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by

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Experimental Determination of Entanglement for Arbitrary Pure States

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We present a way of experimental determination of concurrence in terms of the expectation value of local observables for arbitrary multipartite pure states. In stead of joint measurements on two-copy states needed in the experiment [Nature **440**, 20(2006)] for two-qubit systems, we need only measurements on one-copy state in every single measurement for any arbitrary dimensional multipartite states, which avoids the preparation of twin states or the imperfect copy of pure states.

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Introduction Entanglement is one of the most fascinating features of quantum theory [1]. Thus characterization and quantification of entanglement become an important issue. An important approach to characterize entanglement is the Bell-type inequalities [2–6]. For instance, N. Gisin proved that all two-qubit pure entangled states violate the CHSH inequality [4] and J. Chen *et al.* presented a Bell-type inequality that would be violated by all three-qubit pure entangled states [5]. For general mixed two-qubit states, S. Yu *et al.* [6] proposed a three-measurement setting Bell-type inequality, which gives a sufficient and necessary criterion of separability. Another significant approach is the entanglement witness [7], which could be also implemented experimentally with the present technology [8].

Nevertheless, to detect the entanglement by Bell-type inequalities one involves two or more measurement settings per party. And one has to do infinitely many dichotomic measurements theoretically, let alone we have no necessary and sufficient Bell inequalities to detect the entanglement for general multiqubit systems. While the entanglement witnesses work only for some special states. In fact we have well defined entanglement measures such as entanglement of formation (EoF) [9, 10] and concurrence [11, 12], whereas the concurrence is defined both for bipartite and multipartite states and gives rise to not only the separability, but also the degree of entanglement (at least for arbitrary dimensional bipartite states). The problem is how to use these measures to determine the entanglement for an unknown quantum states experimentally.

In [13] Mintert *et al.* proposed a method to measure the concurrence directly by using joint measurements on a twofold copy of pure states. Latter, S. P. Walborn *et al.* [14, 15] reported the experimental determination of concurrence for two-qubit, provided one has access to two copies of the pure state at every measurement [15].

In this letter, we give a way of experimental determination of concurrence for two-qubit and multi-qubit states, with only one-copy of the state in every single measurement. For the concurrence of two-qubit state in [14, 15],

also only a one set measurement is needed, while the experimental difficult is dramatically reduced. The results are generalized to the case for arbitrary multipartite pure states.

Concurrence for N -qubit system For a N -partite M dimensional pure state $|\psi\rangle = \sum_{i_1, \dots, i_N=0}^{M-1} a_{i_1, \dots, i_N} |i_1, \dots, i_N\rangle$, $a_{i_1, \dots, i_N} \in \mathbb{C}$, the concurrence is given by [16],

$$C_N(|\psi\rangle) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_i \text{tr} \rho_i^2}, \quad (1)$$

where the summation goes over all $2^N - 2$ subsets of the N subsystems.

Up to a constant factor, (1) can be also written as [12],

$$C(|\psi\rangle) = \sqrt{\sum_p \sum_{\{\alpha, \alpha', \beta, \beta'\}}^M |a_{\alpha\beta} a_{\alpha'\beta'} - a_{\alpha\beta'} a_{\alpha'\beta}|^2}, \quad (2)$$

where α and α' (resp. β and β') are subsets of the subindices of a , associated with the same sub-Hilbert spaces but with different summing indices. α (or α') and β (or β') span the whole space of a given subindex of a . \sum_p stands for the summation over all possible combinations of the indices of α and β .

Our main aim is to re-express concurrence in terms of the expectation value of local observables with respect to one-copy of a pure quantum state. In this way we can avoid the preparation of the twin state or imperfect copy of the pure state [13–15] and the experimental determination of concurrence becomes more available.

We first give a general proof that this can be always done: the squared concurrence of N -qubit pure state $|\psi\rangle$, $C^2(|\psi\rangle)$, can be expressed by the real linear summation of $\langle \psi | \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N} | \psi \rangle \langle \psi | \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_N} | \psi \rangle$,

$$C^2(|\psi\rangle) = \sum_{i_1, \dots, i_N, j_1, \dots, j_N=0}^3 x_{i_1, \dots, i_N, j_1, \dots, j_N} \langle \psi | \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N} | \psi \rangle \langle \psi | \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_N} | \psi \rangle. \quad (3)$$

where coefficients $x_{i_1, \dots, i_N, j_1, \dots, j_N}$ are real, σ_0 is the 2×2 identity matrix, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

We only need to show each term in the squared of (2) can be written as the form of right hand side of (3). Note that

$$\begin{aligned} |a_{\alpha\beta}a_{\alpha'\beta'} - a_{\alpha\beta'}a_{\alpha'\beta}|^2 &= |a_{\alpha\beta}a_{\alpha'\beta'}|^2 + |a_{\alpha\beta'}a_{\alpha'\beta}|^2 \\ &- a_{\alpha\beta}^*a_{\alpha'\beta'}^*a_{\alpha\beta'}a_{\alpha'\beta} - a_{\alpha\beta}a_{\alpha'\beta'}a_{\alpha\beta'}^*a_{\alpha'\beta}^*. \end{aligned} \quad (4)$$

$$|a_{\alpha\beta}a_{\alpha'\beta'}|^2 + |a_{\alpha\beta'}a_{\alpha'\beta}|^2 = \sum_{i_1, \dots, i_N, j_1, \dots, j_N=0,3} x_{i_1, \dots, i_N, j_1, \dots, j_N} \langle \psi | \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N} | \psi \rangle \langle \psi | \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_N} | \psi \rangle$$

for some real coefficients $x_{i_1, \dots, i_N, j_1, \dots, j_N}$.

Denote further $A^{(13)} = \frac{1}{\sqrt{2}}(|\alpha\beta\rangle\langle\alpha\beta'| + |\alpha\beta'\rangle\langle\alpha\beta|)$, $A^{(23)} = \frac{1}{\sqrt{2}}(|\alpha'\beta\rangle\langle\alpha'\beta'| + |\alpha'\beta'\rangle\langle\alpha'\beta|)$, $A^{(14)} =$

Set $A^{(11)} = |\alpha\beta\rangle\langle\alpha\beta|$, $A^{(21)} = |\alpha'\beta'\rangle\langle\alpha'\beta'|$, $A^{(12)} = |\alpha\beta'\rangle\langle\alpha\beta|$, $A^{(22)} = |\alpha'\beta\rangle\langle\alpha'\beta|$, then

$$\begin{aligned} |a_{\alpha\beta}a_{\alpha'\beta'}|^2 + |a_{\alpha\beta'}a_{\alpha'\beta}|^2 &= \langle \psi | A^{(11)} | \psi \rangle \langle \psi | A^{(21)} | \psi \rangle \\ &+ \langle \psi | A^{(12)} | \psi \rangle \langle \psi | A^{(22)} | \psi \rangle. \end{aligned} \quad (5)$$

$A^{(ij)}$, $i, j = 1, 2$, obviously has the form $A^{(ij)} = |i_1\rangle\langle i_1| \otimes \cdots \otimes |i_N\rangle\langle i_N|$, where i_1, \dots, i_N take value 0 or 1. As $|0\rangle\langle 0| = \frac{1}{2}(\sigma_0 + \sigma_3)$ and $|1\rangle\langle 1| = \frac{1}{2}(\sigma_0 - \sigma_3)$, we have

$\frac{i}{\sqrt{2}}(|\alpha\beta\rangle\langle\alpha\beta'| - |\alpha\beta'\rangle\langle\alpha\beta|)$, $A^{(24)} = \frac{i}{\sqrt{2}}(|\alpha'\beta\rangle\langle\alpha'\beta'| - |\alpha'\beta'\rangle\langle\alpha'\beta|)$, then

$$-a_{\alpha\beta}^*a_{\alpha'\beta'}^*a_{\alpha\beta'}a_{\alpha'\beta} - a_{\alpha\beta}a_{\alpha'\beta'}a_{\alpha\beta'}^*a_{\alpha'\beta}^* = -(\langle \psi | A^{(13)} | \psi \rangle \langle \psi | A^{(23)} | \psi \rangle + \langle \psi | A^{(14)} | \psi \rangle \langle \psi | A^{(24)} | \psi \rangle). \quad (6)$$

It is clear that $|\alpha\rangle\langle\alpha|$, $|\alpha'\rangle\langle\alpha'|$, $|\beta\rangle\langle\beta'|$ and $|\beta'\rangle\langle\beta|$ are tensor products of $|0\rangle\langle 0| = \frac{1}{2}(\sigma_0 + \sigma_3)$, $|1\rangle\langle 1| = \frac{1}{2}(\sigma_0 - \sigma_3)$, $|0\rangle\langle 1| = \frac{1}{2}(\sigma_1 - i\sigma_2)$ and $|1\rangle\langle 0| = \frac{1}{2}(\sigma_1 + i\sigma_2)$. Without loss of generality we assume $A^{(13)} = |i_1\rangle\langle i_1| \otimes \cdots \otimes |i_s\rangle\langle i_s| \otimes (|i_{s+1}\rangle\langle j_{s+1}| \otimes \cdots \otimes |i_N\rangle\langle j_N| + |j_{s+1}\rangle\langle i_{s+1}| \otimes \cdots \otimes |j_N\rangle\langle i_N|)$, where $1 \leq s < N$, i_k, j_k take values 0 or 1 and $i_k \neq j_k$ for each $s+1 \leq k \leq N$. The

part $|i_1\rangle\langle i_1| \otimes \cdots \otimes |i_s\rangle\langle i_s|$ is the real linear summation of tensor products of σ_0 and σ_3 . While the rest part $T \equiv |i_{s+1}\rangle\langle j_{s+1}| \otimes \cdots \otimes |i_N\rangle\langle j_N| + |j_{s+1}\rangle\langle i_{s+1}| \otimes \cdots \otimes |j_N\rangle\langle i_N|$ can be written as $\frac{1}{2^{N-s}} \bigotimes_{l=s+1}^N (\sigma_1 + i(-1)^{p_l} \sigma_2) + \frac{1}{2^{N-s}} \bigotimes_{l=s+1}^N (\sigma_1 + i(-1)^{1-p_l} \sigma_2)$, here p_l takes values 0 or 1 for each l . T is further of the form

$$\frac{1}{2^{N-s}} \sum_{l=0}^{N-s} \sum_{\{\text{number } l \text{ of } h_j \text{ is } 1, \text{ the others are } 2\}} i^{N-s-l} \bigotimes_{j=1}^{N-s} \sigma_{h_j} ((-1)^{l_m} + (-1)^{N-s-l-l_m}),$$

$0 \leq l_m \leq N-s-l$. If $N-s-l$ is even, then i^{N-s-l} is real and each coefficient before $\bigotimes_{j=1}^{N-s} \sigma_{h_j}$ is real. If $N-s-l$ is odd, then $(-1)^{l_m} + (-1)^{N-s-l-l_m} = 0$ and each coefficient is real too. Hence $A^{(13)}$ is the real linear summation of tensor products of σ_i , $0 \leq i \leq 3$. Similarly one can show that $A^{(14)}$, $A^{(23)}$ and $A^{(24)}$ are real linear summation of tensor products of σ_i , $0 \leq i \leq 3$. Thus $-a_{\alpha\beta}^*a_{\alpha'\beta'}^*a_{\alpha\beta'}a_{\alpha'\beta} - a_{\alpha\beta}a_{\alpha'\beta'}a_{\alpha\beta'}^*a_{\alpha'\beta}^*$ and Eq. (4) can be expressed by real linear summation of $\langle \psi | \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_N} | \psi \rangle \langle \psi | \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_N} | \psi \rangle$.

Therefore the squared concurrence of N -qubit pure states can be expressed as the expectation values of ten-

sor products of σ_i ($0 \leq i \leq 3$) with respect to one copy of the corresponding pure state, though such expressions are not unique (From (5) and (6) one sees that it is possible to find an expression that is invariant under the permutations of the N observables).

a. **Concurrence for two-qubit system** For any two-qubit state $|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$,

$$C^2 = 4|a_{00}a_{11} - a_{01}a_{10}|^2, \quad (7)$$

which can be expressed as

$$C^2 = \frac{1}{2}(1 + \langle \sigma_3 \sigma_3 \rangle^2 - \langle \sigma_3 \sigma_0 \rangle^2 - \langle \sigma_0 \sigma_3 \rangle^2 - \langle \sigma_0 \sigma_1 \rangle^2 + \langle \sigma_3 \sigma_1 \rangle^2 - \langle \sigma_0 \sigma_2 \rangle^2 + \langle \sigma_3 \sigma_2 \rangle^2). \quad (8)$$

For experimental determination of concurrence, one only needs to measure $\langle \sigma_3 \sigma_3 \rangle$, $\langle \sigma_3 \sigma_1 \rangle$ and $\langle \sigma_3 \sigma_2 \rangle$ respectively. An alternative expression with symmetry under the exchange of the two qubits can also be obtained [19].

For states in Schmidt decomposition, $|\psi\rangle = a_0|00\rangle + a_1|11\rangle$, $|a_{00}|^2 + |a_{11}|^2 = 1$, we have

$$C^2 = \frac{1}{8}(1 + \langle \sigma_3 \sigma_3 \rangle^2 - \langle \sigma_0 \sigma_3 \rangle^2 - \langle \sigma_3 \sigma_0 \rangle^2). \quad (9)$$

In this case experimentally we only need to measure $\langle \sigma_3 \sigma_3 \rangle$, or simply count the probability $P(++), P(--)$ of projections $|++\rangle\langle ++|, |--\rangle\langle --|$ with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ respectively, as $C^2 = 16P(++P(--))$. For the state $\alpha|01\rangle + \beta|10\rangle$ used in [14], it is also true that we only need one set local measurement to measure the concurrence. But here we only need one copy of the state in every single measurement, while in [14] joint measurement on two copies of the state are needed in every single measurement.

For small deviation $|\psi'\rangle = \sqrt{1-\epsilon}|\psi\rangle + \sqrt{\epsilon}|\phi\rangle$ from an ideal pure state $|\psi\rangle$ due to imperfect preparation, where $\epsilon \in \mathbb{R}$ and $|\phi\rangle$ is an arbitrary pure state, our protocol shows that the concurrence obtained from the experiment is exact the one of $|\psi'\rangle$. Hence if the parameter ϵ is small enough, the difference of the concurrence be-

tween $|\psi\rangle$ and $|\psi'\rangle$ would be small enough. For a two-crystal type-I down-conversion source, with improper spatial mode matching and spectral filtering, the imperfect preparation procedure could result in mixed states, $\rho = (1-\epsilon)|\psi\rangle\langle\psi| + \epsilon(|\alpha|^2|HH\rangle\langle HH| + |\beta|^2|VV\rangle\langle VV|)$ instead of the ideal pure state $|\psi\rangle = \alpha|HH\rangle + \beta|VV\rangle$, where H and V stand for horizontal and vertical linear polarization respectively. That is, the phase coherence between $|HH\rangle$ and $|VV\rangle$ is reduced by $1-\epsilon$. Therefore the actual concurrence of ρ is smaller than that of $|\psi\rangle$, $C(\rho) = (1-\epsilon)|\alpha\beta|$ [15, 17]. If we still measure the state according to (8) or (9), we have $C(\rho) = |\alpha\beta|$. Thus the relative error due to mixing is linear in ϵ .

In principle one can always use tomography to reconstruct the unknown state. However it requires a large number of measurements. In particular one needs 3^N different settings to reconstruct an arbitrary N -qubit density matrix. To obtain all 16 expectation values of the two-qubit density matrix, nine different settings have to be used [18]. From (8) we only need three different settings to know the entanglement of the state, which is much less than tomography.

b. Concurrence for three-qubit system For any pure three-qubit state $|\psi\rangle = \sum_{i,j,k=0}^1 a_{ijk}|ijk\rangle$, its squared concurrence is of the form,

$$C^2 = 4(|a_{000}a_{111} - a_{001}a_{110}|^2 + |a_{000}a_{111} - a_{010}a_{101}|^2 + |a_{000}a_{111} - a_{011}a_{100}|^2 + |a_{001}a_{110} - a_{010}a_{101}|^2 + |a_{001}a_{110} - a_{011}a_{100}|^2 + |a_{010}a_{101} - a_{011}a_{100}|^2) + 8(|a_{000}a_{011} - a_{001}a_{010}|^2 + |a_{000}a_{101} - a_{001}a_{100}|^2 + |a_{000}a_{110} - a_{010}a_{100}|^2 + |a_{001}a_{111} - a_{011}a_{101}|^2 + |a_{010}a_{111} - a_{011}a_{110}|^2 + |a_{100}a_{111} - a_{101}a_{110}|^2). \quad (10)$$

Up to a constant factor C^2 can be expressed as

$$C^2 = \frac{1}{4}(9 - 5\langle \sigma_0 \sigma_3 \sigma_0 \rangle^2 - 5\langle \sigma_0 \sigma_0 \sigma_3 \rangle^2 - 5\langle \sigma_3 \sigma_0 \sigma_0 \rangle^2 + \langle \sigma_0 \sigma_3 \sigma_3 \rangle^2 + \langle \sigma_3 \sigma_3 \sigma_0 \rangle^2 + \langle \sigma_3 \sigma_0 \sigma_3 \rangle^2 + 3\langle \sigma_3 \sigma_3 \sigma_3 \rangle^2 - 3\langle \sigma_0 \sigma_0 \sigma_1 \rangle^2 - 3\langle \sigma_0 \sigma_1 \sigma_0 \rangle^2 - 3\langle \sigma_1 \sigma_0 \sigma_0 \rangle^2 - \langle \sigma_0 \sigma_3 \sigma_1 \rangle^2 - \langle \sigma_1 \sigma_0 \sigma_3 \rangle^2 - \langle \sigma_3 \sigma_1 \sigma_0 \rangle^2 + 3\langle \sigma_0 \sigma_1 \sigma_3 \rangle^2 + 3\langle \sigma_3 \sigma_0 \sigma_1 \rangle^2 + 3\langle \sigma_1 \sigma_3 \sigma_0 \rangle^2 + \langle \sigma_3 \sigma_3 \sigma_1 \rangle^2 + \langle \sigma_3 \sigma_1 \sigma_3 \rangle^2 + \langle \sigma_1 \sigma_3 \sigma_3 \rangle^2 - 3\langle \sigma_0 \sigma_0 \sigma_2 \rangle^2 - 3\langle \sigma_0 \sigma_2 \sigma_0 \rangle^2 - 3\langle \sigma_2 \sigma_0 \sigma_0 \rangle^2 - \langle \sigma_0 \sigma_3 \sigma_2 \rangle^2 - \langle \sigma_2 \sigma_0 \sigma_3 \rangle^2 - \langle \sigma_3 \sigma_2 \sigma_0 \rangle^2 + 3\langle \sigma_0 \sigma_2 \sigma_3 \rangle^2 + 3\langle \sigma_3 \sigma_0 \sigma_2 \rangle^2 + 3\langle \sigma_2 \sigma_3 \sigma_0 \rangle^2 + \langle \sigma_3 \sigma_3 \sigma_2 \rangle^2 + \langle \sigma_3 \sigma_2 \sigma_3 \rangle^2 + \langle \sigma_2 \sigma_3 \sigma_3 \rangle^2). \quad (11)$$

which is invariant under permutations of the three qubits.

For experimental determination of the concurrence for

arbitrary three-qubit states, seven quantities are needed to be measured: $\langle \sigma_3 \sigma_3 \sigma_3 \rangle$, $\langle \sigma_3 \sigma_3 \sigma_1 \rangle$, $\langle \sigma_3 \sigma_1 \sigma_3 \rangle$, $\langle \sigma_1 \sigma_3 \sigma_3 \rangle$, $\langle \sigma_3 \sigma_3 \sigma_2 \rangle$, $\langle \sigma_3 \sigma_2 \sigma_3 \rangle$, $\langle \sigma_2 \sigma_3 \sigma_3 \rangle$.

In particular for the three-qubit generalized GHZ state, $|\psi\rangle = a_0|000\rangle + a_1|111\rangle$, $|a_0|^2 + |a_1|^2 = 1$, and the generalized W state $|\psi\rangle = a_0|001\rangle + a_1|010\rangle + a_2|100\rangle$, $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$, their squared concurrence are $12|a_{000}a_{111}|^2$ and $8(|a_{001}a_{010}|^2 + |a_{001}a_{100}|^2 + |a_{010}a_{100}|^2)$ respectively. The concurrence of both generalized GHZ states and generalized W states can be measured according to the following formula:

$$C^2 = \frac{1}{4}(9 - 5\langle \sigma_0 \sigma_3 \sigma_0 \rangle^2 - 5\langle \sigma_0 \sigma_0 \sigma_3 \rangle^2 - 5\langle \sigma_3 \sigma_0 \sigma_0 \rangle^2 + \langle \sigma_0 \sigma_3 \sigma_3 \rangle^2 + \langle \sigma_3 \sigma_3 \sigma_0 \rangle^2 + \langle \sigma_3 \sigma_0 \sigma_3 \rangle^2 + 3\langle \sigma_3 \sigma_3 \sigma_3 \rangle^2). \quad (12)$$

(12) shows that for experimental determination of entanglement for these states, we need only one set measurement, $\langle \sigma_3 \sigma_3 \sigma_3 \rangle$.

Similar results can be obtained for multiqubit systems such as N -qubit generalized GHZ state $|\psi\rangle = a_0|0 \cdots 0\rangle + a_1|1 \cdots 1\rangle$, $|a_0|^2 + |a_1|^2 = 1$, or N -qubit generalized W state $|\psi\rangle = a_0|0 \cdots 01\rangle + a_1|0 \cdots 10\rangle + \cdots + a_{N-1}|10 \cdots 0\rangle$, $|a_0|^2 + |a_1|^2 + \cdots + |a_{N-1}|^2 = 1$. For instance for the generalized GHZ state, the concurrence is $|a_0 a_1|$ up to a constant. Its squared concurrence can be expressed as follows:

$$C^2 = 1 + \sum_{\substack{k \leq N \\ \text{is even}}} \langle \sigma_3^{(i_1 \cdots i_k)} \rangle^2 - \sum_{\substack{l \leq N \\ \text{is odd}}} \langle \sigma_3^{(i_1 \cdots i_l)} \rangle^2 + \sum_{\substack{k, k' \leq N \\ \text{is even}}} \langle \sigma_3^{(i_1 \cdots i_k)} \rangle \langle \sigma_3^{(i_1 \cdots i_{k'})} \rangle - \sum_{\substack{l, l' \leq N \\ \text{is even}}} \langle \sigma_3^{(i_1 \cdots i_l)} \rangle \langle \sigma_3^{(i_1 \cdots i_{l'})} \rangle. \quad (13)$$

Here $\langle \sigma_3^{(i_1 \cdots i_k)} \rangle$ denotes the expectation value of the local operators such that the i_1 -th, \cdots , i_k -th are σ_3 and others are identities.

Concurrence for N -partite M -dimensional system Besides qubit systems, our approach can be also used for arbitrary M -dimensional cases. In stead of the Pauli operators, one can use the $SU(M)$ generators as observables:

$$\lambda_0 = \sum_{j=0}^{M-1} |j\rangle \langle j|,$$

$$\lambda_s = \sum_{j=0}^{s-1} |j\rangle \langle j| - s|s\rangle \langle s|, \quad 1 \leq s \leq M-1,$$

$$\lambda_s = |j\rangle \langle k| + |k\rangle \langle j|, \quad s = M, \cdots, \frac{1}{2}(M+2)(M-1),$$

$$\lambda_s = -i(|j\rangle \langle k| - |k\rangle \langle j|), \quad s = \frac{1}{2}(M+1)M, \cdots, M^2 - 1,$$

where $0 \leq j < k \leq M-1$. Note that

$$\begin{aligned} |0\rangle \langle 0| &= \frac{1}{M} \lambda_0 + \frac{1}{M(M-1)} \lambda_{M-1} + \frac{1}{(M-1)(M-2)} \lambda_{M-2} + \cdots + \frac{1}{3*2} \lambda_2 + \frac{1}{2} \lambda_1, \\ |1\rangle \langle 1| &= \frac{1}{M} \lambda_0 + \frac{1}{M(M-1)} \lambda_{M-1} + \frac{1}{(M-1)(M-2)} \lambda_{M-2} + \cdots + \frac{1}{3*2} \lambda_2 - \frac{1}{2} \lambda_1, \\ &\vdots \\ |M-3\rangle \langle M-3| &= \frac{1}{M} \lambda_0 + \frac{1}{M(M-1)} \lambda_{M-1} + \frac{1}{(M-1)(M-2)} \lambda_{M-2} - \frac{1}{M-2} \lambda_{M-3}, \\ |M-2\rangle \langle M-2| &= \frac{1}{M} \lambda_0 + \frac{1}{M(M-1)} \lambda_{M-1} - \frac{1}{M-1} \lambda_{M-2}, \\ |M-1\rangle \langle M-1| &= \frac{1}{M} \lambda_0 - \frac{1}{M} \lambda_{M-1}. \end{aligned}$$

In addition, for arbitrary $0 \leq j < k \leq M - 1$, it has $|j\rangle\langle k| = \frac{1}{2}(\lambda_s + i\lambda_{s'})$ and $|k\rangle\langle j| = \frac{1}{2}(\lambda_s - i\lambda_{s'})$ for some $M \leq s \leq \frac{1}{2}(M + 2)(M - 1)$ and $\frac{1}{2}(M + 1)M \leq s' \leq M^2 - 1$. Similar to the proof of N -

qubit system, it is direct to show that the squared concurrence of N -partite M -dimensional pure state $|\psi\rangle$ can be expressed in terms of real linear summation of $\langle\psi|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N}|\psi\rangle\langle\psi|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_N}|\psi\rangle$:

$$C^2(|\psi\rangle) = \sum_{i_1, \dots, i_N, j_1, \dots, j_N=0}^{M^2-1} x_{i_1, \dots, i_N, j_1, \dots, j_N} \langle\psi|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_N}|\psi\rangle\langle\psi|\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_N}|\psi\rangle, \quad (14)$$

where $x_{i_1, \dots, i_N, j_1, \dots, j_N}$ are real.

Conclusions We have proposed a method for experimental determination of concurrence in terms of the expectation value of local observables, which gives not only sufficient and necessary conditions for separability of the quantum states, but also the relative degree of entanglement. Moreover unlike the case of Bell-inequality where measurements are needed with respect to infinitely many observables, we need only a few mean value of observables. And in stead of joint measurement on two-copy of the state needed in the experiment [13–15] for two-qubit states, we need only the usual measurements on one copy of the state in every single measurement for any arbitrary dimensional multipartite states, which dramatically simply the experiment and reduces the error rates and the imperfectness in the preparation of the states. Compared with entanglement witnesses, for which some a priori knowledge about the states under investigation is needed, we don't need any information before measuring the state in experiment. Moreover our method applies to all pure states.

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- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univesity Press, Cambridge, 2000).
 - [2] J. S. Bell, *Physics* (Long Island, N. Y.) **1**, 195(1964).
 - [3] J. Clauser, M. Horne, A. Shimony and R. Holt, *Phys. Rev. Lett.* **23**, 880(1969).
 - [4] N. Gisin, *Phys. Lett. A* **154**, 201(1991).
 - [5] J. L. Chen, C. F. Wu, L. C. Kwek and C. H. Oh, *Phys.*

- Rev. Lett.* **93**, 140407(2004).
- [6] S. Yu, J. W. Pan, Z. B. Chen and Y. D. Zhang, *Phys. Rev. Lett.* **91**, 21(2003).
- [7] M. Horodecki, P. Horodecki and R. Horodecki, *Phys. Lett. A* **223**, 1(1996); B. M. Terhal, *Phys. Lett. A* **271**, 319(2000).
- [8] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein and A. Sanpera, *Phys. Rev. Lett.* **92**, 087902(2004).
- [9] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, *Phys. Rev. A* **54**, 3824(1996).
- [10] M. Horodecki, *Quantum Inf. Comp.* **1**, 3(2001); D. Bruß, *J. Math. Phys.* **43**, 4237(2002); M. B. Plenio and S. Virmani, *Quantum Inf. Comp.* **7**, 1(2007).
- [11] A. Uhlmann, *Phys. Rev. A* **62**, 032307(2000); P. Rungta, V. Buzek, C. M. Caves, M. Hillery and G. J. Milburn, *Phys. Rev. A* **64**, 042315(2001).
- [12] S. Albeverio and S. M. Fei, *J. Opt. B: Quantum Semi-class. Opt.* **3**, 223(2001).
- [13] F. Mintert, M. Kuś and A. Buchleitner, *Phys. Rev. Lett.* **95**, 260502(2005).
- [14] S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert and A. Buchleitner, *Nature*. **440**, 20(2006).
- [15] S. P. Walborn, P. H. Souto Ribeiro and L. Davidovich, *Phys Rev A* **75**, 032338(2007).
- [16] L. Aolita and F. Mintert, *Phys. Rev. Lett.* **97**, 050501(2006).
- [17] W. K. Wootters, *Phys. Rev. Lett.* **80**, 2245(1998).
- [18] C. F. Roos, G. P. T. Lancaster, M. Riebe, H. Häffner, W. Hänsel, S. Gulde, C. Becher, J. Eschner, F. Schmidt-Kaler and R. Blatt, *Phys. Rev. Lett.* **80**, 2245(1998).
- [19] One can also find an expression that is symmetric under exchanging of the two qubits: $C^2 = \frac{1}{16}(2 + 2\langle\sigma_3\sigma_3\rangle^2 - 2\langle\sigma_3\sigma_0\rangle^2 - 2\langle\sigma_0\sigma_3\rangle^2 - \langle\sigma_0\sigma_1\rangle^2 - \langle\sigma_1\sigma_0\rangle^2 + \langle\sigma_3\sigma_1\rangle^2 + \langle\sigma_1\sigma_3\rangle^2 - \langle\sigma_0\sigma_2\rangle^2 - \langle\sigma_2\sigma_0\rangle^2 + \langle\sigma_3\sigma_2\rangle^2 + \langle\sigma_2\sigma_3\rangle^2)$.