Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds

by

Jürgen Jost, Yi-Hu Yang, and Kang Zuo

Preprint no.: 68 2009

Harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds

Jürgen Jost[*], Yi-Hu Yang[†‡], and Kang Zuo[§¶]

September 28, 2009

Abstract

We construct harmonic metrics on unipotent bundles over quasi-compact Kähler manifolds with carefully controlled asymptotics near the compactifying divisor. This is a key ingredient for a general cohomology theory for quasi-compact Kähler manifolds.

1 Introduction

In this paper, we construct a harmonic metric on a unipotent bundle over a quasi-compact Kähler manifold. A quasi-compact Kähler manifold here is a Kähler manifold that can be compactified to a complex variety by adding a divisor that consists of smooth hypersurfaces with at most normal crossings as singularities. The key point of our construction is the prescribed and carefully controlled asymptotic behavior of the harmonic metric when approaching that compactifying divisor. This control is canonical in some sense. In fact, we have shown existence results for harmonic metrics on unipotent bundles already in our earlier papers [9], but the methods employed there do not yield the control near the compactifying divisor. Therefore, here we shall develop a new and more subtle method.

We now introduce some notation. Thus, $\overline{X}$ is an n-dimensional compact Kähler manifold, D a normal crossing divisor; and $X = \overline{X} \setminus D$ then is the quasi-compact Kähler manifold that we are interested in. Let $\rho : \pi_1(X) \to SL(r, \mathbb{Z}) \subset SL(r, \mathbb{C})$ be a linear representation. Equivalently, we have a flat vector bundle L_ρ over X.

[*]Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany;

[†]Department of Mathematics, Tongji University, Shanghai, China;

[‡]Supported partially by NSF of China (No.10771160);

[§]Department of Mathematics, Mainz University, Mainz, Germany;

[¶]Supported partially by SFB/TR 45 Periods, Moduli Spaces and Arithmetic of Algebraic Varieties of the DFG.

Take $p \in D$ and a small neighborhood in $\overline{X}$ of p, say Δ^n, where Δ is the unit disk; by our normal crossing assumption, we can assume that $\Delta^n \cap X = (\Delta^*)^k \times \Delta^{n-k}$, where Δ^* is the punctured disk. Let $(z_1, z_2, \cdots, z_n)$ be the corresponding local complex coordinate at p covering Δ^n, $z_i = r_i e^{\sqrt{-1}\theta_i} (1 \leq i \leq n)$. Let $\gamma_1, \gamma_2, \cdots, \gamma_k$ be the generators of $\pi_1((\Delta^*)^k \times \Delta^{n-k})$, where γ_i corresponds to the i-th $\Delta^*(1 \leq i \leq k)$. We restrict ρ to $\pi_1((\Delta^*)^k \times \Delta^{n-k})$ (note that the induced map from $\pi_1((\Delta^*)^k \times \Delta^{n-k})$ into $\pi_1(X)$ is not necessarily injective), and denote $\rho(\gamma_i)$ still by γ_i. We note that $\gamma_1, \gamma_2, \cdots, \gamma_k$ commutate, which will be important for our constructions below.

In this note, we always assume that *each* γ_i *is unipotent*; so $N_i = \log \gamma_i$ is nilpotent and the $N_1, \cdots, N_k$ commutate. We call L_ρ a *unipotent bundle*. If L_ρ is endowed with a harmonic metric h with *tame* growth condition (Simpson's terminology [18]) near the divisor, we call (L_ρ, h) a *unipotent harmonic bundle*. For the notion of general harmonic bundles and some related definitions that the present paper uses, we refer the reader to [18].

The notion of a unipotent harmonic bundle is a natural generalization of a variation of Hodge structures. In the study of cohomologies for a variation of polarized Hodge structures [3, 11], it is very important to understand the asymptotic behavior near the divisor of the Hodge metric; likewise, for the study of cohomologies for a unipotent harmonic bundle, we also need to have a good understanding and control of the asymptotic behavior of the endowed tame harmonic metric.

Mochizuki [15], along the way initiated by Cattani-Kaplan-Schmid [2], provides a systematic description for the asymptotic behavior of tame harmonic metrics on a unipotent bundle; however, due to the lack of the analogues in the case of harmonic bundles of the nilpotent orbit theorem and the SL_2-orbit theorem of Hodge theory, we cannot yet directly apply Mochizuki's theory to the study of cohomologies for unipotent harmonic bundles. Based on this consideration, we try to give a new approach to the asymptotic behavior that is more appropriate for the study of cohomology. The key point is that we no longer need the analogues of the nilpotent orbit theorem and the SL_2-orbit theorem – which are unknown presently. This is the objective of this paper.

To this end, we go back to Schmid's earlier paper [16]; there, an equivalent description of the asymptotic behavior of the period mapping (equivalently, the Hodge metric) is developed in terms of an equivariant geodesic embedding of the upper half plane into the period domain—a homogeneous (complex) manifold (although this was not explicitly stated, it clearly is implied); the existence and uniqueness of such a geodesic embedding are obtained by using Jacobson-Morosov's theorem and a result of Kostant [13]. We hope to generalize this description to the case of several variables; consequently, we also need to extend Jacobson-Morosov's theorem and the result of Kostant in some sense. However, we will not directly use such a generalization to obtain general information about the asymptotic behavior of a unipotent harmonic bundle; instead we use this idea to construct an initial metric of finite energy on a unipotent bundle and then deform this initial metric to a harmonic metric of finite energy without changing the behavior near the divisor. So, our tame harmonic metric is a more special one; a general tame harmonic metric may be of infinite energy. In the study of cohomologies, this will be the right harmonic metric to use.

The idea of using a geodesic embedding of the upper half plane as the asymptotic behavior of a unipotent harmonic bundle in the case of one variable has already been successfully used in our previous work [12]—the study of cohomologies for unipotent harmonic bundles over a noncompact curve.

In order to apply Schmid's description of a geodesic embedding of the upper half plane as the asymptotic behavior of a Hodge metric to harmonic metrics in the higher dimensional case, one needs to suitably extend Jacobson-Morosov's theorem and the result of Kostant; we think that such an extension is an appropriate (algebraic) substitute in the harmonic bundles theory of Cattani-Kaplan-Schmid's SL_2-orbit theorem for several variables in the Hodge theory. In §2, we work out this extension; mainly, we use the notions of parabolic subgroups (algebras) and the corresponding horospherical decomposition. Our argument is slightly geometric; we think that there should also exist a purely Lie-theoretic proof.

In Schmid's description, one of the key points is how to get a related geodesic embedding by using one single nilpotent element; this is achieved by finding a semi-simple element by using Jacobson-Morosov's theorem, then Kostant's result implies uniqueness in a certain sense. For several commuting nilpotent elements $N_1, \cdots, N_k$ in the present setting, correspondingly we hope to get a group of semi-simple elements, $Y_1, \cdots, Y_k$, *which are commutative* and such that each pair (N_i, Y_i) can be extended to an $\mathfrak{sl}_2$-triple. (We here remark that in general these triples are not commutative.) The commutativity of $Y_1, \cdots, Y_k$ implies that they are contained in a maximal abelian subspace; this motivates us to use the theory of parabolic subgroups (subalgebras). We also remark that these semisimple elements are unique after fixing a maximal abelain subspace.

After getting the semi-simple elements $Y_1, \cdots, Y_k$, in §3, we are able to construct initial metrics of finite energy; these metrics are not yet harmonic, but they will be some kind of anchor for the harmonic metrics to be constructed subsequently. Let $\mathcal{P}_r$ be the set of positive definite Hermitian symmetric matrices of order r. $SL(r, \mathbb{C})$ acts transitively on $\mathcal{P}_r$ by

$$g \circ H =: gH\bar{g}^t, \ H \in \mathcal{P}_r, g \in SL(r, \mathbb{C}).$$

Obviously, the action has the isotropy subgroup $U(r)$ at the identity I_r. Thus $\mathcal{P}_r$ can be identified with the coset space $SL(r, \mathbb{C})/U(r)$, and can be uniquely endowed with an invariant metric. Our construction at p

then takes the following form

$$H_0(z_1, z_2, \cdots, z_n) = \exp(\frac{1}{2\pi}\sum_{i=1}^{k}\theta_i N_i) \circ \exp(\sum_{i=1}^{k}(\frac{1}{2}\log|\log r_i|)Y_i). \quad (1)$$

Such a construction is compatible for all z_i-directions; namely, for each z_i-direction, it gives an equivariant geodesic embedding of the upper half plane.

In §4, we then deform the initial metric to a harmonic metric with the same asymptotic behavior. In order to make this deformation successful, one needs to impose a geometric condition on the representation ρ, namely semi-simplicity [4, 6, 19]. Under such a condition, we can deform the initial metric to a harmonic one of finite energy. The harmonic metric is pluriharmonic by using Siu's Bochner technique for harmonic map theory. The essential difficulty is to prove that the harmonic metric has the same asymptotic behavior as the initial metric; to this end, we need to patch carefully the above local constructions together so that the initial metric is harmonic on certain complex curves of many open subsets, especially certain punctured disks transversal to the divisor. Restricting the harmonic metric and the initial metric to a suitable punctured disk transversal to the divisor so that they are harmonic, we can then show that the distance function between both metrics is actually bounded on the punctured disk, which accordingly implies that the harmonic metric and the initial metric have the same asymptotic behavior.

We can now state our main result.

Main Theorem. *Let $\overline{X}$ be an n-dimensioanl compact Kähler manifold, D a normal crossing divisor; set $X = \overline{X} \setminus D$. Let $\rho : \pi_1(X) \to SL(r, \mathbb{Z}) \subset SL(r, \mathbb{C})$ be a linear semisimple representation that is unipotent near the divisor, and L_ρ the corresponding unipotent bundle. Then, up to a certain isometry in $\mathcal{P}_r$, there exists uniquely a harmonic metric on L_ρ with the same asymptotic behavior as H_0 (see (1))near the divisor.*

The research of harmonic metrics on noncompact manifolds was initiated by Simpson [18] in the complex one dimensional case under a more algebraic geometric background. The general construction of harmonic metrics on a quasi-compact Kähler manifold was later considered in [9, 10] in a more general setting — equivariant harmonic maps. For the existence, it was assumed that a representation of π_1 be reductive; such a notion applies to more general target manifolds, but when restricted to the present setting, it reduces to the notion of semisimplicity employed here. [9] also obtained a harmonic metric of finite energy; however, due to their construction, the behavior at infinity of the metric is not controlled; in particular, there is no norm estimate for flat sections when translated into the case of harmonic bundles. For such an estimate, we need the present construction. Hopefully, our construction will give a new understanding of asymptotic behavior of variation of Hodge structures or period mapping [2].

This work was begun in February-April of 2004 when the second author was visiting CUHK and HKU in Hong Kong and continued in April-June of 2007 when he was visiting the Universities of Mainz and Essen; its main part was finished when he visited the Max-Planck Institute for Mathematics in the Sciences, Leipzig in Jan.-Feb. and July-August of 2009. He thanks the above institutes for hospitality and good work conditions; especially, he wants to thank Professors N. Mok, H. Esnault and E. Viehweg for their kind invitation.

2 Construction of semisimple elements

In this section, we first recall Morosov-Jacobson's theorem, the theorem of Kostant and the horospherical decomposition associated to a parabolic subgroup; then we present the construction of semisimple elements $Y_1, \cdots, Y_k$.

2.1 Some Lie-theoretic and geometric preliminaries

2.1.1. Morosov-Jacobson's theorem (cf. e.g. [13]): Let G be a noncompact real simple Lie group, $\mathfrak{g}$ the corresponding Lie algebra. Assume N is a nilpotent element in $\mathfrak{g}$. Then, one can extend N to an $\mathfrak{sl}(2,\mathbb{R})$-embedding into $\mathfrak{g}$: $\{N, Y, N^-\} \subset \mathfrak{g}$ satisfying

$$[N, Y] = 2N, \quad [N, Y^-] = -2N^-, \quad [N, N^-] = Y.$$

Geometrically, this means the following: Let X be the corresponding Riemannian symmetric space of G. Then through any fixed point, there exists a geodesic embedding into X of the upper half plane whose horocycles are generated by N and whose geodesics perpendicular to the horocycles are generated by the corresponding semi-simple element (or say the orbits of the corresponding one-parameter groups).

2.1.2. Kostant's theorem (cf. [13]): Let $\mathfrak{g}$ and N be as in Morosov-Jacobson's theorem.
1) The elements in $Im(\mathrm{ad}N) \cap Ker(\mathrm{ad}N)$ are nilpotent;
2) let $\{N, Y, N^-\}$ be an $\mathfrak{sl}_2$-embedding extended by N in $\mathfrak{g}$ as in Morosov-Jacobson's theorem, satisfying

$$[N, Y] = 2N, \quad [N, Y^-] = -2N^-, \quad [N, N^-] = Y.$$

Then, Y is unique up to nilpotent elements in $Im(\mathrm{ad}N) \cap Ker(\mathrm{ad}N)$.

Consequently, we have: Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition and $\mathfrak{a}$ a maximal abelian subalgebra in $\mathfrak{p}$. If Y lies in $\mathfrak{a}$, then, such a Y is unique. Namely, for a fixed maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, $Y \in \mathfrak{a}$ is unique.

Geometrically, this means that, through a fixed point in X, the geodesic embedding in X of the upper half plane corresponding to N in M-J's theorem is unique.

2.1.3. Parabolic subgroups (subalgebras) and the corresponding horospherical decompositions (cf. e.g. Borel and Ji's book): Let $\mathfrak{g}$ be a real semi-simple Lie algebra of noncompact type, G the corresponding noncompact semisimple Lei group, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ Cartan decomposition, and $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{p}$. Relative to $\mathfrak{a}$, one has the restricted root system, denoted by Φ; choose a simple root system, denoted by Δ; denote the corresponding positive root system by Φ^+.

Let I be a subset of Δ. One can construct a corresponding parabolic subalgebra $\mathfrak{p}_I$ (resp. P_I) of $\mathfrak{g}$ (resp. G), called the standard parabolic subalgebra (subgroup) relative to I. Any parabolic subalgebra (subgroup) is conjugate to such a one under G and also under K, the maximal compact subgroup of G corresponding to $\mathfrak{k}$; moreover, for any two distinct subsets I, I' of Δ, $P_I, P_{I'}$ are not conjugate under G.

Let $\mathfrak{m}$ be the centralizer in $\mathfrak{k}$ of $\mathfrak{a}$, $\mathfrak{a}_I = \cap_{\alpha \in I} Ker\alpha \subset \mathfrak{a}$, $\mathfrak{a}^I$ the orthogonal complement in $\mathfrak{a}$ of $\mathfrak{a}_I$; let Φ^I be the set of roots generated by I.
Set

$$\mathfrak{n}_I = \sum_{\alpha \in \Phi^+ - \Phi^I} \mathfrak{g}^\alpha, \quad \mathfrak{m}_I = \mathfrak{m} \oplus \mathfrak{a}^I \oplus \sum_{\alpha \in \Phi^I} \mathfrak{g}^\alpha.$$

Then, $\mathfrak{p}_I = \mathfrak{n}_I \oplus \mathfrak{a}_I \oplus \mathfrak{m}_I$ is the desired parabolic subalgebra, its corresponding subgroup of G denoted by P_I.

Corresponding to the above decomposition of $\mathfrak{p}_I$, one also has the decomposition of P_I in the level of group: Let N_I, A_I, M_I be the Lie subgroups of G having the Lie algebras $\mathfrak{n}_I, \mathfrak{a}_I, \mathfrak{m}_I$ respectively, then

$$P_I = N_I A_I M_I \cong N_I \times A_I \times M_I,$$

where by $\cong$ we mean an analytic diffeomorphism, i.e. the map

$$(n, a, m) \to nam \in P_I \quad (n \in N_I, a \in A_I, m \in M_I)$$

is an analytic diffeomorphism. This is the so-called Langlands decomposition of P_I.
When I is empty, $P_I = P_\emptyset$ is a minimal parabolic subgroup of G, the corresponding decomposition is $P_\emptyset = N_\emptyset AM$, here $A = \exp \mathfrak{a}, M = \exp \mathfrak{m}$.

Let K be the maximal compact subgroup with the Lie algebra $\mathfrak{k}$ of G. The Iwasawa decomposition $G = N_\emptyset AK$ tells us that $G = P_I K$ for any subset I of Δ. So, P_I acts transitively on the symmetric space $X = G/K$. Thus, the Langlands decomposition for P_I induces a decomposition of X associated to P_I, called the horospherical decomposition

$$X \cong N_I \times A_I \times X_I,$$

where $X_I = M_I/(M_I \cap K)$, called the boundary symmetric space associated to P_I; by $\cong$ we again mean an analytic diffeomorphism, i.e. the map

$$(n, a, m(M_I \cap K)) \to namK \in X \quad (n \in N_I, a \in A_I, m \in M_I)$$

is an analytic diffeomorphism.

Let P be a parabolic subgroup of G. As mentioned before, it is conjugate to a unique standard parabolic subgroup P_I under K. Choose $k \in K$ such that under k, P is conjugate to P_I, denoted by $P =^k P_I$. Define

$$N_P =^k N_I, \quad A_P =^k N_I, \quad M_P =^k M_I.$$

Though the choice of k is not unique, the subgroups N_P, A_P, M_P are well-defined. We call N_P, A_P (resp. the corresponding Lie algebras $\mathfrak{n}_P, \mathfrak{a}_P$) the unipotent radical, the split component of P (resp. $\mathfrak{p}$) respectively.
Thus, we can translate the Langlands decomposition of P_I into that of P, namely

$$P = N_P A_P M_P \cong N_P \times A_P \times M_P.$$

Consequently, we also have the horospherical decomposition of $X = G/K$ associated to P

$$X \cong N_P \times A_P \times X_P,$$

where $X_P = M_P/(M_P \cap K)$, called the boundary symmetric space associated with P.

2.2 The construction of semisimple elements

Now, we return to the setting of §1. As mentioned there, $N_1, N, \cdots, N_k$ are some commutative nilpotent matrices. By Engel's theorem, we can assume that all of them are upper triangular with the entries of the diagonal being zero, and hence that $\gamma_1, \gamma_2, \cdots, \gamma_k$ are upper triangular with the entries of the diagonal being 1.
Choose a maximal parabolic subalgebra (resp. subgroup) $\mathfrak{p}$ (resp. P) of $\mathfrak{sl}(r, \mathbb{R})$ (resp. $SL(r, \mathbb{R})$) the unipotent radical of which contains $N_1, N, \cdots, N_k$ (resp. $\gamma_1, \gamma_2, \cdots, \gamma_k$); furthermore, we can choose $\mathfrak{p}$ (resp. P) so that its split component is contained in the set of diagonal matrices of $\mathfrak{sl}(r, \mathbb{R})$ (resp. $SL(r, \mathbb{R})$).
We remark that the set of diagonal matrices in $\mathfrak{sl}(r, \mathbb{R})$ is a maximal abelian subspace contained in the noncompact part of a Cartan decomposition of $\mathfrak{sl}(r, \mathbb{R})$; and that the use of the Engel's theorem shows that we consider the set of diagonal matrices in $\mathfrak{sl}(r, \mathbb{R})$ as such a maximal abelian subspace.
Warning: Such a choice of parabolic subalgebras (resp. subgroups) is not unique, even for a fixed maximal abelian subspace.

Let $P = N_P A_P M_P$ ($\mathfrak{p} = \mathfrak{n}_P \oplus \mathfrak{a}_P \oplus \mathfrak{m}_P$, $N_P = \exp \mathfrak{n}_P, A_P = \exp \mathfrak{a}_P$) be the Langlands decomposition of P, correspondingly, we have the horospherical decomposition

$$SL(r, \mathbb{R})/SO(r) := X = N_P A_P X_P,$$

where $X_P = M_P/(M_P \cap SO(r))$, the boundary symmetric space associated with P.
From the previous choice for $\mathfrak{p}$ (resp. P), we know that $\mathfrak{a}_P$ (resp. A_P), as a set of matrices, is contained in the set of diagonal matrices.

For $N_i \in \mathfrak{n}_P$, $i = 1, \cdots, k$, by the Morosov-Jacobson's theorem, we can extend it to an embedding into $\mathfrak{sl}(r, \mathbb{R})$ of $\mathfrak{sl}(2, \mathbb{R})$, say $\{N_i, Y_i, N_i^-\} \subset \mathfrak{sl}(r, \mathbb{R})$, satisfying

$$[N_i, Y_i] = 2N_i, \quad [N_i, Y_i^-] = -2N_i^-, \quad [N_i, N_i^-] = Y_i.$$

Lemma 1 *For this embedding, one can choose* Y_i *such that it lies in* $\mathfrak{a}_P$*, and hence such a semisimple element is unique.*

In the following, we will show that this can actually be done by using (the geometric interpretations of) the Morosov-Jacobson theorem, the Kostant theorem, and the horospherical decomposition associated to P. It should be an interesting question whether one can give a purely Lie-theoretic proof of this result.

Using the Killing form, we can easily show that the factors $\mathfrak{n}_P, \mathfrak{a}_P, \mathfrak{m}_P$ of $\mathfrak{p}$ are orthogonal to each other, so the orbits of N_P and A_P in the horospherical decomposition are orthogonal and also orthogonal to the boundary symmetric space X_P; in addition, we also can consider X_P as the set of fixed points at infinity of N_P and A_P.

Fix a point $x_0 \in X$. By ∞_0, we denote the (unique) intersection point of the orbit of x_0 under $\exp(tN_i), t \in \mathbb{R}$ (denoted by $\exp(tN_i) \circ x_0$) with X_P. One has then a unique geodesic in X connecting x_0 and ∞_0, denoted by σ_0. Clearly, the orbit $\exp(tN_i) \circ \sigma_0, t \in \mathbb{R}$ is a geodesic embedding into X.
We need to show that the orbit $\exp(tN_i) \circ \sigma_0, t \in \mathbb{R}$ is a geodesic embedding of the upper half plane and hence the orbit of any point in it under $\exp(tN_i)$ is its horocycle.
This is a consequence of the M-J theorem and the Kostant theorem. By the M-J theorem, through x_0, we have a geodesic embedding of the upper half plane into X; the Kostant theorem implies that such an embedding is unique. On the other hand, $\exp(tN_i) \circ x_0$ is contained in this embedding, so the intersection point of this embedding with X_P is also ∞_0. That is to say, this embedding also contains the geodesic σ_0. So, this embedding is just $\exp(tN_i) \circ \sigma_0, t \in \mathbb{R}$.
Furthermore, by the horospherical decomposition associated to P, we can choose a semisimple element Y_i in $\mathfrak{a}_P$, the orbits in this embedding of the one-parameter group of which are geodesics perpendicular to the horocycles. Such a Y_i is just the desired one. This completes the proof of the Lemma.

Remark: From the construction, $Y_1, \cdots, Y_k \in \mathfrak{a}_P$ seem to depend on the choice of the parabolic subalgebra $\mathfrak{p}$. But, by Kostant's theorem, we know that $Y_1, \cdots, Y_k$ only depend on the choice of a maximal abelian

subspace in the noncompact part of a Cartan decomposition; namely they are unique up to some conjugations when we require that all of them be contained in a maximal abelian subspace.
So, when we fix a maximal abelian subspace $\mathfrak{a}$, we can uniquely get semisimple elements $Y_1, \cdots, Y_k \in \mathfrak{a}$ such that $[N_i, Y_i] = 2N_i, i = 1, \cdots, k$.

Remark: The above construction for semisimple elements works for any semisimple Lie algebra, not only for $\mathfrak{sl}(r, \mathbb{R})$; so, this may provide a new way to understand variations of Hodge structures and their degeneration, different from Cattani-Kaplan-Schmid's theory [2].

3 The construction of initial metrics (maps) and their asymptotic behavior

As in the Introduction, let $\overline{X}$ be a compact Kähler manifold, D a normal crossing divisor, $X = \overline{X} \setminus D$. Taking $p \in D$ and a small neighborhood U at p, then $X \cap U$ is of the form $(\Delta^*)^k \times \Delta^{n-k}$. Let $(z_1, z_2, \cdots, z_n)$ be a local complex coordinate on U with

$$(\Delta^*)^k \times \Delta^{n-k} = \{(z_1, z_2, \cdots, z_n) : z_1 \neq 0, z_2 \neq 0, \cdots, z_k \neq 0\}.$$

On $(\Delta^*)^k \times \Delta^{n-k}$, one has the following product metric

$$ds_P^2 = \frac{\sqrt{-1}}{2}\Big[\sum_{i=1}^{k} \frac{dz_i \wedge d\overline{z}_i}{|z_i|^2(\log|z_i|)^2} + \sum_{i=k+1}^{n} dz_i \wedge d\overline{z}_i\Big].$$

In general, one has the following

Proposition 1 *There exists a complete, finite volume Kähler metric on X which is quasi-isometric to the metric of the above form near any point in the divisor D.*

Proof. cf. [5].

Due to the above proposition, when we consider local constructions and various estimates near the divisor in this section, if not involving the derivatives of the Kähler metric (in fact we indeed do not need estimates involving the derivatives), we always use the above local product metric ds_P^2.

3.1 The construction of initial metrics of finite energy

Using the previous constructions for semisimple elements $Y_1, \cdots, Y_n$, we can construct an initial map from the universal covering of $(\Delta^*)^k \times \Delta^{n-k}$ into $SL(r, \mathbb{C})/U(r)$, which is ρ-equivariant; equivalently, we can also consider such a map as a metric on the corresponding flat bundle $L_\rho|_{(\Delta^*)^k \times \Delta^{n-k}}$.
To this end, let us first give some preliminaries. Let $\mathcal{P}_r$ be the set of

positive definite Hermitian symmetric matrices of order r. $SL(r,\mathbb{C})$ acts transitively on $\mathcal{P}_r$ by

$$g \circ H =: gH\bar{g}^t,\ \ H \in \mathcal{P}_r, g \in SL(r,\mathbb{C}).$$

Obviously, the action has the isotropy subgroup $U(r)$ at the identity I_r. Thus $\mathcal{P}_r$ can be identified with the coset space $SL(r,\mathbb{C})/U(r)$, and can be uniquely endowed with an invariant metric[1] up to some constants. In particular, under such a metric, the geodesics through the identity I_r are of the form $\exp(tA)$, $t \in \mathbb{R}$, A being a Hermitian symmetric matrix.

3.1.1 Local construction

Let $(z_1, z_2, \cdots, z_n)$ be the above complex coordinates on $(\Delta^*)^k \times \Delta^{n-k}$, $z_i = r_i e^{\sqrt{-1}\theta_i}$, $0 < r_i < 1, -\infty < \theta_i < \infty$. Set

$$H_0(z_1, z_2, \cdots, z_n) = \exp(\frac{1}{2\pi}\sum_{i=1}^{k}\theta_i N_i) \circ \exp(\sum_{i=1}^{k}(\frac{1}{2}\log|\log r_i|)Y_i),$$

which is independent of $z_{k+1}, \cdots, z_n$. Clearly, it is ρ-equivariant. Similar to [11], one can show that under the product metric ds^2_P, H_0 has finite energy [2].

Geometrically, the finiteness of the energy can be explained as follows. For any fixed $z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n, i \leq k$, H_0 can be considered as a geodesic (and hence harmonic) isometric embedding

$$\exp(\frac{1}{2\pi}\theta_i N_i) \circ \{\exp\big((\frac{1}{4}\log|\log r_i|)Ad_{\exp(\sum_{j\neq i}\frac{1}{2\pi}\theta_i N_i)}Y_i\big)$$
$$\circ[\exp(\frac{1}{2\pi}\sum_{j\neq i}\theta_i N_i) \circ \exp(\sum_{j\neq i}(\frac{1}{2}\log|\log r_i|)Y_i)]\},$$

of a neighborhood of a point at infinity of the upper half plane, say $\{w_i \in \mathbb{C} \mid \Im w_i > \alpha > 0\}$ with $w_i = -\sqrt{-1}\log z_i$, into $\mathcal{P}_r$, which is equivariant with respect to γ_i, as in the 1-dimensional case [12]. This point is also important in the proof in §4; namely, **our local constructions are harmonic on punctured disks transversal to the divisor**.

Since the metric ds^2_P is a product metric on $(\Delta^*)^k \times \Delta^{n-k}$, for the estimate of the energy density (and energy) of H_0, we can consider each $\partial_{z_i}H_0, i \leq k$ separately. Again since the map

$$p : (\{w_i = x_i + \sqrt{-1}y_i \in \mathbb{C} \mid y_i > \alpha > 0\}, \frac{dw_i \wedge d\overline{w}_i}{|\mathrm{Im} w_i|^2}) \to (\Delta^*, \frac{dz_i \wedge d\overline{z}_i}{|z_i|^2(\log|z_i|)^2})$$

[1] In terms of matrices, such an invariant metric can be defined as follows. At the identity I_r, the tangent elements just are symmetric matrices; let A, B be such matrices, then the Riemannian inner product $< A, B >_{\mathcal{P}_r}$ is defined by $tr(AB)$. In general, let $H \in \mathcal{P}_r$, A, B two tangent elements at H, then the Riemannian inner product $< A, B >_{\mathcal{P}_r}$ is defined by $tr(H^{-1}AH^{-1}B)$.

[2] We here remark that, in Proposition 1 of [11], the estimate of $|dh|^2$ should be read as "$|dh|^2 \leq C$" instead of "$|dh|^2 \leq C|\log r|^2$".

where $p(w_i) = z_i = e^{\sqrt{-1}w_i}$, is a Riemannian covering, so restricting to a fundamental domain of p, say $\{x_i + \sqrt{-1}y_i \in \mathbb{C} \mid y_i > \alpha > 0, 0 \leq x_i < 1\}$, we can write H_0 as

$$H_0 = \exp(\frac{1}{2\pi}\sum_{i=1}^{k} x_i N_i) \circ \exp(\sum_{i=1}^{k}(\frac{1}{2}\log y_i)Y_i);$$

and estimating $\partial_{z_i} H_0$ is equivalent to estimating $\partial_{w_i} H_0$. Since for fixed $w_1, \cdots, w_{i-1}, w_{i+1}, \cdots, w_n$, H_0 is a geodesic isometric embedding, we have

$$|\partial_{w_i} H_0| = \text{const.}.$$

On the other hand, the domain $\{x_i + \sqrt{-1}y_i \in \mathbb{C} \mid y_i > \alpha > 0, 0 \leq x_i < 1\}$ has finite volume, so the energy of H_0 is finite.

3.1.2 Patching local constructions together on a tube neighborhood of the divisor

Using the above local construction for H_0 and a finite partition of unity on $\overline{X}$, we can construct a smooth metric on L_ρ which takes the H_0 as asymptotic behavior near the divisor D, still denoted by H_0. So, using the above complete Kähler metric on X, such a metric, as a ρ-equivariant map, has finite energy. However, although our local constructions are harmonic on punctured disks transversal to the divisor, the metric H_0, after patched together, is not necessarily harmonic on such a punctured disk. In order to make sure that the metric H_0 can be deformed to a harmonic metric with the same asymptotic behavior as H_0 at infinity, we need to choose an appropriate partition of unity to patch these local constructions together so that the metric H_0 is harmonic on some open subsets of the divisor at infinity when restricted to a certain small punctured disk transversal to the divisor with the puncture contained in the divisor. This can be done as follows.

For simplicity, we assume in the following that $\dim_{\mathbb{C}} \overline{X} = 3$, $D = D_1 + D_2$ and $D_1 \cap D_2 \neq \varnothing$; so $\dim_{\mathbb{C}} D_1 \cap D_2 = 1$. The discussion for the general case is completely similar.
First, we patch together local constructions near the intersection $D_1 \cap D_2$. Take two enough small tube neighborhoods $U^{12}_{\epsilon'} \subset U^{12}_{\epsilon}$ of $D_1 \cap D_2$ with a holomorphic projection $\pi^{12} : U^{12}_{\epsilon} \to D_1 \cap D_2$. We take U^{12}_{ϵ} enough small so that for any point p of $D_1 \cap D_2$ there exists a neighborhood U in $D_1 \cap D_2$, such that $(\pi^{12})^{-1}(U)$ can be covered by a local complex coordinate (z_1, z_2, z_3) at p of $\overline{X}$ with $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$. Obviously, the fibres of π^{12} are the product of two disks.
Take a *finite* open covering $\{U_\alpha\}$ of $D_1 \cap D_2$ satisfying that each U_α has the property of the above U and that for each U_α one can take a smaller open subset $U'_\alpha \subset U_\alpha$ so that $U'_\alpha \cap U'_\beta = \varnothing$. Corresponding to $\{U'_\alpha \subset U_\alpha\}$, we can choose a partition of unity $\{\phi_\alpha\}$ with $\phi_\alpha|_{U'_\alpha} \equiv 1$; consequently, $\{\phi_\alpha \circ \pi^{12}\}$ is a partition of unity on U^{12}_ϵ with $\phi_\alpha \circ \pi^{12}|_{(\pi^{12})^{-1}(U'_\alpha)} \equiv 1$. Using this partition of unity, we can patch smoothly all local constructions together along $D_1 \cap D_2$ to get a metric of L_ρ on U^{12}_ϵ. We remark that from the above patching process, we can see that the obtained metric is

harmonic on $(\pi^{12})^{-1}(U'_\alpha)$ when restricted to punctured disks transversal to D_1 or D_2.

Using the same way, we can patch all local constructions together along $D_1 - U^{12}_{\epsilon'}$ and $D_2 - U^{12}_{\epsilon'}$. For convenience of notations in the next section, we here give some details. Take two enough small tube neighborhoods $V^i_{\epsilon'} \subset V^i_\epsilon$ of $D_i - U^{12}_{\epsilon'}$, $i = 1, 2$; for our purposes, w.l.o.g we may assume that $V^i_\epsilon \cap U^{12}_{\epsilon'}$ is empty. Then there is a holomorphic projection $\pi^i : V^i_\epsilon \to D_i - U^{12}_{\epsilon'}$. We also assume that V^i_ϵ is small enough so that for any interior point p of $D_i - U^{12}_{\epsilon'}$ there exists a neighborhood V in $D_i - U^{12}_{\epsilon'}$, $(\pi^1)^{-1}(V)$ can be covered by a local complex coordinate (z_1, z_2, z_3) at p of $\overline{X}$ with $D_i = \{z_i = 0\}$. Take a *finite* open covering $\{V_\alpha\}$ of $D_i - U^{12}_{\epsilon'}$ satisfying that each V_α has the property of the above V and that for each V_α one can take a smaller open subset $V'_\alpha \subset V_\alpha$ so that $V'_\alpha \cap V'_\beta = \varnothing$. Corresponding to $\{V'_\alpha \subset V_\alpha\}$, we can choose a partition of unity $\{\psi_\alpha\}$ with $\psi_\alpha|_{V'_\alpha} \equiv 1$; consequently, $\{\psi_\alpha \circ \pi^i\}$ is a partition of unity on V^i_ϵ with $\psi_\alpha \circ \pi^i|_{(\pi^i)^{-1}(V'_\alpha)} \equiv 1$. Using this partition of unity, we can patch smoothly all local constructions together along $D_i - U^{12}_{\epsilon'}$ to get a metric of L_ρ on V^i_ϵ.

Finally, we can patch smoothly the above three metrics together so that the metrics on $U^{12}_{\epsilon'}, V^1_\epsilon - U^{12}_\epsilon, V^2_\epsilon - U^{12}_\epsilon$ are preserved. Thus, near the divisor, we get a smooth metric which is harmonic on certain open subsets, say $(\pi^{12})^{-1}(U'_\alpha) \cap U^{12}_{\epsilon'}$ and $(\pi^i)^{-1}(V'_\alpha)$, when restricted to a punctured disk transversal to the divisor. We can then extend the metric smoothly to all of X to get a metric of L_ρ, still denoted by H_0.

3.2 The norm estimate under H_0 of a flat section of L_ρ

Here, we want to observe what the asymptotic behavior of the norm of a flat section of L_ρ under the metric H_0 is near the divisor. To this end, we continue to restrict ourselves to $(\Delta^*)^k \times \Delta^{n-k}$: Since the θ_i-directions have nothing to do with asymptotic behavior of the norm (actually, we can consider $\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)$ as an isometry on $\mathcal{P}_r$), we only need to observe $\exp(\sum_{i=1}^k (\frac{1}{2}\log|\log r_i|) Y_i)$. By the previous construction for semisimple elements Y_i, they can be diagonalized simultaneously under a suitable basis of L_ρ; assuming this and expanding $\exp(\sum_{i=1}^k (\frac{1}{2}\log|\log r_i|) Y_i)$, one has the following form

$$\begin{pmatrix} \prod_{i=1}^k |\log r_i|^{\frac{a_i^1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & \prod_{i=1}^k |\log r_i|^{\frac{a_i^2}{2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \prod_{i=1}^k |\log r_i|^{\frac{a_i^{r-1}}{2}} & 0 \\ 0 & 0 & \cdots & 0 & \prod_{i=1}^k |\log r_i|^{\frac{a_i^r}{2}} \end{pmatrix}, \tag{2}$$

where

$$Y_i = \begin{pmatrix} a_i^1 & 0 & \cdots & 0 & 0 \\ 0 & a_i^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_i^{r-1} & 0 \\ 0 & 0 & \cdots & 0 & a_i^r \end{pmatrix}. \tag{3}$$

This gives explicitly the asymptotic behavior of the norm of a flat section under the metric H_0.

Remark: In Cattani-Kaplan-Schmid's theory [2], the norm estimates of flat sections under the Hodge metric depend on the order of $N_1, \cdots, N_n$, and consequently one needs to take the corresponding sectors of $(\Delta^*)^k$. However, from our construction, we see that after fixing a maximal abelian subspace in the noncompact part of a Cartan decomposition, although the choice of parabolic subalgebras is not unique, the semisimple elements are unique.

3.3 The behavior of the differential of H_0

Finally, we also need to understand the asymptotic behavior near the divisor of the differential of H_0. This is very different from the estimates of the norm, where one does not need to consider the θ_i-directions. The understanding of this asymptotic behavior is important for the study of cohomologies of harmonic bundles.
First, we do an explicit computation for the differential of H_0. We still restrict to $(\Delta^*)^k \times \Delta^{n-k}$,

$$H_0 = \exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i) \circ \exp(\sum_{i=1}^k (\frac{1}{2}\log|\log r_i|)Y_i).$$

We can consider $\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)$ as an isometry on $\mathcal{P}_r$; so for the $\mathbf{r} := (r_1, \cdots, r_n)$-direction, we have

$$d_{\mathbf{r}} H_0 = \big(\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)\big)_* \big(\sum_{i=1}^k \frac{dr_i}{2r_i \log r_i} Y_i\big).$$

For the $\Theta := (\theta_1, \cdots, \theta_n)$-direction, if we consider the matter at the identity I_r, the differential $d_\Theta H_0$ should be read as

$$\frac{1}{2\pi}\sum_{i=1}^k N_i d\theta_i;$$

so,

$$d_\Theta H_0 = \big(\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)\exp(\sum_{i=1}^k (\frac{1}{4}\log|\log r_i|)Y_i)\big)_* \frac{1}{2\pi}\sum_{i=1}^k N_i d\theta_i,$$

here we consider H_0 as $\big(\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)\exp(\sum_{i=1}^k(\frac{1}{4}\log|\log r_i|)Y_i)\big)\circ I_r$, denoted it by $\sqrt{H_0}\circ I_r$. Translating everything into the complex coordinates $(z_1,\cdots,z_n)$, we have

$$\begin{aligned} d_{\mathbf{r}}H_0 &= \frac{1}{4}\big(\exp(\frac{1}{2\pi}\sum_{i=1}^k \theta_i N_i)\big)_* \sum_{i=1}^k \big(\frac{dz_i}{z_i}+\frac{d\overline{z}_i}{\overline{z}_i}\big)\frac{Y_i}{\log|z_i|}, \\ d_\Theta H_0 &= \frac{1}{4\pi\sqrt{-1}}\big(\sqrt{H_0}\big)_* \sum_{i=1}^k N_i\big(\frac{dz_i}{z_i}+\frac{d\overline{z}_i}{\overline{z}_i}\big). \end{aligned}$$

As showed before, under the product metric ds_P^2 on $(\Delta^*)^k\times\Delta^{n-k}$ and the invariant metric on $\mathcal{P}_r$, $\|dH_0\| = \text{const.}$. So, we have

$$\| d_{\mathbf{r}}H_0 \|^2, \| d_\Theta H_0 \|^2 \le C,$$

for some positive constant C; in particular, $\|N_i\frac{dz_i}{z_i}\|^2 \le C$.
Consequently, since $|\frac{dz_i}{z_i}|^2 = |\log|z_i||^2$ under the positive metric ds_P^2, N_i, as an endomorphism of the bundle L_ρ, has the following point-wise norm estimate[3]

$$\|N_i\|^2 \le C|\log|z_i||^{-2},$$

which is very important in the study of cohomologies of a harmonic bundle, where we always consider $\sum_{i=1}^k N_i\frac{dz_i}{z_i}$ as a 1-form homomorphism on L_ρ.

Remark: We remark that such a norm estimate is not necessarily precise; for example, in the case of $N_i = N_j$, one even has $\|N_i\|^2 \le C|\log|z_i||^{-2}|\log|z_j||^{-2}$.

We now express the above estimates in terms of weight filtrations as follows. This is easy if we note the relation $[N_i, Y_i] = 2N_i$; corresponding to the diagonalisation of Y_i, we can consider the weight filtration of N_i: $\{W_k\}$ satisfying $N_iW_k\subset W_{k-2}$. If v is a flat section lying in W_k, satisfying

$$\| v \|_{H_0}^2 \sim \prod_{j=1}^k |\log r_j|^{a_j}$$

and $N_iv\neq 0$, one then has

$$\| N_iv \|_{H_0}^2 \le C|\log r_i|^{a_i-2}\prod_{j\neq i}|\log r_j|^{a_j}.$$

Remark: By the above discussion, we know that in our study the weight filtration theory is getting not very important. Actually, in the study of cohomologies, the most important aspect is to know what the general form of the norm of flat sections is and how the action of N_i on a flat section changes its norm; all these are clear from the previous discussion.

[3] The norm of a tangent vector of $\mathcal{P}_r$ is equivalent to its norm when it is considered as an endomorphism of L_ρ, cf. [19].

By the above computation, we now have

$$\partial H_0 = \frac{1}{4}\big(\exp(\frac{1}{2\pi}\sum_{i=1}^{k}\theta_i N_i)\big)_*\sum_{i=1}^{k}\frac{dz_i}{z_i}\frac{Y_i}{\log|z_i|} + \frac{1}{4\pi\sqrt{-1}}\big(\sqrt{H_0}\big)_*\sum_{i=1}^{k}N_i\frac{dz_i}{z_i}.$$

Proposition 2 *Using $\frac{dz_i}{z_i}$ as basis, up to some isometries, ∂H_0 is asymptotic to $\sum_{i=1}^{k} N_i \frac{dz_i}{z_i}$ as $|z_i|$ goes to zero. Also, from the previous discussion, we know that when considering ∂H_0 as a 1-form homomorphism on L_ρ, it is point-wise bounded under the complete Kähler metric on X, and hence (in applications) L^2-bounded as an operator between certain L^2-spaces.*

4 Harmonic metrics obtained by deforming the initial metrics H_0

In order to deform the initial metric H_0 into a harmonic one with the same asymptotic behavior as H_0, from now on, we assume that the representation ρ is semisimple. Namely, for any boundary component Σ of $\mathcal{P}_r$, there exists an element $\gamma \in \pi_1(X)$ satisfying $\Sigma \cap \rho(\gamma)(\Sigma) = \emptyset$; in other words, the image of ρ does not fix any boundary component or is not contained in any proper parabolic subgroup [6, 4, 19]. For similar definitions cf. also [8, 14].

Our strategy is first to deform H_0 to a harmonic metric, and then to prove that the harmonic metric has the same asymptotic behavior as H_0. The first step is standard with the assumption that the representation ρ is semisimple; we here give a sketch together with some necessary properties and estimates of the obtained harmonic metric.

4.1. In the following, we consider $X = \overline{X} - D$ as a complete noncompact manifold with the Kähler metric constructed in Proposition 1; sometimes, we also need the Kähler metric to be locally of the product form ds_P^2; we can do this in those cases that do not involve its derivatives. Take a sequence of compact manifolds $\{X_i\}$ (with smooth boundary) such that $X_i \subset X_{i+1}$ and $\cup_{i=1}^{\infty} X_i = X$. Then, according to Hamilton [7] (the theory of Hamilton readily applies to the equivariant setting), we can find a harmonic metric H_i of L_ρ on X_i with $H_i|_{\partial X_i} = H_0|_{\partial X_i}$ and $E(H_i; X_i) \le E(H_0; X_i) \le E(H_0)$. Next, we need to prove that there exists a subsequence of $\{H_i\}$ which converges uniformly on any compact subset of X. Fix a compact subset X_0 of X and a point $p \in X_0$. Then the uniform boundedness of the energies of $\{H_i\}$ imply that the energy densities $e(H_i)$ are also uniformly bounded in i on X_0. Using the semisimplicity of ρ and the uniform boundedness of the energy densities

$e(H_i)$, one can show that $H_i(p) \in \mathcal{P}_r$ are uniformly bounded in i (here, considering H_i as equivariant map into $\mathcal{P}_r$, cf. [19]); so, $\{H_i(X_0) \subset \mathcal{P}_r\}$ are also uniformly bounded in i. Taking a diagonal sequence, we can find a subsequence of $\{H_i\}$, still denoted by $\{H_i\}$, which converges uniformly on any compact subset of X to a harmonic metric H of L_ρ on X. Furthermore, $E(H) \leq E(H_0) < \infty$.

Due to the finiteness of energy of the harmonic metric H, Siu's Bochner technique for harmonic maps [9] (which also applies to the equivariant setting) implies

Fact 1. *The harmonic metric H, as an equivariant map into $\mathcal{P}_r$, is pluriharmonic; namely, when restricted to any complex curve, especially a punctured disk transversal to the divisor D, H is harmonic.*

Next, we want to investigate the derivatives of H near the divisor in the directions "parallel" to smooth divisors. As in §3.1.2, for simplicity of discussion, we here also assume that $\dim_{\mathbb{C}} \overline{X} = 3$, $D = D_1 + D_2$ and $D_1 \cap D_2 \neq \varnothing$ and use the notations there; the general case follows by an obvious induction argument. We first observe the situation in $U_\epsilon^{12} - D_1 \cup D_2$. The (complex) direction parallel to the smooth divisors D_1, D_2 is the direction that $D_1 \cap D_2$ represents. Locally, for $p \in D_1 \cap D_2$, there is a small neighborhood U in $D_1 \cap D_2$ so that there is a local complex coordinate (z_1, z_2, z_3) at p of $\overline{X}$ covering $(\pi^{12})^{-1}(U)$ with $D_1 = \{z_1 = 0\}, D_2 = \{z_2 = 0\}$, and the z_3-direction being the one that $D_1 \cap D_2$ represents. Since $D_1 \cap D_2$ is compact, and under the local coordinate (z_1, z_2, z_3) the Kähler metric of the product form is Euclidean in the z_3-direction, so the finiteness of energy of H implies (cf. [10])

Fact 2. *$|\frac{\partial H}{\partial l^{12}}|$ is bounded on $U_\epsilon^{12} - D_1 \cup D_2$, where l^{12} is any unit direction that $D_1 \cap D_2$ contains.*

We remark that $D_i - U_{\epsilon'}^{12}$ is also compact. So, the argument above also applies to the situation of $V_\epsilon^i - D_i$, the tube neighborhood of $D_i - U_{\epsilon'}^{12}, i = 1, 2$. Thus, one also has the following

Fact 2'. *$|\frac{\partial H}{\partial l^i}|$ is bounded on $V_\epsilon^i - D_i$, where l^i is any unit direction that D_i contains.*

4.2. Now, we can show that H and H_0 have the same asymptotic behavior near the divisor. For simplicity of discussion, we continue to restrict to the case that $\dim_{\mathbb{C}} \overline{X} = 3$, $D = D_1 + D_2$ and $D_1 \cap D_2 \neq \varnothing$ and to use the notations in §3.1.2.

We first consider the situation near $D_1 \cap D_2$, i.e. in $U_{\epsilon'}^{12} - D_1 \cup D_2$. Fix an open set $(p \in U'_\beta \subset) U_\beta$ in the finite open covering $\{U_\alpha\}$ of $D_1 \cap D_2$. By the choice of the open covering, we have a local coordinate (z_1, z_2, z_3) at p of $\overline{X}$ covering $(\pi^{12})^{-1}(U'_\beta) \cap U_{\epsilon'}^{12}$ with $p = (0, 0, 0)$, $D_1 = \{z_1 = 0\}$, and $D_2 = \{z_2 = 0\}$. By the construction of H_0 in §3.1.2, H_0 is harmonic and of finite energy when restricted to a punctured disk (in $(\pi^{12})^{-1}(U'_\beta) \cap U_{\epsilon'}^{12} - D_1 \cup D_2$) transversal to D_1 or D_2; on the other hand, H has the same property by its pluriharmonicity and energy finiteness. Now, we claim

Fact 3. *$dist_{\mathcal{P}_r}(H, H_0)(z_1', z_2', 0)$ [4] is uniformly bounded for $(z_1', z_2', 0) \in (\pi^{12})^{-1}(U_\beta') \cap U_{\epsilon'}^{12} - D_1 \cup D_2$.*

Proof of Fact 3. First, for a fixed z_1', define

$$S_\beta^2(z_1') = \{(z_1', z, 0) \in (\pi^{12})^{-1}(U_\beta') \cap U_{\epsilon'}^{12} - D_1 \cup D_2\};$$

w.l.o.g., we can assume it is a punctured disk which is transversal to D_2 at $(z_1', 0, 0)$. As pointed out above, H_0 and H are harmonic when restricted to $S_\beta^2(z_1')$. We now prove

$$\sup_{x \in S_\beta^2(z_1')} \mathrm{dist}(H, H_0)(x) \leq \sup_{x \in \partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')} \mathrm{dist}(H, H_0)(x).$$

In order to prove the above inequality, we consider the sequence of harmonic metrics $\tilde{H}_i$ on $S_\beta^2(z_1') \cap X_i$ with

$$\tilde{H}_i|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')} = H_i|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')}, \ \tilde{H}_i|_{\partial X_i \cap S_\beta^2(z_1')} = H_0|_{\partial X_i \cap S_\beta^2(z_1')}.$$

We remark that $H_i|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')}$ converges uniformly to $H|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')}$. It is easy to prove that the sequence $\{\tilde{H}_i\}$ (if necessary, go to a subsequence) converges uniformly on any compact subset of $S_\beta^2(z_1')$ to a harmonic metric $\tilde{H}$ on $S_\beta^2(z_1')$ and that, by means of the subharmonicity of $\mathrm{dist}(\tilde{H}_i, H_0)$ [5] on $S_\beta^2(z_1') \cap X_i$ and the maximum principle,

$$\sup_{x \in S_\beta^2(z_1')} \mathrm{dist}(\tilde{H}, H_0)(x) \leq \sup_{x \in \partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')} \mathrm{dist}(\tilde{H}, H_0)(x);$$

and hence $\tilde{H}$ also has finite energy. Thus, on $S_\beta^2(z_1')$, we have two harmonic metrics $H, \tilde{H}$ with finite energy and $\tilde{H}|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')} = H|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')}$. We now want to show $\tilde{H} \equiv H$ on $S_\beta^2(z_1')$, and hence the required inequality.

We observe the distance function $\mathrm{dist}(\tilde{H}, H)$ on $S_\beta^2(z_1')$, denoted by w. It is clear that w is subharmonic and $w|_{\partial U_{\epsilon'}^{12} \cap S_\beta^2(z_1')} = 0$; on the other hand, by the finiteness of energy of $\tilde{H}, H$, a simple computation shows that w is of finite energy. Thus, the problem is reduced to show (for the proof, cf. Appendix)

Fact 4. *Let Δ^* be a puncture disk. Assume that w is a non-negative subharmonic function vanishing on the exterior boundary and of finite energy, then $w \equiv 0$.*

Assume that $\sup_{x \in \partial S_\beta^2(z_1')} \mathrm{dist}(H, H_0)(x)$ is attained at the point $(z_1', z_2^0, 0)$. Define

$$S_\beta^1(z_2^0) = \{(z, z_2^0, 0) \in (\pi^{12})^{-1}(U_\beta') \cap U_{\epsilon'}^{12} - D_1 \cup D_2\};$$

[4] Considering H, H_0 as equivariant maps from the universal covering into $\mathcal{P}_r$, the distance function between H, H_0 is still equivariant, so can be considered as a function on the base manifold.

[5] Since $\tilde{H}_i, H_0$ are harmonic and $\mathcal{P}_r$ has non-positive curvature, a standard computation shows that $\mathrm{dist}(\tilde{H}_i, H_0)$ is subharmonic; cf. e.g. [17].

also assume that it is a punctured disk which is transversal to D_1 at $(0, z_2^0, 0)$. The same argument as above shows

$$\sup_{x\in S^1_\beta(z_2^0)} \mathrm{dist}(H, H_0)(x) \le \sup_{x\in \partial U^{12}_{\epsilon'}\cap S^1_\beta(z_2^0)} \mathrm{dist}(H, H_0)(x).$$

Assuming that $\sup_{x\in \partial U^{12}_{\epsilon'}\cap S^1_\beta(z_2^0)} \mathrm{dist}(H, H_0)(x)$ is attained at the point $(z_1^0, z_2^0, 0)$, then we have

$$\mathrm{dist}(H, H_0)(z_1', z_2', 0) \le \mathrm{dist}(H, H_0)(z_1^0, z_2^0, 0)$$

for $(z_1', z_2', 0) \in (\pi^{12})^{-1}(U_\beta') \cap U^{12}_{\epsilon'} - D_1 \cup D_2$. Clearly, we can choose a fixed compact subset of X, which is independent of $(z_1', z_2', 0)$ and always contains the corresponding $(z_1^0, z_2^0, 0)$. This proves the Fact 3.
The Fact 3 together with the Fact 2 implies that H and H_0 have the same asymptotic behavior at infinity on $U^{12}_{\epsilon'} - D_1 \cup D_2$.

Using a similar argument as above, we can also show that H and H_0 have the same asymptotic behavior at infinity on $V^i_{\epsilon'} - D_i$, i=1, 2. Thus, we have showed that H and H_0 have the same asymptotic behavior at infinity on X.

Now, we turn to the uniqueness problem. This is easy. Assume H_1, H_2 are two harmonic metrics with the same asymptotic behavior as H_0. Then, $\mathrm{dist}_{\mathcal{P}_r}(H_1, H_2)$ is a bounded subharmonic function on X. A well-known fact which says that a bounded positive subharmonic function on a complete Riemannian manifold with finite volume has to be constant implies

$$\mathrm{dist}_{\mathcal{P}_r}(H_1, H_2) = \text{constant}.$$

This shows that H_1 and H_2 are the same up to an isometry of $\mathcal{P}_r$.

Appendix: Since the punctured disk is conformal to a half cylinder, so the Fact 4 is equivalent to the following
Fact 4'. *Let C be a half cylinder with Euclidean metric. Assume that w is a non-negative subharmonic function vanishing on ∂C and of finite energy, then $w \equiv 0$.*

Take a global coordinate (θ, y) on C with $0 \le \theta \le \pi$ and $y \ge 0$; the Euclidean metric is $d\theta^2 + dy^2$. Take a sequence of compact subsets $C_n = \{y \le n\}, n = 1, 2, \cdots$, and consider a sequence of harmonic functions u_n with $u_n|_{\partial C} = 0$ and $u_n|_{y=n} = w_n|_{y=n}$. The maximum principle implies $u \ge w$ on C_n and $E(u_n; C_n) \le E(w; C_n) \le E(w)$. The standard elliptic estimate implies u_n converges to a harmonic function u on C uniformly on any compact subset, which satisfies $u \ge w$, $u|_{\partial C} = 0$, and $E(u) \le E(w)$. So, the proof of the Fact 4' can be reduced to the following fact which is known to analysts but whose proof we include here for completeness.

Lemma. *Let C be a half cylinder with Euclidean metric. Assume that u is a non-negative harmonic function with $u|_{\partial C} = 0$ and finite energy,* *then* $u \equiv 0$.

Proof of Lemma. Take a global coordinate (θ, y) on C with $0 \leq \theta \leq \pi$ and $y \geq 0$; the Euclidean metric is $d\theta^2 + dy^2$. Take a smooth cut-off function $\psi_{r,\rho}$ with $\psi_{r,\rho}(\theta, y) = 1, y \leq r$, $\psi_{r,\rho}(\theta, y) = 0, y \geq r+\rho, \rho > 0$ and $|\nabla \psi_{r,\rho}| \leq \frac{1}{\rho}$.

Compute

$$\begin{aligned} 0 &= -\int_C \Delta u u(\psi_{r,\rho})^2 \\ &= -\int_C \nabla(\nabla u u(\psi_{r,\rho})^2) + \int_C |\nabla u|^2(\psi_{r,\rho})^2 + 2\int_C \nabla u \psi_{r,\rho} \nabla \psi_{r,\rho} u \\ &\geq \int_C |\nabla u|^2(\psi_{r,\rho})^2 - (\frac{1}{2}\int_C |\nabla u|^2(\psi_{r,\rho})^2 + 2\int_C |\nabla \psi_{r,\rho}|^2 u^2). \end{aligned}$$

So, we have

$$\int_C |\nabla u|^2(\psi_{r,\rho})^2 \leq 4\int_C |\nabla \psi_{r,\rho}|^2 u^2.$$

It is clear that $\int_C |\nabla u|^2(\psi_{r,\rho})^2 \to E(u)$, as r goes to infinity. So, if we can choose appropriate $\psi_{r,\rho}$ so that $\int_C |\nabla \psi_{r,\rho}|^2 u^2$ can be arbitrarily small, then $E(u) = 0$ and hence $u = 0$.

Take a sufficiently small ϵ. Since $E(u) < \infty$, $\exists r_0$ so that $\int_{y \geq r} |\nabla u|^2 \leq \epsilon$ for $r \geq r_0$. On the other hand,

$$\int_C |\nabla \psi_{r,\rho}|^2 u^2 \leq \frac{1}{\rho^2}\int_{r \leq y \leq r+\rho} u^2.$$

We now estimate $\int_{r \leq y \leq r+\rho} u^2$. Since u is harmonic, $|\nabla u|^2$ is subharmonic. So by the average value inequality, we have

$$|\nabla u|^2(x) \leq c\int_{r+i-1 \leq y \leq r+i+2} |\nabla u|^2, \ r+i \leq y(x) \leq r+i+1,$$

for $i = 0, 1, \cdots, n-1$, where c is a positive constant independent of r, i.

First, estimate $|u(\theta, r) - u(\theta, r+n)|$ as follows.

$$\begin{aligned} |u(\theta, r) - u(\theta, r+n)|^2 &= |\sum_{i=0}^{n-1}(u(\theta, r+i) - u(\theta, r+i+1))|^2 \\ &\leq \left(\sqrt{c}\sum_{i=0}^{n-1}\sqrt{\int_{r+i-1 \leq y \leq r+i+2} |\nabla u|^2}\right)^2 \\ &\leq 3cn\int_{r-1 \leq y \leq r+n+1} |\nabla u|^2 \\ &\leq 3cn\int_{y \geq r_0} |\nabla u|^2 \leq 3cn\epsilon. \end{aligned}$$

In general, we have

$$|u(\theta, r) - u(\theta, r+s)|^2 \leq 3c\epsilon s.$$

On the other hand, we have $|u(\theta, r+s)|^2 \leq 2|u(\theta, r+s) - u(\theta, r)|^2 + 2|u(\theta, r)|^2$.

Collecting the above, we have

$$\begin{aligned} \frac{1}{\rho^2}\int_{r\leq y\leq r+\rho} u^2 &\leq \frac{1}{\rho^2}\int_{r\leq y\leq r+\rho} (2|u(\theta,y)-u(\theta,r)|^2 + 2|u(\theta,r)|^2) \\ &\leq \frac{1}{\rho^2}\int_{r\leq y\leq r+\rho} (6c\epsilon(y-r) + 2|u(\theta,r)|^2). \end{aligned}$$

Take $r = r_0 + 1$ and ρ sufficiently large, we have

$$\frac{1}{\rho^2}\int_{r\leq y\leq r+\rho} u^2 \leq c'\epsilon,$$

for a certain positive constant. Thus, when r, ρ are sufficiently large, $\int_C |\nabla\psi_{r,\rho}|^2 u^2$ becomes arbitrarily small. The proof of the lemma is finished.

References

[1] A. Borel and Lizhen Ji, *Compactifications of Symmetric and Locally Symmetric Spaces.* Mathematics: Theory and Applications, Birkhäuser, 2006.

[2] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures.* Annales of Mathematics, **123** (1986), 457–535.

[3] E. Cattani, A. Kaplan, and W. Schmid, *Cohomologies of a variation of polarized Hodge structures.* Inventiones Math., **87** (1987), 217–252.

[4] K. Corlette, *Flat G-bundles with canonical metrics.* J. Differential Geometry., **28** (1988), 361-382.

[5] M. Cornalba and P. Griffiths, *Analytic cycles and vector bundles on noncompact algebraic varieties.* Inventiones Math., **28** (1975), 1-106.

[6] S. K. Donaldson, *Twisted harmonic maps.* Proc. London Mathematical Society, **55** (1987), 127-131.

[7] R. Hamilton, *Harmonic maps of manifolds with boundary.* Lecture Notes in Mathematics, Vol. **471**. Springer-Verlag, Berlin-New York, 1975.

[8] J. Jost and S.T. Yau, *harmonic maps and group representations*, in: H.B. Lawson and K. Tenenblat (eds.), *Differential Geometry*, 241-259, Longman Scientific, 1991.

[9] J. Jost and K. Zuo, *Harmonic maps and $SL(r,\mathbb{C})$-representations of fundamental groups of quasiprojective varieties.* Journal of Algebraic Geometry, **5** (1996), 77-106.

[10] J. Jost and K. Zuo, *Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective manifolds.* J. Differential Geometry, **47**, (1997) 469-503.

[11] J. Jost, Y.-H. Yang, and K. Zuo, *The cohomologies of a variation of polarized Hodge structures over quasi-compact Kähler manifolds.* Journal of Algebraic geometry, **16** (2007), 301-344.

[12] J. Jost, Y.-H. Yang, and K. Zuo, *Cohomologies of unipotent harmonic bundles on noncompact curves.* Crelle's Journal, **609** (2007), 137-159.

[13] B. Kostant, *The principal three-dimenasional subgroup and the Betti numbers of a complex simple Lie group.* American Journal of Mathematics, **81** (1959), 973-1032.

[14] F. Labourie, *Existence d'applications harmoniques tordues á valeurs dans les variétés á courbure négative*, Proc. A. M. S., **111** (1991), 877-882.

[15] T. Mochizuki, *Asymptotic behavior of tame nilpotent harmonic bundles with trivial parabolic structure.* Journal of Differential Geometry, **62** (2002), 351-559.

[16] W. Schmid, *Variation of Hodge structure: the sigularities of the period mapping.* Invent. Math., **22** (1973), 211-319.

[17] R. Schoen & S.-T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature.* Topology, **18** (1979), 361-b380..

[18] C. Simpson, *Harmonic bundles on noncompact curves.* J. Amer. Math. Soc., **3** (1990), 513-570.

[19] Yi-Hu Yang, *Meromorphic differentials with twisted coeffcients on comapct Riemannian surface.* Calculus of Variations and PDE, **32** (2008), 411-427.