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and Poisson Structures

by

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Abstract

We construct certain new classes of isochronous dynamical systems based on the recent constructions of Calogero and Leyvraz. We show how a Poisson structure can be ascribed to such equations in \mathbb{R}^3 and indicate their connection with the Nambu structures.

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1 Introduction

A (classical) dynamical system is said to be isochronous if it displays an open region in its phase space in which all its solutions are completely periodic, i.e., periodic in all degrees of freedom, with the same fixed period (see for example [1, 2, 3] for standard review). The linear harmonic oscillator is the prototype of an isochronous system and all other isochronous systems are isoperiodic with the harmonic oscillator. Research on isochronous system has taken a new turn in recent times. A large number of articles have appeared in the last few years inspired by the novel ideas developed by Francesco Calogero in this regard [1, 2]. Indeed, in a series of papers [4, 5, 6], Calogero and Leyvraz have shown that isochronous dynamical systems are not as rare as they were previously thought to be. Prior to Calogero's work a large part of the mathematical literature on isochronous systems was devoted solely to planar systems.

Recently Chalykh and Veselov [7] have shown that among rational potentials only the harmonic oscillator and the isotonic oscillator produce isochronous motions. In fact there exists other classes of isochronous systems described by non-rational potentials, for instance potentials for which the second derivative has a discontinuity.

However, the recent methods introduced by Calogero and Leyvraz allow us to extend any dynamical system, in a manner such that the derived system is either isochronous or asymptotically isochronous or even multi-periodic. The generic solutions of the extended dynamical systems are correspondingly either completely periodic with a fixed period or are asymptotically periodic or become multi-periodic in the asymptotic limit [8, 9]. The procedure devised by Calogero *et al* may be applied to a wide class of systems and is marked by a remarkable degree of simplicity in which novel use is made of the linear harmonic oscillator [4, 1, 5].

Consider a planar dynamical system described by the systems of equations:

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1.1)$$

where P and Q are polynomials with real coefficients. A point (x_0, y_0) is said to be a *critical point* of the system if $P(x_0, y_0) = Q(x_0, y_0) = 0$. We say that a critical point is a *center* if, within some neighborhood, all the orbits surrounding it are closed. The linearized version of the above system is obtained by expanding the functions P and Q about a critical point in a Taylor series:

$$\begin{aligned} \dot{x} &= P_x(x_0, y_0)(x - x_0) + P_y(x_0, y_0)(y - y_0) \\ \dot{y} &= Q_x(x_0, y_0)(x - x_0) + Q_y(x_0, y_0)(y - y_0) \end{aligned} \quad (1.2)$$

where the subscripts denote the usual partial derivatives of the functions. If the linearized vector field has two non-zero eigenvalues then a center is said to be *non-degenerate*. An isochronous center is a center for which all the surrounding orbits have the same period, and it may be proved that such centers are always non-degenerate.

A formal definition of a isochronous vector is as follows:

Definition 1.1 A vector $\mathbf{z}(t; \alpha)$ is isochronous with a period T if all its components are periodic with the same period T independent of the set of parameters $\alpha = \{\alpha_k\}$, i.e.,

$$z_j(t + T; \alpha) = z_j(t; \alpha). \quad (1.3)$$

Note that in this definition the number of parameters need not be equal to the number of components of \mathbf{z} . In case of asymptotically isochronous systems the formal definition is

Definition 1.2 A vector $\mathbf{z}(t; \alpha)$ is asymptotically isochronous if there exists an isochronous vector $\mathbf{z}_0(t; \alpha)$ such that

$$\lim_{t \rightarrow +\infty} \|\mathbf{z}(t; \alpha) - \mathbf{z}_0(t; \alpha)\| = 0, \quad (1.4)$$

where $\|\cdot\|$ denotes some suitable norm and $\alpha = \{\alpha_k\}$.

We recall that a finite dimensional dynamical system is said to have a Poisson structure if it can be written as the following system of ODEs:

$$\dot{x}_i = \sum_{j=1}^n J^{ij} \partial_j H, \quad i = 1, \dots, n$$

where $H(x)$ is the Hamiltonian and J^{ij} are the entries of an $n \times n$ matrix \mathbb{J} - known as the Poisson structure matrix. The latter is a skew-symmetric matrix $J^{ij} = -J^{ji}$ (for all i, j) and is a solution of the Jacobi identity:

$$\sum_{l=1}^n (J^{li} \partial_l J^{jk} + J^{lj} \partial_l J^{ki} + J^{lk} \partial_l J^{ij}) = 0.$$

Given such a Poisson matrix \mathbb{J} we can define a generalization of the classical Hamiltonian systems on which a non canonical Poisson bracket is defined, namely,

$$\{f(x), g(x)\} = \sum_{i,j} \frac{\partial f(x)}{\partial x_i} J_{ij} \frac{\partial g(x)}{\partial x_j},$$

for every pair of smooth functions $f(x)$ and $g(x)$.

The ability to express a finite-dimensional dynamical systems in terms of Hamiltonian and Poisson structure is an open question in general. The main hurdle comes from the computation of the solution of the Jacobi identities. This can be computed relatively simply for three dimensional flows, being a scalar equation. But for higher dimensions due to increasing nonlinearity this is not always manageable. The second problem arises in determining the first integrals. In order to recast the dynamical systems into a Hamiltonian system a first integral or some function of the first integrals often plays the role of the Hamiltonian.

Owing to these two issues one is not always able to express a given dynamical system as a Poisson and/or Hamiltonian system. In this paper we confine ourselves to three dimensional systems. The Jacobi-Poisson formulation for such three dimensional systems has been extensively studied in the last two decades (cf. [10, 11, 12]). Hernandez-Bermejo [13, 14] gave a very large class of the solutions of the Jacobi equation for the Poisson matrix \mathbb{J} and the general solution has been derived by Gürses and his coworkers [15]. We also briefly discuss its connection with the Nambu-Poisson structures [16, 17, 18] and Jacobi's last multiplier (JLM). In our earlier paper [19] we investigate the nature of the connection between the JLM and isochronous non-planar dynamical systems of the Calogero-Leyvraz type. We show that the JLM plays a pivotal role in the construction of isochronous dynamical systems.

The paper is **organized** as follows. In Section 2 we present the main features of the results obtained by Calogero and Leyvraz in [9]. Then in section 3 we construct several new coupled autonomous dynamical systems which exhibit the property of isochronicity. In Section 4 we investigate the solutions of the Jacobi identities to obtain the Poisson matrix \mathbb{J} associated with the isochronous systems obtained in the preceding manner. Finally, we point out the connection between the Jacobi-Poisson structures and the Nambu structure for such systems.

2 Recapitulation of Calogero-Leyvraz' Method

Consider a system of first-order autonomous ODEs

$$\frac{dx_n}{d\tau} = h_n(\mathbf{x}), \quad n = 1, \dots, N. \quad (2.1)$$

This system will be extended by the inclusion of two additional variables $y_i (i = 1, 2)$ such that

$$\dot{x}_n = W_n(\mathbf{x}, y_1, y_2), \quad n = 1, \dots, N \quad (2.2)$$

$$\dot{y}_1 = Y_1(\mathbf{x}, y_1, y_2) \quad (2.3)$$

$$\dot{y}_2 = Y_2(\mathbf{x}, y_1, y_2). \quad (2.4)$$

Here the over dot denotes a derivative with respect to the time t . The connection between (2.1) and the above system is *via* the following identification of the function τ , namely

$$\frac{d\tau}{dt} = \frac{W_n(\mathbf{x}, y_1, y_2)}{h_n(\mathbf{x})} \equiv \phi(\mathbf{x}, y_1, y_2) \quad (2.5)$$

for some function $\phi(\mathbf{x}, y_1, y_2)$, to be specified later. The explicit forms of $Y_i (i = 1, 2)$ will be derived in what follows so as to ensure that the system is isochronous. Equations (2.2)-(2.4) constitute a set of $(N + 2)$ autonomous first-order ODEs. The two additional (or auxiliary) variables $y_i (i = 1, 2)$ are specifically related to the linear harmonic oscillator via the following identification

$$y_1 = \frac{f_1}{F(\mathbf{x})} \quad y_2 = \frac{f_2}{G(\mathbf{x})}, \quad (2.6)$$

where $F(\mathbf{x})$ and $G(\mathbf{x})$ are non-zero real valued functions and f_1 and f_2 satisfy the equation of a linear harmonic oscillator:

$$\dot{f}_1 = -\Omega f_2, \quad \dot{f}_2 = \Omega f_1 \quad \Omega > 0. \quad (2.7)$$

From (2.7) it is obvious that

$$\ddot{f}_i + \Omega^2 f_i = 0, \quad i = 1, 2 \quad (2.8)$$

and the f_i 's admit periodic solutions given by,

$$f_1 = A \sin(\Omega t + \theta) \quad \text{and} \quad f_2 = -A \cos(\Omega t + \theta), \quad (2.9)$$

where A is the amplitude and θ represents the phase difference. From (2.6) and (2.8) it follows that

$$\dot{y}_1 = -\Omega \frac{G}{F} y_2 - \phi(\mathbf{x}, y_1, y_2) y_1 \sum_{n=1}^N \frac{\partial F(\mathbf{x})}{\partial x_n} h_n(\mathbf{x}), \quad (2.10)$$

$$\dot{y}_2 = \Omega \frac{F}{G} y_1 - \phi(\mathbf{x}, y_1, y_2) y_2 \sum_{n=1}^N \frac{\partial G(\mathbf{x})}{\partial x_n} h_n(\mathbf{x}). \quad (2.11)$$

These two equations serve to define the functions $Y_i (i = 1, 2)$ in (2.3) and (2.4) respectively. Therefore (2.1) together with (2.10) and (2.11) constitute a set of $(N + 2)$ autonomous coupled nonlinear ODEs. On the other hand (2.5) implies

$$\tau(t) = \int \phi(\mathbf{x}, y_1, y_2) dt + C, \quad (2.12)$$

while it follows from (2.6) that the auxiliary variables are given by

$$y_1 = \frac{A \sin(\Omega t + \theta)}{F(\mathbf{x}(t))} \quad \text{and} \quad y_2 = -\frac{A \cos(\Omega t + \theta)}{G(\mathbf{x}(t))}, \quad \text{with} \quad \tau(t) = \int^{x_n} \frac{dx'_n}{h(\mathbf{x}')} . \quad (2.13)$$

Note that we have not yet specified the explicit form of the function $\phi(\mathbf{x}, y_1, y_2)$. If $\phi(\mathbf{x}, y_1, y_2) = J(f_1, f_2)$, i.e., it depends on \mathbf{x} and y_1, y_2 implicitly through f_1 and f_2 only, then because the latter are periodic functions with period $2\pi/\Omega$, it follows from (2.12) that $\tau(t)$ must necessarily be periodic and hence x_n as obtained from $\tau(t) = \int^{x_n} \frac{dx'_n}{h(\mathbf{x}')}$ will implicitly be a periodic function with the same period. However $\mathbf{x}(t)$ being periodic implies in turn that $F(\mathbf{x}(t))$ and $G(\mathbf{x}(t))$ are themselves periodic and consequently $y_1(t)$ and $y_2(t)$ are also periodic. Therefore we conclude that the entire extended system of $(N + 2)$ equations is periodic with the same period for all the degrees of freedom, and constitutes an isochronous system.

In [9] the specific choice of $J(f_1, f_2) = f_1$ was dealt with in some detail and it was observed that other combinations of f_1 and f_2 may also be chosen to define the function $\phi(\mathbf{x}, y_1, y_2)$. In the following section we will be concerned with more general forms of $J(f_1, f_2)$. For the sake of simplicity however, we shall assume $N = 1$ and thereby restrict ourselves to three dimensional dynamical systems.

3 New Isochronous Systems

In this section we consider certain specific combinations of f_1 and f_2 to define $\phi(x, y_1, y_2)$ and obtain for each such choice a distinct set of equations in \mathbb{R}^3 which are isochronous by construction. We will also indicate how these coupled equations may be partially decoupled.

Example 3.1 Let $\phi(x, y_1, y_2) = f_1 f_2$

In this case the explicit solution for τ as obtained from

$$\dot{\tau} = \phi(x, y_1, y_2) = f_1 f_2 = -\frac{A^2}{2} \sin(2\Omega t + 2\theta)$$

is

$$\tau(t) = \frac{A^2}{4\Omega} [\cos(2\Omega t + 2\theta) - \cos 2\theta] \quad (3.1)$$

where we have assumed the initial condition $\tau(0) = 0$. From (2.5) we find that

$$\phi(x, y_1, y_2) = f_1 f_2 = y_1 y_2 F(x) G(x). \quad (3.2)$$

Using this in (2.10) and (2.11) we have

$$\begin{aligned} \dot{y}_1 &= -\Omega \frac{G}{F} y_2 - FG y_1^2 y_2 \frac{d \ln F}{dx} h(x), \\ \dot{y}_2 &= \Omega \frac{F}{G} y_1 - FG y_1 y_2^2 \frac{d \ln G}{dx} h(x), \end{aligned} \quad (3.3)$$

along with

$$\dot{x} = y_1 y_2 F G h(x).$$

Suppose we now make the simplifying assumption $F(x) = G(x) = R(x)$. This causes the system (3.3) to reduce to

$$\begin{aligned} \dot{y}_1 &= -\Omega y_2 - y_1^2 y_2 R(x) \frac{dR(x)}{dx} h(x) \\ \dot{y}_2 &= \Omega y_1 - y_1 y_2^2 R(x) \frac{dR(x)}{dx} h(x), \end{aligned} \quad (3.4)$$

while

$$\dot{x} = y_1 y_2 R(x)^2 h(x).$$

We can decouple the system (3.4) from the last equation by demanding

$$R(x) R'(x) h(x) = \lambda \quad (3.5)$$

where λ is any arbitrary constant. This implies in view of the fact that $\tau(t) = \int^x \frac{ds}{h(s)}$

$$|R(x)| = [2\lambda\tau(t)]^{1/2} = \left[\frac{\lambda A^2}{2\Omega} (\cos(2\Omega t + 2\theta) - \cos 2\theta) \right]^{1/2} \quad (3.6)$$

while (3.4) reduces to

$$\dot{y}_1 = -\Omega y_2 - \lambda y_1^2 y_2 \quad (3.7)$$

$$\dot{y}_2 = \Omega y_1 - \lambda y_1 y_2^2 \quad (3.8)$$

$$\dot{x} = \lambda y_1 y_2 \frac{R(x)}{R'(x)}. \quad (3.9)$$

Thus by varying the functional form of $R(x)$ one can obtain a large class of isochronous systems. The solution of the system is easily obtained from the fact that

$$y_1(t) = \frac{A \sin(\Omega t + \theta)}{R(x)}, \quad y_2(t) = -\frac{A \cos(\Omega t + \theta)}{R(x)}$$

where $R^2(x) = 2\lambda\tau(t)$.

Example 3.2 Let $\phi(x, y_1, y_2) = \frac{f_1}{f_2}$

In this case the explicit solution for τ is obtained from

$$\dot{\tau} = -\tan(\Omega t + \theta) \quad (3.10)$$

and is given by

$$\tau(t) = \frac{1}{\Omega} \log \left| \frac{\cos(\Omega t + \theta)}{\cos \theta} \right|, \quad (3.11)$$

where it is assumed that $\tau(0) = 0$. This form of ϕ causes (2.10) and (2.11) to reduce to the following equations, under the simplifying assumption that $F = G = R(x)$, namely

$$\begin{aligned} \dot{y}_1 &= -\Omega y_2 - \frac{y_1^2}{y_2} \frac{d \log R(x)}{dx} h(x) \\ \dot{y}_2 &= \Omega y_1 - y_1 \frac{d \log R(x)}{dx} h(x) \end{aligned} \quad (3.12)$$

along with $dx/d\tau = h(x)$. We may decouple the equations for the y_i 's from x by assuming that $d \log R(x)/dx = 1/h(x)$. This implies

$$R(x) = C \exp \left(\int^x \frac{ds}{h(s)} \right) = C \exp(\tau) \quad (3.13)$$

because $\tau(t) = \int^x ds/h(s)$. Under these conditions the final form of the resulting isochronous system is

$$\dot{y}_1 = -\Omega y_2 - \frac{y_1^2}{y_2}, \quad \dot{y}_2 = \Omega y_1 - y_1, \quad \dot{x} = \frac{y_1}{y_2} \frac{R(x)}{R'(x)}, \quad (3.14)$$

with $R(x) = C \exp(\tau)$. The solution of (3.14) may be obtained by specifying the form of $R(x)$, which determines in turn $h(x)$. The time evolution of y_1 and y_2 are given as usual by

$$y_1(t) = \frac{f_1(t)}{|R(x)|} \quad \text{and} \quad y_2(t) = \frac{f_2(t)}{|R(x)|}$$

respectively. For the sake of concreteness let $R(x) = x$ so that $x(t) = C \exp(\tau(t))$ where $\tau(t)$ is given by (3.11). Then

$$y_1(t) = -\frac{A \sin(\Omega t + \theta)}{|R(x)|} = -\frac{A}{C} e^{-\tau(t)} \sin(\Omega t + \theta),$$

$$y_2(t) = \frac{A \cos(\Omega t + \theta)}{|R(x)|} = \frac{A}{C} e^{-\tau(t)} \cos(\Omega t + \theta).$$

Example 3.3 Let $\phi(x, y_1, y_2) = f_2^2 - f_1^2$

In this case we have $\dot{\tau} = A^2 \cos(2\Omega t + 2\theta)$ whence, assuming $\tau(0) = 0$ we obtain the following solution

$$\tau(t) = \frac{A^2}{2\Omega} [\sin(\Omega t + \theta) - \sin 2\theta].$$

In the general case when $F \neq G$ we are led to the following equations:

$$\dot{y}_1 = -\Omega \frac{G}{F} y_2 - (G^2 y_2^2 - F^2 y_1^2) y_1 \frac{d \log F}{dx} h(x)$$

$$\dot{y}_2 = \Omega \frac{F}{G} y_1 - (G^2 y_2^2 - F^2 y_1^2) y_2 \frac{d \log G}{dx} h(x),$$

together with $dx/d\tau = h(x)$. Furthermore assuming $F = G = R(x)$ and $R^2(x) \frac{d \log R}{dx} = \frac{1}{h(x)}$ we may decouple two of the three equations to obtain

$$\dot{y}_1 = -\Omega y_2 - (y_2^2 - y_1^2) y_1, \quad \dot{y}_2 = \Omega y_1 - (y_2^2 - y_1^2) y_2, \quad \dot{x} = (y_2^2 - y_1^2) \frac{R(x)}{R'(x)}. \quad (3.15)$$

3.1 Further generalization

In this section we attempt to further generalize the class of equations exhibiting isochronicity by assuming that the relation between the auxiliary variables y_1 and y_2 and the functions f_1 and f_2 describing the linear harmonic oscillator are of the following general form, *viz*

$$f_1 = F(x)y_1 + G(x)y_2, \quad f_2 = H(x)y_1 + K(x)y_2.$$

Taking into consideration $\dot{x} = \phi(f_1, f_2)h(x)$ and the conditions $\dot{f}_1 = -f_2$ and $\dot{f}_2 = f_1$ (setting $\Omega = 1$) they imply

$$F(x)\dot{y}_1 + G(x)\dot{y}_2 = -(F'(x)\phi(x, y)h(x) + H(x))y_1 - (G'(x)\phi(x, y)h(x) + K(x))y_2$$

$$H(x)\dot{y}_1 + K(x)\dot{y}_2 = (F(x) - H'(x)\phi(x, y)h(x))y_1 + (G(x) - K'(x)\phi(x, y)h(x))y_2.$$

Setting $H(x) = -G(x)$ and $K(x) = F(x)$ these simplify to become:

$$\begin{aligned}\dot{y}_1 &= -y_2 - \dot{x}y_1 \frac{FF' + GG'}{F^2 + G^2} + \dot{x}y_2 \frac{GF' - FG'}{F^2 + G^2} \\ \dot{y}_2 &= y_1 - \dot{x}y_2 \frac{FF' + GG'}{F^2 + G^2} - \dot{x}y_1 \frac{GF' - FG'}{F^2 + G^2},\end{aligned}\tag{3.16}$$

while the equation for x is still given by $\dot{x} = \phi(x, y_1, y_2)h(x)$. Note that in terms of f_k ($k = 1, 2$) and the functions F and G we can express y_1 and y_2 as follows:

$$y_1 = \frac{f_1F - f_2G}{F^2 + G^2} \quad y_2 = \frac{f_1G + f_2F}{F^2 + G^2}.\tag{3.17}$$

Let us assume the simplest form of the function ϕ , namely $\phi(x, y_1, y_2) = f_1 = y_1F(x) + y_2G(x)$ so that

$$\dot{x} = (y_1F + y_2G)h(x).$$

Consider the simple case when $F(x) = \cos x$ and $G(x) = \sin x$. Since $F^2 + G^2 = \cos^2 x + \sin^2 x = 1$ it follows that $FF' + GG' = 0$. This causes the equations to assume the simplified form

$$\dot{x} = (y_1 \cos x + y_2 \sin x)h(x)\tag{3.18}$$

$$\dot{y}_1 = -y_2 - y_2(y_1 \cos x + y_2 \sin x)h(x), \quad \dot{y}_2 = y_1 + y_2(y_1 \cos x + y_2 \sin x)h(x).\tag{3.19}$$

Assuming that $\phi = d\tau/dt = f_1 = (y_1 \cos x + y_2 \sin x)$, (where as before $\dot{f}_1 = -f_2$ and $\dot{f}_2 = f_1$) we can write

$$\tau(t) = A[\sin(t + \theta) - \sin \theta].\tag{3.20}$$

In arriving at this solution, we assumed that $f_1 = A \cos(t + \theta)$ and $f_2 = A \sin(t + \theta)$ and that $\tau(0) = 0$. Notice that $\tau(t + 2\pi) = \tau(t)$. Consequently, (3.18) can be expressed as $dx/d\tau = h(x)$ and we obtain $\tau = \int^x \frac{dx'}{h(x')}$. For a given function $h(x)$ the latter allows us to express τ in terms of x , which, assuming it is invertible, gives x as a function of τ and hence of t . For instance, the choice $h(x) = \gamma x$ yields $x(\tau) = \exp(\gamma\tau(t))$ with τ given by (3.20). One can easily read off the values of y_1 and y_2 from (3.17)

$$y_1(t) = A \cos(t + x(t) + \theta), \quad \text{and} \quad y_2(t) = A \sin(t + x(t) + \theta),$$

and they are the solutions of the isochronous system (3.18)-(3.19) with $h(x) = \gamma x$.

3.2 Embedding planar isochronous systems within higher dimensional ones

Let us recall the equations (2.10) and (2.11) together with $\dot{x} = \phi(\mathbf{x}, y_1, y_2)h(\mathbf{x})$ and restrict ourselves to the $N = 1$ case only. Our attempt now will be to employ the Calogero-Leyvraz

technique to find a higher-dimensional set of first-order differential equations which are not only themselves isochronous, but include as a subset a planar isochronous system. We illustrate the method with the following well-known planar isochronous system

$$\dot{y}_1 = -y_2 + 2y_1y_2 - ay_1^2y_2, \quad \dot{y}_2 = y_1 + 2y_2^2 - ay_1y_2^2. \quad (3.21)$$

First of all, without loss of generality we may set $\Omega = 1$ and $F = G = R$ in (2.10) and (2.11) to obtain

$$\begin{aligned} \dot{y}_1 &= -y_2 - y_1\phi(x, y_1, y_2)\frac{R'(x)}{R(x)}h(x) \\ \dot{y}_2 &= y_1 - y_2\phi(x, y_1, y_2)\frac{R'(x)}{R(x)}h(x). \end{aligned} \quad (3.22)$$

Comparison with (3.21) clearly indicates a matching of the linear terms. Let us therefore demand that

$$-y_1\phi(x, y_1, y_2)\frac{R'(x)}{R(x)}h(x) = y_1(2y_2 - ay_1y_2).$$

Using the fact that $f_k = R(x)y_k$ ($k = 1, 2$) then allows us to express the last line as

$$\phi(x, y_1, y_2) = \phi(x; f_1, f_2) = \frac{af_1f_2 - 2f_2R(x)}{R(x)R'(x)h(x)}.$$

Let us further assume that the denominator is unity, so that

$$\phi(x; f_1, f_2) = af_1f_2 - 2f_2R(x). \quad (3.23)$$

Note that unlike the forms assumed by Calogero *et al* in [9] and also in the previous section, ϕ here is seen to depend on both f_1, f_2 as well as on $R(x)$. Consequently it is convenient to assume that ϕ may be factorized as

$$\phi(x; f_1, f_2) = \psi(f_1, f_2)\chi(x; f_1, f_2). \quad (3.24)$$

Comparing this with (3.23), we see that here $\psi(f_1, f_2) = f_2$ and $\chi(x; f_1, f_2) = (af_1 - 2R(x))$. As a result the equation for \dot{x} appears as

$$\dot{x} = \psi(f_1, f_2)\chi(x; f_1, f_2)h(x). \quad (3.25)$$

In view of the above we shall define a function τ *via* the following relation: $d\tau/dt = \psi(f_1, f_2)$ which implies $\tau(t) = \int \psi(f_1, f_2)dt + \tau_0$, so that

$$\frac{dx}{d\tau} = \chi(x; f_1, f_2)h(x).$$

Here τ_0 is a constant of integration. If we can express χ in terms of x and τ so that $dx/d\tau = \sigma(x, \tau)h(x)$ then in principle one can integrate this to obtain $x = x(\tau)$ and hence a function of t . In our case since $\psi(f_1, f_2) = f_2$ we find that

$$\tau = \int f_2 dt + \tau_0 = -A \sin(t + \theta) + \tau_0$$

or

$$\tau = -f_1 + \tau_0, \quad \text{which implies } f_1 = (\tau - \tau_0).$$

Under these circumstances $\chi(x; f_1, f_2) = -a(\tau - \tau_0) - 2R(x)$ so that

$$\frac{dx}{d\tau} = -(a(\tau - \tau_0) + 2R(x))h(x).$$

We may eliminate $h(x)$ by using the condition $R(x)R'(x)h(x) = 1$ to get

$$\frac{dx}{d\tau} = -a(\tau - \tau_0)\frac{1}{R(x)R'(x)} - \frac{2}{R'(x)}$$

which leads to the equation

$$\frac{dR(x)}{d\tau} = -a(\tau - \tau_0)\frac{1}{R(x)} - 2. \quad (3.26)$$

We look for a particular solution and assume accordingly $R(x) = k(\tau - \tau_0)$, where k is a constant. Substitution into (3.26) yields the following condition on k , namely

$$k^2 + 2k + a = 0$$

which determines k in terms of the parameter a of the system. It is interesting to note that the explicit form of the function $R(x)$ is largely arbitrary here. In view of these considerations, the equations for y_2 now appears as

$$\dot{y}_2 = y_1 - y_2^2(ay_1 - 2) = y_1 + 2y_2^2 - ay_1y_2^2$$

which is what we sought to obtain. As for the solutions of the system of equations, it is obvious that formally

$$x(t) = R^{-1}(k(\tau - \tau_0)) \quad \text{where } \tau(t) = \tau_0 - A \sin(t + \theta)$$

while

$$y_1(t) = \frac{f_1}{R(x)} = \frac{A \sin(t + \theta)}{k(\tau - \tau_0)} = -\frac{1}{k}$$

$$y_2(t) = \frac{f_2}{R(x)} = -\frac{A \cos(t + \theta)}{k(\tau - \tau_0)} = \frac{1}{k} \cot(t + \theta),$$

with k being the solution of $k^2 + 2k + a = 0$. Clearly the solutions exhibit finite time singularities and one must restrict the range of values of t to avoid them.

4 Poisson structures and nonstandard Hamiltonian formulation of Calogero-Leyvraz type systems

We begin this section with a few important general remarks about Poisson structures on \mathbb{R}^n . Let $\eta = \eta_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ be the Poisson tensor on a manifold M . It is clear in \mathbb{R}^2 the most general Poisson structure is $\eta_{ij} = f(x)\epsilon_{ij}$, where $f(x) \in C^1(\mathbb{R}^2)$. In \mathbb{R}^3 a Poisson tensor must have rank 2 and takes the form

$$\eta_{ij} = f(x)\epsilon_{ijk}\partial_k\xi,$$

where $f(x) \in C^1(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$. Due to antisymmetric property a Poisson tensor in \mathbb{R}^4 must be rank 2 or 4. The rank 2 Poisson tensor is given by

$$\eta_{ij} = f(x)\epsilon_{ijkl}\partial_k\xi_1\partial_l\xi_2,$$

where $g_i \in C^2(\mathbb{R}^4)$.

The Poisson tensors of rank 2 on \mathbb{R}^n are of the following general form

$$\eta_{ij} = f(x)\epsilon_{ijk_1\dots k_{n-2}}\partial_{k_1}\xi_1 \cdots \partial_{k_{n-2}}\xi_{n-2},$$

where $\xi_i \in C^2(\mathbb{R}^n)$, $i = 1, \dots, n-2$ and $\epsilon_{ijk_1\dots k_{n-2}}$ is a fully antisymmetric tensor.

4.1 Poisson structures in \mathbb{R}^3 and Hamiltonians

In this subsection we wish to investigate if the isochronous dynamical systems of the Calogero-Leyvraz type can be endowed with a Poisson structure. Let us start with an explicit representation of the Poisson matrix \mathbb{J} in \mathbb{R}^3 . Any exact Poisson bivector in \mathbb{J}^3 corresponds to a certain function $\zeta(x, y, z)$ is given by

$$\Lambda_\zeta^3 = \frac{\partial\zeta}{\partial x} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial\zeta}{\partial y} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x} + \frac{\partial\zeta}{\partial z} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

This we can express in terms of a Poisson matrix

$$\mathbb{J}_\zeta = \begin{pmatrix} 0 & \frac{\partial\zeta}{\partial z} & -\frac{\partial\zeta}{\partial y} \\ -\frac{\partial\zeta}{\partial z} & 0 & \frac{\partial\zeta}{\partial x} \\ \frac{\partial\zeta}{\partial y} & -\frac{\partial\zeta}{\partial x} & 0 \end{pmatrix} \quad (4.1)$$

If we focus on the Jacobi identity, it is well known that in \mathbb{R}^3 the Jacobi equation for the Poisson structure, is a single scalar equation for the three components of the Poisson structure \mathbb{J} [15]. Note that we stick to usual notation of Poisson matrix. The relevant properties of the matrix $\mathbb{J} = (J)_{ij}$ for the Poisson structure are

$$(i) \quad J_{ij} = -J_{ji} \quad i, j = 1, 2, 3 \quad \text{skew-symmetry}$$

$$(ii) \quad J^{li} \partial_l J^{jk} + J^{lj} \partial_l J^{ki} + J^{lk} \partial_l J^{ij} = 0 \quad i, j, k = 1, 2, 3, \quad (4.2)$$

which is a consequence of the Jacobi identity. If we define $J_{12} = u$, $J_{31} = v$ and $J_{23} = w$ then the Jacobi equation (4.2) yields

$$u \partial_1 v - v \partial_1 u + w \partial_2 u - u \partial_2 w + v \partial_3 w - w \partial_3 v = 0$$

which can also be written as

$$u^2 \partial_1 \left(\frac{v}{u} \right) + w^2 \partial_2 \left(\frac{u}{w} \right) + v^2 \partial_3 \left(\frac{w}{v} \right) = 0. \quad (4.3)$$

We have already discussed the nature of Poisson structures in \mathbb{R}^3 . In fact recently Ay et al. have analysed the nature of the general solution of the Jacobi equation. The following theorem gives the form of the general solution of (4.3).

Theorem 4.1 *All Poisson structures in \mathbb{R}^3 , except at some irregular points, take the form $J_{ij} = \mu \epsilon_{ijk} \partial_k \zeta$ where μ and ζ are some differentiable functions in \mathbb{R}^3 .*

Its proof is given in [15]. Therefore from this theorem we have

$$u = \mu \zeta_z, \quad v = \mu \zeta_y \quad \text{and} \quad w = \mu \zeta_x \quad (4.4)$$

where μ is a function of x, y, z in general and the subscripts denote partial derivatives. In terms of these, the Hamiltonian form of the equations of motion is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & \mu \zeta_z & -\mu \zeta_y \\ -\mu \zeta_z & 0 & \mu \zeta_x \\ \mu \zeta_y & -\mu \zeta_x & 0 \end{pmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix}. \quad (4.5)$$

This equation has nice geometric interpretation. Let $\xi = k_1$ and $H = k_2$ define two surfaces in \mathbb{R}^3 . The intersection of these surfaces define a curve C , whose velocity vector $\dot{\vec{x}}$ is parallel to the cross product of the normal vectors $\nabla \xi$ and ∇H of the surfaces, given by

$$\frac{d\vec{x}}{dt} = -\mu \nabla \xi \times \nabla H,$$

where $\mu \in C^\infty(\mathbb{R}^2)$.

The general structure of the Calogero-Leyvraz isochronous dynamical system in three dimensions, with $\phi(x, y, z) = f_1 = yF(x)$, is

$$\begin{aligned} \dot{x} &= yF(x)h(x) \\ \dot{y} &= z - y^2 F'(x)h(x) \\ \dot{z} &= -y - yz F'(x)h(x) \end{aligned} \quad (4.6)$$

where we have assumed $F(x) = G(x)$ and have set the frequency $\Omega = 1$ for simplicity. Assuming the system admits a Poisson structure we require

$$\dot{x} = \mu(\zeta_z H_y - \zeta_y H_z) = yF(x)h(x), \quad (4.7)$$

$$\dot{y} = \mu(\zeta_x H_z - \zeta_z H_x) = z - y^2 F'(x)h(x), \quad (4.8)$$

$$\dot{z} = \mu(\zeta_y H_x - \zeta_x H_y) = -y - yz F'(x)h(x). \quad (4.9)$$

The linear terms in (4.8) and (4.9) being similar to those of the linear harmonic oscillator suggests that we set $\zeta_x = 1$ and assume

$$\mu H_z = z \quad \mu H_y = y. \quad (4.10)$$

The remaining parts of (4.8) and (4.9) then imply

$$\mu \zeta_z H_x = y^2 F'(x)h(x), \quad (4.11)$$

$$\mu \zeta_y H_x = -yz F'(x)h(x). \quad (4.12)$$

Their ratio yields

$$\frac{\zeta_z}{\zeta_y} = -\frac{y}{z}. \quad (4.13)$$

Next using (4.10) in (4.7) and by eliminating ζ_z using (4.13) we find that

$$\zeta_y = -F(x)h(x)\frac{yz}{y^2 + z^2}, \quad \zeta_z = F(x)h(x)\frac{y^2}{y^2 + z^2}, \quad \text{along with } \zeta_x = 1. \quad (4.14)$$

Consistency of (4.14) requires that $d(Fh)/dx = 0$, so that we may assume without loss of generality $h(x) = 1/F(x)$. Therefore we have finally

$$\zeta_x = 1, \quad \zeta_y = -\frac{yz}{y^2 + z^2}, \quad \zeta_z = \frac{y^2}{y^2 + z^2}. \quad (4.15)$$

Using these values we find from (4.11)/(4.12) that

$$H_x = \frac{F'(x)}{\mu F(x)}(y^2 + z^2). \quad (4.16)$$

The partial derivatives of the Hamiltonian H as given by (4.10) and (4.16) are consistent provided the following condition is satisfied, namely

$$\frac{\mu'}{\mu} = -2\frac{F'(x)}{\mu F(x)}. \quad (4.17)$$

This implies that μ can be taken to be a function of x only. In explicit terms it is given by

$$\mu(x) = \frac{c}{F^2(x)}, \quad (4.18)$$

where c is an arbitrary constant and the Hamiltonian turns out to be

$$H(x, y, z) = \frac{(y^2 + z^2)}{2\mu(x)}. \quad (4.19)$$

Since the Poisson equations of motion are given by

$$\dot{x}^i = \{x^i, H\} = \{x^i, x^j\} \frac{\partial H}{\partial x^j}, \quad i, j = 1, 2, 3.$$

In our case we have

$$\begin{aligned} \dot{x} &= \{x, y\}H_y + \{x, z\}H_z = uH_y - vH_z \\ \dot{y} &= \{y, x\}H_x + \{y, z\}H_z = -uH_x + wH_z \\ \dot{z} &= \{z, y\}H_y + \{z, x\}H_x = vH_x - wH_y \end{aligned}$$

with u, v and w given by (4.4). Using (4.15) and (4.17) we find that they lead to the following system of equations:

$$\dot{x} = y, \quad \dot{y} = z - y^2 \frac{F'(x)}{F(x)}, \quad \dot{z} = -y - yz \frac{F'(x)}{F(x)}, \quad (4.20)$$

where $F(x)$ is any arbitrary non-vanishing function.

4.2 Connection to Nambu structure

The Poisson structure that we have described in the previous section can also be manifested in terms of a Nambu structure. The construction is completely parallel to the Poisson case. Consider a three dimensional system. We fix a volume form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. The volume three form is connected to the existence of the Hamiltonian bivector fields $X^2 \in \lambda^2$, defined as

$$X_H^2 = \frac{\partial H}{\partial x_1} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{\partial H}{\partial x_2} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + \frac{\partial H}{\partial x_3} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}. \quad (4.21)$$

The volume preserving condition yields

$$L_{X^2} \Omega = i_{X^2} d\Omega + d(i_{X^2} \Omega) = 0.$$

Since $d\Omega = 0$, so we conclude $d(i_{X^2} \Omega) = 0$. From Poincaré's lemma it follows that

$$i_{X^2} \Omega = dH, \quad H \in C^\infty(\mathbb{R}^3)$$

is exact. Thus the Nambu bracket is given by

$$i_{X_H^2} (d\zeta \wedge df) = \{H, h, f\},$$

where

$$\{H, \zeta, f\} = \frac{\partial H}{\partial x_1} \{\zeta, f\}_{x_1 x_2} + \frac{\partial H}{\partial x_2} \{\zeta, f\}_{x_3 x_1} + \frac{\partial H}{\partial x_3} \{\zeta, f\}_{x_1 x_2}.$$

The Hamiltonian equation of motion of the form (5.4) can be easily recast into Nambu Hamiltonian system

$$\dot{x}_i = \{x_i, \zeta, H\}. \quad (4.22)$$

This shows $N = 3$ Calogero-Leyvraz has a natural description in terms of Nambu mechanics.

It is known that the Nambu-Poisson includes all the subordinate Poisson structures. Therefore we can construct this Poisson structure via Nambu geometry, i.e., the Poisson structure on \mathbb{R}^3 may be written as

$$\{x_i, x_j\}^F := \{x_i, x_j, F\} \quad x_i = x, y, z. \quad (4.23)$$

It can be easily shown that when the Nambu bracket satisfies the fundamental identity then the corresponding (reduced) Poisson bracket $\{x_i, x_j\}^F$ satisfies the Jacobi identity.

4.2.1 Interpretation of μ

In this section we illucidate the connection between μ and the last multiplier in the Nambu setting. At first we recapitulate the planar case. Consider planar Hamiltonian system

$$\dot{x} = J(x, y)H_y, \quad \dot{y} = -J(x, y)H_x,$$

where J is associated with the symplectic structure. Using last multiplier equation we obtain

$$\frac{d}{dt} \log M + \frac{\dot{J}}{J} = 0,$$

which yield $M = \frac{1}{J}$. Thus Poisson matrix is given by

$$\mathbb{J} = \begin{pmatrix} 0 & M^{-1} \\ -M^{-1} & 0 \end{pmatrix}.$$

We can generalize the Hamiltonization of a non-planar system using Nambu mechanics, as follows. Consider following volume preserving system

$$\dot{x} = \mu \frac{\partial H_1}{\partial y} \frac{\partial H_2}{\partial z}, \quad \dot{y} = -\mu \frac{\partial H_1}{\partial x} \frac{\partial H_2}{\partial z}, \quad \dot{z} = \mu \frac{\partial H_1}{\partial x} \frac{\partial H_2}{\partial y}.$$

Using

$$\frac{d}{dt} \log M + \sum \frac{\partial W_i}{\partial x_i} = 0$$

we obtain

$$M = \frac{1}{\mu}.$$

Therefore the Nambu Hamiltonian equation can be expressed as

$$\dot{x}_i = M^{-1} \epsilon_{ijk} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}. \quad (4.24)$$

Thus the Nambu Hamiltonian system replaces the ordinary Hamiltonian formulation for higher order differential equations and Jacobi's last multiplier is the inverse of the Nambu structure.

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