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by

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Abstract

We show that although we can not distill a singlet from many pairs of bound entangled states, the concurrence and tangle of two entangled quantum states are always strictly larger than that of one, even both entangled quantum states are bound entangled. We present a relation between the concurrence and the fidelity of optimal teleportation. We also give new upper and lower bounds for concurrence and tangle.

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I. INTRODUCTION

Quantum entanglement plays crucial roles in quantum information processing [1]. Quantum entangled states have become the key ingredient in the rapidly expanding field of quantum information science, with remarkable prospective applications such as quantum teleportation [2], quantum cryptography [3], quantum dense coding [4] and parallel computing [5].

However, it has been shown that not all of the entangled quantum states are useful in quantum information. There exist bound entangled states from which no pure entangled states can be distilled under local operation and classical communication (LOCC) [6]. With bound entangled states as the entangled resource teleportation can not be performed better than with a classical channel, even if conclusive teleportation is allowed [7]. It has been shown that bound entangled states can enhance the fidelity of teleportation other non-bound entangled states [8, 9]. However, a bound entangled state can never enhance the teleportation fidelity of another state which is also bound entangled [9].

An important problem in quantum information theory is the detection of quantum entanglement. A series of excellent results have been obtained on separability criteria and evaluation of measures of quantum entanglement such as entanglement of formation (EOF) [11] and concurrence [12, 13].

There have been some (necessary) criteria for separability, the Bell inequalities [14], PPT (positive partial transposition) [15] (which is also sufficient for the cases 2×2 and 2×3 bipartite systems [16]), realignment [17–19] and generalized realignment [20], as well as some necessary and sufficient operational criteria for low rank density matrices [21–23]. Further more, separability criteria based on local uncertainty relation [24–27] and the correlation matrix [28, 29] of the Bloch representation for a quantum state have been derived, which are strictly stronger than or independent of the PPT and realignment criteria. The calculation of entanglement of formation or concurrence is complicated except for 2×2 systems [30] or for states with special forms [31]. For general quantum states with higher dimensions or multipartite case, it seems to be a very difficult problem to obtain analytical formulas. In stead, the lower and the upper bounds of concurrence [32–36] and EOF [37] have been estimated.

In this paper we show in Section II that although we can not distill pure entangled states from any bound entangled states, the concurrence and tangle of two entangled states will be always strictly larger than that of one, even the two entangled states are both bound entangled. We study the relation between the fidelity of optimal teleportation and concurrence in section III, which connects the result in section II to that in [9]. We investigate bounds for concurrence and tangle in Section IV. New lower and upper bounds for concurrence and tangle are derived, which can be used not only for the estimation of entanglement, but also for the investigation of separability. The subadditivity property of concurrence and tangle is proved in Section V. We give concludes and remarks in the last section.

II. CONCURRENCE AND TANGLE OF TWO ENTANGLED QUANTUM STATES

The concurrence and the tangle are two well defined entanglement measures satisfying the standard properties usually regarded as essential for a good entanglement measure (see, for example, [38]).

Let $\mathcal{H}_A$ (resp. $\mathcal{H}_B$) be an M (resp. N)-dimensional complex vector space with $|i\rangle$, $i = 1, \cdots, M$ (resp. $|j\rangle$, $j = 1, \cdots, N$), as an orthonormal basis. We assume $M \leq N$ for convenience. A general pure state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is of the form

$$|\Psi\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}|i\rangle \otimes |j\rangle, \tag{1}$$

where $a_{ij} \in \mathbb{C}$ satisfy the normalization $\sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}a_{ij}^* = 1$.

For a bipartite pure quantum state $|\psi\rangle$ the concurrence is defined by [12]

$$C(|\psi\rangle) = \sqrt{2(1 - Tr\rho_A^2)}, \tag{2}$$

where $\rho_A = Tr_B|\psi\rangle\langle\psi|$, while the tangle is defined by [39]

$$\tau(|\psi\rangle) = C^2(|\psi\rangle) = 2(1 - Tr\rho_A^2). \tag{3}$$

The definition is extended to general mixed states $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ by the convex roof,

$$C(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(\psi_i); \tag{4}$$

$$\tau(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \tau(\psi_i). \tag{5}$$

Let $\rho = \sum_{ijkl} \rho_{ij,kl}|ij\rangle\langle kl| \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma = \sum_{i'j'k'l'} \sigma_{i'j',k'l'}|i'j'\rangle\langle k'l'| \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$. We denote $\rho \otimes \sigma = \sum_{ijkl,i'j'k'l'} \rho_{ij,kl}\sigma_{i'j',k'l'}|ii'\rangle_{AA'}\langle kk'| \otimes |jj'\rangle_{BB'}\langle ll'|$ the bipartite state in the bipartite partition AA' and BB'.

Lemma 1: For pure states $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $|\varphi\rangle \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, the inequalities

$$C(|\psi\rangle \otimes |\varphi\rangle) \geq \max\{C(|\psi\rangle), C(|\varphi\rangle)\} \tag{6}$$

and

$$\tau(|\psi\rangle \otimes |\varphi\rangle) \geq \max\{\tau(|\psi\rangle), \tau(|\varphi\rangle)\} \tag{7}$$

always hold, the equalities hold if and only if at least one of the states, $|\psi\rangle$ and $|\varphi\rangle$, is separable.

Proof: Without loss of generality we assume $C(|\psi\rangle) \geq C(|\varphi\rangle)$. First note that

$$\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle} = \rho_A^{|\psi\rangle} \otimes \rho_{A'}^{|\varphi\rangle}, \tag{8}$$

where $\rho_A^{|\psi\rangle} = Tr_B|\psi\rangle\langle\psi|$, $\rho_{A'}^{|\varphi\rangle} = Tr_{B'}|\varphi\rangle\langle\varphi|$, $\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle} = Tr_{BB'}(|\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|)$. Let $\rho_A^{|\psi\rangle} = \sum_i \lambda_i|i\rangle\langle i|$ and $\rho_{A'}^{|\varphi\rangle} = \sum_j \pi_j|j\rangle\langle j|$ be the spectral decomposition of $\rho_A^{|\psi\rangle}$ and $\rho_{A'}^{|\varphi\rangle}$, with $\sum_i \lambda_i = 1$ and $\sum_j \pi_j = 1$ respectively. By using (8) one obtains that

$$Tr[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2] = \sum \lambda_i\pi_j\lambda_{i'}\pi_{j'}|ij\rangle\langle ij|i'j'\rangle\langle i'j'| = \sum \lambda_i^2\pi_j^2 \tag{9}$$

and

$$Tr[(\rho_A^{|\psi\rangle})^2] = \sum_i \lambda_i^2. \tag{10}$$

From the definition of concurrence and the normalization conditions of λ_i and π_j one immediately gets

$$C(|\psi\rangle \otimes |\varphi\rangle) = \sqrt{2(1 - Tr[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2])} \geq \sqrt{2(1 - Tr[(\rho_A^{|\psi\rangle})^2])} = C(|\psi\rangle). \tag{11}$$

If one of the states $|\psi\rangle$, $|\varphi\rangle$, say $|\varphi\rangle$, is separable, then the rank of $\rho_{A'}^{|\varphi\rangle}$ must be one, which means that there is only one item in the spectral decomposition in $\rho_{A'}^{|\varphi\rangle}$. From the normalization condition of π_j we obtain $Tr[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2] = Tr[(\rho_A^{|\psi\rangle})^2]$. Hence the inequality (11) becomes an equality.

On the other hand, if both $|\psi\rangle$ and $|\varphi\rangle$ are not separable, there must be at least two items in the decomposition of their reduced density matrices $\rho_A^{|\psi\rangle}$ and $\rho_{A'}^{|\varphi\rangle}$ which means that $Tr[(\rho_{AA'}^{|\psi\rangle\otimes|\varphi\rangle})^2]$ is strictly larger than $Tr[(\rho_A^{|\psi\rangle})^2]$.

The inequality (7) also holds because for pure quantum state ρ, $\tau(\rho) = C^2(\rho)$. □

By using the lemma, we have

Theorem 1: For any quantum mixed states $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma \in \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, the inequalities

$$C(\rho \otimes \sigma) \geq \max\{C(\rho), C(\sigma)\} \tag{12}$$

and

$$\tau(\rho \otimes \sigma) \geq \max\{\tau(\rho), \tau(\sigma)\} \tag{13}$$

hold. They become equalities if and only if at least one of the states, ρ and σ, is separable.

Proof: We assume $C(\rho) \geq C(\sigma)$ for convenience. Let $\rho = \sum_i p_i\rho_i$ and $\sigma = \sum_j q_j\sigma_j$ be the optimal decompositions such that $C(\rho\otimes\sigma) = \sum_i p_iq_jC(\rho_i\otimes\sigma_j)$. By using the inequality in Lemma 1 we have

$$C(\rho \otimes \sigma) = \sum_{ij} p_iq_jC(\rho_i \otimes \sigma_j) \geq \sum_{ij} p_iq_jC(\rho_i) = \sum_i p_iC(\rho_i) \geq C(\rho). \tag{14}$$

Case 1. If one of the states ρ and σ, say σ, is separable, i.e. σ can be written as $\sigma = \sum_j q_j\sigma_j$, where $\sum_j q_j = 1$ and σ_j are the density matrices of separable pure states. Suppose $\rho = \sum_i p_i\rho_i$ be the optimal decomposition of ρ such that $C(\rho) = \sum_i p_iC(\rho_i)$. Using Lemma 1 we have

$$C(\rho \otimes \sigma) \leq \sum_{ij} p_iq_jC(\rho_i \otimes \sigma_j) = \sum_{ij} p_iq_jC(\rho_i) = \sum_i p_iC(\rho_i) = C(\rho). \tag{15}$$

Inequalities (14) and (15) show that if σ is separable, then $C(\rho \otimes \sigma) = C(\rho)$.

Case 2: If both ρ and σ are not separable, using Lemma 1 we have

$$C(\rho \otimes \sigma) = \sum_{ij} p_iq_jC(\rho_i \otimes \sigma_j) > \sum_{ij} p_iq_jC(\rho_i) = \sum_i p_iC(\rho_i) \geq C(\rho), \tag{16}$$

i.e. (12) is strictly an inequality.

The inequality (13) for tangle τ can be proved similarly. □

Remark : In [9] the author shew that any entangled state ρ can enhance the teleportation power of a state σ. This holds even if the state ρ is bound entangled. But if ρ is bound entangled, the corresponding state σ must be free entangled (distillable). From Theorem 1, we see that even both ρ and σ are bound entangled states, the concurrence and tangle can be still strictly larger than that of one state.

III. RELATION BETWEEN CONCURRENCE AND FULLY ENTANGLED FRACTION

Quantum entangled states are the key resources in quantum teleportation [40]. As shown in [41], the optimal teleportation fidelity is related to the concurrence of a two-qubit quantum state. For high dimensional case, the optimal fidelity of teleportation with a quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ as an entangled resource, with dimensions $M = N = d$ is given by [42]

$$F(\rho) = \frac{d}{d+1}\mathcal{F}(\rho) + \frac{1}{d+1}, \tag{17}$$

where $\mathcal{F}(\rho)$ is the fully entangled fraction of ρ defined by

$$\mathcal{F}(\rho) = \max_{\phi \in \epsilon} \langle \phi | \rho | \phi \rangle, \tag{18}$$

where ϵ denotes the set of $d \times d$-dimensional maximally entangled states.

Theorem 2: For any bipartite quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$ with dimensions $M = N = d$, we have

$$C(\rho) \geq \sqrt{\frac{2d}{d-1}}[\mathcal{F}(\rho) - \frac{1}{d}]. \tag{19}$$

Proof: It is shown that for any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, the following inequality holds [43]:

$$C(|\psi\rangle) \geq \sqrt{\frac{2d}{d-1}}(max_{|\phi\rangle \in \varepsilon} |\langle \psi | \phi \rangle|^2 - \frac{1}{d}). \tag{20}$$

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ be the optimal decomposition such that $C(\rho) = \sum_i p_i C(|\psi_i\rangle)$.We have

$$\begin{aligned} C(\rho) &= \sum_i p_i C(|\psi_i\rangle) \geq \sum_i p_i \sqrt{\frac{2d}{d-1}}(max_{|\phi\rangle \in \varepsilon} |\langle \psi_i | \phi \rangle|^2 - \frac{1}{d}) \\ &\geq \sqrt{\frac{2d}{d-1}}(max_{|\phi\rangle \in \varepsilon} \sum_i p_i |\langle \psi_i | \phi \rangle|^2 - \frac{1}{d}) \\ &= \sqrt{\frac{2d}{d-1}}(max_{|\phi\rangle \in \varepsilon} \langle \phi | \rho | \phi \rangle - \frac{1}{d}) = \sqrt{\frac{2d}{d-1}}(\mathcal{F}(\rho) - \frac{1}{d}), \end{aligned}$$

which ends the proof. □

The inequality (19) shows a relation between the lower bound of concurrence and the fully entangled fraction (thus the optimal teleportation fidelity), namely the fully entangled fraction of a quantum state ρ is limited by it's concurrence. Moreover (19) also gives a lower bound for concurrence which is obviously closer than that in [43].

IV. ESTIMATION OF CONCURRENCE, TANGLE AND SEPARABILITY

In this section we derive new lower and upper bounds of concurrence and tangle for arbitrary quantum states. Using these bounds we can detect more entangled states. Detailed examples are given to show that the new bounds of concurrence are better than that have been obtained so far.

A. Bounds of Concurrence and Tangle for bipartite quantum systems

We see that (12) and (13) can be regarded as lower bounds for τ and C of a certain state that can be achieved with the help of another state. In fact there have been many lower and upper bounds for concurrence and tangle [32–36, 44–48]. Here we just list several important ones that will be used in the following. In [32] a lower bound for a bipartite state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, with dimensions $M \leq N$, has been obtained,

$$C(\rho) \geq \sqrt{\frac{2}{M(M-1)}}[\max(||\mathcal{T}_A(\rho)||_{KF}, ||R(\rho)||_{KF}) - 1]. \tag{21}$$

where $\mathcal{T}_A$, R and $||\cdot||_{KF}$ stand for the partial transpose, realignment, and the trace norm (sum of the singular values), respectively.

In [34, 44, 45], from the separability criteria related to local uncertainty relation, covariance matrix and correlation matrix, the following lower bounds for bipartite concurrence are obtained:

$$C(\rho) \geq \frac{2||C(\rho)||_{KF} - (1 - Tr\{\rho_A^2\}) - (1 - Tr\{\rho_B^2\})}{\sqrt{2M(M-1)}} \tag{22}$$

and

$$C(\rho) \geq \sqrt{\frac{8}{M^3N^2(M-1)}}(||T(\rho)||_{KF} - \frac{\sqrt{MN(M-1)(N-1)}}{2}), \tag{23}$$

where the entries of the matrices C and T are given by, $C_{ij} = \langle\lambda_i^A \otimes \lambda_j^B\rangle - \langle\lambda_i^A \otimes I_N\rangle\langle I_M \otimes \lambda_j^B\rangle$, $T_{ij} = \frac{MN}{2}\langle\lambda_i^A \otimes \lambda_j^B\rangle$, $\lambda_k^{A/B}$ stands for the normalized generator of $SU(M/N)$ satisfying $Tr\{\lambda_k^{A/B}\lambda_l^{A/B}\} = \delta_{kl}$ and $\langle X\rangle = Tr\{\rho X\}$ stands for the expection value of X. It is shown that the lower bounds (22) and (23) are independent of (21). Besides, in [35], a lower bound for tangle has been derived:

$$\tau(\rho) \geq \frac{8}{MN(M+N)}(||T(\rho)||_{HS}^2 - \frac{MN(M-1)(N-1)}{4}), \tag{24}$$

where $||X||_{HS} = \sqrt{\mathrm{Tr}(XX^\dagger)}$ denotes the Frobenius (Hilbert-Schmidt) norm. Experimentally measurable lower and upper bounds for concurrence have been presented in [36, 47]:

$$\sqrt{2(\mathrm{Tr}[\rho^2] - \mathrm{Tr}[\rho_A^2])} \leq C(\rho) \leq \sqrt{2(1 - \mathrm{Tr}[\rho_A^2])}. \tag{25}$$

Since the convexity of $C^2(\rho)$, we have that $\tau(\rho) \geq C^2(\rho)$ always holds. In [39] the author point out that for two qubits quantum systems, tangle τ is always equal to the square of concurrence C^2, as a decomposition $\{p_i, |\psi_i\rangle\}$ achieving the minimum in Eq. (4) will have the property that $C(|\psi_i\rangle) = C(|\psi_j\rangle)$ for all i, j. But for higher dimensional systems we do not have similar equations. Therefore it is meaningful to derive valid upper bound for tangle and lower bound for concurrence. In the following we derive an effective upper bound for tangle, which can be used to estimate the entanglement of quantum states. We also derive new lower bound for concurrence which is better than that in (21), (22) and (23) for certain quantum states.

Theorem 3: For any quantum state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, we have

$$\tau(\rho) \leq \min\{2(1 - \mathrm{Tr}(\rho_A^2)), 2(1 - \mathrm{Tr}(\rho_B^2))\} \tag{26}$$

$$C(\rho) \geq \sqrt{\frac{8}{MN(M+N)}}(||T(\rho)||_{HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}) \tag{27}$$

where $\rho_{A/B}$ are the reduced matrices of ρ, and $T(\rho)$ is the correlation matrix of ρ defined in (23).

Proof: We assume $1 - \mathrm{Tr}(\rho_A^2) \leq 1 - \mathrm{Tr}(\rho_B^2)$ for convenience. From the definition of τ, we have that for a pure state $|\psi\rangle, \tau(|\psi\rangle) = 2(1 - \mathrm{Tr}(\rho_A^{|\psi\rangle})^2)$. Let $\rho = \sum_i p_i \rho_i$ be the optimal decomposition such that $\tau(\rho) = \sum_i p_i \tau(\rho_i)$. We get

$$\tau(\rho) = \sum_i p_i \tau(\rho_i) = \sum_i p_i 2[1 - \mathrm{Tr}(\rho_A^{|\psi_i\rangle})^2] = 2[1 - \mathrm{Tr} \sum_i p_i (\rho_A^{|\psi_i\rangle})^2] \leq 2[1 - \mathrm{Tr}(\rho_A^2)]. \tag{28}$$

.

To prove (27), first note that in [35] the author obtains, for pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$C(|\psi\rangle) = \sqrt{\frac{8}{MN(M+N)}(||T(|\psi\rangle)||_{HS}^2 - \frac{MN(M-1)(N-1)}{4})}. \tag{29}$$

Using the inequality $\sqrt{a-b} \geq \sqrt{a} - \sqrt{b}$ for any $a \geq b$, we get

$$C(|\psi\rangle) \geq \sqrt{\frac{8}{MN(M+N)}}(||T(|\psi\rangle)||_{HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}). \tag{30}$$

Now let $\rho = \sum_i p_i \rho_i$ be the optimal decomposition such that $C(\rho) = \sum_i p_i C(\rho_i)$. We get

$$\begin{aligned} C(\rho) &= \sum_i p_i C(\rho_i) \geq \sum_i p_i \sqrt{\frac{8}{MN(M+N)}}(||T(\rho_i)||_{HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}) \\ &= \sqrt{\frac{8}{MN(M+N)}}(\sum_i p_i ||T(\rho_i)||_{HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}) \\ &\geq \sqrt{\frac{8}{MN(M+N)}}(||T(\rho)||_{HS} - \frac{\sqrt{MN(M-1)(N-1)}}{2}), \end{aligned}$$

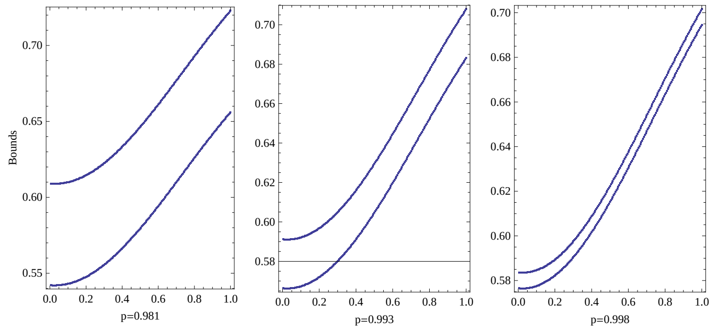

FIG. 1: We take the following 3×3 mixed state as an example: $\rho = \frac{1-p}{9} I_9 + p|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is a randomly generated pure state, I_9 is the 9×9 identity matrix. The upper line is the bound, the lower one is the tangle for pure state $|\psi\rangle$. To compare the validity of the estimation of tangle, we take $p = 0.981$, 0.993 and 0.998 respectively. As seen from the figures, for weakly mixed states (with large p), the bounds provide an excellent estimation for tangle.

which ends the proof. $\square$

The measurable upper bound (26), together with the lower bound in (21), (22), (23), (24) and (25) allow for better estimation of entanglement for arbitrary quantum states. Moreover, since the upper bound is exactly the value of tangle for pure states, the upper bound can be a good estimation when the state is very weakly mixed, see Fig. 1. One can also easily find that the lower bound (27) is obviously stronger than (23) when $||T||_{KF} \approx ||T||_{HS}$.

B. Bounds for Multipartite Concurrence and Separability

For a pure N-partite quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $dim\mathcal{H}_i = d_i$, $i = 1, ..., N$, the concurrence of bipartite decomposition between subsystems $12 \cdots M$ and $M+1 \cdots N$ is defined by

$$C_2(|\psi\rangle\langle\psi|) = \sqrt{2(1 - Tr\{\rho_{12\cdots M}^2\})}, \tag{31}$$

where $\rho_{12\cdots M} = Tr_{M+1\cdots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over subsystems $M+1, \cdots, N$.

On the other hand, the concurrence of $|\psi\rangle$ is defined by [13]

$$C_N(|\psi\rangle\langle\psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_\alpha Tr\{\rho_\alpha^2\}}, \tag{32}$$

where α labels all different reduced density matrices.

For a mixed multipartite quantum state, $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding concurrence of (31) and (32) are then given by the convex roof:

$$C_2(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|), \tag{33}$$

$$C_N(\rho) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|). \tag{34}$$

We now investigate the relation between these two kinds of concurrences.

Lemma 2: For a bipartite density matrix $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, one has

$$1 - Tr\{\rho^2\} \leq 1 - Tr\{\rho_A^2\} + 1 - Tr\{\rho_B^2\}, \tag{35}$$

where $\rho_{A/B} = Tr_{B/A}\{\rho\}$ are the reduced density matrices.

Proof: Let $\rho = \sum\limits_{ij} \lambda_{ij}|ij\rangle\langle ij|$ be the spectral decomposition, where $\lambda_{ij} \geq 0, \sum_{ij}\lambda_{ij} = 1$. Then $\rho_A = \sum_{ij}\lambda_{ij}|i\rangle\langle i|, \rho_B = \sum_{ij}\lambda_{ij}|j\rangle\langle j|$. Therefore

$$\begin{aligned}
& 1 - Tr\{\rho_A^2\} + 1 - Tr\{\rho_B^2\} - 1 + Tr\{\rho^2\} = 1 - Tr\{\rho_A^2\} - Tr\{\rho_B^2\} + Tr\{\rho^2\} \\
= \ & (\sum_{ij}\lambda_{ij})^2 - \sum_{i,j,j'}\lambda_{ij}\lambda_{ij'} - \sum_{i,i',j}\lambda_{ij}\lambda_{i'j} + \sum_{ij}\lambda_{ij}^2 \\
= \ & (\sum_{i=i',j=j'}\lambda_{ij}^2 + \sum_{i=i',j\neq j'}\lambda_{ij}\lambda_{ij'} + \sum_{i\neq i',j=j'}\lambda_{ij}\lambda_{i'j} + \sum_{i\neq i',j\neq j'}\lambda_{ij}\lambda_{i'j'}) - (\sum_{i,j=j'}\lambda_{ij}^2 + \sum_{i,j\neq j'}\lambda_{ij}\lambda_{ij'}) \\
& -(\sum_{i=i',j}\lambda_{ij}^2 + \sum_{i\neq i',j}\lambda_{ij}\lambda_{i'j}) + \sum_{i,j}\lambda_{ij}^2 \\
= \ & \sum_{i\neq i',j\neq j'}\lambda_{ij}\lambda_{i'j'} \geq 0.
\end{aligned}$$

□

Similar result has also been derived in [36, 46] in proving the subadditivity of the linear entropy. Here we just give a simpler and direct proof. From Lemma 2 we have.

Theorem 4: For a multipartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ with $N \geq 3$, the following inequality holds,

$$C_N(\rho) \geq \max 2^{\frac{3-N}{2}} C_2(\rho), \tag{36}$$

where the maximum is taken over all kinds of bipartite concurrence.

Proof: Without lose of generality, we suppose that the maximal bipartite concurrence is attained between subsystems $12\cdots M$ and $(M+1)\cdots N$.

For a pure multipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $Tr\{\rho^2_{12\cdots M}\} = Tr\{\rho^2_{(M+1)\cdots N}\}$. From (35) we have

$$\begin{aligned} C_N^2(|\psi\rangle\langle\psi|) &= 2^{2-N}((2^N-2) - \sum_\alpha Tr\{\rho_\alpha^2\}) \geq 2^{3-N}(N - \sum_{k=1}^N Tr\{\rho_k^2\}) \\ &\geq 2^{3-N}(1 - Tr\{\rho^2_{12\cdots M}\} + 1 - Tr\{\rho^2_{(M+1)\cdots N}\}) \\ &= 2^{3-N} * 2(1 - Tr\{\rho^2_{12\cdots M}\}) = 2^{3-N} C_2^2(|\psi\rangle\langle\psi|), \end{aligned}$$

i.e. $C_N(|\psi\rangle\langle\psi|) \geq 2^{\frac{3-N}{2}} C_2(|\psi\rangle\langle\psi|)$.

Let $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ attain the minimal decomposition of the multipartite concurrence. One has

$$\begin{aligned} C_N(\rho) &= \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|) \geq 2^{\frac{3-N}{2}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|) \\ &\geq 2^{\frac{3-N}{2}} \min_{\{p_i, |\psi_i\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|) = 2^{\frac{3-N}{2}} C_2(\rho). \end{aligned}$$

$\square$

Corollary For a tripartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, the following inequality hold:

$$C_3(\rho) \geq \max C_2(\rho) \tag{37}$$

where the maximum is taken over all kinds of bipartite concurrence.

Now we consider a multipartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ as a bipartite state belonging to $\mathcal{H}^A \otimes \mathcal{H}^B$ with the dimensions of the subsystems A and B being $d_A = d_{s_1} d_{s_2} \cdots d_{s_i}$ and $d_B = d_{s_{i+1}} d_{s_{i+2}} \cdots d_{s_N}$ respectively. By using the corollary, (21), (22), (23) and (27) we have the following lower bound:

Theorem 5: For any N-partite quantum state ρ, we have:

$$C_N(\rho) \geq 2^{\frac{3-N}{2}} \max\{B_1, B_2, B_3, B_4\}, \tag{38}$$

where

$$\begin{aligned} B_1 &= \max_{\{i\}} \sqrt{\frac{2}{M_i(M_i-1)}} \left[\max(||\mathcal{T}_A(\rho^i)||_{KF}, ||R(\rho^i)||_{KF}) - 1\right], \\ B_2 &= \max_{\{i\}} \frac{2||C(\rho^i)||_{KF} - (1 - Tr\{(\rho_A^i)^2\}) - (1 - Tr\{(\rho_B^i)^2\})}{\sqrt{2M_i(M_i-1)}}, \\ B_3 &= \max_{\{i\}} \sqrt{\frac{8}{M_i^3 N_i^2 (M_i-1)}} (||T(\rho^i)||_{KF} - \frac{\sqrt{M_i N_i (M_i-1)(N_i-1)}}{2}), \\ B_4 &= \max_{\{i\}} \sqrt{\frac{8}{M_i N_i (M_i+N_i)}} (||T(\rho^i)||_{HS} - \frac{\sqrt{M_i N_i (M_i-1)(N_i-1)}}{2}), \end{aligned}$$

ρ^is are all possible bipartite decompositions of ρ, and $M_i = \min\{d_{s_1}d_{s_2}\cdots d_{s_i}, d_{s_{i+1}}d_{s_{i+2}}\cdots d_{s_N}\}$, $N_i = \max\{d_{s_1}d_{s_2}\cdots d_{s_i}, d_{s_{i+1}}d_{s_{i+2}}\cdots d_{s_N}\}$.

In [36, 47, 48], it is shown that the upper and lower bound of multipartite concurrence satisfy

$$\sqrt{(4-2^{3-N})Tr\{\rho^2\} - 2^{2-N}\sum_\alpha Tr\{\rho_\alpha^2\}} \le C_N(\rho) \le \sqrt{2^{2-N}[(2^N-2) - \sum_\alpha Tr\{\rho_\alpha^2\}]}. \tag{39}$$

In fact we can obtain a more effective upper bound for multi-partite concurrence. Let $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where $|\psi_i\rangle$s are the orthogonal pure states and $\sum_i \lambda_i = 1$. We have

$$C_N(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_N(|\varphi_i\rangle\langle\varphi_i|) \le \sum_i \lambda_i C_N(|\psi_i\rangle\langle\psi_i|). \tag{40}$$

The right side of (40) gives a new upper bound of $C_N(\rho)$. Since

$$\begin{aligned}\sum_i \lambda_i C_N(|\psi_i\rangle\langle\psi_i|) &= 2^{1-\frac{N}{2}} \sum_i \lambda_i \sqrt{(2^N-2) - \sum_\alpha Tr\{(\rho_\alpha^i)^2\}} \\ &\le 2^{1-\frac{N}{2}} \sqrt{(2^N-2) - \sum_\alpha Tr\{\sum_i \lambda_i (\rho_\alpha^i)^2\}} \\ &\le 2^{1-\frac{N}{2}} \sqrt{(2^N-2) - \sum_\alpha Tr\{(\rho_\alpha)^2\}},\end{aligned}$$

the upper bound obtained in (40) is better than that in (39).

The lower and upper bounds can be used to estimate the value of the concurrence. Meanwhile, the lower bound of concurrence can be used to detect entanglement of quantum states. We now show that our upper and lower bounds can be better than that in (39) by detail examples.

Example 1: Consider the $2\times2\times2$ D*ü*r-Cirac-Tarrach states defined by [49]:

$$\rho = \sum_{\sigma=\pm} \lambda_0^\sigma |\Psi_0^\sigma\rangle\langle\Psi_0^\sigma| + \sum_{j=1}^{3} \lambda_j (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \tag{41}$$

where the orthonormal Greenberger-Horne-Zeilinger (GHZ)-basis $|\Psi_j^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|j\rangle_{12}|0\rangle_3 \pm |(3-j)\rangle_{12}|1\rangle_3)$, $|j\rangle_{12} \equiv |j_1\rangle_1|j_2\rangle_2$ with $j = j_1 j_2$ in binary notation. From theorem 5 we have that the lower bound of ρ is $\frac{1}{3}$. If we mix the state with white noise,

$$\rho(x) = \frac{(1-x)}{8} I_8 + x\rho, \tag{42}$$

by direct computation we have, as shown in FIG. 2, the lower bound obtained in (39) is always zero, while the lower bound in (38) is larger than zero for $0.425 \le x \le 1$, which

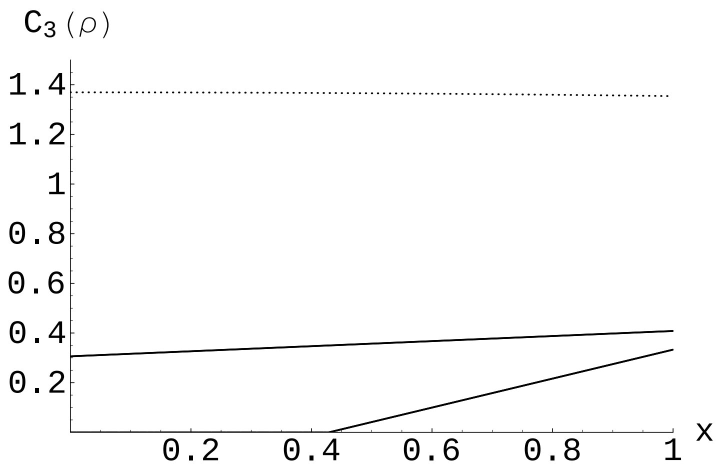

FIG. 2: Our lower and upper bounds of $C_3(\rho)$ from (38), (40) (solid line) and the upper bound obtained in (39) (doted line). The lower bound in (39) is always zero.

shows that $\rho(x)$ is detected to be entangled in this situation. And the upper bound (doted line) in (39) is much larger than the upper bound we have obtained in (40) (solid line).

Example 2: We consider the depolarized state [49]:

$$\rho = \frac{(1-x)}{8} I_8 + x|\psi^+\rangle\langle\psi^+|, \tag{43}$$

where $0 \leq x \leq 1$ represents the degree of depolarization, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. From FIG. 3 one can obviously see that our upper bound is tighter. For $0 \leq x \leq 0.7237$ our lower bound is higher than that in (39), i.e. our lower bound is closer to the true concurrence. Moreover for $0.2 \leq x \leq 0.57735$, our lower bound can detect the entanglement of ρ, while the lower bound in (39) not.

V. SUBADDITIVITY OF CONCURRENCE AND TANGLE

The additivity is an important property of entanglement measures, though it is usually rather difficult to prove. The strong subadditivity of relative entropy has been proved in [50]. In this section, we prove the subadditivity of concurrence and tangle.

Theorem 6: Let ρ and σ be two mixed quantum states in $\mathcal{H}_A \otimes \mathcal{H}_B$. We have

$$C(\rho\otimes\sigma) \leq C(\rho) + C(\sigma) \quad \text{and} \quad \tau(\rho\otimes\sigma) \leq \tau(\rho) + \tau(\sigma). \tag{44}$$

Proof: We first prove the theorem for pure states. Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$. Assume that $\rho_A^{|\psi\rangle} = \sum_i \lambda_i |i\rangle\langle i|$ and $\rho_A^{|\phi\rangle} = \sum_j \pi_j |j\rangle\langle j|$ be the spectral decompo-

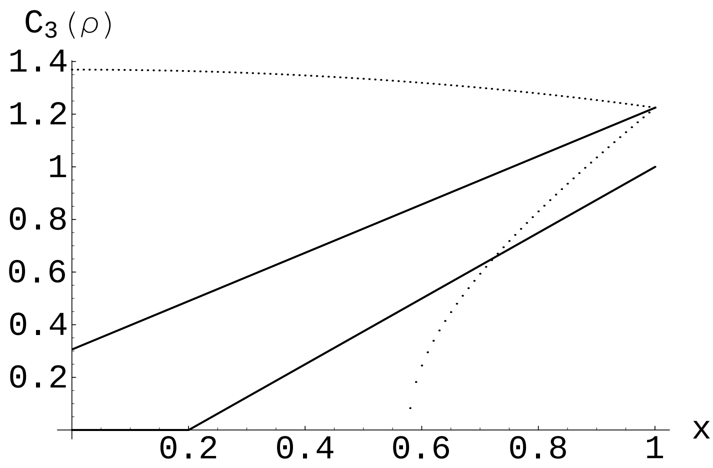

FIG. 3: Our lower and upper bounds of $C_3(\rho)$ from (38) and (40) (solid line) and the bounds obtained in (39) (dot line).

sition of the reduced matrices of $\rho^{|\psi\rangle}$ and $\rho^{|\phi\rangle}$. Then:

$$\begin{aligned}&\frac{1}{2}[C(|\psi\rangle)+C(|\phi\rangle)]^2 \geq 1-Tr[(\rho_A^{|\psi\rangle})^2]+1-Tr[(\rho_A^{|\phi\rangle})^2]\\ =\ & 1-\sum_i \lambda_i^2+1-\sum_j \pi_j^2 \geq 1-\sum_{ij}\lambda_i^2\pi_j^2=\frac{1}{2}C^2(|\psi\rangle\otimes|\phi\rangle).\end{aligned} \tag{45}$$

Namely, $C(|\psi\rangle\otimes|\phi\rangle) \leq C(|\psi\rangle)+C(|\phi\rangle)$. For tangle τ, the following inequality can be obtained similarly by changing the first inequality in (45) to be equality, $\tau(|\psi\rangle\otimes|\phi\rangle) \leq \tau(|\psi\rangle)+\tau(|\phi\rangle)$.

Now let $\rho=\sum_i p_i\rho_i$ and $\sigma=\sum_j q_j\sigma_j$ be two mixed states at optimal decomposition such that $C(\rho)=\sum_i p_iC(\rho_i)$ and $C(\sigma)=\sum_j q_jC(\sigma_j)$. We have

$$C(\rho)+C(\sigma)=\sum_{ij}p_iq_j[C(\rho_i)+C(\sigma_j)] \geq \sum_{ij}p_iq_jC(\rho_i\otimes\sigma_j) \geq C(\rho\otimes\sigma). \tag{46}$$

The inequality for τ can be derived similarly. $\square$

VI. CONCLUSIONS AND REMARKS

We have investigated the concurrence and tangle of quantum states. It has been shown that although one can not distill singlets from many bound entangled state, the concurrence (and tangle) $C(\rho\otimes\sigma)$ (and $\tau(\rho\otimes\sigma)$) is always larger than $\max\{C(\rho),C(\sigma)\}$ ($\max\{\tau(\rho),\tau(\sigma)\}$) respectively. We have derived a relation between concurrence and the optimal fidelity of teleportation, which shows that the optimal fidelity of teleportation is limited by the concurrence. These results show that to improve the fidelity of teleportation, one can use the two bound entangled states directly rather than do distillation first. We have also presented new upper and lower bounds for concurrence and tangle, which give rise to better estimation for entanglement of quantum states. At last we have proved the subadditivity of concurrence and tangled.

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