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**Optimal Securitization of Credit Portfolios via Impulse Control**

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by


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# Optimal Securitization of Credit Portfolios via Impulse Control

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**Abstract**

We study the optimal loan securitization policy of a commercial bank which is mainly engaged in lending activities. For this we propose a stylized dynamic model which contains the main features affecting the securitization decision. In line with reality we assume that there are non-negligible fixed and variable transaction costs associated with each securitization. The fixed transaction costs lead to a formulation of the optimization problem in an impulse control framework. We prove viscosity solution existence and uniqueness for the quasi-variational inequality associated with this impulse control problem. Iterated optimal stopping is used to find a numerical solution of this PDE, and numerical examples are discussed.

**Key words:** Securitization, credit risk, impulse control, viscosity solutions, quasi-variational inequalities, iterated optimal stopping

**JEL Classification:** G11, G21, G31, G33

**Mathematics Subject Classification (2000):** 35B37, 49L25, 49N25, 91B28, 91B70, 93E20

## 1 Introduction

Banks staggered, stock prices plunged, governments had to intervene — the credit crisis starting in 2007 drew the public attention to a specific form of financial derivatives with loans as underlying that had been used to an enormous extent by banks all over the world.

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Complex credit securitization products such as Asset-Backed Securities (ABS) became known to a wider public as investments spreading American subprime home loans all over the world. Notwithstanding this negative connotation, credit securitization has its undeniable benefits: On the macro level, it can help to mitigate concentration risks within the banking sector; on the micro or firm-specific level, securitization is an important risk management tool as it enables an individual bank to reduce its leverage.

In the present paper, we are interested in securitization on the micro level and study the optimal dynamic securitization strategy of a commercial bank which is mainly engaged in lending activities. Transaction costs are an important factor in a securitization decision. We therefore incorporate fixed transaction costs (e.g., rating fees), and variable transaction costs (e.g., price discounts) into our model. In view of the fixed part of the transaction costs, it is natural to formulate and study the optimization problem in an impulse control setting.

**The model.** We consider a bank whose sole business is lending. For simplicity, the bank does not have customer deposits, and therefore refinances itself by debt capital. We assume that this refinancing is short-term, e.g., on the interbank market. The loans issued by the bank are modelled as a discrete portfolio of perpetuities which generate returns proportional to their nominal but may also default. These loans are valued on the bank's balance sheet at their nominal value, minus losses incurred (impairment). If the nominal value of the loans falls below the debt level, then the bank itself defaults. This risk of bank default however implies that the bank's refinancing rate may be higher than the risk-free interest rate.
In reality, loan default probabilities are uncertain and may change with the state of the economy. This leads us to consider a random state of the economy, modelled as a two-state continuous-time Markov chain. Correspondingly, also the market value of the loans and the bank's refinancing cost may change with the economic state.

We study the problem of maximizing the expected utility of the bank's liquidation value at some horizon $T > 0$. For this, the bank has two instruments at its disposal: on the one hand, it can sell loans at market value minus fixed transaction costs; this is modelled as securitization impulse. On the other hand, it can issue new loans; the decision whether to issue new loans is modelled as a standard stochastic control problem. Hence we have to deal with a so-called combined impulse and stochastic control problem. The analysis of this problem is the main technical contribution of this paper.

Our model combines the most important factors affecting a bank's securitization decisions in a dynamic setting: loans may default and thus reduce profitability, or even jeopardize the existence of the bank; a securitization of loans can reduce risks, but the full nominal will probably not be recovered because of fixed and variable transaction costs depending on the current state of the economy; securitization can also be an alternative to refinancing via debt capital, especially if the latter is very expensive due to high refinancing costs. The fixed transaction costs in our model lead to finitely many securitization impulses. This mirrors the relative illiquidity of securitization markets and is in stark contrast to standard continuous-time portfolio optimization models with their assumption of continuous and costless portfolio rebalancing.

In our analysis, we carve out major challenges a bank faces in managing its loan exposure. Despite the complexity of the model, we are able to derive some theoretical results, and compute optimal solutions numerically. These results can serve as guidance for an optimal risk management strategy of a bank which is simultaneously active on the debt market, the securitization market and the retail market.

**PDE approach.** The value function of a combined impulse and stochastic control problem is known to be associated with a certain nonlinear, nonlocal partial differential equation (PDE), called the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI) (for an introduction into the subject, one may consult Øksendal and Sulem [32], or Bensoussan and Lions [2]). Because we are dealing with a three-dimensional impulse control problem until terminal time, we cannot expect to find an analytical solution of the HJBQVI. This is also why standard verification techniques for smooth solutions fail in our case. So we have to consider weak solution concepts, such as viscosity solutions (see Crandall et al. [7] or Fleming and Soner [11]), and to solve the problem by numerical techniques.

In the present paper, we show that the value function of our combined impulse and stochastic control problem is the unique viscosity solution of a suitable HJBQVI, using results from Seydel [39]. Then, we can proceed to the numerical solution of this HJBQVI (which is done by iterated optimal stopping in a finite-difference scheme), and compute optimal impulse strategies for our problem.

**Numerical results.** The overall result from our analysis and numerical computations is that securitization is a valuable tool for a bank's credit risk management, especially if the initial leverage of the bank is high. The higher the bank's refinancing cost, the stronger is this incentive to securitize; this is in line with the general observation in the corporate finance literature that increasing costs to raising new external funds are an important rationale for risk management, see for instance Froot and Stein [15].

Our numerical results also demonstrate that transaction costs (fixed and variable) have a crucial impact in our model: First, different fixed transaction costs can lead to significant changes in the optimal impulse strategy. Second, there is a tendency to perform impulses when (proportional) transaction costs are lower. For our chosen set of parameters, this means that impulses in expansion (where the market value of loans is higher and hence transaction costs lower) are optimal in a relatively large region although loans are profitable in such boom times; such impulses in expansion serve as a provision for bad times. Under the plausible assumption of a strongly procyclical market value of loans, impulses near the default boundary of the bank are simply not admissible in recession, because this would lead to immediate default. The optimal (impulse) strategy in this case is simply to wait for better times. This effect can be observed — although less pronounced — also for a weakly procyclical market value of loans. If the bank decides to do a securitization in recession, then it should only securitize a relatively small amount due to the proportional transaction costs.

**Literature.** The problem of choosing the optimal leverage for a firm is a classical problem in corporate finance, see for instance Leland and Toft [25], Ziegler [40], or Hackbarth et al. [17]; the problem is analyzed specifically for banks in Froot and Stein [15], and an empirical analysis for a commercial bank is carried out in Cebenoyan and Strahan [5]. Here, we concentrate not on this theoretical question, but investigate the problem for a bank from a transaction-based perspective, i.e., "what should the bank optimally do, if in a certain (non-optimal) situation?". Another area of research related to our problem is optimal control for insurers (see, e.g., Schmidli [37]), in particular optimal reinsurance (e.g., Irgens and Paulsen [20] and references therein). Some further background on ABS, securitization and credit risk management can be found in Benvegnu et al. [3], Bluhm et al. [4], Franke and Krahnen [14] and McNeil et al. [27].

The novel features of our control problem (as opposed to standard continuous-time portfolio optimization problems such as Merton [28], [29]) are the inclusion of jumps and the use of impulse control methods. Portfolio optimization with jumps has been studied in Framstad et al. [13], among others; impulse control techniques have previously been used by, e.g., Eastham and Hastings [8] or Korn [22]. Some further references are given in Øksendal and Sulem [32].

**Overview.** The first main section §2 introduces the model in detail, and discusses several aspects of choosing functional forms for market value und transaction costs. In the following §3, the linear boundedness of the value function is shown, and we establish that the value function is the viscosity solution of the HJBQVI. After analyzing several stochastic control simplifications of the model in §4, we describe in §5 the numerical algorithm used for the solution of the HJBQVI, and present and discuss numerical results. The paper is complemented by a conclusion and outlook at the end.

# 2 The model

## 2.1 Basic structure

**The bank.** We consider a commercial bank whose only business is lending. For simplicity we assume that the bank does not have customer deposits so that its balance sheet consists only of equity, debt capital, cash and loans. These four factors then determine success or failure of the bank. Our model is based on the fundamental balance sheet equation

$$\begin{aligned} \text{assets} &= \text{liabilities} \\ \text{cash} + \text{loans} &= \text{equity} + \text{debt capital}. \end{aligned}$$

A bank of this type would normally refinance the issued loans to a large proportion by debt capital (a typical bank actually owns less than 10% of the assets on its balance sheet). As the structure of long-term debt capital typically remains largely unchanged over a longer time horizon, we make here the assumption that the long-term debt capital is constant and - for simplicity - equal to 0. In this simplified setting, the bank refinances itself through a negative cash position, which is interpreted as short-term refinancing, say on the LIBOR interbank market.[1] We stress that negative cash in our model does *not* lead to immediate bankruptcy, but is just an indication that the bank does not own all of the assets on its balance sheet. Relying on short-term funding has been quite a common way for banks to refinance itself, at least until the fall of 2008. Indeed, lending long-term and refinancing short-term is one the *raisons d'être* of banks. The drawback of short-term refinancing however is that the refinancing rate can be quite sensitive to changes in the bank's situation or in the economic environment, or that the bank might even not be able to raise funds at all. This became evident in September and October 2008 when the market for short-term refinancing essentially dried up in reaction to the default of Lehman Brothers.

In the balance sheet equation described above, three factors remain, of which we choose to model the nominal value of loans $L$ and the cash position $C$, and to deduce equity $E = L + C$. As banks cannot take a short position in loans we have $L \geq 0$, whereas the sign of $C$ is not restricted. The bank exists as long as $E \geq 0$, otherwise default occurs. We define the *leverage* of a bank as follows:

$$\text{leverage} = \frac{L}{E} = \frac{L}{L+C} \in [0, \infty].$$

A leverage $> 1$ means that $C < 0$ (refinancing of some of the loans on the short-term debt market) and reversely, a leverage $\in [0, 1]$ means that $C \geq 0$, so that the bank owns all its assets. A high leverage indicates a high riskiness of the bank, should loans default. In this case, we would expect a higher refinancing rate for the bank.

**The dynamic model.** We now present our three-dimensional model step by step, first without securitization. Denote by $X = (L, C, M)^T$ the stochastic process composed of loan value $L$, cash $C$, and state of the economy $M$. We work on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying the usual assumptions.

The **loan portfolio** of the bank is discrete, i.e., at every instant, it consists of finitely many loans. Furthermore, the portfolio is homogeneous, i.e., all loans have the same interest rate, the same risk and the same nominal; without loss of generality we assume for each loan a nominal of 1. Each loan has maturity $\infty$ (perpetuity), and defaults with a certain intensity, independently from the other loans (conditionally on the state of the economy); upon default it is immediately liquidated. The nominal value of the loan exposure $L$ develops in time according to an adapted càdlàg point process with varying intensity:

$$dL_t = -dN_t + \beta_t dP_t, \qquad L_0 \in \mathbb{N}_0. \tag{1.L}$$

Here, $N_t$ is a Poisson process with intensity $\lambda(M_{t-})L_{t-}$ which depends on the state of the economy $M$ (see below) and on the current loan nominal $L$. This process can be derived from the individual defaults of the loans as follows: Loans default with intensity $\lambda(M_{t-})$,

[1]Strictly speaking, this means that cash could be on either side of the balance sheet, depending on whether it is positive or negative.

independent conditionally on $M$. For a total portfolio of $L_{t-}$ loans, the intensity of one loan defaulting is thus $\lambda(M_{t-})L_{t-}$. The process $P$ is an adapted standard Poisson process, independent of $N$, with intensity $\lambda_P \geq 0$, and $\beta$ is a predictable stochastic control process with values in $\{0, 1\}$. This control gives the possibility to increase the loan nominal, should there be an opportunity: a value $\beta = 1$ means green traffic light if a customer comes into the bank and asks for a loan. Note that in this way, we ensure that $L_t \in \mathbb{Z}$ for all $t \geq 0$, i.e., that the loan portfolio stay discrete.

The **cash process** $C$ evolves according to the following SDE (recall $X = (L, C, M)^T$):

$$dC_t = (r_B(X_t)C_t + r_L L_t)\, dt + (1 - \delta(M_{t-}))dN_t - \beta_t dP_t. \tag{1.C}$$

Here, the measurable function $r_B \geq 0$ is the instantaneous refinancing rate of the bank (or interest rate earned on cash if $C_t > 0$). We assume that $r_B$ depends on the riskiness of the bank, in particular on its leverage; see § 2.2 below. In modelling refinancing by an instantaneous cash flow stream instead of the usual three- or six-month horizon on the LIBOR market, we ensure the Markov property of $X$ and thus numerical tractability; in §2.2 we will present examples how to choose the refinancing function. In general, $r_B \geq \rho$ for the risk-free interest rate $\rho \geq 0$. Note that the existence of such a function $r_B$ implies that we assume there is always refinancing available, regardless how risky the bank is. The constant $r_L$ is the continuous rate all customers have to pay for their loans. The remaining terms on the right hand side of (1.C) are already known from the discussion of the loan process: $\delta(M_{t-}) \in [0, 1]$ represents the current loss given default (LGD), so that the term $1 - \delta$ is the recovery rate from the liquidation of a defaulted loan; $\beta_t dP_t$ represents the money that is invested for issuing new loans.

Finally, the **economy process** $M$ is an adapted càdlàg Markov switching process or continuous-time Markov chain with values in $\{0, 1\}$ (expansion, contraction) and switching intensities $\lambda_{01}, \lambda_{10} > 0$, with $\lambda_{01}$ being the intensity to go from 0 to 1. $M$ is assumed to be independent of all other processes encountered so far. Formally, $M$ can be represented as difference of two independent Poisson processes $N^{01}$ and $N^{10}$:

$$dM_t = 1_{\{M_{t-}=0\}} dN_t^{01} - 1_{\{M_{t-}=1\}} dN_t^{10}. \tag{1.M}$$

We consider in this paper only the simple case of two states of the economy; more states (or even a more complex economy process, as long as its stays Markov) can be handled in the same way.

**The bank's interventions.** The bank wants to maximize its expected terminal utility by controlling its loan exposure, which might be too high and thus too risky, or too low to generate significant profits. This maximization can be done either by issuing new loans (control of $\beta$), or it can be done via securitization. Securitization is a means to get loans off the balance sheet, but a securitization comes always with certain fixed costs $c_f > 0$, such as rating agency fees, or legal costs for setting up a special purpose vehicle in a tax haven. Moreover, there may be variable transaction costs, as the securitizing bank may not be able to sell the loans for the value attributed to them on its balance sheet.

Securitization is modelled as impulse control because of the transaction costs. A securitization impulse reduces the loan exposure by $\zeta$, and the cash position is increased by the market value $\eta(x_3, \zeta)$ of the loans minus fixed costs $c_f$. We assume that $\eta(x_3, 0) = 0$, that $\eta \geq 0$, and that $\eta$ is monotonically increasing in the second component. In mathematical terms, the effect of a securitization impulse of $\zeta$ loans is to bring the process $X$ from the state $x \in \mathbb{N}_0 \times \mathbb{R} \times \{0, 1\}$ to the new state

$$\Gamma(x, \zeta) = (x_1 - \zeta, x_2 + \eta(x_3, \zeta) - c_f, x_3)^T, \tag{2}$$

where $^T$ denotes the transpose. The distinction between the nominal value $L$ in (1.L) used in accounting and the market value $\eta$ that investors are willing to pay will be particularly important for our model. A possible choice for the market value function $\eta$ will be presented in §2.2.

Let us denote the impulse control strategy by $\gamma = (\tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots)$, where $\tau_i$ are stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots$, and $\zeta_i$ are $\mathcal{F}_{\tau_i}$-measurable impulses. We admit only impulses $\zeta_i$ that are in the set $\{0, \ldots, L_{\tau_i -}\}$. By $\alpha = (\beta, \gamma) \in A = A(t, x)$, we denote the so-called combined stochastic control, and $A(t, x)$ denotes the set of admissible combined stochastic controls. $A(t, x)$ is chosen such that existence and uniqueness of the SDEs (1.*) holds for all admissible controls. To ensure that the controlled process is Markov, we additionally require that $\alpha \in A$ be Markov in the sense that $\tau_i$ are first exit times of $(t, X_t)_{t \geq 0}$, $\zeta_i \in \sigma(\tau_i, X_{\tau_i})$ and $\beta_t \in \sigma(t, X_{t-})$. In the next section it will be shown that $A$ is non-empty.

The controlled process $X^\alpha = (L^\alpha, C^\alpha, M)$ is determined by the SDEs (1.L), (1.C) and (1.M) between the impulses, and at $\tau_{i+1}$ changed by the impulses:

$$X_{\tau_{i+1}} = \Gamma(\check{X}_{\tau_{i+1}-}, \zeta_{i+1}) \qquad i \in \mathbb{N}_0, \tag{1.I}$$

where the term $\check{X}^\alpha_{\tau_j -}$ denotes the value of the controlled process $X^\alpha$ in $\tau_j$ including a possible jump of the process, but excluding the impulse, i.e., $\check{X}^\alpha_{\tau_j -} = X^\alpha_{\tau_j -} + \Delta X^\alpha_{\tau_j}$.

**The optimization problem.** We consider the optimization problem of the bank on the domain

$$S = \{x : x_1 > -1,\ x_1 + x_2 > 0\} \subset \mathscr{S} := \mathbb{N}_0 \times \mathbb{R} \times \{0, 1\},$$

i.e., as long as the bank does not default, and as the nominal value of the loans is non-negative. The stopping time $\tau_S = \inf\{s \geq t : X^\alpha_s \notin S\}$ denotes the first exit time from $S$. Note that exit from $S$ can only occur on $\{x_1 + x_2 = 0\}$, so shorting loans is not possible. We allow the case $L = 0$ for $C > 0$ although the bank in this case suspends its business; it may continue its business later on by setting $\beta = 1$, i.e., by issuing new loans.

The objective of the bank is to find a strategy $\alpha = (\beta, \gamma) \in A$ that maximizes the expected utility of its liquidation value at some horizon date $T$. Consider a utility function $U : \mathbb{R}^+_0 \to \mathbb{R} \cup \{-\infty\}$ and assume that $U$ is strictly increasing and concave on $[0, \infty)$. Define for $\alpha \in A$, $t \leq T$ and $x \in \mathscr{S}$ the function

$$J^{(\alpha)}(t, x) = \mathbb{E}^{(t,x)} \left[ U \left( \max\{\eta(M_\tau, L^\alpha_\tau) + C^\alpha_\tau, 0\} \right) \right], \quad \text{with } \tau := \tau_S \wedge T. \tag{3}$$

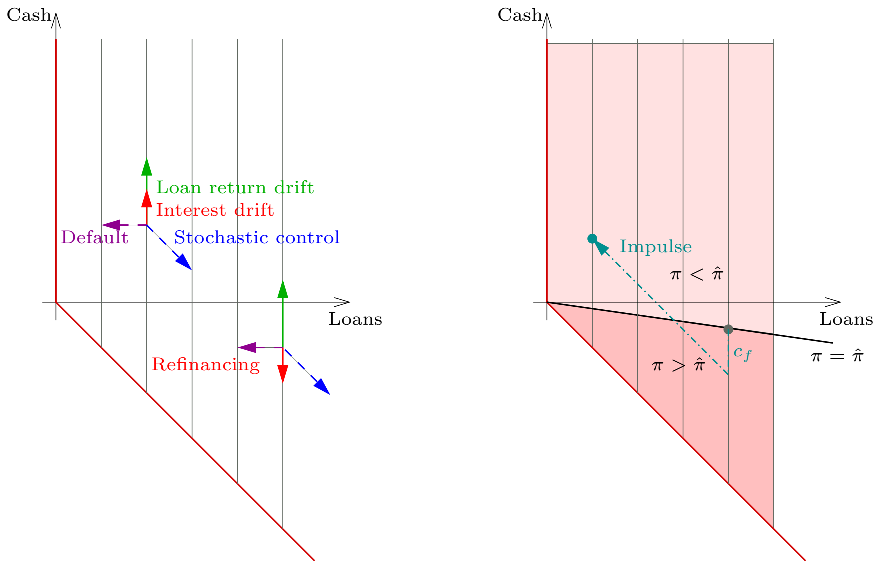


Figure 1: Visualization of the SDE terms for $\delta = 1$ (at left) and impulse graph for $\eta(x_3, \zeta) = \zeta$ (at right). Both are depicted in a $(L, C)$ graph for fixed economy. The shaded regions in the right graph indicate whether the leverage $\pi$ is greater, equal or smaller to the leverage $\hat{\pi}$ at the point of departure

Then the value function $v$ of the bank's optimization problem is defined by

$$v(t, x) = \sup_{\alpha \in A(t,x)} J^{(\alpha)}(t, x). \tag{4}$$

For future use we define $g(x) := U(\max\{\eta(x_3, x_1) + x_2, 0\})$, such that $J^{(\alpha)}(t, x) = \mathbb{E}^{(t,x)}[g(X^\alpha_\tau)]$.

*Remark* 2.1. As economic interpretation, the objective function in the optimization problem (3), $g(X^\alpha_\tau)$, can be viewed as utility of a majority shareholder, when the bank is liquidated at the horizon date $T$.

*Remark* 2.2. "Endogenous bankrupcty" as used in Leland and Toft [25], i.e., the possibility of the shareholders to liquidate the firm at any time, is automatically included in our setting: An impulse to $L = 0$, and then deciding to stay there by $\beta = 0$, terminates the business of the bank; yet still the interest $\rho$ accumulates until $T$.

## 2.2 Refinancing rate $r_B$ and market value $\eta$

In this subsection, we discuss building principles and examples for the refinancing rate $r_B$ and the market value $\eta$. While it is relatively easy to find a good functional form for $\eta$, the discussion on $r_B$ is considerably more involved. Let us emphasize that the functions proposed here are just *ad hoc* choices: they are motivated from the model, but they are not derived from it in a formal way and are therefore not model-endogeneous.

**Market value.** The market value is the amount for which loans can be sold on the secondary market. The starting point for our definition of the market value is what we call the "fundamental" value of one loan. Formally, this quantity is given by $p^{\infty}_{M_t}$ with

$$p^{\infty}_m := \mathbb{E}\left[\int_0^{\tau} e^{-\rho s} r_L ds + e^{-\rho\tau}(1-\delta(M_\tau))\middle| M_0 = m\right],$$

$\tau$ the default time of the loan. We recall that $r_L$ and $\rho$ are the loan interest rate and the risk-free interest rate, and that the functions $\lambda$, $\delta$ represent the relative loan default intensity and loss given loan default, respectively. In the special case where $\lambda$ and $\delta$ are constant, $p^{\infty}_m$ is independent of $m$ and given by $\frac{r_L+(1-\delta)\lambda}{\rho+\lambda}$. In the general case, $p^{\infty}_m$ can be obtained by a simple matrix inversion from the generator matrix of $M$ (see §7.1).

We assume that investors in securitization markets are risk-averse, so that the market value will typically be lower than the fundamental value.

*Example* 2.1. The following form for $\eta$ is used in our numerical examples in §5. In these examples we apply to the risk-neutral value a procyclical factor to reflect risk aversion and cap the resulting value at $\zeta$, so that the bank can not obtain more than the nominal value.

$$\begin{aligned} &(a) \qquad \eta_a(m,\zeta) := \zeta \cdot \min\left(1,\ p^{\infty}_m \cdot (1-(m+1)\delta(m)\lambda(m))\ \right) \\ &(b) \qquad \eta_b(m,\zeta) := \zeta \cdot \min\left(1,\ p^{\infty}_m \cdot (1-\delta(m)\lambda(m))\ \right) \end{aligned}$$

Recall that $m = 0$ in expansion, so the only effect of the factor $(m + 1)$ is to double the proportional deduction in contraction. The procyclical factor can be interpreted as a form of overcollateralization of the ABS, i.e., the bank has to put more loans into the pool such that the expected first-/second-year losses are covered without affecting the investors.

**Refinancing rate.** A constant refinancing rate $r_B$ of the bank would mean that the bank could raise money at a rate independent of its leverage and of the riskiness of its loan portfolio. As this is certainly an unrealistic assumption, we have to think about a functional form of $r_B$ incorporating the main risk factors of the bank in our model. Every reasonable choice for $r_B$ should certainly be monotonically increasing in loan default rates, and also in the leverage of the bank. Furthermore, for $C > 0$ and hence leverage $< 1$, $r_B$ should be equal to the risk-free rate $\rho$, as there is no risk of bank default.

To ensure these properties, we use as point of departure the following basic rule of thumb: On average, the bank's creditors want to earn the annualized risk-free interest $\rho$. Given a lending horizon $h$, they will therefore demand a refinancing rate $r_B$ according to

$$1 + h\rho = PD \cdot (1 - LGD) + (1 - PD) \cdot (1 + hr_B). \tag{5}$$

Here $PD = PD(h)$ and $LGD \in [0, 1]$ represent creditors' perception of the default probability of the bank over the horizon $h$ and of its loss given default.

All quantities in (5) except $\rho$ can be dependent on the current state $(\ell, c, m) \in S$ — in the following, we will mostly omit this argument for ease of notation. Equation (5) leads to the following functional form for $r_B = r_B(\ell, c, m)$

$$r_B := \frac{h\rho + PD \cdot LGD}{h(1 - PD)}. \tag{6}$$

Note that as required, for $PD = 0$, we have $r_B = \rho$. For simplicity we assume that $LGD$ deterministic. Hence the only quantity left to model is the $PD$. Without loss of generality, consider the case $t = 0$. First, for a given loan amount $\ell$ and cash position $c$, the $PD$ can be defined as the probability that loan losses exceed current equity capital $\ell + c$:

$$PD := \mathbb{P}(-\Delta L > \ell + c) = \mathbb{P}\left(\frac{-\Delta L}{\ell} > \frac{\ell + c}{\ell}\right) \tag{7}$$

Hence we need to model the distribution of the $[0, 1]$-valued relative loss $-\Delta L/\ell$ over horizon $h$.

It would be natural to model the relative loss using a (discrete) Bernoulli mixture distribution for the following reason: Given the trajectory of $M$, the loan defaults at a given horizon date $t$ are identically independent Bernoulli distributed, so that $L_t$ in our model follows a Bernoulli mixture model with mixing over the different economy states (cf. McNeil et al. [27], Bluhm et al. [4]). However, it is easier to specify a continuous distribution which does not depend on the granularity of the portfolio; one can further argue that a continuous, or even smooth function $r_B$ is reasonable because in reality the bank's creditor does not have full information about the bank's parameters and current state.

*Example* 2.2. For our numerical examples in §5, we take recourse to the Vasicek portfolio loss distribution. The Vasicek loss distribution arises as limiting case of a probit-normal Bernoulli mixture distribution for an infinitely granular portfolio, that is for $\ell \to \infty$; see Bluhm et al. [4] or the more general Prop. 8.15 in McNeil et al. [27]. Its distribution function is $V_{p,\varrho}(x) = N\left[1/\sqrt{\varrho}(N^{-1}(x)\sqrt{1-\varrho} - N^{-1}(p))\right]$, where $N$ ($N^{-1}$) is the cumulative (inverse) normal distribution function. The parameter $p \in (0, 1)$ has the interpretation of an average default rate, $\varrho \in (0, 1)$ is a correlation parameter that models how much the default rate of a single loan varies with a common factor, such as the economic state $M$. With this choice the default probability of the bank is given by

$$(a) \quad PD := 1 - V_{p,\varrho}\left(\frac{\ell + c}{\ell}\right).$$

$p = p(m)$ will normally be chosen close to the current default intensity in our model, reflecting the short-term horizon of the refinancing. The parameter $\varrho = \varrho(m)$ can be used to model risk aversion on the part of the bank's creditors, arising for instance from incomplete information regarding the current state of the bank. We will use in our numerical examples also another form, which takes into account the proceeds from the loans (assuming a refinancing rate of $\rho$ to avoid a circular reference):

$$(b) \quad PD := 1 - V_{p,\varrho}\left(\frac{(1 + r_L)\ell + (1 + \rho)c}{(1 + r_L)\ell}\right)$$

For the first form, if $\ell + c = 0$, then $PD$ will be 1 and thus $r_B = \infty$. The second form leads for $r_L > \rho$ always to a $PD < 1$ and thus to finite $r_B$.

*Remark* 2.3. The continuous probit-normal mixing distribution underlying the Vasicek distribution in Example 2.2 would correspond to infinitely many economic states in our model. Notably, the effective default intensity implied by the Vasicek distribution is unbounded; in contrast, the mixing distribution in our model with two economic states only assumes values in two states determined by the two possible default intensities.

We stress that already via the mere *existence* of a refinancing function $r_B$, we assume that there is always refinancing available. If refinancing were not available (e.g., because $PD$ is dependent on $r_B$, and there is no solution to (5)), then default would occur not at the boundary $\partial S$, but already inside $S$. This would then give rise to an endogenous default definition via backward induction, and thus further complicate matters.

# 3 Properties of the value function

This section collects a few technical properties of the model and the value function.

First of all, we note that existence and uniqueness of the SDE defined in (1.L), (1.C), (1.M) for constant $\beta$ follows from Theorem V.3.7 in Protter [34], provided that (process) Lipschitz conditions on $r_B C$ are satisfied: A Poisson process with state-dependent intensity (without explosion time) is a semimartingale, so $L$ is a well-defined semimartingale, too; the same holds for $M$. The process $C$ is well-defined on $S$ if

$$\sup_{c \geq -L_{t-}+\varepsilon} |r_B(L_{t-}, c, M_{t-})| \tag{8}$$

is an adapted càglàd process for each $\varepsilon > 0$, which is true as long as the sup in (8) exists for constant $L$, $M$ (because $L$ and $M$ are step processes). In particular, the condition is satisfied for a constant $r_B$ and the $r_B$ examples given in §2.2, Example 2.2.
We can conclude that $A(t, x)$ is non-empty.

The value function of a combined stochastic and impulse control problem is known to be associated with a certain partial integro-differential equation (PIDE), called the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI); see §3.2 for a more precise statement. Let $S_T := [0, T) \times S$, and define its parabolic "boundary" $\partial^+ S_T := ([0, T) \times S^c) \cup (\{T\} \times \mathscr{S})$, where the complement is taken in $\mathscr{S} = \mathbb{N}_0 \times \mathbb{R} \times \{0, 1\}$. Then the HJBQVI in our setting takes the form

$$\begin{aligned} \min(-\sup_{\beta\in\{0,1\}} \{u_t + \mathcal{L}^\beta u\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S_T \\ \min(u - g, u - \mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{9}$$

where $\mathcal{L}^\beta$ is the infinitesimal generator of the state process $X$ defined by the SDE (1.*): with $\tilde{x} := (x_1, x_2)$,

$$\begin{aligned} \mathcal{L}^\beta u(x) = \left(u(\tilde{x} + \begin{pmatrix} -1 \\ 1-\delta(x_3) \end{pmatrix}, x_3) - u(x)\right) \lambda(x_3) x_1 + \left(u(\tilde{x} + \begin{pmatrix} \beta \\ -\beta \end{pmatrix}, x_3) - u(x)\right) \lambda_P \\ + (u(\tilde{x}, 1 - x_3) - u(x))\, \lambda_{x_3,(1-x_3)} + (r_B(x) x_2 + r_L x_1) u_{x_2}. \end{aligned}$$

Finally, the impulse intervention operator $\mathcal{M} = \mathcal{M}^{(t,x)}$ is defined to be

$$\mathcal{M}u(t, x) = \sup\{u(t, \Gamma(x, \zeta)) : \zeta \in \{0, \ldots, x_1\}\}. \tag{10}$$

Intuitively, the condition $v - \mathcal{M}v \geq 0$ means that an impulse can not improve the value function $v$. The inequality $\sup_{\beta\in\{0,1\}}\{v_t + \mathcal{L}^\beta v\} \leq 0$ then suggests that under all possible strategies, $v(t, X_t^\alpha)$ is a supermartingale (so decreases in expectation). In any point $(t, x) \in S_T$, either $v = \mathcal{M}v$ has to hold (an impulse takes place), or $\sup_{\beta\in\{0,1\}}\{v_t + \mathcal{L}^\beta v\} = 0$ (the stochastic process evolves according to SDEs (1.L), (1.C), (1.M)).

The PDE (9), corresponding to the full problem as exposed in §2, has no known analytical solution. First and foremost, this is because impulse control until a terminal time is very difficult, if not impossible to solve explicitly. In our case, the high dimensionality makes it very unlikely for such strategies to succeed, even in the time-independent or elliptic case.

## 3.1 Bounds for the value function

We want to prove that the value function is bounded (linearly) from below and above. While the first statement is immediate if $U$ is bounded from below, the second statement necessitates some work. In the following, we will use that if we admit general adapted controls, then this will not change our value function, i.e., it suffices to consider Markov controls. For proofs of this fact in stochastic control, we refer to Øksendal [30], Haussmann [18] or El Karoui et al. [9].

**Proposition 3.1.** *The function $c \mapsto v(t, \ell, c, m)$ is increasing for all $t \in [0, T]$, $(\ell, c, m) \in S$; it is strictly increasing if $r_B > 0$.*

**Proof:** For a given admissible combined control strategy $\alpha$, we fix this strategy dependent on the events of $X^{\alpha,t,x}$ started in $X_t = x \in S$. Consider $X^{\alpha,t,y}$ for a $y$ with all components equal to $x$, but $y_2 > x_2$ (more cash). As a concatenation of (strictly) increasing functions (SDE, impulses, and $U$), $g(X_\tau^{\alpha,t,y}) \geq (>)\ g(X_\tau^{\alpha,t,x})$. Note that $\alpha$ is in general not a Markov strategy of $X$ started in $y$, but only adapted to $(\mathcal{F}_s)_{s\leq t}$ (this is why the optimality of Markov controls is needed as prerequisite). □

To be able to prove that the value function is bounded, we need an upper bound on the (optimal) leverage $\pi$. In the original setting of (1.*), this problem is elegantly resolved by the "business arrival process" $P$ with its finite intensity. On the one hand, this means that the leverage can only be increased if $P$ jumps. On the other hand, we will see that this implies the linear boundedness of the value function: The initial leverage is automatically reduced by loan proceeds (which accumulate in the cash account) – the finite intensity of $P$ makes sure that there is a natural upper bound to shifting back these proceeds. In business terms, this may be interpreted as the potential demand of the customer base being finite.

We denote in the following by $S^c$ the complement of the domain $S \subset \mathscr{S}$, and $S^+ := \{x \in S, x_2 > 0\}$, $S^- := \{x \in S, x_2 < 0\}$.

**Proposition 3.2.** *The value function $v$ is linearly bounded from above if $\tilde{\rho} := \sup_{x\in S^+} r_B(x) < \infty$, $\tilde{\rho} \leq \sup_{x\in S^-} r_B(x)$ (roughly: refinancing cost greater than risk-free interest rate), and $\eta(\cdot, \zeta) \leq b\zeta$ for some $b > 0$.*

**Proof:** We bound the impulse control value function by the value function of a stochastic control problem on $S$; the upper bound for $v$ on $S^c$ then immediately follows.

Without loss of generality, we assume $r_L > \tilde{\rho}$. The original impulse control value function is (by Prop. 3.1) bounded by the value function of the problem without fixed or proportional transaction costs (i.e., $c_f = 0$ and $\eta(m,\zeta) \geq \zeta$), and $r_B \equiv \tilde{\rho}$ (which is typically the risk-free interest rate $\rho$). Without transaction costs, it is clear that we will obtain another upper bound if we place ourselves in expansion without loan defaults. A further upper estimate can be obtained if we allow impulses up to the amount of loans, but without deducting the securitized amount from $L$. The resulting optimally controlled process follows the SDE (starting wlog in $t = 0$):

$$\begin{aligned} dL_t &= dP_t, \qquad L_0 = \ell \\ dC_t &= (r_L L_t + \tilde{\rho} C_t)dt + b\,dP_t, \qquad C_0 = c + b\ell, \end{aligned} \tag{11}$$

where of course the optimal strategy was to have maximal leverage (by construction of (11), all possible impulse benefits are already included at no cost). Now we can assume that the $P$ jumps happen immediately in 0, and the value function can be bounded as follows (with the definition $\tilde{U}(x) := U(\max\{x,0\})$):

$$\begin{aligned} v(0,\ell,c) &= \mathbb{E}[\tilde{U}(\eta(0,L_T) + C_T)] \\ &\leq \sum_{q=0}^{\infty} \mathbb{E}\left[\tilde{U}\left(\eta(0,\ell+q) + (c+b\ell+bq)\exp(\tilde{\rho}T) + (\ell+q)\frac{r_L}{\tilde{\rho}}(\exp(\tilde{\rho}T)-1)\right)\right]\mathbb{P}(P_T = q) \\ &\leq \sum_{q=0}^{\infty} \tilde{U}\left(C_1\ell + C_2 c + C_3 q\right)\frac{(\lambda_P T)^q}{q!} \\ &\leq C_1\ell + C_2 c + C_3 \end{aligned}$$

for generic constants $C_i$ dependent on $T$, where we have used increasingness and concavity of $U$. □

It is trivial that $v$ is bounded from below if $U$ is bounded from below. If $U$ is not bounded from below (e.g., $U(0) = -\infty$ as in case of a log-utility function), then the existence of a lower bound is a question of controllability. We can make sure that $v(t,x) > -\infty$ if there is an $\alpha \in A(t,x)$ and an $\varepsilon = \varepsilon(t,x) > 0$ such that $\mathbb{P}^{(t,x)}(L^\alpha_\tau + C^\alpha_\tau \leq \varepsilon) = 0$. This is the case if there is an impulse control that immediately puts the bank permanently out of danger. Boundedness from below thus holds if these strategies exist with uniform $\varepsilon > 0$, which can only be the case if $\eta(x_3,\zeta) \geq \zeta$.

*Remark* 3.1. For the proof of Prop. 3.2, we could also have used a verification theorem in the style of Øksendal and Sulem [32], Theorem 8.1. We chose the above approach because the strict increasingness property is useful in itself and less abstract.

## 3.2 Viscosity solution property

In this subsection, we will state that the value function of our combined stochastic and impulse control problem (4) is the unique viscosity solution of the HJBQVI (9) from

§3. The proof (in the appendix) consists mainly in checking that the assumptions of the general results in Seydel [39], Theorem 2.2 are satisfied. See the references in this paper, or [32], [11] for more information on viscosity solutions in connection with stochastic control.

A viscosity solution of (9) is defined pointwise by replacing the solution with suitable $C^2$ differentiable functions. Since the definition of viscosity solution is rather involved, and not needed elsewhere in this paper, we refer to Seydel [39] for its precise statement (and relegate the technical proof to the appendix).

**Theorem 3.3.** *Assume that $c \mapsto r_B(\ell, c, m)$ is continuous $\forall(\ell, c, m) \in S$, and that $U$ is continuous and bounded from below. Further assume that $\liminf_{c\downarrow -\ell} r_B(\ell, c, \cdot) > r_L$ for $\ell > 0$, that $\tilde{\rho} := \sup_{x\in S^+} r_B(x) < \infty$, $\tilde{\rho} \leq \sup_{x\in S^-} r_B(x)$ (roughly: refinancing cost greater than risk-free interest rate), and $\eta(\cdot, \zeta) \leq \zeta$. Then the value function $v$ in (4) is the unique viscosity solution of (9) in the class of linearly bounded functions, and it is continuous on $[0, T] \times \mathscr{S}$ (i.e., continuous in time and in cash).*

# 4 Frictionless markets

We investigate in this section stochastic control models related to our original model that help us to understand better the model in a few special cases. Without transaction costs (i.e., for $\eta(\cdot, \zeta) \equiv \zeta$ and $c_f = 0$), the model can be reduced in dimension, and the controls boil down to one scalar control variable representing the leverage of the bank. If we define $\pi_t := \frac{L_{t-}}{L_{t-}+C_t}$, then the dynamics for the equity value $Y_t := L_t + C_t$ reads as follows:

$$\begin{aligned} dY_t &= -\delta(M_{t-})dN_t + (r_B(X_t)C_t + r_L L_t)\,dt \\ &= -\delta(M_{t-})dN_t + ((1-\pi_t)r_B(\pi_t Y_t, (1-\pi_t)Y_t, M_t) + \pi_t r_L)\,Y_t dt \end{aligned} \tag{12}$$

Note that in the original model setting, $\pi_t$ cannot be chosen freely by the controller: While it is possible to reduce immediately $\pi_t$ to 0 (impulses in the original model), the possible increase $\Delta\pi_t$ in time depends on the "new business arrival process" $P$, and the previous leverage $\pi_t$. To obtain meaningful results, we leave all these restrictions aside, and analyze the Hamilton-Jacobi-Bellman (HJB) equation of stochastic control for $\pi \in [0, K]$ for some $K > 0$:[2]

$$\begin{aligned} \sup_{\pi\in[0,K]} \{u_t + \mathcal{L}^\pi u\} &= 0 \qquad \text{in } S_T \\ u &= g \qquad \text{in } \partial^+ S_T \end{aligned} \tag{13}$$

where this time, $S = (0, \infty) \times \{0, 1\}$, and the infinitesimal generator $\mathcal{L}^\pi$ on $S$ has the form

$$\begin{aligned} \mathcal{L}^\pi u(y, m) &= (u(y - \delta(m), m) - u(y, m))\,\lambda(m)\pi y \\ &\quad + (u(y, 1-m) - u(y, m))\,\lambda_{m,(1-m)} + ((1-\pi)r_B(\pi y, (1-\pi)y, m) + \pi r_L)y u_y(y, m). \end{aligned}$$

[2] $K$ may be interpreted as some upper bound regulators impose on the bank's leverage.

If (13) has a suitably differentiable solution, then verification results say (see, e.g., Øksendal and Sulem [32], Theorem 3.1) that this solution is equal to the value function, and a maximizer in (13) yields an optimal stochastic control. Let us assume that this is the case for our stochastic control value function $\tilde{v}$, and for simplicity that $r_B$ is constant. Then for $\delta = \delta(m)$, $\lambda = \lambda(m)$

$$\pi \mapsto \pi y \left[ (\tilde{v}(t, y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B) v_y(t,y) \right]$$

has to be maximized (separately for each $m$), with the solutions

$$\hat{\pi} = \begin{cases} 0 & \text{if } (\tilde{v}(t,y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B)\tilde{v}_y(t,y) < 0, \\ K & \text{if } (\tilde{v}(t,y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B)\tilde{v}_y(t,y) > 0, \\ [0,K] & \text{else.} \end{cases}$$

Which of the conditions is satisfied, depends very much on the boundary values in $S^c$ and on their propagation inside $S$.

Our above analysis shows that quite trivial optimal controls (either no loans, or highest possible leverage) can be expected in this simple setting; these results are confirmed in §5. More interesting results can be expected if we introduce a risk-dependent refinancing function for $r_B$, as done in our model. If $r_B$ depends only on the leverage (in our case $\pi$), then one can derive criteria $r_B$ has to satisfy to ensure that the maximum in (13) is attained in $(0, K)$.

We would like to stress that these trivial stochastic controls were derived under several assumptions (notably the smoothness of the corresponding value function), and are only optimal if $\pi$ can be set in an arbitrary way in the interval $[0, K]$. In general (including the Markov-switching economy, and restrictions on the control process $\pi$), the picture it so no clear anymore: there may be an incentive to keep loans although they are not profitable in the momentary economic situation. In the impulse control case, this can be observed in the numerical results of §5.

**Large portfolio approximation**

Next, we consider an approximation with an infinitely granular portfolio. In the limit, the randomness related to individual defaults disappears, and the economy process $M$ remains the only risk factor. If we increase for constant loan nominal the portfolio granularity to $n \in \mathbb{N}$, then loan defaults are more frequent, but have a smaller proportional effect. This is reflected in the generator of the $n$-granular SDE (for $\tilde{x} = (x_1, x_2)$):

$$\begin{aligned} \mathcal{L}^{n,\beta} u(x) = \left( u(\tilde{x} + \frac{1}{n} \begin{pmatrix} -1 \\ 1 - \delta(x_3) \end{pmatrix}, x_3) - u(x) \right) \lambda(x_3) n x_1 + \left( u(\tilde{x} + \frac{1}{n} \begin{pmatrix} \beta \\ -\beta \end{pmatrix}) - u(x) \right) \lambda_P n \\ + \left( u(\tilde{x}, 1 - x_3) - u(x) \right) \lambda_{x_3,(1-x_3)} + (r_B(x) x_2 + r_L x_1) u_{x_2}, \end{aligned}$$

For $u \in C^1(S)$, the generator $\mathcal{L}^{n,\beta} u$ converges uniformly on each compact for $n \to \infty$ to:

$$\begin{aligned} \mathcal{L}^{\infty,\beta} u(x) = -\lambda(x_3) x_1 u_{x_1}(x) + (1 - \delta(x_3))\lambda(x_3) x_1 u_{x_2}(x) + \beta \lambda_P u_{x_1}(x) - \beta \lambda_P u_{x_2}(x) \\ + \left( u(\tilde{x}, 1 - x_3) - u(x) \right) \lambda_{x_3,(1-x_3)} + (r_B(x) x_2 + r_L x_1) u_{x_2}(x), \end{aligned}$$

Following Jacod and Shiryaev [21], ch. IX.4, this implies the weak convergence in law of the $n$-granular SDE solution of (1.L), (1.C) to the solution of

$$\begin{aligned} dL_t^\infty &= (-\lambda(M_t)L_t^\infty + \beta_t\lambda_P)\,dt, \qquad L_0^\infty \in \mathbb{R}_0^+ \\ dC_t^\infty &= (r_B(X_t^\infty)C_t^\infty + r_L L_t^\infty + (1-\delta(M_t))\lambda(M_t)L_t^\infty - \beta_t\lambda_P)\,dt \end{aligned} \tag{14}$$

with the still unchanged Markov switching process $M$. Here, the dynamics for the equity value $Y_t^\infty := L_t^\infty + C_t^\infty$ with controlled leverage $\pi$ (and no transaction costs) is:

$$dY_t^\infty = (-\pi_t\delta(M_t)\lambda(M_t) + (1-\pi_t)r_B(\pi_t Y_t^\infty, (1-\pi_t)Y_t^\infty, M_t) + \pi_t r_L)\,Y_t^\infty dt \tag{15}$$

If we assume that we are able to control freely the leverage $\pi \in [0, K]$, the corresponding HJB equation is again (13), but with the infinitesimal generator $\mathcal{L}^\pi$ equal to

$$\begin{aligned} \mathcal{L}^\pi u(y,m) = (-\pi\delta(m)\lambda(m) + (1-\pi)r_B(\pi y, (1-\pi)y, m) + \pi r_L)\,y u_y(y,m) \\ + (u(y,1-m) - u(y,m))\,\lambda_{m,(1-m)}. \end{aligned}$$

It is clear from (15) that the optimal strategy is obtained by maximizing the instantaneous return for each economy state separately. Under the assumption that $r_B = r_B(\cdot, m)$ only depends on the leverage $\pi$ (and on the economy state), the instantaneous return $R$ to maximize is ($\delta = \delta(m)$, $\lambda = \lambda(m)$)

$$R(\pi) := -\pi\delta\lambda + (1-\pi)r_B(\pi) + \pi r_L$$

with derivative

$$R'(\pi) = -\delta\lambda + r_L - r_B(\pi) + (1-\pi)r_B'(\pi).$$

We assume for the moment $r_B \in C^2$, $r_B \geq \rho$, $r_B \equiv \rho$ on $\{\pi \leq 1\}$, $r_B'(\pi) > 0$ on $\{\pi > 1\}$ and $r_B'(\pi) > \delta$ on $\{\pi > 1 + 1/\delta\}$ for some $\delta > 0$. Then the maximizer $\hat{\pi}$ is 0 if loans are not profitable on average ($r_L - \rho - \delta\lambda < 0$). If loans are profitable ($r_L - \rho - \delta\lambda > 0$), then there is a maximizer $\hat{\pi} \in (1, \infty)$, and $R'(\hat{\pi}) = 0$. Typical $R$ and $R'$ are shown in Figure 2 for $r_B$ as proposed in §2.2, Example 2.2.

As the maximal rate of return $R$ is deterministic for each state of the economy, the corresponding value function then depends only on how long the bank spends in each state of the economy until $T$. An explicit representation of the value function in form of a matrix exponential can then be given using the results in the appendix (§7.1). The reader will certainly agree that the assumption of being able to manipulate freely the proportion of loans in the bank's portfolio is quite unrealistic. In reality, issuing loans will be a slow process, and reducing loan exposure may be quick, but costly – so we would expect some sort of interplay between the economic states. We see that transaction costs and/or control restrictions are keys to a good model, because otherwise the result can be as unrealistic as for the large portfolio approximation. Furthermore, the large portfolio approximation shows that the discreteness of our portfolio is necessary to have risk other than economy switching.

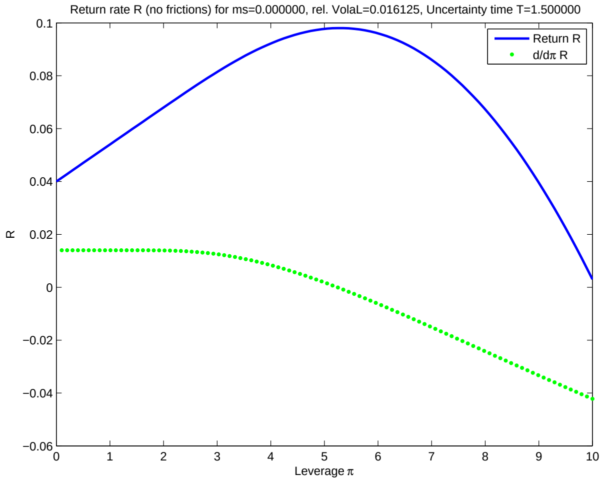


Figure 2: Return rate $R$ (solid line) and $R'$ (dotted) dependent on leverage $\pi$ in large portfolio approximation. Example parameters are as used in §5 for expansion: $\delta = 1$, $\lambda = 0.026$, $r_L = 0.08$, $\rho = 0.04$. The variable refinancing rate $r_B$ is based on a Vasicek loss distribution with default probability $p = 1.5\delta\lambda$, correlation $\varrho = 0.2$ and $LGD = 0.4$ (PD according to Example 2.2, form (b))

# 5 Numerical results

This section starts with a short description of the numerical scheme used to solve the PDE (9) and thus the combined impulse and stochastic control problem. Then, numerical results are presented and discussed from an economic point of view.

## 5.1 Finite Difference scheme

The main problem with solving the HJBQVI (9) is that the impulse intervention operator $\mathcal{M}$ introduces a circular reference to the value function: We need the value function for $\mathcal{M}v$ to be able to solve the PDE for the value function.[3] A common approach to circumvent this problem is the method of iterated optimal stopping, which we employ here. The idea is simple: First calculate the value function $v^0$ without impulses, then compute $\mathcal{M}v^0$. In the second iteration, find the solution $v^1$ of the optimal stopping problem "Either do not stop at all, or stop to get payoff $\mathcal{M}v^0$". This means that $v^1$ includes already one optimally placed impulse. Defining recursively in the same way $v^j$ for $j \geq 2$, we can hope that $v^j$ converges to the value function $v$ of the impulse control problem. Further information and proofs that this method really works can be found in Bensoussan and Lions [2], Øksendal and Sulem [32]. For alternative methods, the reader may consult again [32], or Chen and Forsyth [6].

The corresponding impulse control strategy after $j$ iterations is to do the first impulse ac-

[3] This is why it is called *quasi*-variational inequality (*quasi* refers to the fact that the obstacle $\mathcal{M}v$ is dependent on the value function $v$ itself).

cording to “Jump in points where $v^j \leq \mathcal{M}v^{j-1}$”, and the $j$-th impulse according to “Jump in points where $v^1 \leq \mathcal{M}v^0$” (see [32]). The approximately optimal strategy depicted in our graphs follows the rule “Jump in points where $v^j \leq \mathcal{M}v^{j-1}$” (perhaps infinitely often); it can be proved that this strategy is at least as good as if the true optimal strategy were applied at most $j$ times.

Computations were carried out in MATLAB. The initial PDE iteration and the optimal stopping problems are solved using a finite difference scheme on a rectangular space grid. The optimal control to use in each time step is calculated using the value function from the previous timestep (explicit), the rest of the timestepping is done in a $\theta$-scheme with $\theta = 0.5$ (Crank-Nicolson); see for instance Quarteroni et al. [35] or [38] for details. The (discrete) optimal stopping problem is solved using PSOR (projected successive over-relaxation) with adaptive relaxation parameter. We used a bespoke optimization routine for the impulse maximization; the destination of a potential impulse from $(t, x)$ is determined by the maximizer in $\mathcal{M}v^{j-1}(t, x)$. To handle boundary values at $\infty$, the computational domain was enlarged in all iterations, and Neumann boundary values equal to the derivative of the discounted utility function were applied at the cutoff boundary.

## 5.2 Numerical examples

In the numerical examples we used the following parameter values: The utility function is of CRRA-type (constant relative risk aversion) and given by $U(x) = \sqrt{x}$. The Markov chain intensities for the economy are set to be equal to $\lambda_{10} = \lambda_{01} = 0.3$, the default intensities per loan are 2.6% in expansion and 4.7% in contraction (which seems to be a rather conservative estimate for the changes between different economic states), with no loan default recovery ($\delta \equiv 1$). The risk-free rate $\rho$ is constant 0.04, the loan interest rate is set to 0.08. We used the finite variable refinancing cost in §2.2, Example 2.2 with $LGD = 0.4$ and $PD$ (b), based on a Vasicek loss distribution with $p = 1.5\lambda$, and correlation $\varrho = 0.2$ (0.4) in expansion (contraction). The resulting $r_B$ is the green dotted line in Figure 9. (This refinancing cost may seem relatively high, however this reflects that our bank’s only assets are risky loans.) The fixed transaction cost was 0.5, while the market value of securitized loans was chosen according to form (a) (strongly procyclical form; see §2.2, Example 2.1), which results in proportional transaction costs of 0% in expansion and of about 6.5% in contraction.

It will be shown below in Figure 11 that the stochastic control variable $\beta$ has only a small impact on the value function and on the optimal impulse control strategy; unless stated otherwise, we will therefore take $\beta \equiv 0$.

The first Figure 3 shows the value function, i.e., the expected terminal utility under optimal impulses, for start in expansion or in contraction. Figure 4 demonstrates the benefit of controlling the loan exposure: the utility indifference graph displayed in that figure shows the cash value of impulses (compare also Figure 5). At the risk of oversimplification, this quantity can be interpreted as the maximum salary the bank should pay its risk manager for implementing the optimal impulse strategy (compared to no securitization). For our chosen parameters, this benefit is greater in good economic times and reaches up

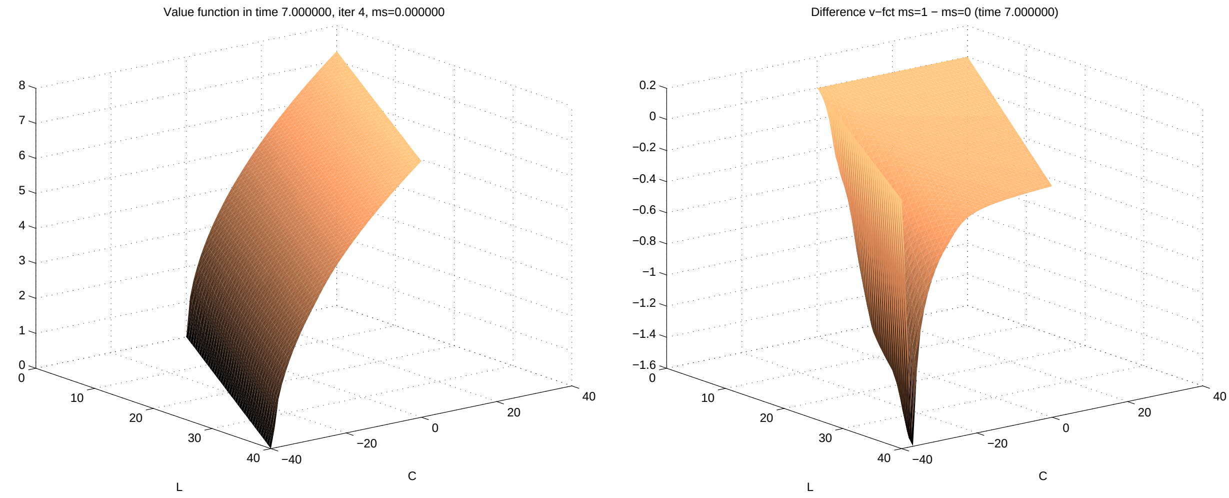


Figure 3: Value function in expansion time (left) and difference value function in contraction minus value function in expansion (right), for $T = 7$. The $x$ coordinate is the loan exposure, the $y$ coordinate the cash. Parameters for this example are as described in the text. The timestep for this numerical simulation was 0.5 years, 5 optimal stopping iterations were carried out

to about 10% of the loan exposure. This is mainly due to the lower proportional transaction costs during expansion.[4] The cash value of impulses is lower in recession, simply because the essence of the optimal strategy is to wait for the next boom (compare Figure 6 and explanation below).

The form of the optimal impulse control strategy is depicted in Figures 6 and 7. Again we see that securitization in good times is more beneficial than in bad times, essentially because the high proportional transaction costs of around 6.5% in contraction keep the bank from acting near $\{x+y=0\}$. This is remarkable as the high leverage and the resulting default risk and refinancing costs endanger the bank's existence. In such a situation, it is optimal for the bank to wait for better times. This lack of admissible impulses is compensated in better economic times: here the loan exposure of the problematic region is reduced to practically 0 as a provision for contraction, which amounts to a (temporary) liquidation of the bank.
If the market value of loans is less procyclical, then a lot less interventions take place in expansion, and more in contraction; this can be seen in the right column of Figure 12, where different function choices for refinancing cost and market value are compared. We can conclude that (proportional) transaction costs seem to be a crucial input into our model.

In the impulse graphs dependent on time to maturity (Figure 7; only in expansion), we see that immediately prior to $T$, the fixed transaction costs make it optimal to wait rather than to transact. For comparison, we have also included the same impulses-over-time

[4] If the market value is chosen according to form (b), the benefit is smaller in expansion, because then proportional transaction costs are low for both economic states.

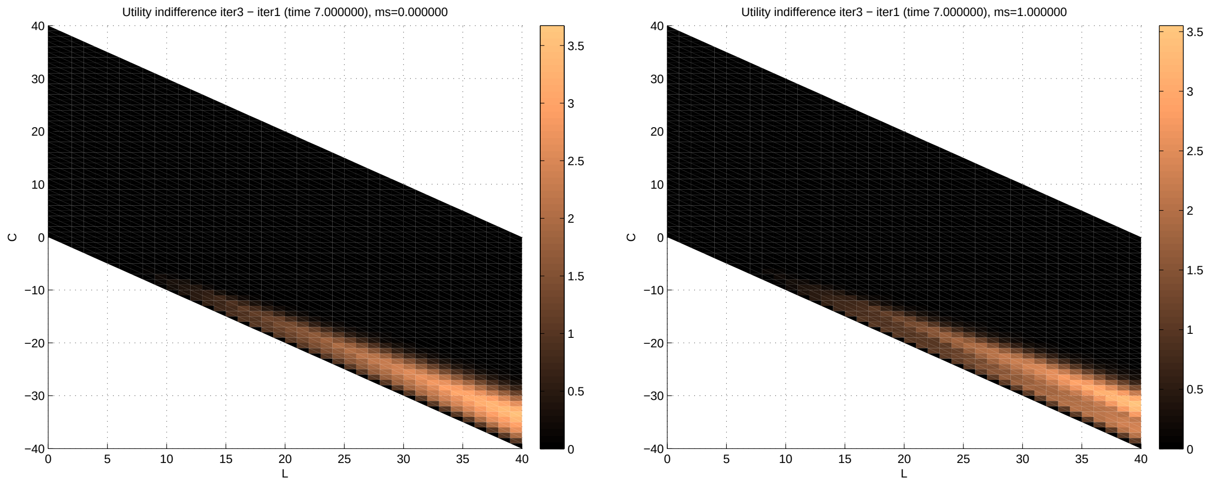


Figure 4: Cash value of impulses in expansion (left) and contraction (right) for $T = 7$ (in a bird's view; height according to colour code on the right). Shown for each point $x$ is the value $a$ such that $v_3(x_1, x_2 - a) = v_1(x_1, x_2)$ ($v_3$ being the value function with impulses, $v_1$ without), i.e., the cash the impulse-controlled bank can pay out while still being better off than the uncontrolled bank in the same situation. The cash value of impulses is practically 0 in the large dark region, and it is maximal in the lower right corner. Same data as in Figure 3

graph for lower fixed transaction costs $c_f = 0.2$ (Figure 8). We observe in Figure 8 that for $T = 3$, the transaction region is larger than for $T = 1$ — here the fear that at terminal time the bank may end up in contraction with the corresponding low liquidation value of loans dominates the desire to get a higher return rate until terminal time, and dominates also the reluctancy to pay the (fixed) transaction costs now. We note that naturally, this last effect does not affect impulses in contraction.

Further findings from our analysis and numerical results for our chosen set of parameters are:

(i) A bank has an incentive to reduce a high leverage — the higher the refinancing cost $r_B$ or the loan default rate $\lambda$, the stronger the incentive. On the one hand, this can be inferred from the analysis in frictionless markets (§4). For the original setting, a comparison of the different function choices for refinancing cost and market value can be found in Figure 12.

(ii) In expansion, the loan exposure should be reduced to 0 if the initial leverage is sufficiently high (see Figure 6) — this hinges on the absence of proportional transaction costs in our expansion case, and also on the high transaction costs in recession (compare to right column of Figure 12). Positive proportional transaction costs, on the other hand, lead to significantly smaller impulses, as can be seen from the right hand side (contraction) of Figure 6.

(iii) We note without graph that the impact of a different / more risk-averse utility

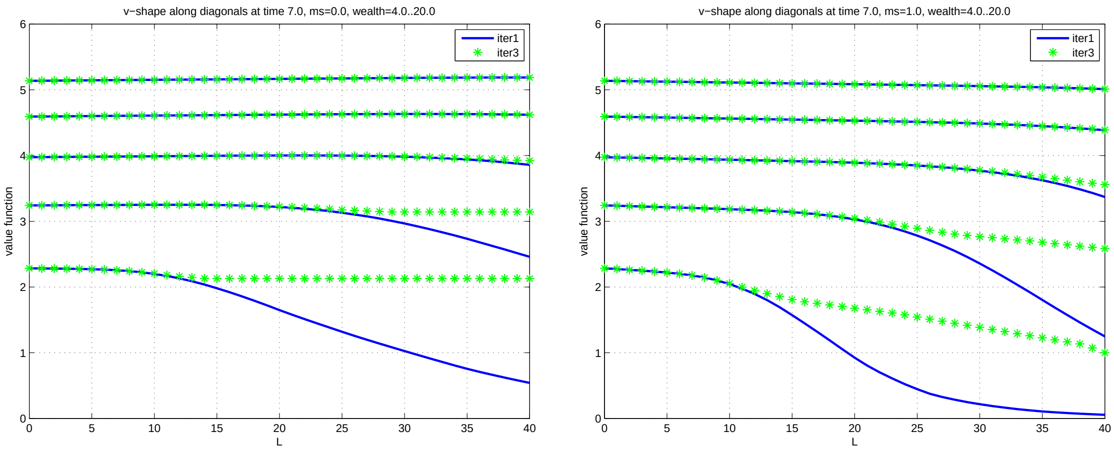


Figure 5: Value function with (green stars) and without impulses (blue line) in expansion (left) and contraction (right) for $T = 7$. Each (dotted or solid) line shows the value function along a diagonal with constant equity capital $L + C$. In the above graph, the $x$ coordinate is the loan exposure; here, a high loan exposure value corresponds to a high leverage of the bank. Roughly, an impulse is optimal when the starred line is significantly above the solid line — the larger the difference, the more valuable an impulse is. The starred line in expansion is in large parts constant because there impulses always end in $L = 0$, independent of the starting point; this line is decreasing in contraction essentially because the proportional transaction costs lead to a different equity capital level. The selected equity capital levels range from 4 to 20. Same data as in Figure 3

function is not substantial.

# 6 Conclusion and Outlook

We have presented and analyzed in this work a new model centered around optimal (impulse) control of the leverage of a bank. As many input functions complicated the problem, we resorted to numerical methods to find an approximate solution.

The first and most obvious question for a bank is: Should it reduce its leverage? This question can be answered by considering the default probabilities, the parameters of the economy process, and the refinancing rate. But then the next question arises: Is the bank actually able to reduce its leverage? This is not the case if proportional transaction costs are too high: the wedge or cone from the origin, spanned by $(1, -\eta(1))$ and $(1, -1)$, is a region where transactions can not reduce leverage. The – at first view surprising – result is that banks should not securitize in a contraction if transaction costs are high, but rather wait for better times. Either the leverage is too high, and a transaction only worsens the situation, or the leverage is not high enough to justify an intervention. This reluctancy to securitize in contraction is compensated by more impulses in expansion times.

Then why would banks sell loans (or ABS) at such large discounts, as observed in the

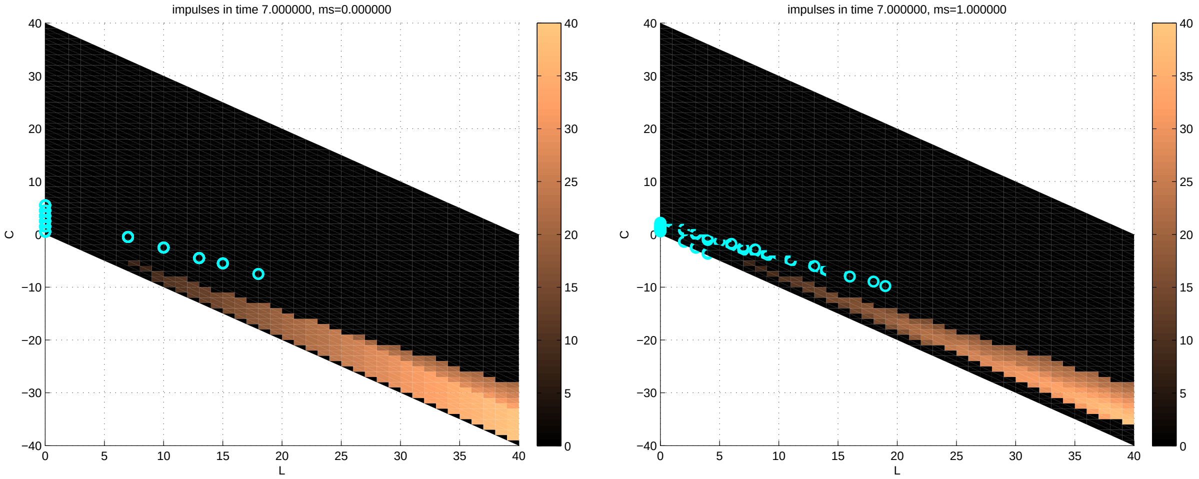


Figure 6: Optimal impulses in expansion (left) and contraction (right) for $T = 7$. The light areas mark the impulse departure points (with the lightness indicating how far to the left the impulses goes, i.e., how many loans are sold), the cyan circles represent the corresponding impulse arrival points. There are no impulses in the large dark region, and the largest impulses occur in the lower right corner. Same data as in Figure 3

recent credit crisis? One reason may be that the true value (discounted expected earnings) justified the discount; another may be that there was a high risk of the economic situation worsening – in other words, there could be more economic states than just two (this is a straightforward extension of our model).

As well, regulatory aspects have been neglected in our model. If the bank has in its portfolio also assets other than loans (e.g., government bonds with non-zero risk weight), then the conclusions might change under the Basel regulation. This could be another explanation for some recent sell-offs during the credit crisis.

Even if a complete sale of loans is not admissible, it may still be possible to offset risks by a synthetic transaction, such as a credit default swap. This introduces an additional running cost, and will certainly reduce profitability of the bank — but this type of transaction should still be possible in the transaction-free wedge.

A natural and interesting extension of our model would be to handle loan portfolios dependent on several (economic) factors. As this would increase the dimensionality of the problem, numerical results would be however more difficult to obtain. The conclusions of our study might change also if we admit injections of capital into the bank — here we would have to deal with an additional impulse.

**Acknowledgements.** The second author wishes to express his thanks to his fellow PhD student André Schlichting for several fruitful discussions on death processes, and to Carolin Mohm for her patient explanation of practical aspects of securitization.

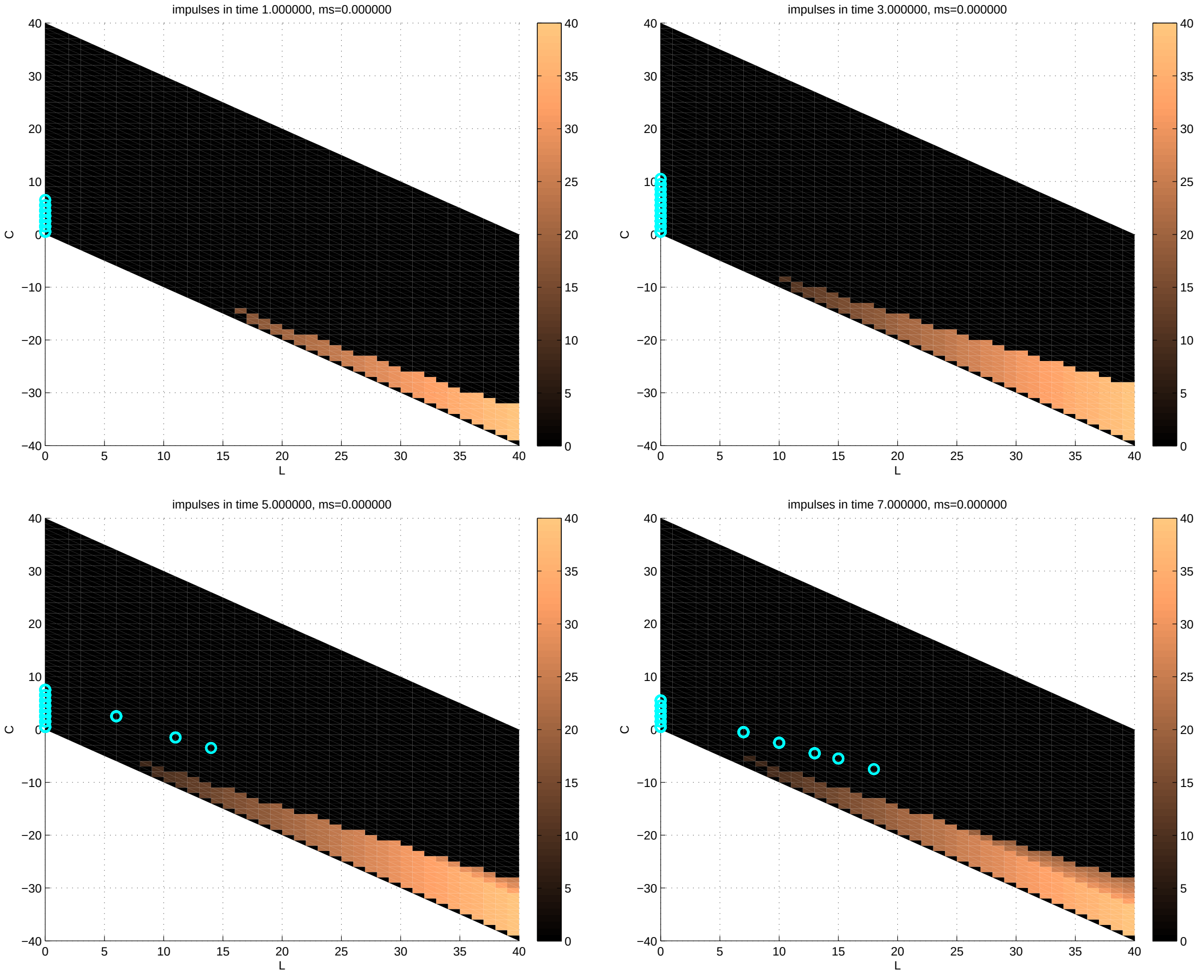


Figure 7: Optimal impulses in expansion for different $T$: top left $T = 1$, top right $T = 3$, bottom left $T = 5$ and bottom right $T = 7$. For the colour code, see the explanations in Figure 6. Same data as in Figure 3

# 7 Appendix

## 7.1 Pricing in a Markov-switching economy

We shortly describe here how the price of infinite-maturity loans in a Markov-switching economy can be derived, assuming that all parameters are risk-neutral; this price was used in the discussion of §2.2. First we calculate the risk-neutral valuation formulas for a loan with maturity $T > 0$. If $\tau$ is the default time of the loan, then its price $p_i^T(0)$ if we

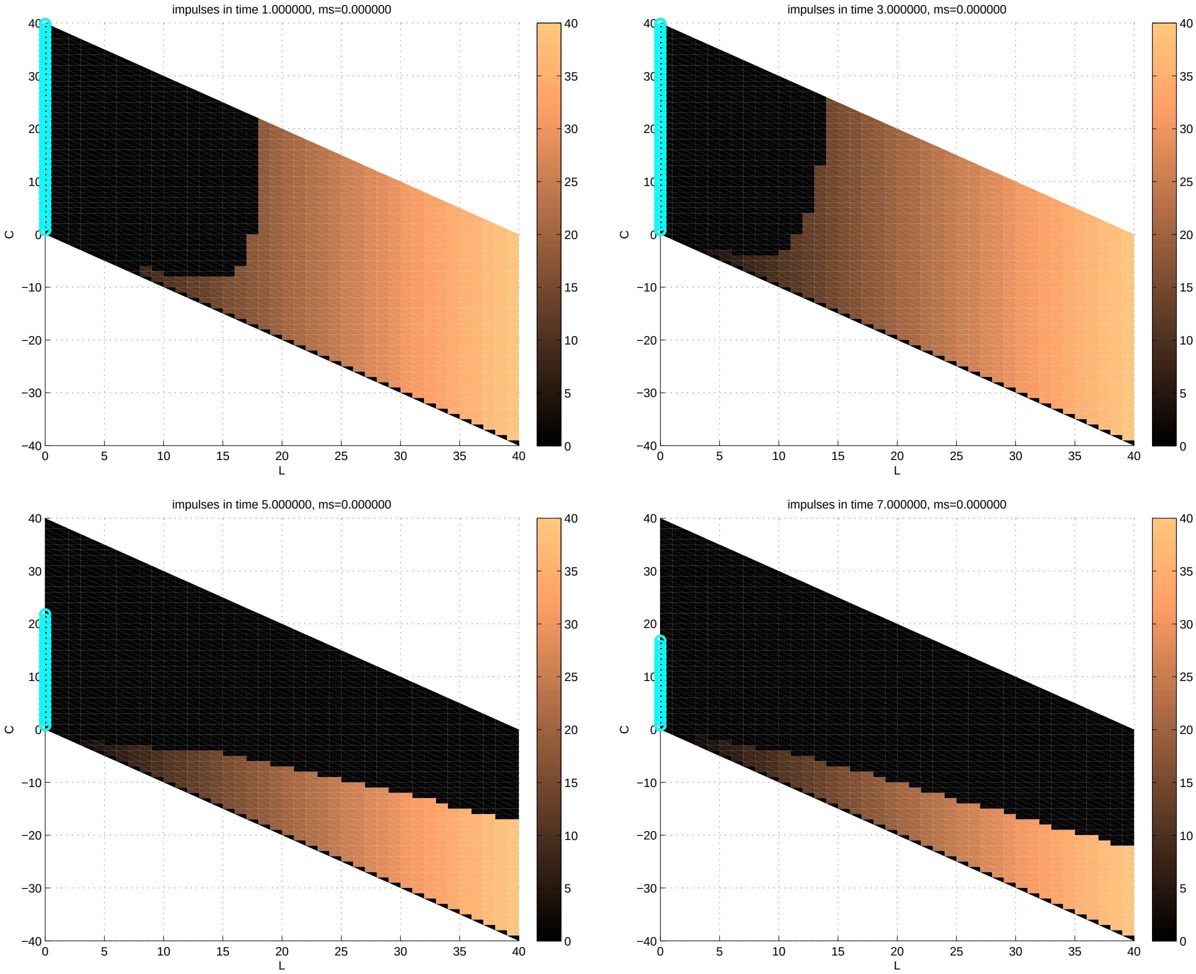


Figure 8: Optimal impulses in expansion for different $T$, fixed transaction costs $c_f = 0.2$: top left $T = 1$, top right $T = 3$, bottom left $T = 5$ and bottom right $T = 7$. For the colour code, see the explanations in Figure 6. Apart from $c_f$, same data as in Figure 3

start in $t = 0$ with economy state $i$ is determined by:

$$\begin{aligned} p_i^T(0) &= \mathbb{E}^{(0,i)}\left[\int_0^{T\wedge\tau} e^{-\rho t} r_L dt + (1-\delta(M_\tau))e^{-\rho\tau}1_{\tau\leq T} + e^{-\rho T}1_{\tau>T}\right] \\ &= \int_0^T \mathbb{E}^{(0,i)}[e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho t} r_L dt + \int_0^T \mathbb{E}^{(0,i)}[(1-\delta(M_t))\lambda(M_t)e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho t}dt \\ &\quad + \mathbb{E}^{(0,i)}[e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho T} \end{aligned} \tag{16}$$

where the last equality was obtained by conditioning on the filtration generated by the economy process (see, e.g., Th. 9.23 in McNeil et al. [27]), and interchanging integration and expectation. We see that we have to determine for some function $f$ and $T > t$ the expectation $v_i(t, x) := \mathbb{E}[f(M_T)e^{-X_T}|M_t = i, X_t = x]$ for $dX_s = \lambda(M_s)ds$. $v$ is the unique

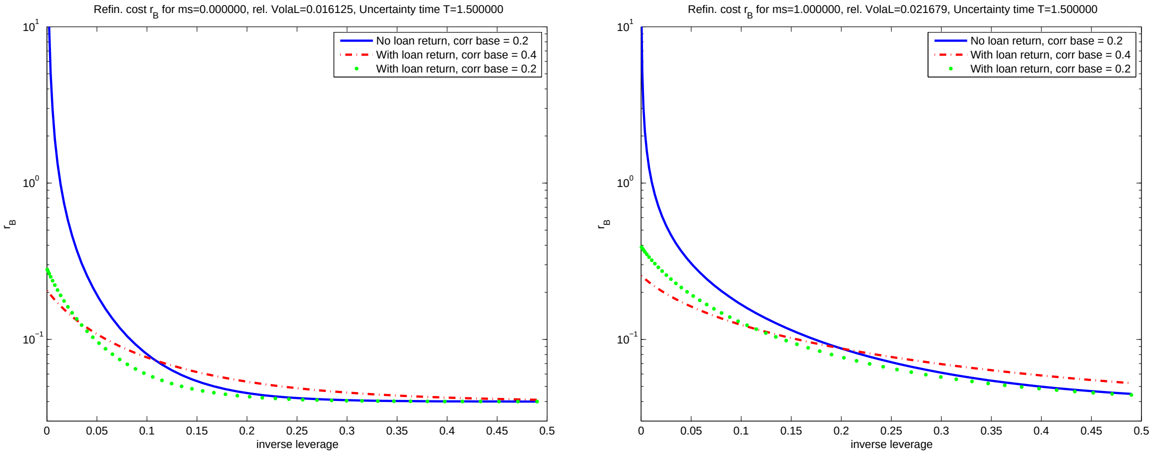


Figure 9: Variable refinancing rates in expansion (left) and contraction (right) used in Figure 12, based on a Vasicek loss distribution with default probability $p = 1.5\delta\lambda$ and $LGD = 0.4$. The rates are shown as a function of the inverse leverage $\frac{L+C}{L}$ in a logarithmic scale. The green dotted line is $r_B$ with loan return (PD according to Example 2.2, form (b)) and correlation $\varrho = 0.2$ $(0.4)$ in expansion (contraction); the red dash-dotted line shows the same $r_B$ for $\varrho$ increased by 0.2; the blue line is $r_B$ according to form (a) (infinite at $\{x + y = 0\}$) for correlation $\varrho = 0.2$ $(0.4)$

solution to the parabolic PDE

$$v_t + \begin{pmatrix} \lambda(0) & 0 \\ 0 & \lambda(1) \end{pmatrix} v_x + \begin{pmatrix} -\lambda_{01} & \lambda_{01} \\ \lambda_{10} & -\lambda_{10} \end{pmatrix} v = 0, \qquad v(T, x) = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} e^{-x} \tag{17}$$

on $(0, T) \times \mathbb{R}$. Because we know that $v(t, x) = e^{-x} v(t, 0)$ and thus $v_x = -v$, we have to solve the standard ODE

$$v' = A_\lambda v, \qquad v(T) = \boldsymbol{f} e^{-x}$$

for $A_\lambda = \begin{pmatrix} \lambda(0)+\lambda_{01} & -\lambda_{01} \\ -\lambda_{10} & \lambda(1)+\lambda_{10} \end{pmatrix}$ and $\boldsymbol{f} = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}$, which has the general solution $v(t, x) = \exp(-A_\lambda(T - t))\boldsymbol{f} e^{-x}$. Coming back to our original problem (16), by formal integration of the matrix exponential, we obtain

$$p^T(0) = A_{\lambda,\rho}^{-1}(I - \exp(-A_{\lambda,\rho}T)) \left( r_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1 - \delta(0))\lambda(0) \\ (1 - \delta(1))\lambda(1) \end{pmatrix} \right) + \exp(-A_{\lambda,\rho}T), \tag{18}$$

where $A_{\lambda,\rho} := A_\lambda + \rho I$ for the unity matrix $I$. The corresponding formula for an infinite-maturity loan can be obtained by $T \to \infty$:

$$p^\infty(0) = A_{\lambda,\rho}^{-1} \left( r_L \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1 - \delta(0))\lambda(0) \\ (1 - \delta(1))\lambda(1) \end{pmatrix} \right). \tag{19}$$

## 7.2 Proof of viscosity solution property

The conditions in Seydel [39] for $v$ to be a (unique) viscosity solution of (9) can be roughly summarized as follows: (Lipschitz) continuity of functions involved ((V*), (B*) and (E2) conditions), polynomial boundedness of the value function (E1), and continuity of $v$ at the boundary (E3). Furthermore, we need the existence of a strict supersolution $w$.

We note that our setting here is slightly different from the setting in [39] in two main respects: (a) discrete state variables, and (b) state-dependent intensity. The proofs in [39] however adapt readily to (a) with effectively no continuity requirements in the discrete variables. A state-dependent intensity fits into the random measure driven SDE in [39] as follows: For a Poisson random measure $N$ with intensity measure $\nu(dz) = 1_{[0,\infty)} Leb(dz)$, the process $\int_0^t \int_{\mathbb{R}} 1_{z \le \lambda_s} N(dz, ds)$ has a time-dependent intensity $\lambda_s$ (and the jump measure effectively has bounded support $[0, \lambda_s]$). The indicator function in the integral does not satisfy the continuity requirements; however the proofs in [39] can be carried out in the same way for a state-dependent intensity.

**Proof of Th. 3.3:** In general, continuity requirements have to hold only in $(t, x_2)$ (time and cash), because loans and economy are discrete state variables.

(V1), (B1) hold because of discreteness, (V3), (E2) and (B2) by discreteness and assumption. (V2) is satisfied because the Hausdorff convergence in discrete loan dimension does not have to hold (non-emptyness holds wlog because for $x_1 < 0$, we can set the intervention set to $\{0\}$ without affecting the value function).
(U1), (U2) do not need to hold because the jump measures are finite. (V4) holds trivially again because of the finiteness of the jump measures; the set $\mathcal{PB}$ can be defined with an arbitrary polynomial. (E4) holds, e.g., by setting $\hat{\beta} := 10$. (E1) holds because of Proposition 3.2.

(E3) only needs to hold for $t_n \to t \in (0, T]$, $c_n \downarrow -\ell$ due to the loan discreteness. Wlog, control / interventions are not possible anymore for $n \to \infty$, as $\eta(\cdot, \zeta) \le \zeta$. For $\ell > 0$, the (deterministic) explicit solution of

$$dC_t = (r_B(X_t) C_t + r_L \ell)\, dt, \qquad C_{t_n} = c_n$$

converges to $-\ell$ for $n \to \infty$ in arbitrarily short time by assumption $r_B > r_L$, leading to $g(\ell, -\ell, x_3)$ as payout. Possible loan defaults would not change this result, and lead to the same payout $g(S^c) \equiv U(0)$. For $\ell = 0$, the boundedness of $r_B(x)$ for $x_2 > 0$ proves the result.

Finally, we have to find a nonnegative function $w$ as strict supersolution that increases faster than $v$ for $|x| \to \infty$, e.g., super-linearly in view of Prop. 3.2. As first criterion, this $w$ has to satisfy $\sup_{\beta \in \{0,1\}} \{w_t + \mathcal{L}^\beta w\} \le -\kappa$ for a $\kappa > 0$ in $[0, T] \times S$. Consider for some $b > 1$, $a > 0$, and a $\tilde{\kappa}$ to be specified:

$$w(t, x) := \exp(-\tilde{\kappa} t) \left( 1_{bx_1 + x_2 \ge 0} (bx_1 + x_2)^2 + bx_1 + x_2 + a \right)$$

($C^1$ continuity is sufficient, we do not need to consider a suitably smoothed version). Its generator on $S$ has the following form:

$$\exp(\tilde{\kappa} t) \mathcal{L}^\beta w = C_1 x_1^2 + C_2 x_2^2 + C_3 x_1 x_2 + C_4 x_1 + C_5 x_2 + C_6 + x_2 (2x_2 + 2bx_1 + 1) r_B(x) \quad (20)$$

for suitable constants $C_i = C_i(x_3, \beta) \in \mathbb{R}$. Note that $bx_1 + x_2$ is a norm on the cone $\{x_1 + x_2 \geq 0, x_1 \geq 0\}$, as is easily checked. By equivalence of all norms, we see that the first part in (20) (without $r_B$) can be bounded by $C\,(1 + (bx_1 + x_2)^2)$. For $\{x_2 < 0\} \cap S$, the factor $x_2(2x_2 + 2bx_1 + 1)$ in front of $r_B$ is negative, so that $\tilde{\kappa}$ only needs to depend on $\sup_{x_2>0, x_3} r_B(x)$ and other constants to achieve our desired goal.

One checks easily that thanks to the fixed costs, $w - \mathcal{M}w \geq \kappa$ in $[0,T] \times \mathscr{S}$ (for another $\kappa > 0$), provided that $\eta(\cdot, \zeta) \leq b\zeta$ (i.e., an impulse can increase the equity value at most by a factor of $b - 1$).
The function $w$ as defined above does not yet satisfy $|w(t,x)| \to \infty$ for $|x| \to \infty$ on $S^c$, the complement of $S$. However, we have to take care that modifying $w$ on $S^c$ does not negatively affect the property $w - \mathcal{M}w \geq \kappa$ in $[0,T] \times \mathscr{S}$. The idea is to adapt the impulse function $\Gamma$ on $S^c$ so that impulses go in the direction of a minimum point $(0, -p, x_3)$ with $p > 0$ (for each fixed economy), and to introduce a function $K \leq 0$ in the problem formulation to take care of the fixed costs, with

$$\mathcal{M}w(t,x) = \sup_{\zeta \in \{0,\dots,x_1\}} \{w(t, \Gamma(x,\zeta)) + K(t,x,\zeta)\}; \tag{21}$$

correspondingly, the sum of the fixed costs $K$ over all impulses effected is added in the objective function. All these changes in $\Gamma$ and $K$ do not affect the value function, because it is impossible to get back to $S$ once $S^c$ is reached, and thus the value function is constant on $S^c$. What is more, for any starting point $x \in \mathscr{S}$, we may wlog modify the trajectory $\Gamma(x, \{0, \dots, x_1\}) \cap S^c$ because it is never optimal to jump to $S^c$. We define on $S^c$ the function

$$\tilde{w}(t,x) := \kappa_1(t)|(x_1 - 0, x_2 - (-p))| + \kappa_2(t)$$

a function whose contour lines form concentric circles around the point $(0, -p)$. We choose $\kappa_2(t)$ such that $\tilde{w}(t, (0, -p, x_3)) = w(t, (0, -p, x_3))$. We take as new function $\hat{w} := \max(w, \tilde{w})$ in a suitably smoothed form. For a small enough $\kappa_1 > 0$, the intersections where $w = \tilde{w}$ are curves completely within the interior of $S^c$ (see Figure 10). Denote $R := \{x : \tilde{w}(t,x) > w(t,x)\} \subset S^c$, and $NR := \{x : \tilde{w}(t,x) < w(t,x)\}$. We define the distance to $R$ for each $x_1$ separately as $d(t,x,R) := \max(x_2 - \sup_{y \in R, y_1 = x_1} y_2, 0)$. For any starting value $x \in NR$, we modify the trajectory of $\Gamma$ on $S^c$ such that $\Gamma(x, \{0, \dots, x_1\}) \subset NR$, while still respecting $w(t, \Gamma(x,\zeta)) \leq w(t,x)$. For $x - (0, c_f, 0)^T \notin NR$ (which we can take wlog in $S^c$), we start modifying $K$:

$$K(t,x,\zeta) := \min(-c_f + d(t,x,R), 0)$$

For any starting point $x \in R$ with $x_1 > 0$, we set

$$\Gamma(x,\zeta) := \frac{x_1 - \zeta}{x_1} x + \frac{\zeta}{x_1}(0, -p, x_3)^T,$$

such that impulses go towards $(0, -p, x_3)$ and the impulse direction is perpendicular to the contour lines of $\tilde{w}$; if $x_1 \leq 0$, changing $\Gamma$ is not necessary. Wlog, we can choose $\Gamma$ continuous on $\overline{R} \cap \overline{NR}$ because of the modification of its trajectory in $NR \cap S^c$. We conclude that with the modified $\Gamma, K$ and $\mathcal{M}$ as defined in (21), $\hat{w} - \mathcal{M}\hat{w} \geq \kappa$ in $[0,T] \times \mathscr{S}$ for some $\kappa > 0$.

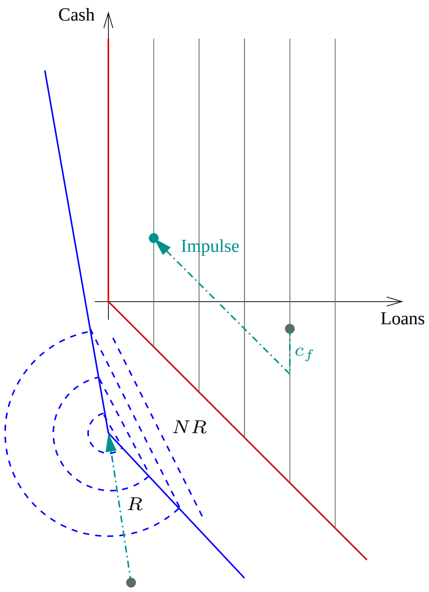


Figure 10: Contour lines of $w$, $\tilde{w}$ in the proof of Theorem 3.3 in a $(L, C)$ graph for fixed economy

The inequality $\hat{w} - g \geq \kappa$ holds in $[0, T] \times S^c$ if we set the constants in $w$, $\tilde{w}$ large enough. □

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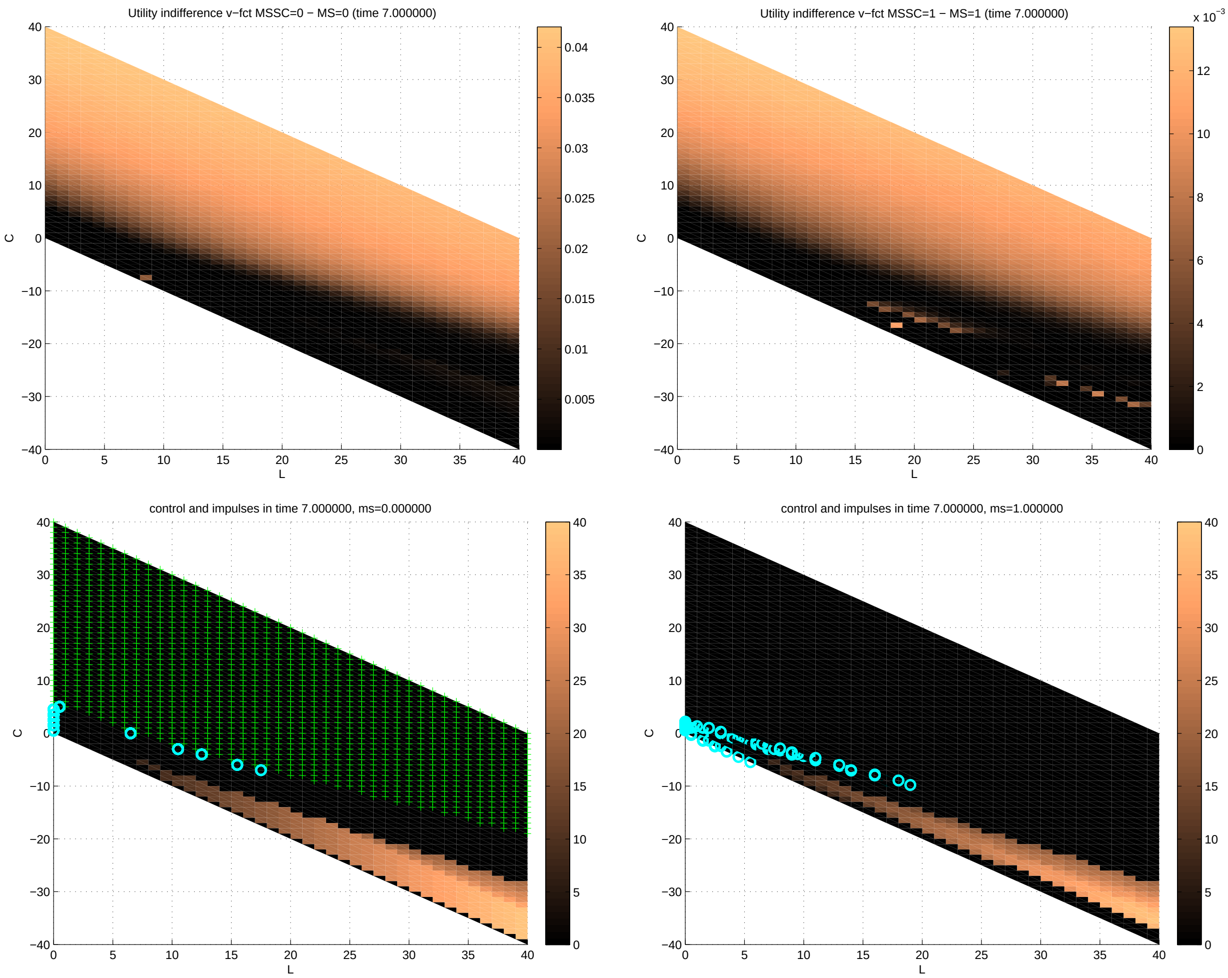


Figure 11: Impulse and stochastic control: Cash value of additional stochastic control (top row), and optimal strategy (bottom row) in expansion (left) and contraction (right), for $T = 7$. The cash value shows (as in Figure 4) the value $a$ such that $v_{SC}(x_1, x_2 - a) = v(x_1, x_2)$ ($v_{SC}$ being the value function including stochastic control, $v$ only with impulse control). The impulses are plotted in the same way as in Figure 6, points with positive stochastic control are marked with a green + (green light for customers). Business arrival intensity $\lambda_P = 2$, otherwise same data as in Figure 3

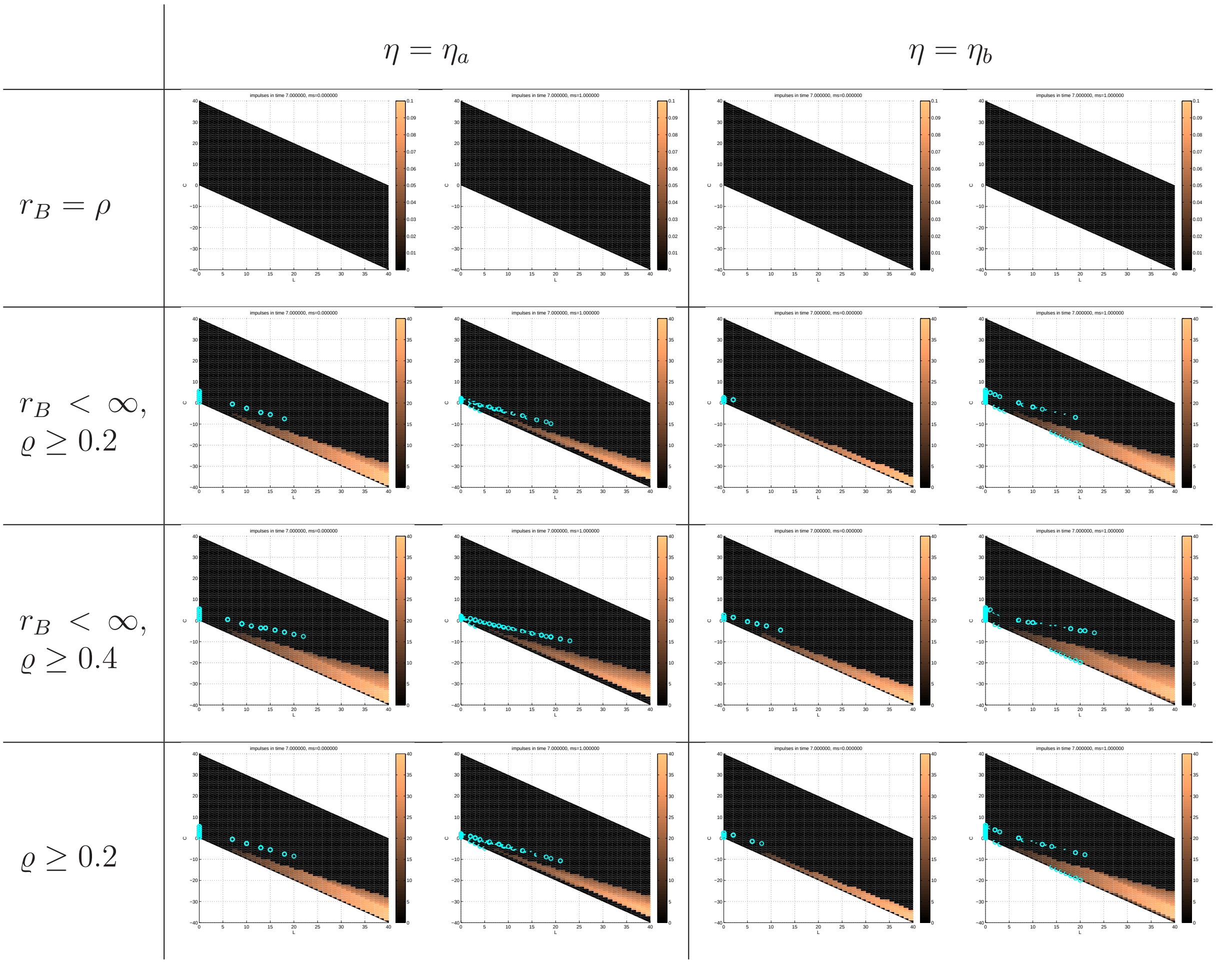


Figure 12: Optimal impulses for different refinancing functions (rows) and different market values (columns) for $T = 7$. In each cell, impulses in expansion are on the left, and impulses in contraction on the right. Refinancing cost from top to bottom: (1) $r_B$ equal to the risk-free rate $\rho$; (2) $r_B$ based on Vasicek loss distribution with loan return (Example 2.2, form (b)) for $p = 1.5\delta\lambda$ and correlation $\varrho = 0.2$ (0.4) in expansion (contraction); (3) $r_B$ with the same form (b), but correlation $\varrho = 0.4$ (0.6); (4) $r_B$ according to form (a) (infinite at $\{x + y = 0\}$) for correlation $\varrho = 0.2$ (0.4). Market values from left to right: (1) Market value $\eta$ according to Example 2.1, form (a) (procyclical), corresponding to no proportional transaction costs in expansion, and $\approx 6.5\%$ in contraction; (2) market value $\eta$ according to form (b), corresponding to about 0% (1.7%) proportional transaction costs in expansion (contraction). Otherwise, same data as in Figure 3