0. \tag{3.54}$$

Here

$$\frac{1}{2}\phi_x = -\frac{2}{3}\left\{\frac{1}{x} + \frac{1}{x-1}\right\}. \tag{3.55}$$

Consequently we have

$$K(x) = e^{\frac{\phi}{2}} = \frac{1}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}} \tag{3.56}$$

$$K_1(x) = -3x^{1/3}(-1)^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x), \tag{3.57}$$

where ${}_2F_1(a,b;c;x)$ denotes the standard hypergeometric function. It follows that $F(t,x) = -\frac{3}{t}(-x)^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x)$ and $G = \frac{1}{t^2}$ as before. The time independent first integral, which matches that given in Ince's book is

$$I_2 = \frac{\dot{x}}{x^{2/3}(x-1)^{2/3}}, \tag{3.58}$$

while the time dependent one is given by

$$I_1 = \frac{t\dot{x}}{x^{\frac{2}{3}}(x-1)^{\frac{2}{3}}} + 3(-1)^{1/3}x^{1/3}\ {}_2F_1(1/3, 2/3; 4/3; x). \tag{3.59}$$

3.2.7 Painlevé-Gambier equation XLIII

The equation is

$$\ddot{x} - \frac{3}{4}\Big\{\frac{1}{x} + \frac{1}{x-1}\Big\}\dot{x}^2 = 0. \tag{3.60}$$

Here

$$\frac{1}{2}\phi_x = -\frac{3}{4}\Big\{\frac{1}{x} + \frac{1}{x-1}\Big\} \tag{3.61}$$

So that

$$K(x) = e^{\frac{\phi}{2}} = \frac{1}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}} \tag{3.62}$$

and

$$K_1(x) = -4x^{1/4}(-1)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x). \tag{3.63}$$

Assuming $F = \frac{K_1(x)}{t}$, for the sake of simplicity, the latter may be expressed in terms of the above hypergeometric function and with $G = \frac{1}{t^2}$, the time independent first integral is given by

$$I_2 = \frac{\dot{x}}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}}. \tag{3.64}$$

On the other hand the time dependent first integral has the following form:

$$I_1 = \frac{t\dot{x}}{x^{\frac{3}{4}}(x-1)^{\frac{3}{4}}} + 4x^{1/4}(-1)^{1/4}\ {}_2F_1(3/4, 1/4; 5/4; x). \tag{3.65}$$

4 Further special cases of the Jacobi equation

In this section we shall consider further cases of the Jacobi equation, in particular when $\phi_t = 0 = B_t$ but $B(x) \neq 0$.
As before, there are a number of equations of the Painlevé-Gambier classification, which belong to this category. Our prototype equation for this section has the generic form

$$\ddot{x} + \frac{1}{2}\phi_x\dot{x}^2 + B(x) = 0. \tag{4.1}$$

The objective being to construct a generalized Sundman transformation (2.2) (GST) such that (4.1) is mapped to the following equation

$$X'' + a_0(X) = 0, \tag{4.2}$$

where $X' = \frac{dX}{dT}$. The exact form of $a_0(X)$ will be specified later. This will be true, if the following conditions (i.e., the Sundman determining equations) on the coefficients of (4.1) hold good:

$$\frac{1}{2}\phi_x = \frac{F_{xx}}{F_x} - \frac{G_x}{G} \tag{4.3}$$

$$0 = 2\frac{F_{xt}}{F_x} - \frac{G_x}{G}\frac{F_t}{F_x} - \frac{G_t}{G} \tag{4.4}$$

$$B(x) = \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + a_0(F)\frac{G^2}{F_x} \tag{4.5}$$

from (4.3) we have

$$\ln F_x - \ln G = \int \frac{1}{2}\phi_x dx - \ln b(t).$$

Here $b(t)$ is an arbitrary constant of integration. It follows that

$$G(t,x) = b(t)e^{-\phi/2}F_x. \tag{4.6}$$

Substituting G from (4.6) to (4.5) we have

$$\frac{F_{tt}}{F_x} - \frac{F_{xt}F_t}{F_x^2} - \frac{\dot{b(t)}}{b(t)}\frac{F_t}{F_x} + a_0(F)b(t)^2 e^{-\phi}F_x = B(x). \tag{4.7}$$

If we set $b(t) = \beta$, i.e., a constant independent of t and assume

$$\frac{\partial}{\partial t}\Big(\frac{F_t}{F_x}\Big) = 0 \tag{4.8}$$

then (4.7) implies

$$a_0(F)\beta^2 e^{-\phi}F_x = B(x). \tag{4.9}$$

Instead of trying to determine the form of F first, it is more convenient to stipulate $a_0(F)$ and see whether with such a choice of $a_0(F)$ we can satisfy the remaining equations. To this end we choose

$$a_0(F) = \pm F. \tag{4.10}$$

Then (4.9) yields

$$F^2 = \pm\frac{2}{\beta^2}\int B(x)e^{\phi}dx. \tag{4.11}$$

Thus F is a function of x only and as a result it is obvious that (4.8) is trivially satisfied. It remains to verify whether such an expression for F is consistent with (4.4). Since $b(t) = \beta$ is a constant, we have from (4.6),

$$G(t,x) = \beta e^{-\phi/2}F_x = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}, \tag{4.12}$$

which is clearly independent of t and hence $G_t = 0$. Consequently since F and G are only functions of x it follows that (4.4) is clearly satisfied. In summary we therefore have the following form of the GST mapping (4.1) to the equation $X'' \pm X = 0$, *viz*

$$X = F(x) = \left(\pm\frac{2}{\beta^2}\int B(x)e^{\phi(x)}dx\right)^{1/2}, \quad dT = \frac{B(x)e^{\phi/2}}{\left(\pm 2\int B(x)e^{\phi}dx\right)^{1/2}}dt. \tag{4.13}$$

The latter being obviously a nonlocal transformation.

4.1 The Sundman symmetry

The Sundman symmetry associated with (4.1) is not difficult to deduce. As before, for notational convenience we shall denote

$$\hat{F} = F(\tilde{t}, \tilde{x}) \quad \text{and} \quad \hat{G} = G(\tilde{t}, \tilde{x}).$$

To ensure invariance of the Sundman determining equations, namely (4.3)-(4.5) we shall assume

$$\hat{F} = M(F) \ \text{ and } \ \hat{G} = G(t, x)\psi(F). \tag{4.14}$$

The functional forms of M and ψ will be determined by demanding invariance of the Sundman determining equations. Invariance of (4.3) leads to

$$\psi(F) = K\frac{dM(F)}{dF},$$

where K is a constant of integration, which may be set to unity, so that

$$\psi(F) = M'(F). \tag{4.15}$$

Invariance of (4.5) then leads to the equation

$$\frac{dM}{dF} = \frac{a_0(F)}{a_0(M)}$$

whence it follows, with $a_0(F) = \pm F$, that

$$M = \pm\sqrt{F^2 + c}, \tag{4.16}$$

where c is a constant of integration. Note that if $c = 0$ then we get a trivial symmetry. The functional form of ψ is therefore given by

$$\psi(F) = \pm\frac{F}{\sqrt{F^2 + c}}. \tag{4.17}$$

With M and ψ given by (4.16) and (4.17) respectively, one can easily verify that the final Sundman determining equation, namely (4.4) is identically satisfied. Thus in summary we have the following Sundman symmetry for (4.1)

$$F(\tilde{t}, \tilde{x}) = \pm\sqrt{F^2(t, x) + c} \ \text{ and } \ G(\tilde{t}, \tilde{x})d\tilde{t} = \pm G(t, x)\frac{F}{\sqrt{F^2 + c}}dt. \tag{4.18}$$

In the following we shall consider only the case, in which the GST maps autonomous equations of the Painlevé-Gambier classification belonging to the class of (4.1) to a harmonic oscillator equation

$$X'' + X = 0. \tag{4.19}$$

4.2 Painlevé-Gambier equation XVIII

The equation No. 18 reads

$$\ddot{x} - \frac{1}{2x}\dot{x}^2 - 4x^2 = 0. \tag{4.20}$$

Here $\phi_x = -\frac{1}{2x}$ implying $\phi = \ln x$ and $B(x) = -4x^2$. As a result from (4.11), taking the positive square root we find $F(x) = \frac{2ix}{\beta}$ and it turns out that $G = \beta i\sqrt{x}$. Hence the Sundman transformation has the explicit form

$$X = \frac{2ix}{\beta}, \quad dT = \beta i\sqrt{x}dt. \tag{4.21}$$

A first integral for (4.19) is

$$I_1 = X'^2 + X^2$$

and its evaluation in terms of the preceding transformation does indeed reproduce the result given in [14], namely

$$\dot{x}^2 = 4x(I_1 + x^2)$$

upon setting $\beta = 2$. Another first integral for (4.19)is

$$I_2 = \frac{1}{2i}(X' - iX)e^{iT}$$

and its evaluation in terms of the preceding transformation gives

$$I_2 = \frac{1}{2i}\Big(\frac{\dot{x}}{2x^{1/2}} + x\Big)e^{-\int 2x^{1/2}dt}. \tag{4.22}$$

Using Sundman transformations, one can easily construct a solution of the differential equation under consideration. Since the nonlocal transformation maps the ODE to the linear harmonic oscillator equation, namely $X'' + X = 0$, we may use the fact that $X(T) = c_1e^{iT}$ (c_1 being a constant) is a solution of this differential equation. Now as $X = F(t, x)$, we have from the first part of (4.13)

$$X = ix, \tag{4.23}$$

or in other words

$$x = \frac{X}{i}. \tag{4.24}$$

On the other hand since

$$G(t, x) = 2i\sqrt{x}, \tag{4.25}$$

we can use the second expression in (4.13) and the definition $dT = Gdt$ to get

$$dT = 2(iX)^{1/2}dt = 2(ic_1)^{1/2}e^{\frac{iT}{2}}dt,$$

whence the variables may be separated (use having been made of the solution $X = c_1e^{iT}$):

$$e^{iT} = \frac{-4}{(c_2 + 2(ic_1)^{1/2}t)^2}. \tag{4.26}$$

From $X = c_1 e^{iT}$ by using the above expression for e^{iT} and also (4.24) we obtain finally a particular solution of the ODE as

$$x(t) = \frac{4ic_1}{(c_2 + 2(ic_1)^{1/2}t)^2}, \tag{4.27}$$

or in other words a solution of the form:

$$x(t) = \frac{1}{(\gamma + t)^2} \tag{4.28}$$

where $\gamma = \frac{c_2}{2(ic_1)^{1/2}}$ is an arbitrary constant.
Applying a similar procedure to the other linearly independent solution $X(T) = c_3 e^{-iT}$ of (4.19) and using the the general Sundman transformation (4.23) and (4.25) for this ODE and the definition $dT = Gdt$, one obtains

$$dT = 2(iX)^{1/2}dt = 2(ic_3)^{1/2}e^{\frac{-iT}{2}}dt$$

whence the variables may again be separated to obtain

$$e^{iT} = -\frac{1}{4}\{c_4 + 2(ic_3)^{1/2}t\}^2. \tag{4.29}$$

By using the expression for e^{-iT} from (4.29) and also using (4.23) we get finally another solution of the ODE in the form

$$x(t) = \frac{4ic_3}{(c_4 + 2(ic_3)^{1/2}t)^2}. \tag{4.30}$$

In other words

$$x(t) = \frac{1}{(\delta + t)^2}. \tag{4.31}$$

Here $\delta = \frac{c_4}{2(ic_3)^{1/2}}$ is an arbitrary constant. This particular solution being of exactly the same nature as (4.28), it appears that the final outcome is independent of the choice of the particular solution of the transformed ODE (4.19).

To deduce the Sundman symmetry of this equation we use (4.18) (taking the positive sign). The Sundman symmetries are then given by

$$\tilde{x} = -i\sqrt{c - x^2}, \quad \tilde{t} = A + \int \frac{(x)^{3/2}}{i^{1/2}(c - x^2)^{3/4}}dt, \tag{4.32}$$

where A is an arbitrary constant.

4.3 Painlevé-Gambier equation XXII

The equation number 22 reads

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 + 1 = 0. \tag{4.33}$$

Here $\phi_x = -\frac{3}{4x}$ implying $\phi = \ln x^{-3/2}$ and $B(x) = 1$. As a result from (4.13), taking the positive square root and $\beta = 1$ we find $F(x) = 2ix^{-1/4}$ and it turns out that $G = -\frac{i}{2}x^{-1/2}$. Hence the Sundman transformations has the explicit form

$$X = F(x) = 2ix^{-1/4}, \;\; dT = -\frac{i}{2}x^{-1/2}dt. \tag{4.34}$$

The first integral for (4.19) is well known to be

$$I_1 = X'^2 + X^2$$

and its evaluation in terms of the preceding transformation does indeed reproduce the result given in [14], namely

$$I_1 = \frac{\dot{x}^2}{x^{3/2}} - \frac{4}{x^{1/2}}.$$

Another first integral for (4.19) is

$$I_2 = \frac{1}{2i}(X' - iX)e^{iT}$$

and its evaluation in terms of the preceding transformation gives

$$I_2 = \frac{1}{2i}\Big(\frac{\dot{x}}{x^{\frac{3}{4}}} + 2x^{\frac{-1}{4}}\Big)e^{\frac{1}{2}\int x^{-1/2}dt}. \tag{4.35}$$

Once again using Sundman transformations, one can easily construct a solution of the differential equation under consideration. Since the nonlocal transformation maps the ODE to the linear harmonic oscillator equation, namely $X'' + X = 0$, we make use of the fact that $X(T) = c_1e^{iT}$ is a solution of this differential equation (here c_1 is a constant). But by definition $X = F(t, x)$, consequently using the results of (4.13) we may write

$$X = 2ix^{-1/4} \tag{4.36}$$

whence it follows that

$$x = \frac{16}{X^4}. \tag{4.37}$$

On the other hand using

$$G(t, x) = -\frac{i}{2x^{1/2}} \tag{4.38}$$

together with (4.13) and the definition $dT = Gdt$, one gets

$$dT = -\frac{iX^2}{8}dt = -\frac{i}{8}c_1^2e^{2iT}dt$$

whence the variables may be separated (use having been made of the solution $X = c_1 e^{iT}$):

$$e^{iT} = \frac{1}{(\sqrt{2i}(\frac{c_1^2 i}{8}t - c_2)^{1/2}}. \tag{4.39}$$

From $X = c_1 e^{iT}$ by using the above expression for e^{iT} and (4.36) we get finally a solution of the ODE in the form

$$x(t) = -\frac{64(\frac{c_1^2 i}{8}t - c_2)^2}{c_1^4}. \tag{4.40}$$

That is in other words

$$x(t) = (t + \xi)^2. \tag{4.41}$$

Here $\xi = \frac{-8c_2}{c_1^2 i}$ is an arbitrary constant.
If we consider the other linearly independent solution $X(T) = c_3 e^{-iT}$ of (4.19) and use the the general Sundman transformation (4.36) and (4.38) for this ODE along with the definition $dT = Gdt$, we obtain

$$dT = -\frac{iX^2}{8}dt = -\frac{i}{8}c_1^2 e^{-2iT}dt,$$

whence the variables may be separated to get

$$e^{iT} = \pm\frac{c_3}{2}(c_4 + t)^2. \tag{4.42}$$

By using the expression for e^{-iT} from (4.29) and also using (4.23) we get finally a particular solution of the ODE in the form

$$x(t) = (c_4 + t)^2. \tag{4.43}$$

Here c_4 is an arbitrary constant. Once again we notice that this solution is similar in structure to (4.41). This leads us to conclude that a solution obtained in the above manner is independent of the choice of the particular solution of the transformed ODE (4.19).

To deduce Sundman symmetry we use (4.18) (taking the positive sign). The Sundman symmetry in this case is given by:

$$\tilde{x} = \frac{16}{(c - 2x^{-1/2})}, \qquad \tilde{t} = C + \int \frac{8ix^{-3/4}}{(c - 2x^{-1/2})\sqrt{c - 4x^{-1/2}}}dt, \tag{4.44}$$

where C is an arbitrary constant.

4.4 Painlevé-Gambier equation XXI

The equation no. 21 reads

$$\ddot{x} - \frac{3}{4x}\dot{x}^2 - 3x^2 = 0. \tag{4.45}$$

Here $\phi_x = -\frac{3}{4x}$ implying $\phi = \ln x^{-3/2}$ and $B(x) = -3x^2$. As a result from (4.13), taking the positive square root and $\beta = 1$ we find $F(x) = 2ix^{\frac{3}{4}}$ and it turns out that $G = \frac{3i}{2}x^{\frac{1}{2}}$. Hence the Sundman transformation has the explicit form

$$X = F(x) = 2ix^{\frac{3}{4}}, \;\; dT = \frac{3i}{2}x^{\frac{1}{2}}dt. \tag{4.46}$$

The first integral for (4.19) is well known to be

$$I_1 = X'^2 + X^2$$

and its evaluation in terms of the preceding transformation does indeed reproduce the result given in [14] namely

$$I_1 = \frac{\dot{x}^2}{x^{3/2}} - 4x^{3/2}.$$

Another first integral for (4.19) is

$$I_2 = \frac{1}{2i}(X' - iX)e^{iT}$$

and its evaluation in terms of the preceding transformation gives

$$I_2 = \frac{1}{2i}\Big(\frac{\dot{x}}{x^{\frac{3}{4}}} + 2x^{\frac{3}{4}}\Big)e^{-\frac{3}{4}\int x^{1/2}dt}. \tag{4.47}$$

Using Sundman transformations, one can easily construct a solution of the differential equation under consideration. Proceeding in the same manner as in the earlier examples we get the solution of the ODE in the form

$$x(t) = \frac{3^2 c_1^{\frac{4}{3}} i^{2/3}}{(2)^{10/3}(\frac{3c_1^{2/3}i^{1/3}}{2^{5/3}}t + c_2)^2}. \tag{4.48}$$

In other words $x(t) = \frac{1}{(t+\ell)^2}$ where $\ell = \frac{c_2 2^{5/3}}{3c_1^{2/3}i^{1/3}}$ is an arbitrary constants.
As in the previous two examples of Painlevé-Gambier equations 18 and 22 we can show that this particular solution is independent of the choice of the particular solution of the transformed equation (4.19).
To deduce Sundman symmetry we use (4.18) (and take the positive sign). The Sundman symmetry in this case is given by:

$$\tilde{x} = \frac{(c - 4x^{3/2})^{2/3}}{(2i)^{4/3}}, \qquad \tilde{t} = B + \int \frac{(2i)^{5/3}x^{5/4}}{(c - 4x^2)^{1/3}\sqrt{c - 2x^{3/2}}}dt, \tag{4.49}$$

where B is an arbitrary constant.

5 Conclusion

In this paper we have studied generalized Sundman transformation for the Jacobi equation and have obtained from special cases of it, the Sundman transformation for a number of second-order autonomous ODEs of the Painlevé-Gambier classification. Interestingly, we are able to derive certain new first integrals for these equations, which are not contained in the standard text by Ince. This certainly adds to the existing body of known results for an important class of ODEs. In addition we have also shown how one can derive particular solutions for the equations, together with their associated Sundman symmetries by systematically using the Sundman transformation. This clearly indicates the powerful nature of such a transformation and points to the necessity for a more extensive study of its applications.

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