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Standard Model Action

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Real \mathbb{Z}_2 –Bi-Gradings, Majorana Modules and the Standard Model Action

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Abstract

The action functional of the Standard Model of particle physics is intimately related to a specific class of first order differential operators called Dirac operators of Pauli type (“Pauli-Dirac operators”). The aim of this article is to carefully analyze the geometrical structure of this class of Dirac operators on the basis of real Dirac operators of simple type. On the basis of simple type Dirac operators, it is shown how the Standard Model action (STM action) may be viewed as generalizing the Einstein-Hilbert action in a similar way the Einstein-Hilbert action is generalized by a cosmological constant. Furthermore, we demonstrate how the geometrical scheme presented allows to naturally incorporate also Majorana mass terms within the Standard Model. For reasons of consistency these Majorana mass terms are shown to dynamically contribute to the Einstein-Hilbert action by a “true” cosmological constant. Due to its specific form, this cosmological constant can be very small. Nonetheless, this cosmological constant may provide a significant contribution to dark matter/energy. In the geometrical description presented this possibility arises from a subtle interplay between Dirac and Majorana masses.

Keywords: Dirac Type Differential Operators, (Real) Clifford Modules,
General Relativity, Gauge Theories, Majorana Masses, Cosmological Constant

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1 Introduction

The dynamical description of fermions and bosons is usually based upon different geometrical schemes. The fermionic actions are always defined in terms of Dirac type operators. In contrast, the gravitational and the Yang-Mills functionals are defined in terms of the respective curvatures associated with connections. Accordingly, the following “ambiguity” in the definition of the fermionic action is not taken into account. Let $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ be a section of the Hermitian Clifford module

$$(\mathcal{E}, \gamma_{\mathcal{E}}) \rightarrow (M, g_M) \quad (1)$$

over an oriented (semi-)Riemannian manifold of even dimension and arbitrary signature. Also, let $\mathcal{D}_{\mathcal{E}}$ be a Dirac (type) operator (see below) that acts on $\mathfrak{Sec}(M, \mathcal{E})$. The fermionic action is defined in terms of the smooth function: $\langle \psi, \mathcal{D}_{\mathcal{E}} \psi \rangle_{\mathcal{E}}$, with $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ being the Hermitian form on \mathcal{E} . Clearly, nothing changes when a (tricky) null is added, i.e.

$$\begin{aligned} \langle \psi, \mathcal{D}_{\mathcal{E}} \psi \rangle_{\mathcal{E}} &\equiv \langle \psi, \mathcal{D}_{\mathcal{E}} \psi \rangle_{\mathcal{E}} + \langle \psi, \Phi_{\mathcal{E}} \psi \rangle_{\mathcal{E}} - \langle \psi, \Phi_{\mathcal{E}} \psi \rangle_{\mathcal{E}} \\ &=: \left\langle \begin{pmatrix} \psi \\ \psi \end{pmatrix}, \begin{pmatrix} \mathcal{D}_{\mathcal{E}} & -\Phi_{\mathcal{E}} \\ \Phi_{\mathcal{E}} & \mathcal{D}_{\mathcal{E}} \end{pmatrix} \begin{pmatrix} \psi \\ \psi \end{pmatrix} \right\rangle_{2_{\mathcal{E}}}. \end{aligned} \quad (2)$$

Here, $\Phi_{\mathcal{E}} \in \mathfrak{Sec}(M, \text{End}(\mathcal{E}))$ denotes an arbitrary zero-order operator and ${}^2\mathcal{E} := \mathcal{E} \oplus \mathcal{E}$ the “doubling” of \mathcal{E} with an appropriately induced Hermitian form and Clifford structure.

This apparently trivial observation may become meaningful, actually, if the bosonic action is also defined in terms of Dirac operators. In fact, it has been shown that both the fermionic part and the bosonic part of the Standard Model action – the latter also including the Einstein-Hilbert functional – can be geometrically described in terms of a single Dirac operator (c.f., for instance, in [AT '96] and [Tol '98] with respect to the combined Einstein-Hilbert-Yang-Mills and the Einstein-Hilbert-Yang-Mills-Higgs action in terms of the non-commutative residue):

$$\begin{aligned} \mathcal{P}_{\text{D}} &= \begin{pmatrix} i\mathcal{D}_{\text{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} & -\mathcal{F}_{\text{D}} \\ \mathcal{F}_{\text{D}} & i\mathcal{D}_{\text{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \end{pmatrix} \\ &\equiv i\mathcal{D}_{\text{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} + \mathcal{I}_{\mathcal{E}} \circ \mathcal{F}_{\text{D}}. \end{aligned} \quad (3)$$

Here, respectively, the Dirac operator

$$\mathcal{D}_{\mathcal{E}} \equiv i\mathcal{D}_{\text{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{E}} \quad (4)$$

belongs to the distinguished class of Dirac operators of simple type on the Clifford module (1) and \mathcal{F}_{D} is the “quantized” relative curvature of (4). The details of these and the following notions will be summarized in the next section.

The specific class of Dirac operators (4) will play a crucial role in what follows (see also, for example, [Qui '85], [Bis '86] for the role of simple type Dirac operators in the case of the family index theorem and [Con '94] of non-commutative geometry). When evaluated with respect to (3), the “total Dirac action” (see below)

$$\mathcal{I}_{\text{D,tot}} := \int_M \left(\langle \Psi, \mathcal{P}_{\text{D}} \Psi \rangle_{2_{\mathcal{E}}} + \text{tr}_{\gamma}(\text{curv}(\mathcal{P}_{\text{D}}) - \varepsilon \text{ev}_g(\omega_{\text{D}}^2)) \right) \text{dvol}_M \quad (5)$$

decomposes into the various parts of the Standard Model action, including gravity described in terms of the Einstein-Hilbert functional. In particular, the fermionic part reduces to the usual Dirac-Yukawa action:

$$\int_M \langle \Psi, \not{P}_D \Psi \rangle_{2\mathcal{E}} dvol_M = \int_M \langle \psi, (i\not{\partial}_A + \phi_\mathcal{E})\psi \rangle_\mathcal{E} dvol_M, \quad (6)$$

provided the sections $\Psi \in \mathfrak{Sec}(M, {}^2\mathcal{E})$ on the doubled Clifford module

$$({}^2\mathcal{E} \equiv \mathcal{E} \oplus \mathcal{E}, \tau_{2\mathcal{E}} \equiv \tau_\mathcal{E} \ominus \tau_\mathcal{E}, \gamma_{2\mathcal{E}} \equiv \gamma_\mathcal{E} \oplus \gamma_\mathcal{E}) \rightarrow (M, g_M) \quad (7)$$

are restricted to “diagonal sections” $\Psi = (\psi, \psi)$ and the sections $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ are restricted, furthermore, to the “physical sub-bundle” $\mathcal{E}_{\text{phys}} \hookrightarrow \mathcal{E} \rightarrow M$ of the underlying Clifford module (c.f. [TT ’06a] and the corresponding references therein).

The specific form of the Dirac operator (3), acting on the sections of the doubled Clifford module, parallels the first order differential operator

$$i\not{\partial}_A - m - i\not{F}_A, \quad (8)$$

with F_A being the electromagnetic field strength that was introduced to account for the anomalous magnetic moment of the proton at a time when it was not yet clear that the proton is a composite of quarks but considered as “elementary” (see, for example, Chapter 2-2-3 in [IZ ’87]). However, when the quarks entered the stage of particle physics the *Pauli term* $i\not{F}_A$ became superfluous. Moreover, the additional fermionic interaction caused by the Pauli term rendered the quantum field theory based upon (8) non-renormalizable.

It is a remarkable feature of “Dirac type gauge theories” that the complete Standard Model action (including gravity) can be geometrically described in terms of the “Pauli type Dirac operator” (3). It has been shown that this description of the Standard Model allows to make a prediction for the value of the mass of the Higgs boson which is consistent with all the otherwise known data from the Standard Model. In other words, the geometrical description of the Standard Model based upon the geometry of \not{P}_D renders the Standard Model even more predictive than it is the case with respect to its usual description (c.f. [TT ’06b], where one can also find a brief comparison to similar results presented in [CCM ’06], see also [CM ’07]). For this matter it seems worth investigating more closely the specific form of Pauli type Dirac operators and the restrictions made with respect to the fermionic sector that guarantee the Pauli like term $\mathcal{I}_\mathcal{E} \circ \not{P}_D$ to only contribute to the bosonic part of the total Dirac action (5).

In this paper, we carefully discuss the fact that in the bosonic part of (5) only curvature terms enter, whereas the fermionic part is determined by connections, only. This subtle interplay between the fermionic and the bosonic part of the total Dirac action permits to geometrically regard the Yang-Mills action as a “covariant generalization” of the Einstein-Hilbert action and the Standard Model action as a natural generalization of the Einstein-Hilbert action with cosmological constant. Moreover, the geometrical analysis of the operator (3) permits to also naturally include the notion of Majorana masses within the scheme of Dirac type gauge theories. It will be shown that the thus described Majorana masses dynamically contribute to the bosonic part of (5) in the form of Einstein’s “biggest blunder”.

Some of the features presented seem close to the geometrical description of the Standard Model in terms of A. Connes' non-commutative geometry (c.f., for example, [Con '95], [Con '96], [CCM '06] and [CM '07]). However, the geometrical setup presented is different in various respects. For example, the relation between Dirac operators and connections is based upon the canonical first order decomposition of any Dirac (type) operator (c.f. Section 2). As a consequence, the Higgs boson is intimately tied to gravity in the setup presented. Indeed, the Higgs boson is shown to generalize the Yang-Mills connection via the metric. Furthermore, the bosonic part of the total Dirac action (5) is based upon the canonical second order decomposition of any Dirac (type) operator. This, indeed, provides a canonical generalization of the Einstein-Hilbert action with cosmological constant (c.f. Section 3). These basic features of Dirac type gauge theories will be the starting point of everything that follows.

The paper is organized as follows: The following section provides a summary of some of the basic notions already used in the introduction. Also, some motivation for the ensuing constructions are presented. In the third section, we present the geometrical picture that underlies Dirac type gauge theories. In particular we discuss the Einstein-Hilbert action from the point of view of Dirac operators. In the fourth section, we discuss Pauli type Dirac operators in view of “real, \mathbb{Z}_2 -bi-graded Clifford modules” (“real Clifford modules”, for short, see the work [ABS '64], which may serve as a kind of standard reference). We present some examples of particular interest. In the fifth section, we discuss the geometrical description of Majorana masses within Dirac type gauge theories. In particular, we discuss a generalization of the STM action when Majorana masses are taken into account. The sixth section is devoted to a discussion of the Standard Model (STM) action in terms of real Dirac operators of simple type. This will provide a new geometrical picture of the STM action and how the latter is related to the Einstein-Hilbert functional of General Relativity. Finally, the last section summarizes the main conclusions. Before we get started, however, it might be worth presenting a brief summary of the main results obtained.

The presented geometrical discussion of the operators (3), defining the bosonic part of the total Dirac action (5), is based upon a careful analysis of the geometrical background of the Dirac equation and the Majorana equation:

$$i\cancel{\partial}\chi = m_D\chi \quad \Leftrightarrow \quad \begin{cases} i\cancel{\partial}\chi_R = m_D\chi_L, \\ i\cancel{\partial}\chi_L = m_D\chi_R, \end{cases} \quad (9)$$

$$i\cancel{\partial}\chi = m_M\chi^{cc} \quad \Leftrightarrow \quad \begin{cases} i\cancel{\partial}\chi_R = m_M\chi_R^{cc}, \\ i\cancel{\partial}\chi_L = m_M\chi_L^{cc}. \end{cases} \quad (10)$$

Here, respectively, χ_R, χ_L are the “chiral” eigen sections, m_D is the “Dirac mass”, m_M the “Majorana mass” and “ cc ” has the physical meaning of “charge conjugation”. It will be shown how a specific interplay between the two basic \mathbb{Z}_2 -gradings, realized in nature by chirality and charge conjugation, allows one to overcome the issue of fermion doubling. The latter is usually encountered in the description of the fermionic action in terms of simple type Dirac operators. Furthermore, the interplay between parity and charge conjugation will also give the Pauli-Dirac operators their geometrical meaning. The geometrical background of Pauli-Dirac operators in terms of real Clifford modules has been partially discussed before in

[Tol '09]. There, however, only the reduced Dirac action:

$$\mathcal{I}_{\text{D,red}} := \int_M \text{tr}_\gamma \text{curv}(\mathcal{D}) \, d\text{vol}_M \quad (11)$$

was used. Also, the requirements imposed on “particle-anti-particle modules” (c.f. Definition 3 in loc. site) turn out to be too restrictive and do not allow to geometrically describe, for example, Dirac’s first order differential operator $i\mathcal{D} - m_{\text{D}}$ in terms of simple type Dirac operators. It thus does not account for the issue of the fermion doubling already mentioned. This drawback is remedied in this work in terms of Dirac modules associated with Majorana modules (c.f. Section 4). Also, we take the opportunity to generalize formulae 110 and 113 of Lemma 1 in loc. site, which hold true only in the special case of $\Phi \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{E}))$ (c.f. the corresponding formulae 110 and 111 of Lemma 4.1).

In this work emphasis is put on *real Dirac operators of simple type*, which turn out to yield an appropriate geometrical description of both the fermionic and the bosonic action of the Standard Model. Indeed, on the basis of real Dirac operators of simple type, the Standard Model action will be shown to be described by the Einstein-Hilbert action with a “cosmological constant” term (c.f. Section 6):

$$\mathcal{I}_{\text{EHYMH}} = \int_M \text{tr}_\gamma \left[\text{curv}(\mathcal{D}_A) + \Phi_{\text{YMH}}^2 \right] \, d\text{vol}_M. \quad (12)$$

This may be regarded as a generalization of Lovelock’s Theorem (c.f. [Lov '72] and Section 3). In contrast to this theorem, however,

$$\Lambda_{\text{YMH}} := \text{tr}_\gamma \left(\Phi_{\text{YMH}}^2 \right) \quad (13)$$

also depends on the metric and, in fact, is shown to coincide with the (Hodge dual of the) STM Lagrangian density $\mathcal{L}_{\text{YMH}} \in \Omega^n(M)$ plus a “true” cosmological constant term that is determined by Dirac and Majorana masses of an otherwise non-interacting species of particles, collectively called “cosmological neutrinos” (c.f. Section 4):

$$\begin{aligned} \Lambda_{\text{YMH}} &= * \mathcal{L}_{\text{YMH}} - \Lambda_{\text{DM},\nu}, \\ \Lambda_{\text{DM},\nu} &\equiv a' \text{tr}_{\mathcal{W}_\nu} m_{\text{D},\nu}^4 - b' \text{tr}_{\mathcal{W}_\nu} m_{\text{D},\nu}^2 + a' \text{tr}_{\mathcal{W}_\nu} m_{\text{M},\nu}^4 - b' \text{tr}_{\mathcal{W}_\nu} m_{\text{M},\nu}^2 \\ &\quad - 2a' \text{tr}_{\mathcal{W}_\nu} (m_{\text{D},\nu} \circ m_{\text{M},\nu})^2. \end{aligned} \quad (14)$$

Here, $a', b' > 0$ are numerical constants that are determined by the dimension of the underlying (space-time) manifold (c.f. Sections 4 and 5). Though not discussed in detail in this work, the point to be emphasized here is that, due to its specific form, the cosmological constant $\Lambda_{\text{DM},\nu}$ can be arbitrarily small, though, for example, the contribution of the Majorana masses $m_{\text{M},\nu}$ to the “dark matter/energy” of the universe can be very high.

2 Preliminaries

For the sake of self-consistency and for the convenience of the reader, we summarize some facts about general Clifford modules although later on we shall be mainly concerned with the

case of twisted Grassmann bundles. Nonetheless, it seems worth starting with the general case to clarify the general scheme. Afterwards, we shall introduce some facts concerning the case of twisted Grassmann bundles (resp. sub-bundles thereof).

The bundle of Grassmann and Clifford algebras with respect to (M, g_M) are supposed to be generated by the cotangent bundle of M . In what follows, however, we shall be mainly concerned with their complexifications. In particular, all Clifford modules are considered as complex vector bundles.

2.1 General Clifford modules

To get started let $(\mathcal{E}, \gamma_{\mathcal{E}}) \rightarrow (M, g_M)$ be a general bundle of Clifford modules over a smooth, orientable (semi-)Riemannian manifold of even dimension $n = p + q$ and signature $s = p - q$. Let $Cl_M \rightarrow M$ be the algebra bundle of (complexified) Clifford algebras with respect to the (semi-)metric g_M that is generated by the cotangent bundle $T^*M \rightarrow M$. The mapping $T^*M \xrightarrow{\gamma_{\mathcal{E}}} \text{End}(\mathcal{E})$ denotes a Clifford mapping:

$$\gamma_{\mathcal{E}}(\alpha)^2 = \varepsilon g_M(\alpha, \alpha) \text{id}_{\mathcal{E}}, \quad (15)$$

for all $\alpha \in T^*M$. Here, the use of $\varepsilon \in \{\pm 1\}$ allows to treat both signatures $\pm s$ simultaneously. Especially, it takes into account that both signatures are physically indistinguishable.

By abuse of notation, Clifford mappings also denote the induced representations of the Clifford bundle on the corresponding algebra bundles of endomorphisms $\text{End}(\mathcal{E}) \rightarrow M$. Also, we do not distinguish between (semi-)metrics of signature $s = p - q$ on M and sections of the ‘‘Einstein-Hilbert bundle’’

$$\mathcal{E}_{\text{EH}} := \mathcal{F}_M \times_{\text{GL}(n)} \text{GL}(n)/\text{SO}(p, q) \rightarrow M \quad (16)$$

that is associated with the frame bundle $\mathcal{F}_M \rightarrow M$ of M . Finally, the scalar products on the tangent and the cotangent bundle of M are also denoted by g_M .

On every Clifford module there exists a canonical one-form $\Theta \in \Omega^1(M, \text{End}(\mathcal{E}))$, which locally reads:

$$\Theta \stackrel{\text{loc.}}{=} \frac{\varepsilon}{n} e^k \otimes \gamma_{\mathcal{E}}(e_k^{\flat}). \quad (17)$$

Here, (e_1, \dots, e_n) is a local (orthonormal) basis of $TM \rightarrow M$ and (e^1, \dots, e^n) its dual. The mappings: $\flat/\sharp : TM \rightleftharpoons T^*M$, are the ‘‘musical’’ isomorphisms induced by g_M .

The canonical one-form (17) is thus the (normalized) soldering form of the frame bundle of M lifted to $\mathcal{E} \rightarrow M$. It also plays a basic role in the definition of the twistor operator in conformal geometry¹ (c.f. page 164, Lecture 6 in [Bra '04]). Indeed, the canonical one-form provides a right inverse of the restriction of the canonical mapping:

$$\begin{aligned} \delta_{\gamma} : \Omega^*(M, \text{End}(\mathcal{E})) &\longrightarrow \mathfrak{Sec}(M, \text{End}(\mathcal{E})) \\ \omega \otimes \chi &\longmapsto \gamma_{\mathcal{E}}(\sigma_{\text{Ch}}^{-1}(\omega)) \circ \chi \end{aligned} \quad (18)$$

¹The author would like to thank M. Schneider for pointing him out this relation.

to one-forms via

$$\begin{aligned} ext_{\Theta} : \Omega(M, \text{End}(\mathcal{E})) &\longrightarrow \Omega^1(M, \text{End}(\mathcal{E})) \\ \Phi &\mapsto \Theta \wedge \Phi. \end{aligned} \quad (19)$$

Here,

$$\begin{aligned} \sigma_{\text{Ch}} : Cl_M &\longrightarrow \Lambda_M \\ a &\mapsto \gamma_{\text{Ch}}(a)1_{\Lambda} \end{aligned} \quad (20)$$

denotes Chevalley's canonical linear isomorphism between the Clifford bundle and the Grassmann bundle $\Lambda_M \rightarrow M$ of M . It is based upon the Clifford mapping:

$$\begin{aligned} \gamma_{\text{Ch}} : T^*M &\longrightarrow \text{End}(\Lambda_M) \\ \alpha &\mapsto \begin{cases} \Lambda_M &\longrightarrow \Lambda_M \\ \omega &\mapsto \varepsilon \text{int}_g(\alpha)\omega + ext(\alpha)\omega. \end{cases} \end{aligned} \quad (21)$$

Here, respectively, $\text{int}_g(\alpha)\omega$ is the contraction (inner derivative) of ω with respect to $\alpha^{\sharp} \in TM$ and $ext(\alpha)\omega$ denotes the exterior multiplication of ω with respect to $\alpha \in T^*M$.

The isomorphism (20) is referred to as ‘‘symbol map’’ and its inverse as ‘‘quantization map’’. Although misleading from a physical point of view, we shall still use this common term and call the section $q \equiv \delta_{\gamma}(\alpha) \in \mathfrak{Sec}(M, \text{End}(\mathcal{E}))$ the ‘‘quantization’’ of the ‘‘non-commutative super-field’’ $\alpha \in \Omega^*(M, \text{End}(\mathcal{E})) = \bigoplus_{k=0}^n \mathfrak{Sec}(M, \Lambda^k T^*M \otimes \text{End}(\mathcal{E}))$.

On the affine set $\mathcal{A}(\mathcal{E})$ of all (linear) connections on $\mathcal{E} \rightarrow M$ there exists a distinguished affine subset, consisting of what is called *Clifford connections*. This subset may be characterized as follows:

$$\mathcal{A}_{\text{Cl}}(\mathcal{E}) := \{\partial_A \in \mathcal{A}(\mathcal{E}) \mid \partial_A^{T^*M \otimes \text{End}(\mathcal{E})} \Theta \equiv 0\}. \quad (22)$$

We call a first order differential operator \mathcal{D} , acting on sections of $\mathcal{E} \rightarrow M$, of *Dirac type*, provided it fulfills:

$$[\mathcal{D}, f] = \gamma_{\mathcal{E}}(df), \quad (23)$$

for all $f \in C^{\infty}(M)$. The set of all such operators is denoted by $\mathcal{D}(\mathcal{E})$.

An odd Dirac type operator $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ on a \mathbb{Z}_2 -graded Clifford module bundle $(\mathcal{E}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}})$ is called a *Dirac operator*. Here, $\tau_{\mathcal{E}} \in \text{End}(\mathcal{E})$ denotes the underlying grading involution, such that $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ is a Dirac operator provided it satisfies: $\mathcal{D} \circ \tau_{\mathcal{E}} = -\tau_{\mathcal{E}} \circ \mathcal{D}$.

At this point, we have to warn the reader. Usually, every Dirac type operator is assumed to be odd. For reasons that will become clear in the next section, however, we want to distinguish between Dirac type operators and Dirac operators on a Clifford module. Clearly, every Dirac operator is of Dirac type. Moreover, every Dirac type operator may be written as

$$\mathcal{D} = \mathcal{D}_A + \Phi_A, \quad (24)$$

where $\not\partial_A \equiv \delta_\gamma \circ \partial_A$ and $\Phi_A \in \mathfrak{Sec}(M, \text{End}(\mathcal{E}))$, in general, will depend on the choice of the Clifford connection $\partial_A \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$.

Every Dirac type operator $\not\mathcal{D} \in \mathcal{D}(\mathcal{E})$ has a unique first order and second order decomposition:

$$\not\mathcal{D} = \not\partial_B + \Phi_D, \quad (25)$$

$$\not\mathcal{D}^2 = \Delta_B + V_D. \quad (26)$$

Here, $\not\partial_B \equiv \delta_\gamma \circ \partial_B$ is the quantization of the *Bochner connection* $\partial_B \in \mathcal{A}(\mathcal{E})$ that is uniquely defined by $\not\mathcal{D}$ via

$$2 \text{ev}_g(df, \partial_B \psi) := \varepsilon \left([\not\mathcal{D}^2, f] - \delta_g df \right) \psi, \quad (27)$$

for all $f \in C^\infty(M)$ and $\psi \in \mathfrak{Sec}(M, \mathcal{E})$. The second order operator:

$$\Delta_B := \varepsilon \text{ev}_g \left(\partial_B^{T^*M \otimes \mathcal{E}} \circ \partial_B \right), \quad (28)$$

is the induced *Bochner-Laplacian* (or “trace/connection Laplacian”, see, for example, in [BG ’90] and [Gil ’95], as well as in Chapter 2.1 in [BGV ’96]).

With every Dirac type operator $\not\mathcal{D} \in \mathcal{D}(\mathcal{E})$ there is naturally associated with a connection $\partial_D \in \mathcal{A}(\mathcal{E})$, such that $\not\partial_D \equiv \delta_\gamma \circ \partial_D = \not\mathcal{D}$. This *Dirac connection* is given by (c.f. [TT ’06a])

$$\partial_D := \partial_B + \text{ext}_\Theta(\Phi_D). \quad (29)$$

We call the one-form $\omega_D := \text{ext}_\Theta(\not\mathcal{D} - \not\partial_B)$, uniquely associated with $\not\mathcal{D} \in \mathcal{D}(\mathcal{E})$, the *Dirac form* and the tangent vector field: $\xi_D := -\varepsilon(\text{tr}_\mathcal{E} \omega_D)^\sharp$, the *Dirac vector field*.

It follows that

$$\text{tr}_\mathcal{E} V_D = \text{tr}_\gamma \left(\text{curv}(\not\mathcal{D}) - \varepsilon \text{ev}_g(\omega_D^2) \right) + \text{div} \xi_D, \quad (30)$$

where

$$\text{curv}(\not\mathcal{D}) := \partial_D \wedge \partial_D \in \Omega^2(M, \text{End}(\mathcal{E})) \quad (31)$$

is the curvature of the Dirac type operator $\not\mathcal{D} \in \mathcal{D}(\mathcal{E})$ and $\text{tr}_\gamma \equiv \text{tr}_\mathcal{E} \circ \delta_\gamma$ is the “quantized trace”.

We call the Hodge dual of the smooth function (30) the *universal Dirac-Lagrangian*:

$$\mathcal{L}_D := * \text{tr}_\mathcal{E} V_D. \quad (32)$$

Its cohomology class is generated by the top-form: $\text{tr}_\gamma(\text{curv}(\not\mathcal{D}) - \varepsilon \text{ev}_g(\omega_D^2)) \text{dvol}_M$.

It follows that two Dirac type operators $\not\mathcal{D}, \not\mathcal{D}' \in \mathcal{D}(\mathcal{E})$ define the same Bochner connection, provided $\not\mathcal{D}' - \not\mathcal{D}$ anti-commutes with the Clifford connection (c.f. Corollary 4.1). Therefore, on every \mathbb{Z}_2 -graded Clifford module there is a distinguished class of Dirac type operators, depending on whether $\Phi_D = \not\mathcal{D} - \not\partial_B$ even or odd. In particular, we call a Dirac operator of *simple type* if it reads:

$$\not\mathcal{D} = \not\partial_A + \tau_\mathcal{E} \circ \phi_\mathcal{E}, \quad (33)$$

where $\phi_{\mathcal{E}} \in \text{Sec}(M, \text{End}_{\gamma}^{-}(\mathcal{E}))$. Here, $\text{End}_{\gamma}(\mathcal{E}) \hookrightarrow \text{End}(\mathcal{E}) \rightarrow M$ is the algebra sub-bundle of all $\gamma_{\mathcal{E}}$ -invariant endomorphisms and $\tau_{\mathcal{E}}$ is the underlying grading involution on $\mathcal{E} \rightarrow M$.

The specific form (33) of simple type Dirac operators is determined by the condition that the corresponding Bochner connections are given by Clifford connections. Of course, every $\not{D}_{\Lambda} \in \mathcal{D}(\mathcal{E})$ is of simple type.

Likewise, one may consider Dirac type operators of the form

$$\begin{aligned} \not{D} &:= \not{D}_{\Lambda} + \Phi_{\text{H}}, \\ \Phi_{\text{H}} &\in \text{Sec}(M, \text{End}_{\gamma}(\mathcal{E})). \end{aligned} \quad (34)$$

These Dirac type operators have the property that their Dirac connections read:

$$\begin{aligned} \partial_{\text{D}} &= \partial_{\text{B}} + \text{ext}_{\Theta}(\Phi_{\text{D}}) \\ &= \partial_{\Lambda} + \text{ext}_{\Theta}(n\varepsilon \Phi_{\text{H}}) + \text{ext}_{\Theta}[(1 - n\varepsilon)\Phi_{\text{H}}] \\ &= \partial_{\Lambda} + \Phi_{\text{H}}\Theta \\ &\equiv \partial_{\Lambda} + H. \end{aligned} \quad (35)$$

We call the one-form

$$\begin{aligned} H &:= \Phi_{\text{H}}\Theta \\ &\stackrel{\text{loc.}}{=} \varepsilon_n e^k \otimes \gamma_{\mathcal{E}}(e_k^b) \circ \Phi_{\text{H}} \end{aligned} \quad (36)$$

the *Higgs gauge potential* and the connections: $\partial_{\text{YMH}} := \partial_{\Lambda} + H \stackrel{\text{loc.}}{=} \partial + A + H$, *Yang-Mills-Higgs connections* on the Clifford module $\mathcal{E} \rightarrow M$.

We remark that for $\Phi_{\text{H}} \in \text{Sec}(M, \text{End}_{\gamma}^{+}(\mathcal{E}))$, the Yang-Mills-Higgs connections are odd connections. They have the property that the (locally defined) Yang-Mills gauge potentials $A \in \Omega^1(M, \text{End}_{\gamma}^{+}(\mathcal{E}))$ provide connections which respect the sub-bundles $\mathcal{E}^{\pm} \rightarrow M$. The Yang-Mills part of a Yang-Mills-Higgs connection is thus “chirality preserving”. In contrast, the Higgs gauge potential $H \in \Omega^1(M, \text{End}^{-}(\mathcal{E}))$ provides a connection between these sub-bundles of $\mathcal{E} \rightarrow M$ and thus constitutes the “chirality violating” part of the Yang-Mills-Higgs gauge potential. This is similar to the geometrical interpretation of the connections constructed within the original Connes-Lott description of the Yang-Mills-Higgs sector of the Standard Model (c.f. [CL ’90], [GV ’93], [Con ’94], [SZ ’95], [KS ’96]; see also [MO ’94] and [MO ’96] in the case of alternative approaches). However, in contrast to non-commutative geometry, where mainly the algebraic structure of connections is taken into account, Dirac connections are always related to the underlying geometry that is encoded within the canonical one-form (17). In particular, within the scheme presented, the Higgs gauge potential is intimately related to gravity.

Definition 2.1 A Clifford module bundle $(\mathcal{E}, \gamma_{\mathcal{E}}) \rightarrow (M, g_{\text{M}})$ is said to be “flat”, provided there is a Clifford connection $\not{D}_{\Lambda} \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$ fulfilling

$$\text{curv}(\not{D}_{\Lambda}) = \mathbb{R}\text{iem}(g_{\text{M}}), \quad (37)$$

where $\mathbb{R}\text{iem}(g_{\text{M}}) \in \Omega^2(M, \text{End}(\mathcal{E}))$ is the Riemann curvature with respect to g_{M} lifted to the Clifford module.

We call a Clifford connection satisfying (37) a *flat Clifford connection* and denote it by $\partial \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$.

Let $(\mathcal{E}, \gamma_{\mathcal{E}}, \gamma_{\mathcal{E}, \text{op}}) \rightarrow M$ be a *Clifford bi-module*. Besides the Clifford left action, provided by the Clifford mapping $\gamma_{\mathcal{E}} : T^*M \rightarrow \text{End}(\mathcal{E})$, there is also a right action of $Cl_M \rightarrow M$ on $\mathcal{E} \rightarrow M$. Accordingly, there is a Clifford left action of the bundle of *opposite Clifford algebras*: $Cl_M^{\text{op}} \rightarrow M$ that is induced by a Clifford mapping $\gamma_{\mathcal{E}, \text{op}} : T^*M \rightarrow \text{End}(\mathcal{E})$. This left action of Cl_M^{op} on \mathcal{E} is again denoted by $\gamma_{\mathcal{E}, \text{op}}$. Note that

$$\gamma_{\mathcal{E}, \text{op}}(\mathbf{a}) \in \text{End}_{\gamma}(\mathcal{E}) \quad (38)$$

for all $\mathbf{a} \in Cl_M^{\text{op}}$.

On a Clifford bi-module there exists a distinguished class of connections.

Definition 2.2 *Let $(\mathcal{E}, \gamma_{\mathcal{E}}, \gamma_{\mathcal{E}, \text{op}}) \rightarrow M$ be a Clifford bi-module. A connection $\nabla^{\mathcal{E}} \in \mathcal{A}(\mathcal{E})$ is called “S–reducible”, provided it is a right Clifford connection:*

$$\begin{aligned} \nabla_{\xi}^{\text{End}(\mathcal{E})} \gamma_{\mathcal{E}, \text{op}}(\mathbf{a}) &:= [\nabla_{\xi}^{\mathcal{E}}, \gamma_{\mathcal{E}, \text{op}}(\mathbf{a})] \\ &= \gamma_{\mathcal{E}, \text{op}}(\nabla_{\xi}^{\text{Cl}^{\text{op}}} \mathbf{a}), \end{aligned} \quad (39)$$

for all $\mathbf{a} \in \mathfrak{Sec}(M, Cl_M^{\text{op}})$ and $\xi \in \mathfrak{Sec}(M, TM)$. The (affine) sub-space of S–reducible connections is denoted by $\mathcal{A}_S(\mathcal{E}) \subset \mathcal{A}(\mathcal{E})$.

We make use of the following (common) terminology: A “Clifford module” generically means a Clifford left module. Accordingly, “Clifford connections” always refer to the appropriate left action. Likewise, the notion of “Dirac (type) operators” also refers to this left action. However, on a Clifford bi-module every Dirac (type) operator $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ may be considered to act on $\mathfrak{Sec}(M, \mathcal{E})$ either from the left, or from the right. That is, one has to distinguish between “left-Dirac (type) operators” and “right-Dirac (type) operators”. In the sequel, “Dirac (type) operators” always mean left-Dirac (type) operators. Note that for every right-Dirac (type) operator there is a unique Dirac (type) operator $\mathcal{D}_{\text{op}} \in \mathcal{D}(\mathcal{E})$, which acts from the left on $\mathfrak{Sec}(M, \mathcal{E})$ via $\gamma_{\mathcal{E}, \text{op}}$. Clearly, every $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ uniquely defines an appropriate $\mathcal{D}_{\text{op}} \in \mathcal{D}(\mathcal{E})$ and vice versa. We thus call \mathcal{D}_{op} the *opposite Dirac (type) operator* associated with $\mathcal{D} \in \mathcal{D}(\mathcal{E})$.

Note that S–reducible connections on a Clifford bi-module may also be characterized by the requirement

$$\nabla^{T^*M \otimes \text{End}(\mathcal{E})} \Theta_{\text{op}} = 0. \quad (40)$$

Here, $\Theta_{\text{op}} \stackrel{\text{loc.}}{=} \frac{\varepsilon}{n} e^k \otimes \gamma_{\mathcal{E}, \text{op}}(e_k^{\flat})$ is the canonical one-form represented on the Clifford bi-module via $\gamma_{\mathcal{E}, \text{op}}$. Hence, $\delta_{\gamma_{\text{op}}} \circ \text{ext}_{\Theta_{\text{op}}} = \frac{\varepsilon}{n} \gamma_{\mathcal{E}, \text{op}}(e_k^{\flat} e^k) = \text{id}_{\mathcal{E}}$.

Clearly, the Grassmann bundle provides the archetypical example of a Clifford bi-module.

Definition 2.3 *A Dirac (type) operator $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ on a Clifford bi-module is called “S–reducible”, provided its Dirac connection is S–reducible: $\partial_{\mathcal{D}} \in \mathcal{A}_S(\mathcal{E})$. The set of all S–reducible Dirac (type) operators is denoted by $\mathcal{D}_S(\mathcal{E})$.*

In this section, we summarized some basic notions with respect to general Clifford (bi-) modules. In the sequel, we shall restrict our further discussion mainly to the more specific case of twisted Grassmann bundles. On the one hand, this case is broad enough to geometrically describe most of the cases encountered in physics. On the other hand, it is topologically less restrictive than the case of a twisted spinor bundle.

2.2 Twisted Grassmann bundles

Basically, the advantage in restricting to twisted Grassmann bundles is provided by the fact that each section of the Einstein-Hilbert bundle then yields a natural Clifford action and thus turns the twisted Grassmann bundle into a Clifford module bundle. This Clifford action, of course, is given by Chevalley's canonical isomorphism (20), which only takes the metric structure into account.

Therefore, let $E \rightarrow M$ be any Hermitian and (maybe trivially) \mathbb{Z}_2 -graded complex vector bundle of finite rank. We then consider

$$\mathcal{S} := \Lambda_M \otimes_{\mathbb{C}} E \rightarrow M. \quad (41)$$

Any section $g_M \in \mathfrak{Sec}(M, \mathcal{E}_{\text{EH}})$ turns (41) into a bundle of Clifford left modules according to the action:

$$\begin{aligned} Cl_M \times_M \mathcal{S} &\rightarrow \mathcal{S} \\ (a, \omega \otimes \chi) &\mapsto \sigma_{\text{Ch}}(a\sigma_{\text{Ch}}^{-1}(\omega)) \otimes \chi, \end{aligned} \quad (42)$$

where the Clifford multiplication is denoted by juxtaposition. The underlying Clifford mapping is denoted, again, by γ_{Ch} and no distinction is made between the Clifford mapping and its induced homomorphism.

Note that

$$\text{End}_{\gamma}(\mathcal{S}) = Cl_M^{\text{op}} \times_M \text{End}(E), \quad (43)$$

with $Cl_M^{\text{op}} \rightarrow M$ acting from the right on $\mathcal{S} \rightarrow M$. Also, note that any $g_M \in \mathfrak{Sec}(M, \mathcal{E}_{\text{EH}})$ turns $\mathcal{S} \rightarrow M$ into a Hermitian vector bundle:

$$\langle \omega_1 \otimes \chi_1, \omega_2 \otimes \chi_2 \rangle_{\mathcal{S}} = g_{\Lambda M}(\omega_1, \omega_2) \langle \chi_1, \chi_2 \rangle_{\text{E}}. \quad (44)$$

Here, $\langle \cdot, \cdot \rangle_{\text{E}}$ is the Hermitian product on $E \rightarrow M$ and $g_{\Lambda M}$ is the induced (semi-)metric on the Grassmann bundle $\Lambda_M \rightarrow M$.

A first order differential operator \mathcal{D} , acting on sections of $\mathcal{S} \rightarrow M$, is called of Dirac type, if there is a section $g_M \in \mathfrak{Sec}(M, \mathcal{E}_{\text{EH}})$ such that

$$[\mathcal{D}, f] = \gamma_{\text{Ch}}(df), \quad (45)$$

for all $f \in C^{\infty}(M)$. The set of all of these operators is denoted by $\mathcal{D}(\mathcal{S})$. Similar to the general case, a Dirac operator on a twisted Grassmann bundle is defined to be an odd Dirac type operator with respect to the particular grading involution:

$$\tau_{\mathcal{S}} = \tau_M \circ \iota_{\mathcal{S}},$$

$$\iota_{\mathcal{S}} := \text{id}_{\Lambda} \otimes \tau_{\mathbb{E}}. \quad (46)$$

Here, $\tau_{\mathbb{E}}$ is a grading involution on $E \rightarrow M$ that can also be trivial. The *chirality* involution $\tau_{\mathbb{M}} \in \text{End}(\mathcal{E})$ is defined by

$$\tau_{\mathbb{M}} := \sqrt{(-1)^{n(n-1)/2+q}} \delta_{\gamma}(dvol_{\mathbb{M}}), \quad (47)$$

with $dvol_{\mathbb{M}} \in \Omega^n(M)$ being the metric induced volume form. We call $\iota_{\mathcal{S}} \in \text{End}_{\gamma}(\mathcal{S})$ the “inner involution”.

The *universal Dirac action* is (formally) defined by the functional²:

$$\begin{aligned} \mathcal{I}_{\mathbb{D}} : \mathcal{D}(\mathcal{S}) &\longrightarrow \mathbb{C} \\ \mathcal{D} &\longmapsto \langle\langle [M], [\mathcal{L}_{\mathbb{D}}] \rangle\rangle \equiv \int_M \mathcal{L}_{\mathbb{D}}. \end{aligned} \quad (48)$$

The *total Dirac action* is given by the functional:

$$\begin{aligned} \mathcal{I}_{\mathbb{D},\text{tot}} : \mathfrak{Sec}(M, \mathcal{S}) \times \mathcal{D}(\mathcal{S}) &\longrightarrow \mathbb{C} \\ (\psi, \mathcal{D}) &\longmapsto \langle\langle \psi, \mathcal{D}\psi \rangle\rangle + \mathcal{I}_{\mathbb{D}}(\mathcal{D}), \end{aligned} \quad (49)$$

with

$$\langle\langle \psi, \mathcal{D}\psi \rangle\rangle := \int_M \langle \psi, \mathcal{D}\psi \rangle_{\mathcal{S}} dvol_{\mathbb{M}}. \quad (50)$$

For every (symmetric) $\mathcal{D} \in \mathcal{D}(\mathcal{S})$ the functional (50) is considered as a (real-valued) quadratic form on $\mathfrak{Sec}(M, \mathcal{S})$. It is called the *fermionic* part of the total Dirac action. Accordingly, the universal Dirac action (48) is referred to as the *bosonic* part of the total Dirac action.

It follows that the *gauge group of the total Dirac action* is given by the semi-direct product:

$$\mathcal{G}_{\mathbb{D}} = \text{Diff}(M) \ltimes \mathcal{G}_{\mathcal{S}}, \quad (51)$$

with $\mathcal{G}_{\mathcal{S}}$ being the gauge group of $\mathcal{S} \rightarrow M$. For every section $g_{\mathbb{M}}$ this gauge group explicitly reads:

$$\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{\text{EH}} \times \mathcal{G}_{\text{YM}} \quad (52)$$

Here, $\mathcal{G}_{\text{YM}} \simeq \text{Aut}_{\gamma}(\mathcal{S})$ is the subgroup of all bundle automorphisms on $\mathcal{S} \rightarrow M$ which are γ -invariant and \mathcal{G}_{EH} is the gauge group of the $\text{SO}(p,q)$ -reduced frame bundle.

In contrast, the *gauge group of the universal Dirac action* is provided by the affine group:

$$\mathcal{P}_{\mathbb{D}} = \mathcal{G}_{\mathbb{D}} \ltimes \mathcal{T}_{\mathbb{D}}, \quad (53)$$

with the *translational group* being given by

$$\mathcal{T}_{\mathbb{D}} \simeq \Omega^1(M, \text{End}_{\gamma}(\mathcal{S})). \quad (54)$$

Its action on $\mathcal{D}(\mathcal{S})$ reads: $\mathcal{D} \mapsto \mathcal{D} + \mathcal{A}$. We stress that the universal Dirac-Lagrangian is invariant with respect to this action.

We close this section with the following remarks concerning the case of *twisted spinor bundles*³. For this let M be an even-dimensional, orientable spin-manifold. In this case,

²In general, the domain of definition of the universal Dirac action is an appropriate subset of $\mathcal{D}(\mathcal{E})$, only, for M is not supposed to be compact.

³The author would like to thank V. Soucek for appropriate remarks.

every Clifford module bundle $\mathcal{E} \rightarrow M$ is equivalent to a twisted spinor bundle $\mathcal{S} \otimes \mathcal{W} \rightarrow M$. Here, the (total space of the) vector bundle $\mathcal{W} := \text{Hom}_\gamma(\mathcal{S}, \mathcal{E}) \rightarrow M$ is defined by the γ -equivariant homomorphisms (c.f. [ABS '64] and Sec. 3.3 in [BGV '96]). Basically, this follows from Wedderburn's structure theorems about invariant linear mappings (see, for example, Chap. 11 in [Gre '78]). Accordingly, the above mentioned equivalence is provided by the evaluation map. Although the spinor module carries a canonical Clifford action according to the identification $Cl_M \otimes \mathbb{C} \simeq \text{End}(\mathcal{S})$, there are usually different (i.e. inequivalent) spin-structures for given $g_M \in \mathfrak{Sec}(M, \mathcal{E}_{\text{EH}})$ (see, for example, Chap. 3 in [BGV '96] and Sec. 1.8 in [Jos '98]). This makes the actual domain of definition of the Dirac action geometrically more interesting, for one may ask how the Dirac action changes with a change of the spin-structure (c.f., for example, [Bau '81] and [Fri '84]). One may also take into account what is called "generalized spin structures", or "canonically generalized spin structures" (c.f., for example, [AI '80], [Hes '94] and [Hes '96]). In fact, in the case of real representations the appropriate discussions presented in [Hes '96] seem to fit with the discussion presented in this work. Clearly, if the spin-structure is basically unique, then the case of twisted spinor bundles can be similarly treated to the case of twisted Grassmann bundles.

3 The geometrical picture of Dirac type operators and the Einstein-Hilbert action

In this section, we briefly discuss the geometrical picture that underpins the Einstein-Hilbert action, \mathcal{I}_{EH} , when the latter is expressed in terms of Dirac type operators. From the usual Lichnerowicz/Schrödinger decomposition of \not{D}_A^2 (c.f. [Schr '32] and [Lich '63]):

$$\begin{aligned} \not{D}_A^2 - \varepsilon e v_y(\partial_A^{T^*M \otimes S} \circ \partial_A) &= \delta_\gamma(\text{curv}(\not{D}_A)) \\ &= -\frac{\varepsilon}{4} \text{scal}(g_M) + \delta_\gamma(E_A), \end{aligned} \quad (55)$$

it follows that

$$\mathcal{I}_{\text{EH}}(g_M) \sim \int_M * \text{tr}_\gamma(\text{curv}(\not{D}_A)). \quad (56)$$

Note that the "relative curvature": $E_A := \text{curv}(\not{D}_A) - \mathcal{R}iem(g_M)$, of Clifford connections⁴ has the peculiar property that $E_A \in \Omega^2(M, \text{End}_\gamma^+(\mathcal{S}))$. Therefore, $\text{tr}_\gamma(E_A) \equiv 0$.

This description of the Einstein-Hilbert action allows us to point out a subtle difference between the fermionic and the bosonic actions, usually not taken into account. This difference provides the geometrical origin of the difference between the respective gauge groups of the fermionic and the bosonic part of the total Dirac action. The discussion presented in this section will eventually yield some motivation for the "Pauli map" that is introduced in the next section, which permits to interpret the Yang-Mills and the Standard Model action as natural generalizations of the Einstein-Hilbert action with a "cosmological constant term".

Let $\mathcal{S}_D := \mathcal{E}_{\text{EH}} \times_M \text{End}(\mathcal{S}) \rightarrow M$. We consider the quotient

$$\Gamma_D := \mathfrak{Sec}(M, \mathcal{S}_D) / \mathcal{T}_D, \quad (57)$$

⁴In the case of Clifford connections, the relative curvature of \not{D}_A is also called "twisting curvature".

with the equivalence relation given by

$$(g_M, \Phi) \sim (g'_M, \Phi') \quad :\Leftrightarrow \quad \begin{cases} g'_M &= g_M, \\ \Phi' &= \Phi + \varphi. \end{cases} \quad (58)$$

It follows that $\Gamma_D \simeq \mathcal{D}(\mathcal{S})/\mathcal{T}_D$. Therefore, the restriction to S -reducible Dirac connections yields a principal fibering

$$\begin{aligned} \mathcal{D}_S(\mathcal{S}) &\twoheadrightarrow \Gamma_D \\ \mathcal{D} = \not\partial_A + \Phi_A &\mapsto [(g_M, \Phi_A)], \end{aligned} \quad (59)$$

with typical fiber given by the abelian group $\Omega^1(M, \text{End}(E))$.

The principal fibering (59) is clearly trivial but only in a non-canonical way unless the twisting part of $\mathcal{S} \twoheadrightarrow M$ is given by the trivial bundle $E = M \times \mathbb{C}^N \twoheadrightarrow M$. This holds true also in the case where $\mathcal{S} \twoheadrightarrow M$ is supposed to be flat for every g_M . Indeed, every choice of a connection on $E \twoheadrightarrow M$ yields a trivializing section:

$$\begin{aligned} \sigma_A : \Gamma_D &\longrightarrow \mathcal{D}_S(\mathcal{S}) \\ [(g_M, \Phi)] &\mapsto \not\partial_A + \Phi. \end{aligned} \quad (60)$$

It follows that $\sigma_A^* \mathcal{I}_D$ is independent of the choice of the trivializing section, because of the translational invariance of the universal Dirac action. In particular, when restricted to the distinguished subset:

$$\begin{aligned} \Gamma_{\text{EH}} &:= \{ [(g_M, \Phi)] \in \Gamma_D \mid \Phi \sim \varphi \} \\ &\simeq \text{Sec}(M, \mathcal{E}_{\text{EH}}), \end{aligned} \quad (61)$$

every trivializing section (60) yields the Einstein-Hilbert functional:

$$\begin{aligned} \sigma_A^* \mathcal{I}_D : \text{Sec}(M, \mathcal{E}_{\text{EH}}) &\longrightarrow \mathbb{C} \\ g_M &\mapsto \mathcal{I}_{\text{EH}}(g_M). \end{aligned} \quad (62)$$

The sections $g_M \in \text{Sec}(M, \mathcal{E}_{\text{EH}})$ are thus geometrically represented on $\mathcal{D}_S(\mathcal{S})$ by the trivializing sections (60):

$$\sigma_A(g_M) = \not\partial_A = d_A + \varepsilon \delta_{g, A}. \quad (63)$$

Here, $\delta_{g, A}$ denotes the formal adjoint of the exterior covariant derivative d_A that is defined with respect to some Clifford connection.

Accordingly, the geometrical meaning of these gauge sections is to make the metric on M “covariant” on \mathcal{S} . This geometrical view of the gauge sections becomes most apparent for flat modules (i.e. for flat $E \twoheadrightarrow M$). In this case, one gets:

$$\begin{aligned} \sigma_A(g_M) &= \not\partial + \not\mathcal{A} \\ &= d + \varepsilon \delta_g + \not\mathcal{A}, \end{aligned} \quad (64)$$

with the Gauss-Bonnet-Hodge-de Rham operator, $d + \varepsilon \delta_g$, being determined by the (semi-)metric g_M .

We emphasize that this geometrical picture of the (semi-)metric is provided by the translational invariance of the universal Dirac-Lagrangian. Finally, any Dirac (type) operator $\mathcal{D} \in \mathcal{D}(\mathcal{S})$ may be locally regarded as a “generalized covariance” of its underlying (semi-)Riemannian metric g_M :

$$\sigma_{\Phi_A}(g_M) \equiv \sigma_A([(g_M, \Phi)]) \stackrel{\text{loc.}}{=} d + \varepsilon \delta_g + \Phi_A, \quad (65)$$

where locally: $\Phi_A := \Phi + \mathcal{A}$.

So far, we discussed the geometrical picture of the Einstein-Hilbert action, when the latter is described in terms of the universal Dirac action. It is natural to ask for the appropriate substitute of the Einstein-Hilbert action with a cosmological constant $\Lambda \in \mathbb{R}$:

$$\mathcal{I}_{\text{EH}, \Lambda}(g_M) = \int_M *(scal(g_M) + \Lambda). \quad (66)$$

To answer this question, we take into account (56) that expresses the Einstein-Hilbert action in terms of the curvature of the quantized Yang-Mills connection $\partial_{\text{YM}} \equiv \partial_A \stackrel{\text{loc.}}{=} \partial + A$. It may thus not come as a big surprise that the functional (66) turns out to be expressible in terms of the curvature of the quantized Yang-Mills-Higgs connection $\partial_{\text{YMH}} \stackrel{\text{loc.}}{=} \partial + A + H$:

$$\begin{aligned} \mathcal{I}_{\text{EH}, \Lambda}(g_M) &\sim \int_M *\text{tr}_\gamma(\text{curv}(\partial_{\text{YMH}}) - \varepsilon ev_g(\omega_D^2)) \\ &= \int_M *(\text{tr}_\gamma \text{curv}(\partial_{\text{YM}}) - \Lambda_H), \end{aligned} \quad (67)$$

whereby the “cosmological constant” reads:

$$\begin{aligned} \Lambda \equiv \Lambda_H &:= \lambda \text{tr}_g H^2 \\ &= \lambda' \text{tr}_S \Phi_H^2. \end{aligned} \quad (68)$$

Here, $\lambda, \lambda' \in \mathbb{R}$ are numerical constants determined by the dimension of M . Notice that the right-hand side of (68) is indeed independent of the metric although the Higgs gauge potential itself is metric dependent.

The point to be emphasized is that the Einstein equations do not demand the Higgs gauge potential $H = ext_\Theta(\Phi_H)$ itself to be constant but only to take values on the sphere bundle of radius Λ/λ' . Consequently, if the Yang-Mills gauge group $\mathcal{G}_{\text{YM}} \subset \mathcal{P}_D$ is supposed to act transitively on the sphere bundle (like in the case of the ordinary Higgs potential), then the Higgs gauge potential becomes gauge equivalent to the one-form $im_D \Theta$, with $m_D \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{S}))$ being a constant section of length Λ/λ' . Clearly, such a section exists if and only if the Yang-Mills gauge group is reducible to the isotropy group of m_D . The rank of the reduced gauge group is determined by the co-dimension of the sphere bundle depending on the representation of the Yang-Mills gauge group on the Clifford module. This completely parallels the usual Higgs mechanism used in the Standard Model description of particle physics and indicates how spontaneous symmetry breaking can be described when gravity is taken into account. In fact, we claim that the Higgs potential is only needed to provide the Higgs boson itself with mass but the symmetry reduction is triggered by gravity

in the way indicated. As mentioned earlier, this geometrical interpretation of spontaneous symmetry breaking is based upon the intimate relation between gravity and the Higgs that is formally provided by the geometrical construction of Dirac connections in terms of the canonical one-form. We also point to the fact that the Higgs mass-term (68) is of the same physical dimension as the scalar curvature, as opposed to the fourth order term in the usual Higgs potential, which is dimensionless like the quadratic Yang-Mills-Lagrangian (in four dimensions). The same holds true for the “kinetic term” of the Higgs. Hence, from a geometrical point of view one may regard the Higgs sector of the Standard Model as the sum of various terms having different geometrical origin. This is most clearly exhibited when the Einstein-Hilbert action with cosmological constant is expressed in terms of the Yang-Mills-Higgs connection ∂_{YMH} and when also the *Yang-Mills-Higgs curvature*:

$$\begin{aligned} F_{\text{YMH}} &:= \text{curv}(\partial_{\text{YMH}}) - \mathcal{R}\text{iem}(g_M) \\ &= F_{\text{YM}} + d_A H + H \wedge H \end{aligned} \quad (69)$$

is taken into account, where $F_{\text{YM}} \equiv F_A \in \Omega^2(M, \text{End}_\gamma^+(\mathcal{E}))$ is the usual *Yang-Mills curvature* (“twisting curvature”).

Note that

$$d_A H + H \wedge H = \left(d_A \Phi_H + \Phi_H^2 \Theta \right) \wedge \Theta. \quad (70)$$

Therefore, the Yang-Mills-Higgs connection is not flat, in general, even if the Yang-Mills connection is supposed to be flat and $\Phi_H = im_D$ is a constant section, for in this case

$$F_{\text{YMH}} = -m_D^2 \Theta \wedge \Theta. \quad (71)$$

The (square of the) Dirac mass may be geometrically interpreted as curvature.

To summarize: We briefly discussed how the Einstein-Hilbert action and the Einstein-Hilbert action with a cosmological constant can be geometrically described in terms of, respectively, Yang-Mills and Yang-Mills-Higgs connections. Because of the translational invariance of the universal Dirac-Lagrangian, the latter does not depend on the chosen Yang-Mills connection. The Higgs part of the Yang-Mills-Higgs connection may serve to provide a symmetry reduction of the underlying Yang-Mills gauge group and thus contributes only by a constant section. Therefore, the bosonic part of the total Dirac action does not explicitly depend on the choice of the Yang-Mills part of the Yang-Mills-Higgs connection. It is (up to gauge) locally determined by

$$\partial_{\text{YMH}} \stackrel{\text{loc.}}{=} d + \varepsilon \delta_g + im_D, \quad (72)$$

which is but the general relativistic analogue of Dirac’s original first order differential operator $i\partial - m$. We point out that the Dirac connection of (72) is basically identical to the notion of the “extended connection” in terms of a “frame field” as discussed, for example, in [CF ’08] and the corresponding references cited therein. In fact, the local term $-i\gamma_\mu M/4$ (c.f. the beginning of page 547 in loc. site) is but a special case of a (locally defined) Dirac form $\omega_D = \text{ext}_\Theta(\mathcal{D} - \mathcal{D}_B)$ (c.f. Sec. 2.1).

When the fermionic part of the total Dirac action is taken into account, the translational symmetry of the bosonic part is broken. As a quadratic form that is determined by $\mathcal{D} \in \mathcal{D}(\mathcal{S})$, this gauge reduction occurs since the fermionic part of the total action only depends on the choice of the Dirac operator. In contrast, the universal Dirac action is defined in terms of the corresponding curvature of the chosen Dirac operator. This subtle interplay between the fermionic and the bosonic part of the total Dirac action will be geometrically analyzed more carefully in the following section in terms of “real Clifford modules” and the “Pauli map”.

4 Real Clifford modules and the Pauli map

Let $(\mathcal{E}, \gamma_{\mathcal{E}}) \rightarrow (M, g_M)$ be a Hermitian Clifford module. The Hermitian product is denoted by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$.

Definition 4.1 *A Hermitian Clifford module is called a “real \mathbb{Z}_2 -bi-graded Hermitian Clifford module” (“real Clifford module” for short), if it is endowed, in addition, with a \mathbb{C} -linear involution $\tau_{\mathcal{E}}$, making $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \rightarrow M$ \mathbb{Z}_2 -graded, and a \mathbb{C} -anti-linear involution $J_{\mathcal{E}}$, making $\mathcal{E} = \mathcal{M}_{\mathcal{E}} \otimes \mathbb{C} \rightarrow M$ real, such that*

$$\begin{aligned} \tau_{\mathcal{E}} \circ \gamma_{\mathcal{E}}(\alpha) &= -\gamma_{\mathcal{E}}(\alpha) \circ \tau_{\mathcal{E}}, \\ J_{\mathcal{E}} \circ \gamma_{\mathcal{E}}(\alpha) &= \pm \gamma_{\mathcal{E}}(\alpha) \circ J_{\mathcal{E}}, \\ J_{\mathcal{E}} \circ \tau_{\mathcal{E}} &= \pm \tau_{\mathcal{E}} \circ J_{\mathcal{E}}, \\ \langle J_{\mathcal{E}}(z), J_{\mathcal{E}}(w) \rangle_{\mathcal{E}} &= \pm \langle w, z \rangle_{\mathcal{E}}, \end{aligned} \tag{73}$$

for all $\alpha \in T^*M$ and $z, w \in \mathcal{E}$.

A real Clifford module is called a “Majorana module”, provided that

$$J_{\mathcal{E}} \circ \tau_M = -\tau_M \circ J_{\mathcal{E}}. \tag{74}$$

We make use of the following abbreviation: $B^{\text{cc}} \equiv J_{\mathcal{E}} \circ B \circ J_{\mathcal{E}}$, for all $B \in \text{End}(\mathcal{E})$. Similarly, $\mathcal{D}_{\mathcal{E}}^{\text{cc}} \equiv J_{\mathcal{E}} \circ \mathcal{D}_{\mathcal{E}} \circ J_{\mathcal{E}}$, for all $\mathcal{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$. An operator $B \in \text{End}(\mathcal{E})$ is called “real” (resp. “imaginary”), if $B^{\text{cc}} = B$ (resp. $B^{\text{cc}} = -B$). We denote by $\mathcal{D}_{\text{real}}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E})$ the subset of all real Dirac (type) operators: $\mathcal{D}_{\mathcal{E}}^{\text{cc}} = \mathcal{D}_{\mathcal{E}}$.

Let

$$(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}}) \tag{75}$$

be a real Clifford module bundle over (M, g_M) , such that

$$\tau_{\mathcal{E}}^{\text{cc}} = \pm \tau_{\mathcal{E}}, \tag{76}$$

$$\gamma_{\mathcal{E}}^{\text{cc}} = +\gamma_{\mathcal{E}}. \tag{77}$$

We denote by

$$(\mathcal{P}, \langle \cdot, \cdot \rangle_{\mathcal{P}}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}) \tag{78}$$

the doubling of the Clifford module $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}})$. That is,

$$\mathcal{P} := {}^2\mathcal{E} \equiv \begin{array}{c} \mathcal{E} \\ \oplus \\ \mathcal{E} \end{array} = \mathcal{E} \otimes \mathbb{C}^2, \quad (79)$$

$$\langle \cdot, \cdot \rangle_{\mathcal{P}} := \frac{1}{2}(\langle \cdot, \cdot \rangle_{\mathcal{E}} + \langle \cdot, \cdot \rangle_{\mathcal{E}}), \quad (80)$$

$$\tau_{\mathcal{P}} := \tau_{\mathcal{E}} \otimes \tau_2, \quad (81)$$

$$\gamma_{\mathcal{P}} := \gamma_{\mathcal{E}} \otimes \mathbf{1}_2. \quad (82)$$

Here and in the sequel: $\mathbf{1}_2 \in \mathbb{C}(2)$ and $\tau_2, \varepsilon_2, \mathbf{I}_2 \in \mathbb{C}(2)$ denote, respectively, the two-by-two unit matrix and

$$\tau_2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{I}_2 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (83)$$

The real structure on \mathcal{E} then allows to also introduce a real structure on the doubled Clifford module $\mathcal{P} = {}^2\mathcal{E}$:

$$J_{\mathcal{P}} := J_{\mathcal{E}} \otimes \varepsilon_2, \quad (84)$$

such that

$$(\mathcal{P}, \langle \cdot, \cdot \rangle_{\mathcal{P}}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}, J_{\mathcal{P}}) \quad (85)$$

becomes, again, a real Clifford module over (M, g_M) . It follows that

$$\tau_{\mathcal{P}}^{\text{cc}} = \pm \tau_{\mathcal{P}} \Leftrightarrow \tau_{\mathcal{E}}^{\text{cc}} = \mp \tau_{\mathcal{E}}, \quad (86)$$

$$\gamma_{\mathcal{P}}^{\text{cc}} = +\gamma_{\mathcal{P}}. \quad (87)$$

With respect to the real structure $J_{\mathcal{P}}$ the doubled Clifford module may be regarded as the complexification of the real vector bundle:

$$\mathcal{M}_{\mathcal{P}} := \left\{ \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z}^{\text{cc}} \end{pmatrix} \in \mathcal{P} \mid \mathfrak{z} \in \mathcal{E} \right\} \rightarrow M. \quad (88)$$

This real vector bundle contains a distinguished real sub-vector bundle:

$$\mathcal{V}_{\mathcal{P}} := \left\{ \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix} \in \mathcal{P} \mid \mathfrak{z} \in \mathcal{M}_{\mathcal{E}} \right\} \rightarrow M, \quad (89)$$

whoses complexification $\mathcal{V}_{\mathcal{P}}^{\mathbb{C}}$ of the total space may be identified with the diagonal embedding

$$\begin{array}{ccc} \mathcal{E} & \hookrightarrow & {}^2\mathcal{E} \\ \mathfrak{z} & \mapsto & \begin{pmatrix} \mathfrak{z} \\ \mathfrak{z} \end{pmatrix}. \end{array} \quad (90)$$

Here, $\mathcal{M}_{\mathcal{E}} := \{\mathfrak{z} \in \mathcal{E} \mid J_{\mathcal{E}}(\mathfrak{z}) = \mathfrak{z}\} \subset \mathcal{E}$ is the (total space) of the induced real sub-vector bundle, such that $\mathcal{E} = \mathcal{M}_{\mathcal{E}}^{\mathbb{C}}$.

A general real Dirac operator on the real Clifford module (85) reads (for a proof see, for example, in [Tol '09], Theorem 1):

$$\mathcal{D}_{\mathcal{P}} = \begin{pmatrix} \mathcal{D}_{\mathcal{E}} & \phi_{\mathcal{E}} - \mathcal{F}_{\mathcal{E}} \\ \phi_{\mathcal{E}} + \mathcal{F}_{\mathcal{E}} & \mathcal{D}_{\mathcal{E}}^{\text{cc}} \end{pmatrix}. \quad (91)$$

Here, respectively, $\mathcal{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$ is any Dirac operator on (75) and

$$\begin{aligned} \phi_{\mathcal{E}}^{\text{cc}} &= +\phi_{\mathcal{E}}, \\ \mathcal{F}_{\mathcal{E}}^{\text{cc}} &= -\mathcal{F}_{\mathcal{E}} \end{aligned} \quad (92)$$

are general sections of $\text{End}^+(\mathcal{E}) \rightarrow M$.

The (affine) set of all real Dirac operators on the doubled Clifford module (85) contains a distinguished (affine) sub-set, consisting of those Dirac operators where in addition $\mathcal{D}_{\mathcal{E}}^{\text{cc}} = \mathcal{D}_{\mathcal{E}}$ is a real Dirac operator on (75). In particular, the Dirac operators

$$\begin{aligned} \mathcal{D}_{\mathcal{P}} &= \begin{pmatrix} \mathcal{D}_{\mathcal{E}} & \phi_{\mathcal{E}} \\ \phi_{\mathcal{E}} & \mathcal{D}_{\mathcal{E}} \end{pmatrix} \\ &= \mathcal{D}_{\mathcal{E}} \otimes \mathbf{1}_2 + \phi_{\mathcal{E}} \otimes \varepsilon_2 \end{aligned} \quad (93)$$

also preserve $\mathfrak{Sec}(M, \mathcal{V}_{\mathcal{P}})$.

In contrast, one may consider the distinguished class of Dirac operators on the doubled real Clifford module (85) which are determined already by the real Dirac operators on (75):

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} &:= \begin{pmatrix} \mathcal{D}_{\mathcal{E}} & -\mathcal{F}_{\mathcal{E}} \\ \mathcal{F}_{\mathcal{E}} & \mathcal{D}_{\mathcal{E}} \end{pmatrix} \\ &= \mathcal{D}_{\mathcal{E}} \otimes \mathbf{1}_2 + \mathcal{F}_{\mathcal{E}} \otimes \mathbf{I}_2, \end{aligned} \quad (94)$$

with $\mathcal{F}_{\mathcal{E}}$ being defined by the (relative) curvature of $\mathcal{D}_{\mathcal{E}}$:

$$\begin{aligned} \mathcal{F}_{\mathcal{E}} &:= \mathcal{F}_{\mathcal{D}} \equiv i\delta_{\gamma}(\text{curv}(\mathcal{D}_{\mathcal{E}}) - \mathcal{Riem}(g_M)) \\ &= i\mathcal{F}_{\mathcal{D}}. \end{aligned} \quad (95)$$

Note that $\mathcal{F}_{\mathcal{D}} \in \mathfrak{Sec}(M, \text{End}^+(\mathcal{E}))$ is even and real for real (or imaginary) Dirac operators $\mathcal{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$. Whence, $\mathcal{F}_{\mathcal{D}}^{\text{cc}} = -\mathcal{F}_{\mathcal{D}}$.

By a slight abuse of notation, we rewrite the *Pauli type* Dirac operators (94) as

$$\mathcal{P}_{\mathcal{D}} := \mathcal{D}_{\mathcal{E}} + \iota\mathcal{F}_{\mathcal{D}} \quad (96)$$

to bring them most closely to Dirac's first order operator including the Pauli term $i\mathcal{F}$. Here,

$$\iota\mathcal{F}_{\mathcal{D}} \equiv \begin{pmatrix} 0 & -\text{id}_{\mathcal{E}} \\ \text{id}_{\mathcal{E}} & 0 \end{pmatrix} \circ \begin{pmatrix} \mathcal{F}_{\mathcal{D}} & 0 \\ 0 & \mathcal{F}_{\mathcal{D}} \end{pmatrix}. \quad (97)$$

In the sequel, we shall consider Pauli type Dirac operators on the doubled Clifford module (85) as mappings:

$$\begin{aligned} \mathcal{P}_{\mathcal{D}} : \mathfrak{Sec}(M, \mathcal{V}_{\mathcal{P}}^{\mathbb{C}}) &\longrightarrow \mathfrak{Sec}(M, \mathcal{P}) \\ {}^2\psi = \begin{pmatrix} \psi \\ \psi \end{pmatrix} &\mapsto \begin{pmatrix} \mathcal{D}_{\mathcal{E}}\psi - \mathcal{F}_{\mathcal{D}}\psi \\ \mathcal{D}_{\mathcal{E}}\psi + \mathcal{F}_{\mathcal{D}}\psi \end{pmatrix}. \end{aligned} \quad (98)$$

Therefore, the restriction of our original Pauli type Dirac operators to the diagonal embedding $\mathcal{E} \hookrightarrow {}^2\mathcal{E}$ may formally be interpreted as the restriction of the real Dirac operators (94) to the sections of the distinguished sub-bundle

$$\mathcal{V}_P^{\mathbb{C}} \hookrightarrow \mathcal{P} \rightarrow M. \quad (99)$$

For this matter, we also call this bundle the *Pauli bundle* associated with the real Clifford bundle (75).

We put emphasize on the following fact: The Lagrangian density that is defined by the smooth function $\langle \Psi, \not{P}_D \Psi \rangle_{\mathcal{P}}$ reduces to $\langle \psi, \not{D}_{\mathcal{E}} \psi \rangle_{\mathcal{E}}$, when the Pauli type Dirac operators are restricted to the sections of the Pauli bundle. Therefore, the two fermionic functionals:

$$\begin{aligned} \mathcal{I}_{D,\text{ferm}} : \mathfrak{Sec}(M, \mathcal{E}) \times \mathcal{D}(\mathcal{E}) &\longrightarrow \mathbb{C} \\ (\psi, \not{D}_{\mathcal{E}}) &\longmapsto \int_M \langle \psi, \not{D}_{\mathcal{E}} \psi \rangle_{\mathcal{E}} d\text{vol}_M, \end{aligned} \quad (100)$$

$$\begin{aligned} \mathcal{I}'_{D,\text{ferm}} : \mathfrak{Sec}(M, \mathcal{E}) \times \mathcal{D}(\mathcal{E}) &\longrightarrow \mathbb{C} \\ (\psi, \not{D}_{\mathcal{E}}) &\longmapsto \int_M \langle {}^2\psi, \not{P}_D {}^2\psi \rangle_{\mathcal{P}} d\text{vol}_M \end{aligned} \quad (101)$$

contain the same information, actually.

The (generalized) Pauli term thus does not alter the fermionic action. In particular, the fermionic action is fully determined by the (Dirac) connections on the vector bundle $\mathcal{E} \rightarrow M$ and not, in addition, by the curvature of these (Dirac) connections. As mentioned earlier, this fact is known to play a fundamental role in quantizing the fermionic action. Of course, when the functional $\mathcal{I}'_{D,\text{ferm}}$ is actually regarded as being a functional on $\mathfrak{Sec}(M, \mathcal{V}_P^{\mathbb{C}})$, then a stationary point of this functional has to satisfy the more restrictive condition:

$$\psi \in \ker(\not{D}_{\mathcal{E}}) \cap \ker(\not{F}_D). \quad (102)$$

The equivalence of the two fermionic actions $\mathcal{I}_{D,\text{ferm}}$ and $\mathcal{I}'_{D,\text{ferm}}$ (when both are regarded as being functionals on the same domain) is very basic for the structure of Dirac type gauge theories. Indeed, these equivalent geometrical descriptions of the fermionic action seem to provide a deep relation between the fermionic part of the total Dirac action and its corresponding bosonic part.

To formalize the above discussed equivalence of the functionals $\mathcal{I}_{D,\text{ferm}}$ and $\mathcal{I}'_{D,\text{ferm}}$, we introduce the following

Definition 4.2 *Let $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ be a real Clifford module bundle over (M, g_M) satisfying the requirements imposed on (75). Also, let $\mathcal{D}_{\text{real}}(\mathcal{E}) \subset \mathcal{D}(\mathcal{E})$ be the (affine) set of real Dirac operators acting on $\mathfrak{Sec}(M, \mathcal{E})$.*

We call the mapping

$$\begin{aligned} \mathcal{P}_D : \mathcal{D}_{\text{real}}(\mathcal{E}) &\longrightarrow \mathcal{D}_{\text{real}}(\mathcal{P}) \\ \not{D}_{\mathcal{E}} &\longmapsto \not{P}_D, \end{aligned} \quad (103)$$

which associates with every real Dirac operator on $\mathcal{E} \rightarrow M$ the appropriate Pauli type Dirac operator on the doubled Clifford module $\mathcal{P} = {}^2\mathcal{E} \rightarrow M$, the “Pauli map”.

Geometrically, one may regard the fermionic action as a mapping from $\mathcal{D}(\mathcal{E})$ into the quadratic forms on $\mathfrak{Sec}(M, \mathcal{E})$:

$$\begin{aligned} \mathcal{I}_{\mathcal{D},\text{ferm}} : \mathcal{D}(\mathcal{E}) &\longrightarrow \mathfrak{Map}(\mathfrak{Sec}(M, \mathcal{E}), \mathbb{C}) \\ \mathcal{D}_{\mathcal{E}} &\mapsto \begin{cases} \mathfrak{Sec}(M, \mathcal{E}) &\longrightarrow \mathbb{C} \\ \psi &\mapsto \mathcal{I}_{\mathcal{D},\text{ferm}}(\psi, \mathcal{D}_{\mathcal{E}}) \end{cases} \end{aligned} \quad (104)$$

When restricted to $\mathcal{D}_{\text{real}}(\mathcal{E})$, the Pauli map thus allows to lift the quadratic form $\mathcal{I}_{\mathcal{D},\text{ferm}}$ on $\mathfrak{Sec}(M, \mathcal{E})$ to the quadratic form $\mathcal{I}'_{\mathcal{D},\text{ferm}}$ on $\mathfrak{Sec}(M, \mathcal{P})$:

$$\mathcal{I}'_{\mathcal{D},\text{ferm}} = \mathcal{I}_{\mathcal{D},\text{ferm}} \circ \mathcal{P}_{\mathcal{D}}, \quad (105)$$

such that $\mathcal{P}_{\mathcal{D}}$ acts like the identity when $\mathcal{I}'_{\mathcal{D},\text{ferm}}$ is restricted to $\mathfrak{Sec}(M, \mathcal{V}_{\mathcal{P}}^{\mathbb{C}}) \subset \mathfrak{Sec}(M, \mathcal{P})$.

As mentioned earlier, on every Clifford module there exists a distinguished class of Dirac operators called of simple type. Explicitly, they read:

$$\mathcal{D}_{\mathcal{E}} = \mathcal{D}_{\mathcal{A}} + \tau_{\mathcal{E}} \circ \phi_{\mathcal{D}} \quad (106)$$

where $\phi_{\mathcal{D}} \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{-}(\mathcal{E}))$. In general, however, these Dirac operators are not real. Therefore, our original Pauli type Dirac operators (3) fail to be real and our geometrical understanding of this class of Dirac operators in terms of the Pauli map (103) is not yet complete.

Of course, this flaw may most straightforwardly be remedied by giving up the restriction of the Pauli map to real Dirac operators. However, this will then not yield any new insight concerning the structure of the original Pauli type operators (3). Even worse, one loses significant information as will be shown in the next section. Indeed, there it will be shown that the Pauli map (103) allows to naturally include the geometrical description of ‘‘Majorana masses’’ in terms of real Dirac operators of simple type.

4.1 Majorana masses and real Dirac operators of simple type

Let $(\mathcal{S}, \langle \cdot, \cdot \rangle_{\mathcal{S}}, \tau_{\mathcal{S}}, \gamma_{\mathcal{S}}, J_{\mathcal{S}})$ be a real Clifford module bundle over (M, g_M) . We put

$$\mathcal{E} := {}^2\mathcal{S} = \mathcal{S} \otimes \mathbb{C}^2, \quad (107)$$

$$\langle \cdot, \cdot \rangle_{\mathcal{E}} := \frac{1}{2}(\langle \cdot, \cdot \rangle_{\mathcal{S}} + \langle \cdot, \cdot \rangle_{\mathcal{S}}), \quad (108)$$

$$\tau_{\mathcal{E}} := \begin{pmatrix} \tau_{\mathcal{S}} & 0 \\ 0 & -\tau_{\mathcal{S}} \end{pmatrix} = \tau_{\mathcal{S}} \otimes \tau_2, \quad (109)$$

$$\gamma_{\mathcal{E}} := \begin{pmatrix} \gamma_{\mathcal{S}} & 0 \\ 0 & \gamma_{\mathcal{S}}^{\text{cc}} \end{pmatrix}, \quad (110)$$

$$J_{\mathcal{E}} := \begin{pmatrix} 0 & J_{\mathcal{S}} \\ J_{\mathcal{S}} & 0 \end{pmatrix} = J_{\mathcal{S}} \otimes \varepsilon_2. \quad (111)$$

It follows that

$$\begin{aligned} \tau_{\mathcal{E}}^{\text{cc}} &= \pm \tau_{\mathcal{E}} \quad \Leftrightarrow \quad \tau_{\mathcal{S}}^{\text{cc}} = \mp \tau_{\mathcal{S}}, \\ \gamma_{\mathcal{E}}^{\text{cc}} &= +\gamma_{\mathcal{E}}. \end{aligned} \quad (112)$$

Definition 4.3 Let $\mathcal{D}_S \in \mathcal{D}(\mathcal{S})$ be a Dirac type operator on the real Clifford module $\mathcal{S} \rightarrow M$. The real Dirac type operator on the induced real Clifford module $\mathcal{E} \rightarrow M$:

$$\begin{aligned} \mathcal{D}_\mathcal{E} &:= \begin{pmatrix} \mathcal{D}_S & 0 \\ 0 & \mathcal{D}_S^{\text{cc}} \end{pmatrix}, \\ &\equiv \mathcal{D}_S \oplus \mathcal{D}_S^{\text{cc}}, \end{aligned} \quad (113)$$

is called the “real form” of \mathcal{D}_S .

Proposition 4.1 The most general real Dirac operator of simple type, acting on $\mathfrak{Sec}(M, \mathcal{E})$, explicitly reads:

$$\mathcal{D}_\mathcal{E} = \mathcal{D}_A + \tau_\mathcal{E} \circ \phi_\mathcal{E}, \quad (114)$$

whereby $\mathcal{D}_A := \mathcal{D}_A \oplus \mathcal{D}_A^{\text{cc}}$ is the real form of \mathcal{D}_A and

$$\phi_\mathcal{E} := \begin{pmatrix} \chi_S & \pm \phi_S^{\text{cc}} \\ -\phi_S & \mp \chi_S^{\text{cc}} \end{pmatrix}, \quad (115)$$

depending on whether $\tau_S^{\text{cc}} = \pm \tau_S$. Moreover, $\phi_S \in \mathfrak{Sec}(M, \text{End}_\gamma^+(\mathcal{S}))$ is explicitly given by

$$\phi_S \equiv \begin{cases} \chi'_S + \tau_S \circ \delta_\gamma(\sigma_S), & \text{for } \gamma_S^{\text{cc}} = +\gamma_S, \\ \tau_S \circ \mu_M + \delta_\gamma(\sigma_S), & \text{for } \gamma_S^{\text{cc}} = -\gamma_S. \end{cases} \quad (116)$$

Here, $\mu_M, \chi'_S \in \Omega^0(M, \text{End}_\gamma^+(\mathcal{S}))$, $\chi_S \in \Omega^0(M, \text{End}_\gamma^-(\mathcal{S}))$ and $\sigma_S \in \Omega^1(M, \text{End}_\gamma^-(\mathcal{S}))$.

The proof of the above statement is based upon the following statement and a corollary thereof. Both of which are interesting in its own and will be useful also later on.

Lemma 4.1 Let $(\mathcal{E}, \gamma_\mathcal{E}) \rightarrow (M, g_M)$ be a general Clifford module over a smooth (semi-)Riemannian manifold. Also, let $\mathcal{D}_k + \Phi_k \in \mathcal{D}(\mathcal{E})$ ($k = 1, 2$) be two Dirac type operators, acting on $\mathfrak{Sec}(M, \mathcal{E})$. The Laplace type operator

$$H := (\mathcal{D}_1 + \Phi_1) \circ (\mathcal{D}_2 + \Phi_2) \quad (117)$$

has the explicit Lichnerowicz decomposition: $H = \Delta_H + V_H$, where the second order part is defined in terms of the connection:

$$\begin{aligned} \partial_H &:= \partial_B + \alpha_H, \\ \alpha_H(v) &:= \frac{\varepsilon}{2} \left(\gamma_\mathcal{E}(v^b) \circ \Phi_2 + \Phi_1 \circ \gamma_\mathcal{E}(v^b) + (\mathcal{D}_1 - \mathcal{D}_2) \circ \gamma_\mathcal{E}(v^b) \right), \end{aligned} \quad (118)$$

for all $v \in TM$. The zero order part explicitly reads:

$$V_H :=$$

$$V_D + \delta_\gamma(\partial_B \Phi_2) - \varepsilon \text{ev}_g(\partial_H \alpha_H) - \varepsilon \text{ev}_g(\alpha_H^2) + \Phi_D \circ \Phi_2 + (\Phi_1 + (\mathcal{D}_1 - \mathcal{D}_2)) \circ (\Phi_2 + \Phi_D). \quad (119)$$

Here, $\partial_B \in \mathcal{A}(\mathcal{E})$ denotes the Bochner connection that is defined by $\mathcal{D}_2 \equiv \mathcal{D}$ and

$$\begin{aligned} V_D &:= \mathcal{D}^2 - \Delta_B, \\ \Phi_D &:= \mathcal{D} - \mathcal{D}_B. \end{aligned} \quad (120)$$

Proof: First, we again put $\mathcal{D} \equiv \mathcal{D}_2$ and abbreviate $\Phi_{12} \equiv \mathcal{D}_1 - \mathcal{D}_2$ to re-write H as

$$H = \mathcal{D}^2 + [\mathcal{D}, \Phi_2] + (\Phi_1 + \Phi_2 + \Phi_{12}) \circ \mathcal{D} + (\Phi_1 + \Phi_{12}) \circ \Phi_2. \quad (121)$$

It follows that for all $f \in C^\infty(M)$:

$$[[\mathcal{D}, \Phi_2], f] = [\delta_\gamma(df), \Phi_2] \quad (122)$$

and thus

$$[H, f] = [\mathcal{D}^2, f] + \delta_\gamma(df) \circ \Phi_2 + \Phi_1 \circ \delta_\gamma(df) + \Phi_{12} \circ \delta_\gamma(df). \quad (123)$$

This yields the explicit formula (118) for the connection ∂_H .

The explicit formula (119) for the zero order term is then obtained from the identity $V_H = H - \Delta_H$ and

$$\begin{aligned} \Delta_H &\equiv \varepsilon ev_g(\partial_H \circ \partial_H) \\ &= \Delta_D + \varepsilon ev_g(\partial_H \alpha_H) + \varepsilon ev_g(\alpha_H^2) + 2\varepsilon ev_g(\alpha_H, \partial_B). \end{aligned} \quad (124)$$

This proves the statement. \square

For later convenience we consider $V_H \equiv V_D$ in the case where $H = \mathcal{D}^2$ and $\mathcal{D} = \mathcal{D}_A + \Phi$. From Lemma 4.1 it follows for $\mathcal{D}_1 = \mathcal{D}_2 \equiv \mathcal{D}_A$ and $\Phi_1 = \Phi_2 \equiv \Phi$ that

$$V_D = \delta_\gamma(\text{curv}(\mathcal{D}_A)) + \delta_\gamma(\partial_A \Phi) + \Phi^2 - \varepsilon ev_g(\alpha_D^2) - \varepsilon ev_g(\partial_B \alpha_D), \quad (125)$$

whereby $\partial_B = \partial_A + \alpha_D$ and

$$\alpha_D(v) := \frac{\varepsilon}{2} \left\{ \gamma_\varepsilon(v^b), \Phi \right\}. \quad (126)$$

Clearly, Lemma (4.1) generalizes the well-known formula by Lichnerowicz/Schrödinger (55) with respect to \mathcal{D}_A^2 to general Laplacians which can be factorized by arbitrary Dirac type operators. The next statement yields an easy characterization of simply type Dirac operators.

Corollary 4.1 *A Dirac operator \mathcal{D} on a \mathbb{Z}_2 -graded Clifford module $(\mathcal{E}, \gamma_\varepsilon) \rightarrow (M, g_M)$ is of simple type if and only if*

$$\{\mathcal{D} - \mathcal{D}_B, \gamma_\varepsilon(\alpha)\} \equiv 0, \quad (127)$$

for all $\alpha \in T^*M$. Here, $\mathcal{D}_B \equiv \delta_\gamma \circ \partial_B$ is the quantized Bochner connection that is defined by $\mathcal{D} \in \mathcal{D}(\mathcal{E})$.

Proof: It follows from Lemma 4.1 that two Dirac type operators $\mathcal{D}', \mathcal{D} \in \mathcal{D}(\mathcal{E})$ share the same Bochner connection if and only if the zero-order operator $\mathcal{D}' - \mathcal{D}$ anti-commutes with the Clifford action (c.f. formula (126)). Whence, \mathcal{D} and \mathcal{D}_B have the same Bochner connection ∂_B if and only if $\mathcal{D} - \mathcal{D}_B$ anti-commutes with the Clifford action. However, Clifford connections

$\partial_A \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$ are the only connections with the property that the three notions of Dirac connection, Clifford connection and Bochner connection coincide, i.e.:

$$\partial_D = \partial_A = \partial_B. \quad (128)$$

Whence, the Dirac type operator $\not{\partial}_B \in \mathcal{D}(\mathcal{E})$ yields the Bochner connection ∂_B if and only if $\partial_B \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$. This proves the statement. \square

We now turn back to the proof of Proposition 4.1.

Proof of Proposition 4.1: The most general real Dirac operator, acting on $\mathfrak{Sec}(M, \mathcal{E})$, reads:

$$\not{D}'_{\mathcal{E}} = \begin{pmatrix} \not{D}_S & \Phi_S^{\text{cc}} \\ \Phi_S & \not{D}_S^{\text{cc}} \end{pmatrix}, \quad (129)$$

whereby $\Phi_S \in \mathfrak{Sec}(M, \text{End}^+(\mathcal{S}))$. We may re-write this real Dirac operator as

$$\not{D}'_{\mathcal{E}} = \not{D}_{\mathcal{E}} + \Phi'_{\mathcal{E}} \quad (130)$$

with $\not{D}_{\mathcal{E}}$ being the real form of \not{D}_S and

$$\Phi'_{\mathcal{E}} \equiv \begin{pmatrix} 0 & \Phi_S^{\text{cc}} \\ \Phi_S & 0 \end{pmatrix}. \quad (131)$$

Let, respectively, $\partial_{B'}$ and ∂_B be the Bochner connections of $\not{D}'_{\mathcal{E}}$ and $\not{D}_{\mathcal{E}}$. Then, Lemma 4.1 implies that

$$\begin{aligned} \partial_{B'} &= \partial_B + \alpha_{D'}, \\ \alpha_{D'}(v) &= \frac{\varepsilon}{2} \left\{ \gamma_{\mathcal{E}}(v^{\flat}), \Phi'_{\mathcal{E}} \right\}. \end{aligned} \quad (132)$$

By assumption $\partial_{B'} \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$. We show that also ∂_B is a Clifford connection and thus $\alpha_{D'}$ has to commute with the Clifford action. This condition will eventually give us the explicit form of the zero order operator $\Phi'_{\mathcal{E}}$.

Indeed, it follows that

$$\begin{aligned} \not{D}'_{\mathcal{E}} &= \not{\partial}_{B'} + \Phi_{D'} \\ &= \not{\partial}_B + \not{\alpha}_{D'} + \Phi_{D'} \\ &= \not{\partial}_B + \Phi_D + \Phi'_{\mathcal{E}}, \end{aligned} \quad (133)$$

where $\Phi_D = \not{D}_{\mathcal{E}} - \not{\partial}_B$. Therefore,

$$\begin{aligned} \Phi_{D'} &= \tau_{\mathcal{E}} \circ \phi_{D'} \\ &= \Phi_D + \Phi'_{\mathcal{E}} - \not{\alpha}_{D'}. \end{aligned} \quad (134)$$

Whence,

$$\phi_{D'} = \tau_{\mathcal{E}} \circ \Phi_D + \tau_{\mathcal{E}} \circ (\Phi'_{\mathcal{E}} - \not{\alpha}_{D'}) \quad (135)$$

and the condition $\phi_{D'} \in \mathfrak{Sec}(M, \text{End}_\gamma^-(\mathcal{E}))$ yields the equivalence:

$$[\phi_{D'}, \gamma_\varepsilon(\alpha)] = 0 \quad \Leftrightarrow \quad \begin{cases} \{\Phi_D, \gamma_\varepsilon(\alpha)\} & = 0, \\ \{(\Phi'_\varepsilon - \phi_{D'}), \gamma_\varepsilon(\alpha)\} & = 0, \end{cases} \quad (136)$$

for all $\alpha \in T^*M$.

According to Corollary 4.1, the first relation of (136) implies that also $\partial_B \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$. Whence, \mathcal{D}_ε is of simple type:

$$\mathcal{D}_\varepsilon = \begin{pmatrix} \partial_A + \tau_S \circ \chi_S & 0 \\ 0 & (\partial_A + \tau_S \circ \chi_S)^{\text{cc}} \end{pmatrix}, \quad (137)$$

with $\chi_S \in \mathfrak{Sec}(M, \text{End}_\gamma^-(\mathcal{S}))$.

Moreover, being the difference of two Clifford connections it follows that

$$[\alpha_{D'}(v), \gamma_\varepsilon(\alpha)] \equiv 0, \quad (138)$$

for all $v \in TM$ and $\alpha \in T^*M$. Taking into account the explicit form of $\alpha_{D'}$, the condition (138) is seen to be equivalent to

$$[[\Phi_S, \gamma_S(\alpha_1)]_\pm, \gamma_S(\alpha_2)]_\mp \equiv 0, \quad (139)$$

for all $\alpha_1, \alpha_2 \in T^*M$. Here, $[x, y]_\pm \equiv xy \pm yx$, with the relative sign referring to $\gamma_S^{\text{cc}} = \pm\gamma_S$.

It follows that

$$\Phi_S = \begin{cases} \delta_\gamma(\sigma_S) + \tau_S \circ \chi'_S, & \text{for } \gamma_S^{\text{cc}} = +\gamma_S, \\ \mu_M + \tau_S \circ \delta_\gamma(\sigma_S), & \text{for } \gamma_S^{\text{cc}} = -\gamma_S, \end{cases} \quad (140)$$

with $\chi'_S, \mu_M \in \mathfrak{Sec}(M, \text{End}_\gamma^+(\mathcal{S}))$ and $\sigma_S \in \Omega^1(M, \text{End}_\gamma^-(\mathcal{S}))$.

For reasons of consistency we still have to verify the second relation of (136) in order to complete the proof of Proposition 4.1. However, this is done straightforwardly taking the explicit solution (140) of (138) into account. \square

The significance of Proposition (4.1) is given by generalizing the notion of simple type Dirac operators to those which are also real. These are certainly distinguished Dirac operators on the real Clifford module $\mathcal{E} = {}^2\mathcal{S} \rightarrow M$ on which one may then apply the Pauli map (103). Even more, these real simple type Dirac operators also allow to incorporate Majorana masses within the scheme of Dirac type gauge theories. For this, let (\mathcal{S}, ∂) be a flat Majorana module with an imaginary Clifford action and grading involution. The stationary points of the fermionic action $\mathcal{I}_{D, \text{ferm}}$, which is defined by the real Dirac operator of simple type

$$\mathcal{D}_M := \begin{pmatrix} \partial & i\mu_M \\ -i\mu_M & -\partial \end{pmatrix} \in \mathcal{D}_{\text{real}}(\mathcal{E}) \quad (141)$$

with $\mu_M \in \mathfrak{Sec}(M, \text{End}_\gamma^+(\mathcal{S}))$ being real, fulfil the Majorana equations:

$$\begin{aligned} i\partial\chi &= \mu_M\chi^{\text{cc}}, \\ i\partial\chi^{\text{cc}} &= \mu_M\chi. \end{aligned} \quad (142)$$

We note that the (total space of the) real sub-vector bundle $\mathcal{M}_\mathcal{E} \rightarrow M$, whose complexification equals $\mathcal{E} \rightarrow M$, reads:

$$\mathcal{M}_\mathcal{E} = \left\{ \begin{pmatrix} z \\ z^{cc} \end{pmatrix} \in \mathcal{E} \mid z \in \mathcal{S} \right\}. \quad (143)$$

Hence, \mathcal{D}_M leaves the real module $\mathfrak{Sec}(M, \mathcal{M}_\mathcal{E})$ invariant.

The equations (142) are diagonal with respect to the grading involution $\tau_\mathcal{S}$. In particular, they are diagonal with respect to the chirality involution τ_M :

$$\mathcal{D}_M \psi = 0 \quad \Leftrightarrow \quad \begin{cases} i\mathcal{D}\chi_R &= \mu_M \chi_R^{cc}, \\ i\mathcal{D}\chi_L &= \mu_M \chi_L^{cc}, \end{cases} \quad (144)$$

plus the corresponding conjugate equations. Here, we have put $\psi = \begin{pmatrix} \chi \\ \chi^{cc} \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{M}_\mathcal{E})$ and the chiral eigen sections of τ_M are again denoted by $\chi_R, \chi_L \in \mathfrak{Sec}(M, \mathcal{S})$, such that $\chi = \chi_R + \chi_L$.

In this section, we discussed how the Majorana equations can be described in terms of real Dirac operators of simple type on real Clifford modules. We turn now to the corresponding discussion of the *Dirac-Yukawa equation*:

$$i\mathcal{D}_A \chi = \varphi_D \chi \quad \Leftrightarrow \quad \begin{cases} i\mathcal{D}_A \chi_R &= \varphi_D \chi_L, \\ i\mathcal{D}_A \chi_L &= \varphi_D \chi_R. \end{cases} \quad (145)$$

The *Yukawa (coupling) term* φ_D generalizes in a gauge covariant manner the usual mass term m_D of the Dirac equation (9) with help of the Higgs field.

4.2 Dirac masses and real Dirac operators of simple type

In the last section we have shown how Majorana masses can be geometrically described in terms of a real Clifford module if the latter is considered as being the doubling of a Majorana module. In order to also geometrically describe Dirac masses within Dirac type gauge theories we have to consider special Majorana modules $\mathcal{S} \rightarrow M$, called *Dirac modules*. More precisely, we make the following

Definition 4.4 *A real Clifford module*

$$(\mathcal{S}, \langle \cdot, \cdot \rangle_\mathcal{S}, \tau_\mathcal{S}, \gamma_\mathcal{S}, \mathcal{J}_\mathcal{S}) \quad (146)$$

is called a “Dirac module”, provided there is a Majorana module $(\mathcal{W}, \langle \cdot, \cdot \rangle_\mathcal{W}, \tau_\mathcal{W}, \gamma_\mathcal{W}, \mathcal{J}_\mathcal{W})$ over (M, g_M) , such that

$$\mathcal{S} = {}^2\mathcal{W} = \mathcal{W} \otimes \mathbb{C}^2 \quad (147)$$

and

$$\tau_\mathcal{S} = \begin{pmatrix} \text{id}_\mathcal{W} & 0 \\ 0 & -\text{id}_\mathcal{W} \end{pmatrix} = \text{id}_\mathcal{W} \otimes \tau_2, \quad (148)$$

$$\gamma_S = \begin{pmatrix} 0 & \gamma_W \\ \gamma_W & 0 \end{pmatrix} = \gamma_W \otimes \varepsilon_2, \quad (149)$$

$$J_S = \begin{pmatrix} 0 & J_W \\ J_W & 0 \end{pmatrix} = J_W \otimes \varepsilon_2. \quad (150)$$

Finally,

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_S = \langle u_1, v_2 \rangle_W \pm \langle v_1, u_2 \rangle_W, \quad (151)$$

depending on whether $\langle J_W(u), J_W(v) \rangle_W = \pm \langle v, u \rangle_W$, for all $u, v \in \mathcal{W}$.

It follows that

$$\begin{aligned} \tau_S^{\text{cc}} &= -\tau_S, \\ \gamma_S^{\text{cc}} &= \pm \gamma_S \Leftrightarrow \gamma_W^{\text{cc}} = \pm \gamma_W. \end{aligned} \quad (152)$$

Let $\not\partial_A \in \mathcal{D}(\mathcal{W})$ and $\varphi_D \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}))$. Furthermore, we assume that $\not\partial_A \pm i\varphi_D$ are S -reducible if $\mathcal{W} \hookrightarrow \Lambda_M \otimes E \rightarrow M$. We put

$$\not\mathcal{D}_D := \begin{pmatrix} 0 & \not\partial_A - i\varphi_D \\ \not\partial_A + i\varphi_D & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \quad (153)$$

which is easily seen to be of simple type. In fact, one may rewrite $\not\mathcal{D}_D$ as

$$\not\mathcal{D}_D = \not\partial_A + i\mu_D, \quad (154)$$

with

$$\begin{aligned} \mu_D &\equiv -\tau_S \circ \phi_D, \\ \phi_D &:= \varphi_D \otimes \varepsilon_2 \in \mathfrak{Sec}(M, \text{End}_\gamma^-(\mathcal{S})). \end{aligned} \quad (155)$$

Here, by a slight abuse of notation we identify $\partial_A \in \mathcal{A}_{\text{Cl}}(\mathcal{W})$ with

$$\partial_A = \begin{pmatrix} \partial_A & 0 \\ 0 & \partial_A \end{pmatrix} \in \mathcal{A}_{\text{Cl}}(\mathcal{S}), \quad (156)$$

such that always $\not\partial_A = \delta_\gamma \circ \partial_A$ and, respectively, $\not\partial_A \in \mathcal{D}(\mathcal{W})$, or $\not\partial_A \in \mathcal{D}(\mathcal{S})$, depending on whether “ γ ” denotes either γ_W , or γ_S .

Note that the simple type Dirac operator $\not\mathcal{D}_D \in \mathcal{D}(\mathcal{S})$ is not real. Also, the first order operators $\not\partial_A \pm i\varphi_D \in \mathcal{D}(\mathcal{W})$ are not Dirac operators, in general. Clearly, for constant sections $\varphi_D = m_D$, these two operators are but the complex factors of the *Klein-Gordon operator*: $\not\mathcal{D}_D^2 = \not\partial_A^2 + m_D^2$.

Also note that the most general Dirac operators on a Dirac module read:

$$\not\mathcal{D}_S := \begin{pmatrix} 0 & \not\mathcal{D}_{\mathcal{W},1} \\ \not\mathcal{D}_{\mathcal{W},2} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \quad (157)$$

where, respectively, $\mathcal{D}_{\mathcal{W},1}, \mathcal{D}_{\mathcal{W},2} \in \mathcal{D}(\mathcal{W})$ are of Dirac type. In particular, the most general real Dirac operators on a Dirac module are given by

$$\mathcal{D}_{\mathcal{S}} := \begin{pmatrix} 0 & \mathcal{D}_{\mathcal{W}}^{\text{cc}} \\ \mathcal{D}_{\mathcal{W}} & 0 \end{pmatrix} \in \mathcal{D}_{\text{real}}(\mathcal{S}), \quad (158)$$

In either case, the Dirac operators on a Dirac module are thus parameterized by general first order differential operators, acting on $\mathfrak{Sec}(M, \mathcal{W})$, such that their principal symbols are determined by the Clifford action of the underlying Majorana module. Then, our Lemma (4.1) provides an explicit (global) formula for the corresponding Lichnerowicz/Schrödinger decomposition of any such Dirac operator $\mathcal{D}_{\mathcal{S}} \in \mathcal{D}(\mathcal{S})$ in terms of the underlying Dirac operators $\mathcal{D}_{\mathcal{W},1}, \mathcal{D}_{\mathcal{W},2} \in \mathcal{D}(\mathcal{W})$.

Finally, the most general Dirac operator of simple type, acting on sections of a Dirac module, takes the form

$$\mathcal{D}_{\mathcal{S}} := \begin{pmatrix} 0 & \mathcal{D}_{\Lambda} - \Phi_{\mathcal{W}} \\ \mathcal{D}_{\Lambda} + \Phi_{\mathcal{W}} & 0 \end{pmatrix} \in \mathcal{D}(\mathcal{S}), \quad (159)$$

with $\Phi_{\mathcal{W}} \in \mathfrak{Sec}(M, \text{End}_{\gamma}(\mathcal{W}))$ being a general section. Indeed, for

$$\mu_{\mathcal{S}} := \Phi_{\mathcal{W}} \otimes \mathbb{I}_2 \in \mathfrak{Sec}(M, \text{End}(\mathcal{S})) \quad (160)$$

one obtains that for all $\alpha \in T^*M$:

$$\begin{aligned} \{\gamma_{\mathcal{S}}(\alpha), \mu_{\mathcal{S}}\} &= \gamma_{\mathcal{W}}(\alpha) \circ \Phi_{\mathcal{W}} \otimes \{\varepsilon_2, \mathbb{I}_2\} \\ &= 0. \end{aligned} \quad (161)$$

Therefore, $\mathcal{D}_{\mathcal{S}} \equiv \mathcal{D}_{\Lambda} + \mu_{\mathcal{S}}$ is of simple type, whereby

$$\begin{aligned} \mu_{\mathcal{S}} &:= -\tau_{\mathcal{S}} \circ \phi_{\mathcal{S}}, \\ \phi_{\mathcal{S}} &:= \Phi_{\mathcal{W}} \otimes \varepsilon_2 \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{-}(\mathcal{S})). \end{aligned} \quad (162)$$

Note that real Dirac operators on a Dirac module cannot be of simple type and vice versa.

To clarify how the Dirac-Yukawa equation (145) may arise from the Dirac functional $\mathcal{I}'_{\mathcal{D},\text{ferm}}$, one simply considers the real form of the (symmetric) simple type Dirac operator $\mathcal{D}_{\mathcal{D}} = \mathcal{D}_{\Lambda} + i\mu_{\mathcal{D}}$ on the Dirac module $\mathcal{S} \rightarrow M$, thereby defining a real Dirac operator of simple type on the associated real Clifford module $\mathcal{E} \rightarrow M$. Clearly,

$$\mathcal{I}_{\mathcal{D},\text{ferm}}(\mathcal{P}_{\mathcal{D}}(\mathcal{D}_{\mathcal{D}} \oplus \mathcal{D}_{\mathcal{D}}^{\text{cc}}))({}^2\Psi) = \mathcal{I}_{\mathcal{D},\text{ferm}}(\psi, \mathcal{D}_{\mathcal{D}}). \quad (163)$$

Here,

$$\psi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{S}), \quad \Psi = \begin{pmatrix} \psi \\ \psi^{\text{cc}} \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{E}), \quad (164)$$

where $\chi_1, \chi_2 \in \mathfrak{Sec}(M, \mathcal{W})$ are arbitrary sections, which are in one-to-one correspondence with arbitrary eigen sections of the involution $\tau_{\mathcal{S}}$ (not of $\tau_{\mathcal{W}}$!). Hence, to recover the Dirac-Yukawa equation (145) one may restrict to the eigen sections of $\tau_{\mathcal{S}}$, corresponding to the eigen value equal to +1. That is,

$$i\mathcal{D}_{\Lambda}\chi = \varphi_{\mathcal{D}}\chi \quad \Leftrightarrow \quad \begin{cases} \mathcal{D}_{\mathcal{D}}\psi = 0, \\ \tau_{\mathcal{S}}\psi = \psi. \end{cases} \quad (165)$$

This “solves” the issue of fermion doubling already mentioned in the introduction (and carefully discussed, for example, in [LMMS '96], [LMMS '97], [GIS '98] and [TT '06a]; see also [CM '07]).

For $\varphi_D \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}))$ the Dirac-Yukawa equation is odd with respect to the chirality involution (47):

$$i\mathcal{D}_A\chi = \varphi_D\chi \quad \Leftrightarrow \quad \begin{cases} i\mathcal{D}_A\chi_R &= \varphi_D\chi_L, \\ i\mathcal{D}_A\chi_L &= \varphi_D\chi_R. \end{cases} \quad (166)$$

Again, $\chi_R, \chi_L \in \mathfrak{Sec}(M, \mathcal{W})$ denote the chiral eigen sections: $\tau_M\chi_{R/L} = \pm\chi_{R/L}$, such that $\chi = \chi_R + \chi_L$.

For $\varphi_D \in \mathfrak{Sec}(M, \text{End}_\gamma^+(\mathcal{W}))$ the Dirac-Yukawa equation is gauge covariant only if the chiral eigen sections carry the same representation of the underlying gauge group (i.e. the fermions are considered “left-right gauge symmetric”). Otherwise, the Yukawa coupling term has to be odd: $\varphi_D \in \mathfrak{Sec}(M, \text{End}_\gamma^-(\mathcal{W}))$.

The first order differential operators $\mathcal{D}_A \pm i\varphi_D$ on the Majorana module $\mathcal{W} \rightarrow M$ are of Dirac type, in general. In contrast, the induced first order operator $\mathcal{D}_D \in \mathcal{D}(\mathcal{S})$ is always a Dirac operator (of simple type) on the corresponding Dirac module. We stress once more that in any case both Dirac type operators $\mathcal{D}_A \pm i\varphi_D$ are needed, actually, to define a simple type Dirac operator on the Dirac module, thereby excluding the reality of \mathcal{D}_D . Finally, the Dirac-Yukawa equation is clearly diagonal with respect to the action of “charge conjugation” $J_{\mathcal{W}}$.

In the next section, we discuss the combined Dirac-Yukawa-Majorana equation and its implication for the Dirac action. We also briefly discuss the bundle structure that allows to regard the Majorana masses as constant sections of the Dirac module bundle associated with a Majorana module.

5 The Pauli map of the combined Dirac-Yukawa and Dirac-Majorana operator

In the following, let (\mathcal{W}, ∂) be a (partially) flat Majorana module over (M, g_M) , such that $\gamma_{\mathcal{W}}^{\text{cc}} = -\gamma_{\mathcal{W}}$.

We may complement the Majorana operator $\mathcal{D}_M \in \mathcal{D}(\mathcal{E})$ by the replacement of the real Dirac operator $\mathcal{D} \ominus \mathcal{D}$ on the Dirac module $\mathcal{E} \rightarrow M$ by the real form of $\mathcal{D}_D = \mathcal{D}_A + i\mu_D \in \mathcal{D}(\mathcal{S})$ to obtain the following real Dirac operator of simple type:

$$\begin{aligned} \mathcal{D}_{\text{YM}} &:= \begin{pmatrix} \mathcal{D}_A + i\mu_D & i\mu_M \\ -i\mu_M & (\mathcal{D}_A + i\mu_D)^{\text{cc}} \end{pmatrix} \\ &\equiv \mathcal{D}_A + i\mu_{\text{YM}} \in \mathcal{D}_{\text{real}}(\mathcal{E}). \end{aligned} \quad (167)$$

Here, respectively,

$$\mathcal{D}_A := \mathcal{D}_A \oplus \mathcal{D}_A^{\text{cc}} \in \mathcal{D}_{\text{real}}(\mathcal{E}) \quad (168)$$

is the real form of $\not\partial_A \in \mathcal{D}(\mathcal{S})$ and

$$\mu_{\text{YM}} := \tau_{\mathcal{E}} \circ \phi_{\text{YM}}, \quad (169)$$

$$\begin{aligned} \phi_{\text{YM}} &:= \begin{pmatrix} \tau_{\mathcal{S}} \circ \mu_{\text{D}} & \tau_{\mathcal{S}} \circ \mu_{\text{M}} \\ \tau_{\mathcal{S}} \circ \mu_{\text{M}} & -(\tau_{\mathcal{S}} \circ \mu_{\text{D}})^{\text{cc}} \end{pmatrix} \\ &\equiv \begin{pmatrix} -\phi_{\text{D}} & \phi_{\text{M}} \\ \phi_{\text{M}} & \phi_{\text{D}}^{\text{cc}} \end{pmatrix} \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{-}(\mathcal{E})). \end{aligned} \quad (170)$$

We call in mind that the *Majorana mass operator* $\mu_{\text{M}} \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{+}(\mathcal{S}))$ is supposed to be real. In contrast, no such reality assumption is imposed on the *Dirac mass operator* $\mu_{\text{D}} \in \mathfrak{Sec}(M, \text{End}^{-}(\mathcal{S}))$, which has to fulfil the requirement:

$$\{\mu_{\text{D}}, \gamma_{\mathcal{S}}(\alpha)\} = 0, \quad (171)$$

for all $\alpha \in T^*M$.

We call $\not\mathcal{D}_{\text{YM}} = \not\partial_A + i\mu_{\text{YM}} \in \mathcal{D}_{\text{real}}(\mathcal{E})$ the *Dirac-Yukawa-Majorana operator (DYM)*.

Let $\chi \in \mathfrak{Sec}(M, \mathcal{W})$ and $\psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{S})$ be the associated eigen section of $\tau_{\mathcal{S}}$ that corresponds to the eigen value equal to +1. Also let

$$\not\mathcal{P}_{\text{DYM}} := \mathcal{P}_{\text{D}}(\not\mathcal{D}_{\text{YM}}) \in \mathcal{D}_{\text{real}}(\mathcal{P}) \quad (172)$$

and $\Psi = \begin{pmatrix} \psi \\ \psi^{\text{cc}} \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{E})$. Note that $\psi^{\text{cc}} = \begin{pmatrix} 0 \\ \chi^{\text{cc}} \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{S})$.

Clearly,

$$\langle {}^2\Psi, \not\mathcal{P}_{\text{DYM}} {}^2\Psi \rangle_{\mathcal{P}} = \langle \Psi, \not\mathcal{D}_{\text{YM}} \Psi \rangle_{\mathcal{E}} \quad (173)$$

and

$$\not\mathcal{D}_{\text{YM}} \Psi = \begin{pmatrix} (\not\partial_A + i\mu_{\text{D}})\psi + i\mu_{\text{M}}\psi^{\text{cc}} \\ (\not\partial_A + i\mu_{\text{D}})^{\text{cc}}\psi^{\text{cc}} - i\mu_{\text{M}}\psi \end{pmatrix}. \quad (174)$$

Whence,

$$\begin{aligned} \langle {}^2\Psi, \not\mathcal{P}_{\text{DYM}} {}^2\Psi \rangle_{\mathcal{P}} &= \frac{1}{2}(\langle \psi, (\not\partial_A + i\mu_{\text{D}})\psi \rangle_{\mathcal{S}} + \langle \psi, i\mu_{\text{M}}\psi^{\text{cc}} \rangle_{\mathcal{S}} + \\ &\quad + \langle \psi^{\text{cc}}, (\not\partial_A + i\mu_{\text{D}})^{\text{cc}}\psi^{\text{cc}} \rangle_{\mathcal{S}} - \langle \psi^{\text{cc}}, i\mu_{\text{M}}\psi \rangle_{\mathcal{S}}) \\ &= \frac{1}{2}(\langle \chi, (\not\partial_A + i\mu_{\text{D}})\chi \rangle_{\mathcal{W}} + \langle \chi, im_{\text{M}}\chi^{\text{cc}} \rangle_{\mathcal{W}} + \\ &\quad + \langle \chi^{\text{cc}}, (\not\partial_A + i\mu_{\text{D}})^{\text{cc}}\chi^{\text{cc}} \rangle_{\mathcal{W}} - \langle \chi^{\text{cc}}, im_{\text{M}}\chi \rangle_{\mathcal{W}}), \end{aligned} \quad (175)$$

where we have put

$$\mu_{\text{M}} \equiv \begin{pmatrix} m_{\text{M}} & 0 \\ 0 & m_{\text{M}} \end{pmatrix}, \quad m_{\text{M}} \in \mathfrak{Sec}(M, \text{End}_{\gamma}(\mathcal{W})) \text{ real and constant} \quad (176)$$

according to the definition (and the physical interpretation) of the Majorana mass operator.

Thus, the quadratic form $\mathcal{I}_{\text{D,ferm}}(\mathcal{P}_{\text{DYM}})$ on $\mathfrak{Sec}(M, \mathcal{E})$ yields the Euler-Lagrange equations:

$$i\mathcal{D}_A\psi = \mu_{\text{D}}\psi + \mu_{\text{M}}\psi^{\text{cc}}, \quad (177)$$

$$(i\mathcal{D}_A\psi)^{\text{cc}} = \mu_{\text{D}}^{\text{cc}}\psi^{\text{cc}} + \mu_{\text{M}}\psi. \quad (178)$$

When restricted to $\tau_{\mathfrak{S}}\psi = \psi$, these equations become equivalent to:

$$i\mathcal{D}_A\chi = \varphi_{\text{D}}\chi + m_{\text{M}}\chi^{\text{cc}}, \quad (179)$$

$$(i\mathcal{D}_A\chi)^{\text{cc}} = \varphi_{\text{D}}^{\text{cc}}\chi^{\text{cc}} + m_{\text{M}}\chi. \quad (180)$$

In order to geometrically describe the Yukawa coupling term (and thus the “Dirac mass” after spontaneous symmetry breaking) in terms of (real) Dirac operators of simple type one simply has to go from the underlying Majorana module to the corresponding Dirac module quite similar to how the “Pauli matrices” are lifted to the “Dirac matrices” (the latter being considered in the Majorana representation). The doubling of the Dirac module then allows to also geometrically describe the characteristic “particle-anti-particle” coupling that arises by the Majorana mass term, in terms of real, simple type Dirac operators. In fact, this is where the real structure necessarily enters the scheme, thereby turning the Clifford module into a real Clifford module. Finally, the Pauli map of the DYM, which describes both the “left-right” coupling and the “particle-anti-particle” coupling on the same geometrical footing, then allows to geometrically describe the Pauli term in a way that does not alter the fermionic action. For reasons of renormalization, this is actually necessary.

Before we discuss the bosonic part of the full Dirac action with respect to the real Dirac operator \mathcal{P}_{DYM} , we still comment on the gauge invariance of the equations (179–180). Of course, this is related to the dynamical discrepancy between the fermionic left-right coupling, provided by the Dirac mass, and the particle-anti-particle coupling that is invoked on the fermions by the Majorana mass.

For the sake of gauge invariance, the underlying Majorana module $\mathcal{W} \rightarrow M$ has to be *partially flat* when Majorana masses are taken into account. In this case: $\partial_A \neq \partial$ only for $\chi \in \ker(m_{\text{M}})$. In geometrical terms this and the constant Majorana mass operator may be described by the assumption that the Majorana module splits:

$$\mathcal{W} = \begin{array}{c} \mathcal{W}_\nu \\ \oplus \\ \mathcal{W}_e \end{array} \rightarrow M, \quad (181)$$

where the sub-bundle $\mathcal{W}_\nu \rightarrow M$ carries the trivial representation of the Yang-Mills gauge group $\mathcal{G}_{\text{YM}} \subset \mathcal{G}_{\text{D}}$ and

$$m_{\text{M}} \equiv \begin{pmatrix} m_{\text{M},\nu} & 0 \\ 0 & 0 \end{pmatrix}. \quad (182)$$

Accordingly, $\partial_A \in \mathcal{A}_{\text{Cl}}(\mathcal{W})$ and $\varphi_{\text{D}} \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}))$ may be decomposed as

$$\begin{aligned} \partial_A &\equiv \begin{pmatrix} \partial & 0 \\ 0 & \partial_A \end{pmatrix}, \\ \varphi_{\text{D}} &\equiv \begin{pmatrix} m_{\text{D},\nu} & 0 \\ 0 & \varphi_e \end{pmatrix}, \end{aligned} \quad (183)$$

with $m_{M,\nu}, m_{D,\nu} \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}_\nu))$ being real with respect to J_W and constant. Furthermore, $\varphi_e \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}_e))$.

The combined Dirac-Majorana equations (179–180) become equivalent to

$$i\hat{\mathcal{D}}\nu = m_{D,\nu}\nu + m_{M,\nu}\nu^{cc}, \quad (184)$$

$$i\hat{\mathcal{D}}_A e = \varphi_e e, \quad (185)$$

together with the corresponding complex (or charge) conjugate equations. Here, we have put $\chi \equiv (\nu, e) \in \mathfrak{Sec}(M, \mathcal{W}_\nu \oplus \mathcal{W}_e)$ for the “uncharged sections” and the “charged sections”, respectively, of the Majorana module $\mathcal{W} \rightarrow M$. Generically, the uncharged sections $\nu \in \mathfrak{Sec}(M, \mathcal{W}_\nu)$ are referred to as “cosmological neutrinos”. They are carriers of Dirac and/or Majorana masses or are massless, depending on $\ker(m_{D,\nu})$ and $\ker(m_{M,\nu})$. Clearly, in the case of *Majorana neutrinos*: $\nu^{cc} = \nu \in \mathfrak{Sec}(M, \mathcal{M}_{W,\nu}) \subset \mathfrak{Sec}(M, \mathcal{M}_W)$ (whereby $\mathcal{W} = \mathcal{M}_{W,\nu}^C$), the notions of Dirac and Majorana masses coincide and (184) reduces to

$$i\hat{\mathcal{D}}\nu = m_\nu \nu, \quad (\nu^{cc} = \nu). \quad (186)$$

Only the sub-module

$$\ker(m_M) = \mathcal{W}_e \hookrightarrow \mathcal{W} \rightarrow M \quad (187)$$

of the Majorana module carries a non-trivial representation of the Yang-Mills gauge sub-group of \mathcal{G}_D .

In the case of the Standard Model, the cosmological neutrinos should not be confounded with the electrically neutral (left-handed) component of $e \in \mathfrak{Sec}(M, \mathcal{W}_e)$ after the mechanism of spontaneous symmetry break has been established. Indeed, the sections $\nu \in \mathfrak{Sec}(M, \mathcal{W}_\nu)$ represent a kind of new species of particles which do not contribute to any yet known kind of interaction besides gravity. This “ghost like species” of particles may thus serve as candidates for “dark matter” (resp. “dark energy”). Of course, the masses of the cosmological neutrinos cannot be dynamically generated by the mechanism of spontaneous symmetry breaking since the cosmological neutrinos only carry the trivial representation of the Yang-Mills gauge group. This is certainly unsatisfying but may change with the upcoming experiments made at the Large Hadron Collider (LHC) at CERN/Swiss.

The Dirac mass matrix is known to only couple particles of different chirality but respects the particle-anti-particle grading. This is opposed to the Majorana mass matrix. Since the latter is “non-dynamical” one may wonder to what extent the Majorana masses may nonetheless dynamically contribute, for example, to the Standard Model? A partial answer to this question within Dirac type gauge theories will be discussed next.

5.1 The Dirac action concerning DYM

So far, we have carefully discussed the fermionic action of the total Dirac action. In this section we discuss the bosonic part of the latter with respect to the corresponding DYM. Since the Dirac-Yukawa-Majorana operator $\mathcal{D}_{YM} = \hat{\mathcal{D}}_A + i\mu_{YM} \in \mathcal{D}_{\text{real}}(\mathcal{E})$ is of simple type it lifts to $\mathcal{D}_{\text{real}}(\mathcal{P} = {}^2\mathcal{E})$ via the Pauli map. It thereby generalizes the operator (3). The

latter operator has been shown earlier to yield the Standard Model (STM) action including gravity (c.f. [Tol '98], [TT '06a] and [TT '06b]). This time, however, also Majorana masses are taken into account. We therefore summarize the basic steps allowing to express the Lagrangian density

$$\mathcal{L}_{\text{DYM}} := * \text{tr}_\gamma \left(\text{curv}(\mathcal{F}_{\text{DYM}}) - \varepsilon \text{ev}_g(\omega_{\text{D}}^2) \right) \quad (188)$$

in terms of the sections given by the metric g_{M} , the Yang-Mills gauge field A , the Higgs field φ_{D} (resp. φ_e) and the Majorana (Dirac) masses m_{M} (m_{D}) which altogether parameterize the Dirac-Yukawa-Majorana operator $\mathcal{D}_{\text{YM}} \in \mathcal{D}_{\text{S,real}}(\mathcal{E})$.

Following the calculation, it will be shown that this density is automatically real and thus takes values in $\Omega^n(M)$. Furthermore, the calculation will also allow to reveal a subtle relation between simple type Dirac operators and the “kinetic term” of the Higgs field within the STM action.

To get started, we put $\mathcal{F}_{\text{DYM}} = \mathcal{D}_A + i\Phi_{\text{DYM}}$, where

$$\Phi_{\text{DYM}} \equiv \begin{pmatrix} \mu_{\text{YM}} & -\mathcal{F}_{\text{DYM}} \\ \mathcal{F}_{\text{DYM}} & \mu_{\text{YM}} \end{pmatrix}, \quad \mu_{\text{YM}} := \begin{pmatrix} \mu_{\text{D}} & \mu_{\text{M}} \\ -\mu_{\text{M}} & -\mu_{\text{D}}^{\text{cc}} \end{pmatrix} \quad (189)$$

and $\mathcal{F}_{\text{DYM}} \in \mathfrak{Sec}(M, \text{End}^+(\mathcal{E}))$ is the (quantized) relative curvature of \mathcal{D}_{YM} . Since the latter is of simple type, it follows that

$$F_{\text{DYM}} = F_A - (d_A(i\mu_{\text{YM}}) + (i\mu_{\text{YM}})^2\Theta) \wedge \Theta \in \Omega^2(M, \text{End}^+(\mathcal{E})). \quad (190)$$

Here, d_A is the exterior covariant derivative with respect to the (real) Clifford connection $\partial_A \in \mathcal{A}_{\text{Cl}}(\mathcal{E})$ and $F_A \in \Omega^2(M, \text{End}_\gamma^+(\mathcal{E}))$ its twisting curvature. Hence,

$$\mathcal{F}_{\text{DYM}} = \mathcal{F}_A + \frac{n-1}{n} \left(\delta_\gamma(d_A(i\mu_{\text{YM}})) + (i\mu_{\text{YM}})^2 \right). \quad (191)$$

We may then take advantage of Lemma (4.1) to obtain:

$$\text{tr}_{\mathcal{P}} V_{\text{D}} = \text{tr}_\gamma(\text{curv}(\mathcal{D}_A)) - \text{tr}_{\mathcal{P}} \Phi_{\text{DYM}}^2 + \frac{\varepsilon}{4} g_{\text{M}}(e_i, e_j) \text{tr}_{\mathcal{P}} \left(\{\gamma_{\mathcal{P}}(e^i), \Phi_{\text{DYM}}\} \{\gamma_{\mathcal{P}}(e^j), \Phi_{\text{DYM}}\} \right), \quad (192)$$

where we have neglected an appropriate boundary term and $e_1, \dots, e_n \in TM$ is any local (g_{M} -orthonormal) basis with dual basis $e^1, \dots, e^n \in T^*M$.

It follows that

$$\begin{aligned} \text{tr}_{\mathcal{P}} \Phi_{\text{DYM}}^2 &= 2 \text{tr}_{\mathcal{E}} \left(\mu_{\text{YM}}^2 - \mathcal{F}_{\text{DYM}}^2 \right), \\ \text{tr}_{\mathcal{P}} \left(\{\gamma_{\mathcal{P}}(e^i), \Phi_{\text{DYM}}\} \{\gamma_{\mathcal{P}}(e^j), \Phi_{\text{DYM}}\} \right) &= 2 \text{tr}_{\mathcal{E}} \left(\{\gamma_{\mathcal{E}}(e^i), \mu_{\text{YM}}\} \{\gamma_{\mathcal{E}}(e^j), \mu_{\text{YM}}\} \right) - \\ &\quad - 2 \text{tr}_{\mathcal{E}} \left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{F}_{\text{DYM}}\} \{\gamma_{\mathcal{E}}(e^j), \mathcal{F}_{\text{DYM}}\} \right). \end{aligned} \quad (193)$$

Furthermore,

$$\text{tr}_{\mathcal{E}} \mathcal{F}_{\text{DYM}}^2 = -\frac{1}{2} \text{tr}_g F_A^2 + \varepsilon \left(\frac{n-1}{n} \right)^2 \text{tr}_g (\partial_A \mu_{\text{YM}})^2 + \left(\frac{n-1}{n} \right)^2 \text{tr}_{\mathcal{E}} \mu_{\text{YM}}^4 \quad (194)$$

and

$$\mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mu_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j), \mu_{\mathrm{YM}}\}\right) = 0, \quad (195)$$

$$\begin{aligned} \mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{F}_{\mathrm{DYM}}\}\{\gamma_{\mathcal{E}}(e^j), \mathcal{F}_{\mathrm{DYM}}\}\right) &= \mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{F}_{\mathcal{A}}\}\{\gamma_{\mathcal{E}}(e^j), \mathcal{F}_{\mathcal{A}}\}\right) + \\ &+ \left(\frac{1-n}{n}\right)^2 \mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{Q}_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j), \mathcal{Q}_{\mathrm{YM}}\}\right) + \\ &+ 4\left(\frac{1-n}{n}\right)^2 \mathrm{tr}_{\mathcal{E}}\left(\gamma_{\mathcal{E}}(e^i)\gamma_{\mathcal{E}}(e^i)\mu_{\mathrm{YM}}^4\right), \end{aligned} \quad (196)$$

where we abbreviated $\mathcal{Q}_{\mathrm{YM}} \equiv \delta_{\gamma}(d_{\mathcal{A}}(i\mu_{\mathrm{YM}}))$. Also,

$$\frac{\varepsilon}{4} g_{\mathrm{M}}(e_i, e_j) \mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{F}_{\mathcal{A}}\}\{\gamma_{\mathcal{E}}(e^j), \mathcal{F}_{\mathcal{A}}\}\right) = \frac{2-n}{2} \mathrm{tr}_g(F_{\mathcal{A}}^2), \quad (197)$$

$$\left(\frac{1-n}{n}\right)^2 \frac{\varepsilon}{4} g_{\mathrm{M}}(e_i, e_j) \mathrm{tr}_{\mathcal{E}}\left(\{\gamma_{\mathcal{E}}(e^i), \mathcal{Q}_{\mathrm{YM}}\}\{\gamma_{\mathcal{E}}(e^j), \mathcal{Q}_{\mathrm{YM}}\}\right) = -\varepsilon \frac{(1-n)^3}{n^2} \mathrm{tr}_g(\partial_{\mathcal{A}} \mu_{\mathrm{YM}})^2, \quad (198)$$

$$\varepsilon \left(\frac{1-n}{n}\right)^2 g_{\mathrm{M}}(e_i, e_j) \mathrm{tr}_{\mathcal{E}}\left(\gamma_{\mathcal{E}}(e^i)\gamma_{\mathcal{E}}(e^i)\mu_{\mathrm{YM}}^4\right) = n\left(\frac{1-n}{n}\right)^2 \mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^4). \quad (199)$$

Finally, one ends up with

$$\begin{aligned} -\mathrm{tr}_{\mathcal{P}}\Phi_{\mathrm{DYM}}^2 + \frac{\varepsilon}{4} g_{\mathrm{M}}(e_i, e_j) \mathrm{tr}_{\mathcal{P}}\left(\{\gamma_{\mathcal{P}}(e^i), \Phi_{\mathrm{DYM}}\}\{\gamma_{\mathcal{P}}(e^j), \Phi_{\mathrm{DYM}}\}\right) &= \\ -2\mathrm{tr}_{\mathcal{E}}\mu_{\mathrm{YM}}^2 + 2\mathrm{tr}_{\mathcal{E}}\mathcal{F}_{\mathrm{DYM}}^2 - \frac{\varepsilon}{2} g_{\mathrm{M}}(e_i, e_j) \mathrm{tr}_{\mathcal{E}}\left\{\gamma_{\mathcal{E}}(e^i), \mathcal{F}_{\mathrm{DYM}}\right\}\left\{\gamma_{\mathcal{E}}(e^j), \mathcal{F}_{\mathrm{DYM}}\right\} &= \\ (n-3) \mathrm{tr}_g(F_{\mathcal{A}}^2) - 2\varepsilon(n-2)\left(\frac{n-1}{n}\right)^2 \mathrm{tr}_g(\partial_{\mathcal{A}} \mu_{\mathrm{YM}})^2 - 2\frac{(n-1)^3}{n^2} \mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^4) - 2\mathrm{tr}_{\mathcal{E}}(\mu_{\mathrm{YM}}^2), \end{aligned} \quad (200)$$

which for $\varepsilon := +1$ and anti-Hermitian μ_{YM} has the form of the Standard Model Lagrangian. We stress that the “kinetic term” of the Higgs, $\mathrm{tr}_g(\partial_{\mathcal{A}} \mu_{\mathrm{YM}})^2$, drops out, if $\mathcal{F}_{\mathrm{DYM}}$ were not of simple type.

The explicit form of the combined Dirac-Majorana mass operator $\mu_{\mathrm{YM}} \in \mathfrak{Sec}(M, \mathrm{End}(\mathcal{E}))$ yields:

$$\mathrm{tr}_g(\partial_{\mathcal{A}} \mu_{\mathrm{YM}})^2 = -4\mathrm{Re} \mathrm{tr}_g(\partial_{\mathcal{A}} \varphi_e)^2, \quad (201)$$

$$a \mathrm{tr}_{\mathcal{E}}\mu_{\mathrm{YM}}^4 + \mathrm{tr}_{\mathcal{E}}\mu_{\mathrm{YM}}^2 = 4\mathrm{Re} \left(a \mathrm{tr}_{\mathcal{W}_e}\varphi_e^4 - \mathrm{tr}_{\mathcal{W}_e}\varphi_e^2 + \Lambda_{\mathrm{DM},\nu} \right), \quad (202)$$

whereby $a \equiv 2\frac{(n-1)^3}{n^2}$ and

$$\begin{aligned} \Lambda_{\mathrm{DM},\nu} &\equiv a \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{D},\nu}^4 - \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{D},\nu}^2 + a \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{M},\nu}^4 - \mathrm{tr}_{\mathcal{W}_\nu} m_{\mathrm{M},\nu}^2 \\ &- 2a \mathrm{tr}_{\mathcal{W}_\nu} (m_{\mathrm{D},\nu} \circ m_{\mathrm{M},\nu})^2 \end{aligned} \quad (203)$$

is the “true cosmological constant”, which naturally occurs in the Einstein-Hilbert action when Majorana masses are taken into account within the geometrical frame of Dirac type gauge theories. Its possible phenomenological consequences, for instance, with respect to the mass of the Higgs boson and the cosmological issue of “dark matter”, will be discussed separately in a forthcoming paper. However, because of the significance of the cosmological constant, we summarize the basic steps to obtain the result (203). This will also enlighten the subtle interplay between simple type Dirac operators and the peculiar form of $\Lambda_{\mathrm{DM},\nu}$ as the sum of two Higgs potentials and an “interaction term” for the Dirac and Majorana masses.

First, it follows that μ_{YM}^2 structurally reads:

$$\begin{aligned}\mu_{\text{YM}}^2 &= \begin{pmatrix} u & z \\ z^{\text{cc}} & u^{\text{cc}} \end{pmatrix}, \\ u &\equiv \mu_{\text{D}}^2 - \mu_{\text{M}}^2, \\ z &\equiv \mu_{\text{D}} \circ \mu_{\text{M}} - \mu_{\text{M}} \circ \mu_{\text{D}}^{\text{cc}}.\end{aligned}\tag{204}$$

Because of the explicit form of the sections $\varphi_{\text{D}} \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}_\nu \oplus \mathcal{W}_e))$ and $m_{\text{M}} \in \mathfrak{Sec}(M, \text{End}_\gamma^+(\mathcal{W}_\nu \oplus \mathcal{W}_e))$, one gets

$$m_{\text{M}} \circ \varphi_{\text{D}}^{\text{cc}} = m_{\text{M}} \circ \varphi_{\text{D}} \quad \Rightarrow \quad \mu_{\text{M}} \circ \mu_{\text{D}}^{\text{cc}} = -\mu_{\text{M}} \circ \mu_{\text{D}}.\tag{205}$$

Therefore,

$$\begin{aligned}a \text{tr}_\mathcal{E} \mu_{\text{YM}}^4 + \text{tr}_\mathcal{E} \mu_{\text{YM}}^2 &= 4\text{Re}[a \text{tr}_\mathcal{E} \varphi_{\text{D}}^4 - \text{tr}_\mathcal{E} \varphi_{\text{D}}^2 + a \text{tr}_\mathcal{E} m_{\text{M}}^4 - \text{tr}_\mathcal{E} m_{\text{M}}^2 \\ &\quad - 2a \text{tr}_\mathcal{E} (\varphi_{\text{D}} \circ m_{\text{M}})^2],\end{aligned}\tag{206}$$

where the occurrence of the Higgs potentials of φ_{D} and m_{M} are due to the fact that the Dirac-Yukawa-Majorana operator $\mathcal{D}_{\text{YM}} = \mathcal{D}_{\text{A}} + i\mu_{\text{YM}} \in \mathcal{D}_{\text{real}}(\mathcal{E})$ is of simple type.

Note that also $\text{tr}_g(F_{\text{A}}^2) = 4\text{Re} \text{tr}_g(F_{\text{A}}^2)$. Hence, the total Dirac action with respect to the real Dirac operator $\mathcal{D}_{\text{DYM}} \in \mathcal{D}(\mathcal{P})$ is a real-valued functional, actually. This is independent of whether the section $\varphi_e \in \mathfrak{Sec}(M, \text{End}_\gamma(\mathcal{W}_e))$ and the simple type Dirac operator $\mathcal{D}_{\text{A}} \in \mathcal{D}(\mathcal{W})$ are supposed to be Hermitian or anti-Hermitian.

6 Real Clifford bi-modules and the “ π_D -map”

The Pauli map (103) is defined for general real Clifford modules. In the previous section we demonstrated how the Pauli map of the simple type Dirac operator defined in terms of a Yang-Mills-Higgs connection on a Majorana module encodes the full STM action functional.

In this section we discuss once again the STM action in view of the Dirac operators (3), this time, however, in the case where the underlying Majorana module is supposed to have the structure of a Clifford bi-module. The discussion of the previous section exhibited the importance of simple type Dirac operators. However, the Pauli map does not preserve the structure of simple type Dirac operators. Although the Yukawa-coupling term and the Pauli term are geometrically treated almost in the same manner, there is yet a basic asymmetry between these two terms. Basically, this is because $i\mu_{\text{YM}} \in \mathfrak{Sec}(M, \text{End}^-(\mathcal{P}))$ anti-commutes with the Clifford action in contrast to $i\mathcal{F}_{\text{D}} \in \mathfrak{Sec}(M, \text{End}^-(\mathcal{P}))$. This apparent asymmetry, however, may be easily overcome in the case where the underlying Majorana module is (embedded into) a Clifford bi-module. This will yield a straightforward geometrical interpretation of the STM action in terms of the Einstein-Hilbert action including a “cosmological constant” term.

Definition 6.1 *Let $(\mathcal{E}, \langle \cdot, \cdot \rangle_\mathcal{E}, \tau_\mathcal{E}, \gamma_\mathcal{E}, \gamma_{\mathcal{E}, \text{op}}, J_\mathcal{E})$ be a real Clifford bi-module over (M, g_{M}) . The mapping*

$$\begin{aligned}\pi_{\text{D}} : \mathcal{D}_{\text{real}}(\mathcal{E}) &\longrightarrow \mathcal{D}_{\text{real}}(\mathcal{P}) \\ \mathcal{D} &\longmapsto \mathcal{D}_{\text{D}} := \mathcal{D} + i\mathcal{F}_{\text{D}}\end{aligned}\tag{207}$$

is called the “ π_D -map”. Here,

$$\begin{aligned}\mathcal{F}_D &:= \begin{pmatrix} 0 & -\tau_\mathcal{E} \circ \mathcal{F}_D \\ \tau_\mathcal{E} \circ \mathcal{F}_D & 0 \end{pmatrix} \\ &\equiv -\tau_\mathcal{P} \circ \mathcal{F}_D \in \text{Sec}(M, \text{End}^-(\mathcal{E})),\end{aligned}\tag{208}$$

$$\mathcal{F}_D := \mathcal{F}_{D,\text{op}} \otimes \varepsilon_2 \in \text{Sec}(M, \text{End}_\gamma^-(\mathcal{E}))\tag{209}$$

and

$$\mathcal{F}_{D,\text{op}} \in \text{Sec}(M, \text{End}_\gamma^+(\mathcal{E}))\tag{210}$$

is the relative curvature of \mathcal{D}_{op} , quantized with respect to $\gamma_{\mathcal{E},\text{op}}$.

Apparently, the real Pauli-like Dirac operator that is defined by (207) is most analogous to the real, simple type Dirac operator (153). In particular, if $\mathcal{D} \in \mathcal{D}(\mathcal{E})$ is of simple type, then so is its associated Pauli-like operator $\mathcal{P}_D \in \mathcal{D}(\mathcal{P} = {}^2\mathcal{E})$. In other words: In contrast to the Pauli map (103), the map (207) preserves the distinguished structure of simple type Dirac operators. According to its definition, however, the π_D -map does not preserve S -reducibility, as opposed to the Pauli map \mathcal{P}_D .

Starting again with a Yang-Mills-Higgs connection $\partial_{\text{YMH}} \in \mathcal{A}(\mathcal{W})$ on the Majorana module $\mathcal{W} \rightarrow M$, we may consider the real Dirac operator of simple type:

$$\begin{aligned}\mathcal{P}_{\text{DYM}} &:= \pi_D(\mathcal{D}_A + i\mu_{\text{YM}}) \\ &\equiv \mathcal{D}_A + i(\mu_{\text{YM}} + \mathcal{F}_{\text{DYM}}),\end{aligned}\tag{211}$$

with $\mathcal{F}_{\text{DYM}} := -\tau_\mathcal{P} \circ (\mathcal{F}_{\text{DYM,op}} \otimes \varepsilon_2)$ being defined by the ($\gamma_{\mathcal{E},\text{op}}$ -quantized) relative curvature of $\mathcal{D}_{\text{YM,op}} = \mathcal{D}_{A,\text{op}} + i\mu_{\text{YM}}$. Notice that $\mathcal{D}_{\text{YM,op}} \notin \mathcal{D}_S(\mathcal{E})$, in contrast to \mathcal{D}_{YM} . However, the \mathcal{D}_{YM} -induced Pauli-like curvature term $i\mathcal{F}_{\text{DYM}}$ does not contribute to the fermionic part of the total Dirac action. Instead, the latter is fully determined by the Dirac connection

$$\begin{aligned}\partial_{\text{DYM}} &:= \mathcal{D}_A + i\text{ext}_\Theta(\mu_{\text{YM}}) \\ &\equiv \partial_{\mathcal{YMH}} + i\text{ext}_\Theta(\mu_M)\end{aligned}\tag{212}$$

of the simple type Dirac operator $\mathcal{D}_{\text{YM}} = \delta_\gamma \circ \partial_{\text{DYM}} \in \mathcal{D}_{S,\text{real}}(\mathcal{E})$. Here, $\partial_{\mathcal{YMH}} \in \mathcal{A}_S(\mathcal{E})$ denotes the real form of the Dirac connection of the simple type Dirac operator (153) on the Dirac module $\mathcal{S} \rightarrow M$ that is induced by the Yang-Mills-Higgs connection $\partial_{\text{YMH}} \in \mathcal{A}_S(\mathcal{W})$ on the underlying Majorana module $\mathcal{W} \rightarrow M$.

Since the real Dirac operator \mathcal{P}_{DYM} is of simple type it becomes straightforward to express the Dirac action

$$\mathcal{I}_{\text{DYM}} := \int_M * \text{tr}_\gamma(\text{curv}(\mathcal{P}_{\text{DYM}}) - \varepsilon \text{ev}_g(\omega_D^2))\tag{213}$$

in terms of the sections parameterizing $\mathcal{D}_{\text{YM}} \in \mathcal{D}_{S,\text{real}}(\mathcal{E})$.

First, we mention that the Dirac vector field $\xi_D \in \mathfrak{Sec}(M, TM)$ of any simple type Dirac operator vanishes identically. This holds true for arbitrary Clifford modules $(\mathcal{E}, \gamma_\mathcal{E}) \rightarrow (M, g_M)$. Therefore,

$$\text{tr}_\mathcal{E}(\mathcal{D}^2 - \Delta_B) = \text{tr}_\gamma(\text{curv}(\mathcal{D}) - \varepsilon \text{ev}_g(\omega_D^2))\tag{214}$$

does not hold true only up to boundary terms but is an identity for simple type Dirac operators $\mathcal{D} \in \mathcal{D}(\mathcal{E})$.

Similar to the last section, we put $\Phi_{\text{DYM}} := i(\mu_{\text{YM}} + \mathcal{F}_{\text{DYM}}) \in \mathfrak{Sec}(M, \text{End}^-(\mathcal{P}))$ and apply once more Lemma 4.1. This time, however, we may take advantage of $\{\gamma_{\mathcal{P}}(\alpha), \Phi_{\text{DYM}}\} \equiv 0$, for all $\alpha \in T^*M$. Consequently, $\Theta \wedge \Phi_{\text{DYM}} = -\Phi_{\text{DYM}} \wedge \Theta$ and thus

$$\begin{aligned} ev_g(\omega_{\text{D}}^2) &= -g_{\text{M}}(e^i, e^j) \Theta(e_i) \circ \Theta(e_j) \circ \Phi_{\text{DYM}}^2 \\ &= -\frac{\varepsilon^2}{n^2} g_{\text{M}}(e_i, e_j) \gamma_{\mathcal{P}}(e^i) \circ \gamma_{\mathcal{P}}(e^j) \circ \Phi_{\text{DYM}}^2 \\ &= -\frac{\varepsilon}{n} \Phi_{\text{DYM}}^2. \end{aligned} \quad (215)$$

Furthermore,

$$\begin{aligned} \mathcal{I}_{\text{DYM}} &= \int_M \left[\text{tr}_{\gamma}(\text{curv}(\mathcal{D}_{\mathcal{A}}) - (d_{\mathcal{A}}\Phi_{\text{DYM}} + \Phi_{\text{DYM}}^2 \Theta) \wedge \Theta) + \frac{1}{n} \text{tr}_{\mathcal{P}} \Phi_{\text{DYM}}^2 \right] d\text{vol}_M \\ &= \int_M \left[\text{tr}_{\gamma} \text{curv}(\mathcal{D}_{\mathcal{A}}) + \text{tr}_{\mathcal{P}} \Phi_{\text{DYM}}^2 \right] d\text{vol}_M. \end{aligned} \quad (216)$$

The Dirac action with respect to $\mathcal{P}_{\text{DYM}} = \pi_{\text{D}}(\mathcal{D}_{\mathcal{A}} + i\mu_{\text{YM}}) \in \mathcal{D}_{\text{real}}(\mathcal{P})$ thus dynamically generalizes the Einstein-Hilbert action (67 – 68) with the cosmological constant induced by the Yang-Mills-Higgs connection, whose quantization (together with the Majorana masses) defines the fermionic action⁵. This time, however, the “cosmological constant” does depend on the metric as opposed to (68). Indeed, according to the explicit form of Φ_{DYM} , it follows that

$$\begin{aligned} \text{tr}_{\mathcal{P}} \Phi_{\text{DYM}}^2 &= 2 \text{tr}_{\mathcal{E}} \left(\mathcal{F}_{\text{DYM,op}}^2 - \mu_{\text{YM}}^2 \right) \\ &= -\text{tr}_g F_{\mathcal{A}}^2 + 2\varepsilon \left(\frac{n-1}{n} \right)^2 \text{tr}_g (\partial_{\mathcal{A}} \mu_{\text{YM}})^2 + 2 \left(\frac{n-1}{n} \right)^2 \text{tr}_{\varepsilon} \mu_{\text{YM}}^4 - 2 \text{tr}_{\varepsilon} \mu_{\text{YM}}^2. \end{aligned} \quad (217)$$

For $\varepsilon := -1$ and Hermitian μ_{YM} the “cosmological constant term” has the form of the usual Lagrangian of the Standard Model, such that \mathcal{I}_{DYM} , again, takes the form of the combined Einstein-Hilbert-Yang-Mills-Higgs action⁶. This functional basically coincides with what has been derived from the Pauli-Dirac operator $\mathcal{P}_{\text{D}}(\mathcal{D}_{\mathcal{A}} + i\mu_{\text{YM}}) \in \mathcal{D}_{\text{S,real}}(\mathcal{P})$ in the previous section. Note, however, that there is a difference concerning the conditions imposed on $(\varepsilon, \mu_{\text{YM}})$. Also note that there is a significant difference between \mathcal{P}_{D} and π_{D} in dimension two.

7 Conclusion

In this article we discussed the geometrical structure of Pauli-type Dirac operators which encode the STM action including gravity. This has been done by carefully analyzing the corresponding structure of the Dirac equation and the Majorana equation in terms of real Clifford (bi-)modules and Dirac operators of simple type. It has been shown how the geometrical frame presented allows to overcome the issue of “fermion doubling” and how the combined Einstein-Hilbert-Yang-Mills-Higgs (EHYMH-) action can be derived from the distinguished

⁵It makes no sense to take the Majorana masses into account directly on the Majorana module $\mathcal{W} \rightarrow M$. Indeed, for this one has to make use of the induced Dirac module.

⁶For $\varepsilon := +1$ one gets the corresponding Lagrangian with respect to the Euclidean signature of g_{M} .

class of real Dirac operators of simple type. The latter description allows to geometrically recast the EHYMH-action into a form which formally looks identical to the Einstein-Hilbert action with a cosmological constant. On this basis, we have demonstrated how Majorana masses are naturally included within the geometrical frame of Dirac type gauge theories and how they dynamically contribute to the combined EHYMH-action in terms of a peculiar cosmological constant. This cosmological constant may have interesting phenomenological consequences with respect to dark matter/energy and the mass of the Higgs boson to be discussed in a forthcoming work.

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Appendix

In this appendix, we briefly introduce a specific class of Majorana modules. The latter will be appropriate to geometrically describe the (minimal) Standard Model in terms of Dirac type gauge theories when also Majorana masses are taken into account. The inclusion of massive Dirac neutrinos within the (minimal) Standard Model has been discussed already in [TT '06b].

Let M be an orientable, connected and simply connected four-dimensional spin manifold admitting a Lorentzian structure. For each Lorentz metric $g_M \in \mathfrak{Sec}(M, \mathcal{E}_{EH})$, the corresponding Lorentz manifold (M, g_M) is also supposed to be time orientable. For every choice of a spin-structure let

$$\mathcal{W}_e := S \otimes_{\mathbb{C}} E \rightarrow M \quad (218)$$

be a twisted spinor bundle. The Hermitian vector bundle $E = E_R \oplus E_L \rightarrow M$ is assumed to be associated with a G -principal bundle $G \hookrightarrow \mathcal{P}_G \rightarrow M$. In the case of the Standard Model: $G := SU(3) \times SU(2) \times U(1)$. According to Geroch's Theorem, the frame bundle $\mathcal{F}_M \rightarrow M$ of M is trivial, provided M is open (c.f. [Ger '68] and [Ger '70]). Also, the topological structure of the electroweak gauge sub-bundle: $\mathcal{P}_{SU(2) \times U(1)} \hookrightarrow \mathcal{P}_{SU(3)} \times_M \mathcal{P}_{SU(2) \times U(1)} \rightarrow M$, is fully determined by the moduli space of ground states of the Higgs boson. In particular, the electroweak gauge bundle is trivial, provided the electrically charged weak vector bosons W^\pm are considered as being charge conjugate to each other (c.f. [Tol '05]). In the case of the Standard Model, the Hermitian vector bundle $E \rightarrow M$ is defined by the fermionic representation of G (c.f., for example, Sec. 3.1 in [TT '06b]).

We put, respectively, for the uncharged and charged sector of the Majorana module

$\mathcal{W} = \mathcal{W}_\nu \oplus \mathcal{W}_e \rightarrow M$:

$$\mathcal{W}_\nu := \begin{array}{l} \mathcal{W}_{\nu,R} \equiv \mathcal{W}_{\nu,RR} \oplus \mathcal{W}_{\nu,RL} \\ \oplus \\ \mathcal{W}_{\nu,L} \equiv \mathcal{W}_{\nu,LR} \oplus \mathcal{W}_{\nu,LL} \end{array}, \quad \mathcal{W}_e := \begin{array}{l} \mathcal{W}_{e,R} \equiv \mathcal{W}_{e,RR} \oplus \mathcal{W}_{e,RL} \\ \oplus \\ \mathcal{W}_{e,L} \equiv \mathcal{W}_{e,LR} \oplus \mathcal{W}_{e,LL} \end{array}. \quad (219)$$

Here,

$$\begin{array}{ll} \mathcal{W}_{\nu,RR} & := S_R \otimes V_R, & \mathcal{W}_{\nu,LL} & := S_L \otimes V_L, \\ \mathcal{W}_{\nu,RL} & := S_R \otimes V_L, & \mathcal{W}_{\nu,LR} & := S_L \otimes V_R, \end{array} \quad (220)$$

$$\begin{array}{ll} \mathcal{W}_{e,RR} & := S_R \otimes E_R, & \mathcal{W}_{e,LL} & := S_L \otimes E_L, \\ \mathcal{W}_{e,RL} & := S_R \otimes E_L, & \mathcal{W}_{e,LR} & := S_L \otimes E_R. \end{array} \quad (221)$$

The Hermitian vector space $V = V_R \oplus V_L$ carries the trivial representation of G . Its dimension may be arbitrarily chosen.

Note that

$$\mathcal{W} \simeq S \otimes_{\mathbb{C}} (V \oplus E). \quad (222)$$

Let, respectively, τ_V and τ_E be the corresponding grading involutions of $M \times V_R \oplus V_L \rightarrow M$ and $E = E_R \oplus E_L \rightarrow M$. According to the above decomposition, the grading involution reads:

$$\tau_{\mathcal{W}} := \begin{pmatrix} \tau_{\mathcal{W}_\nu} & 0 \\ 0 & \tau_{\mathcal{W}_e} \end{pmatrix}, \quad (223)$$

whereby

$$\tau_{\mathcal{W}_\nu} := \begin{pmatrix} \tau_V & 0 \\ 0 & -\tau_V \end{pmatrix}, \quad \tau_{\mathcal{W}_e} := \begin{pmatrix} \tau_E & 0 \\ 0 & -\tau_E \end{pmatrix}. \quad (224)$$

By abuse of notation, we do not distinguish between τ_V and $\text{id}_S \otimes \tau_V$. Likewise, τ_E is identified with $\text{id}_S \otimes \tau_E$.

It follows that, for example,

$$\mathcal{W}_{\nu,LR} = \{\nu \in \mathcal{W}_\nu \mid \tau_M \nu = -\nu, \tau_V \nu = +\nu\} \subset \mathcal{W}_\nu, \quad (225)$$

$$\mathcal{W}_{e,RL} = \{e \in \mathcal{W}_e \mid \tau_M e = +e, \tau_E e = -e\} \subset \mathcal{W}_e, \quad \text{etc.} \quad (226)$$

In particular, the (total spaces of the) respective eigen bundles of $\tau_{\mathcal{W}_\nu}$ and $\tau_{\mathcal{W}_e}$ read:

$$\begin{array}{l} \mathcal{W}_\nu \simeq \mathcal{W}_\nu^+ \oplus \mathcal{W}_\nu^-, \\ \mathcal{W}_\nu^+ := \mathcal{W}_{\nu,RR} \oplus \mathcal{W}_{\nu,LL}, \\ \mathcal{W}_\nu^- := \mathcal{W}_{\nu,RL} \oplus \mathcal{W}_{\nu,LR}; \end{array} \quad (227)$$

$$\begin{array}{l} \mathcal{W}_e \simeq \mathcal{W}_e^+ \oplus \mathcal{W}_e^-, \\ \mathcal{W}_e^+ := \mathcal{W}_{e,RR} \oplus \mathcal{W}_{e,LL}, \\ \mathcal{W}_e^- := \mathcal{W}_{e,RL} \oplus \mathcal{W}_{e,LR}. \end{array} \quad (228)$$

The Clifford action is defined in terms of the Clifford mapping:

$$\gamma_{\mathcal{W}} := \begin{pmatrix} \gamma_{\mathcal{W}_\nu} & 0 \\ 0 & \gamma_{\mathcal{W}_e} \end{pmatrix}, \quad \gamma_{\mathcal{W}_\nu} = \gamma_{\mathcal{W}_e} := \begin{pmatrix} 0 & \gamma_{\text{Ch}} \\ \gamma_{\text{Ch}} & 0 \end{pmatrix}, \quad (229)$$

whereby the Clifford mapping $\gamma_{\text{Ch}} : T^*M \rightarrow \text{End}_{\mathbb{C}}(S)$ acts trivially on the sub-bundles $E_{\text{R}}, E_{\text{L}} \subset E \rightarrow M$ and on the Hermitian vector spaces $V_{\text{R}}, V_{\text{L}}$.

The real structure is defined by

$$J_{\mathcal{W}} := \begin{pmatrix} J_{\mathcal{W}_\nu} & 0 \\ 0 & J_{\mathcal{W}_e} \end{pmatrix}, \quad J_{\mathcal{W}_\nu} = J_{\mathcal{W}_e} := \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad (230)$$

whereby $J \circ \gamma_{\text{Ch}}(\alpha) = -\gamma_{\text{Ch}}(\alpha) \circ J$ and $J \circ \tau_M = -\tau_M \circ J$ is assumed to hold for all $\alpha \in T^*M$.

The Dirac type operator $\not{\partial}_{\text{A}} + i\varphi_{\text{D}}$ decomposes as follows:

- On the uncharged sector: $m_{\text{D},\nu} \in \mathfrak{Sec}(M, \text{End}_{\gamma}(\mathcal{W}_\nu))$ (real and constant)

$$\not{\partial} + im_{\text{D},\nu} \equiv \begin{pmatrix} im_{\text{D},\nu} & \not{\partial} \\ \not{\partial} & im_{\text{D},\nu} \end{pmatrix}; \quad (231)$$

- On the charged sector:

$$\not{\partial}_{\text{A}} + i\varphi_e \equiv \begin{pmatrix} i\varphi_e & \not{\partial}_{\text{A}} \\ \not{\partial}_{\text{A}} & i\varphi_e \end{pmatrix}, \quad (232)$$

where either $\varphi_e \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{-}(\mathcal{W}_e))$, or $\varphi_e \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{+}(\mathcal{W}_e))$.

As mentioned already, the latter case holds true only for left-right symmetric gauge theories. It does not hold true in the usual (minimal) Standard Model where parity is maximally violated.

The block matrix notation used for the Dirac type operators refers to the embedding

$$\begin{aligned} \mathcal{W}_{e,\text{R}} &\hookrightarrow \mathcal{W}_e \\ e_{\text{R}} &\mapsto \begin{pmatrix} e_{\text{R}} \\ 0 \end{pmatrix}, \end{aligned} \quad (233)$$

where, for example, on the left-hand side $e_{\text{RR}} \in \{e \in \mathcal{W}_e \mid \tau_{\text{M}}e = +e, \tau_{\text{E}}e = +e\} = \mathcal{W}_{e,\text{RR}} \subset \mathcal{W}_e$, etc.

Accordingly, the Majorana mass matrix $m_{\text{M},\nu} \in \mathfrak{Sec}(M, \text{End}_{\gamma}^{+}(\mathcal{W}_\nu))$ (real and constant) reads:

$$m_{\text{M},\nu} \equiv \begin{pmatrix} m_{\text{M},\nu} & 0 \\ 0 & m_{\text{M},\nu} \end{pmatrix}, \quad (234)$$

such that the combined Dirac-Yukawa-Majorana equations (184–185) explicitly takes the form:

$$i\not{\partial}\nu = m_{\text{D},\nu}\nu + m_{\text{M},\nu}\nu^{\text{cc}} \Leftrightarrow \begin{cases} i\not{\partial}\nu_{\text{R}} = m_{\text{D},\nu}\nu_{\text{L}} + m_{\text{M},\nu}\nu_{\text{R}}^{\text{cc}}, \\ i\not{\partial}\nu_{\text{L}} = m_{\text{D},\nu}\nu_{\text{R}} + m_{\text{M},\nu}\nu_{\text{L}}^{\text{cc}}, \end{cases} \quad (235)$$

$$i\not{\partial}_{\text{A}}e = \varphi_e e \Leftrightarrow \begin{cases} i\not{\partial}_{\text{A}}e_{\text{R}} = \varphi_e e_{\text{L}}, \\ i\not{\partial}_{\text{A}}e_{\text{L}} = \varphi_e e_{\text{R}}. \end{cases} \quad (236)$$

In the case of the Standard Model a “charged state” is geometrically represented by a section of $\mathcal{W}_e^+ \subset \mathcal{W}_e \rightarrow M$:

$$e = \begin{pmatrix} e_{RR} \\ e_{LL} \end{pmatrix} \in \mathfrak{Sec}(M, \mathcal{W}_e^+). \quad (237)$$

Whence,

$$i\mathcal{D}_A e = \varphi_e e \quad \Leftrightarrow \quad \begin{cases} i\mathcal{D}_A e_{RR} = \varphi_{e,RL} e_{LL}, \\ i\mathcal{D}_A e_{LL} = \varphi_{e,LR} e_{RR}, \end{cases} \quad (238)$$

where

$$\begin{aligned} \varphi_{e,RL} &\equiv \pi_{E,R} \circ \varphi_e \circ \pi_{E,L} \in \mathfrak{Sec}(M, \text{Hom}_\gamma(\mathcal{W}_{e,L}, \mathcal{W}_{e,R})), \\ \varphi_{e,LR} &\equiv \pi_{E,L} \circ \varphi_e \circ \pi_{E,R} \in \mathfrak{Sec}(M, \text{Hom}_\gamma(\mathcal{W}_{e,R}, \mathcal{W}_{e,L})), \end{aligned} \quad (239)$$

and $\pi_{E,R/L} := (\text{id}_E \pm \tau_E)/2$ are the complementary idempotents with respect to the \mathbb{Z}_2 -grading $E = E_R \oplus E_L \rightarrow M$.

Finally, in the case of the (minimal) Standard Model, the section $\varphi_e \in \mathfrak{Sec}(M, \text{End}_\gamma^-(\mathcal{W}_e))$ is related to the usual Higgs field via the “Yukawa mapping”:

$$\varphi_{e,LR} := G_Y(\varphi) \equiv \begin{pmatrix} (\mathbf{g}'^q \otimes \varphi, -\mathbf{g}^q \otimes \mathbf{I}_2 \otimes \varphi^{cc}) & 0 \\ 0 & \mathbf{g}^l \otimes \varphi \end{pmatrix}, \quad (240)$$

$$\varphi_{e,RL} := \varphi_{e,LR}^\dagger. \quad (241)$$

Here, respectively, $\mathbf{g}'^q, \mathbf{g}^q \in \mathbb{C}(N)$ and $\mathbf{g}^l \in \mathbb{C}(N)$ are the matrices of the “Yukawa coupling constants” of the quarks of electrical charge $-1/3$ and $+2/3$ and the leptons of electrical charge equal to -1 . The section $\varphi \in \mathfrak{Sec}(M, E_H)$ geometrically describes the (semi-classical state of the) Higgs field. According to the minimal Standard Model the Higgs boson carries a rank two sub-representation $E_H \hookrightarrow E \rightarrow M$ of the fermionic representation $E \rightarrow M$. This sub-representation is fixed by the “hyper-charge relations” between the hyper-charges carried by the quarks and leptons (see again, for example, Sec. 3.1 in loc. site; for a geometrical discussion of the Yukawa mapping: $G_Y : E_H \hookrightarrow E$, see also [TT '06a]). When these relations are known, the hyper-charge of the Higgs boson is fixed by the demand that the Dirac type operators $\mathcal{D}_A \pm i\varphi_e \in \mathcal{D}(\mathcal{W}_e)$ transform with respect to the adjoint representation of the Yang-Mills gauge sub-group of \mathcal{G}_D .

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