Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

The regularity of harmonic maps into spheres and applications to Bernstein problems

by

Jürgen Jost, Yuanlong Xin, and Ling Yang

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THE REGULARITY OF HARMONIC MAPS INTO SPHERES AND APPLICATIONS TO BERNSTEIN PROBLEMS

J. JOST, Y. L. XIN AND LING YANG

Abstract. We show the regularity of, and derive a-priori estimates for (weakly) harmonic maps from a Riemannian manifold into a Euclidean sphere under the assumption that the image avoids some neighborhood of a half-equator. The proofs combine constructions of strictly convex functions and the regularity theory of quasi-linear elliptic systems.

We apply these results to the spherical and Euclidean Bernstein problems for minimal hypersurfaces, obtaining new conditions under which compact minimal hypersurfaces in spheres or complete minimal hypersurfaces in Euclidean spaces are trivial.

1. Introduction

Harmonic maps into spheres need not be regular. The basic example is due to [21]: The map

$$\frac{x}{|x|} : \mathbb{R}^n \to S^{n-1} \tag{1.1}$$

has a singularity at the origin 0, while having finite energy on finite balls for $n \geq 3$ (it thus is a so-called weakly harmonic map). This example can be modified by embedding S^{n-1} as an equator into S^n, and the composed map then is a singular harmonic map from $\mathbb{R}^n$ to the sphere S^n with image contained in an equator. This equator is the boundary of a closed hemisphere. In contrast to this phenomenon, Hildebrandt-Kaul-Widman [21] proved the regularity of weakly harmonic maps whose image is contained in some compact subset of an open hemisphere. Hildebrandt-Jost-Widman [22] then derived a-priori estimates for harmonic maps in that situation. The example then shows that these results are optimal in the sense that the open hemisphere cannot be replaced by a closed one. It was then the general opinion that for general harmonic maps (not necessarily energy minimizing, in which case the method of [36] yields additional results, see for instance [41]), this is the best that one can do.

Here, we show that one can do substantially better. In fact, we shall show that weakly harmonic maps into a sphere are regular, and satisfy a-priori estimates, under the condition that their image be contained in a compact subset of the complement

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of half of an equator, that is, in the complement of half of a totally geodesic $(n-1)$-dimensional subsphere. Of course, our condition still rules out the counterexample of [21].

Our condition is presumably optimal, for the following reason. The basic principle underlying the regularity theory for harmonic maps is the fact that the composition of a harmonic map with a convex function on the target yields a subharmonic function, and in the case of weakly harmonic maps, we obtain a weakly subharmonic function. One then exploits the maximum principle for such a (weakly) subharmonic function, or in more refined schemes, Moser's Harnack inequality to derive estimates for the original (weakly) harmonic map. This obviously depends on the careful utilization of the geometric properties of the convex function. In fact, for the full regularity scheme, it is not sufficient to have a single convex function, but we rather need a family of such convex functions. More precisely, for each point in the target, we need a strictly convex function that assumes its minimum at that particular point. (In the original scheme of [21, 22], the authors worked with squared distance functions from points in the target. Therefore, in an open hemisphere, one could only have such functions that were strictly convex only on some part of that hemisphere, depending on where their minimum was located. This necessitated an iteration scheme whose idea was to show that the image of a (weakly) harmonic map gets smaller in a controlled manner when one decreases its domain. Some simplification can be achieved by the construction of Kendall [29] of strictly convex functions on arbitrary compact subsets of an open hemisphere with a minimum at some prescribed point.)

Thus, an essential part of the scheme developed in the present paper consists in the construction of such strictly convex functions on arbitrary compact subsets of the complement of a half-equator. This is rather subtle. In fact, the functions we construct will depend on the compact set K in question, and none of them will be convex on the entire open complement $\mathbb{V}$ of the half-equator. (In fact, there is no $D \subset S^n$ that is a maximal domain of definition of a strictly convex function, see [2].) We just fine-tune them in such a way that intersections of their level sets with K are convex, while these level sets are allowed to be concave on $\mathbb{V} \setminus K$, so as to turn around the boundary of the half-equator.

The reason why our result is presumably optimal then is that as soon we enlarge the open set $\mathbb{V}$, it will contain a closed geodesic. Since strictly convex functions are strictly monotonic along geodesic arcs, a set containing a closed geodesic cannot carry a strictly convex function. Therefore, our construction will no longer work then. Since the presence of strictly convex functions is essentially necessary for harmonic map regularity, this then seems to prohibit any general regularity result, let alone an explicit estimate.

Still, even with those strictly convex functions, the regularity theory is difficult and subtle, and we need to utilize the most advanced tools available in the literature. In particular, we use the Green test function technique and image shrinking method employed in [21, 22] and the generalization of that scheme in [17], the estimates for Green functions of [19, 4] that depend on Moser's Harnack inequality [31], the telescoping trick of [15, 16], and the Harnack inequality method of [26] that converts convexity assumptions on the target into energy and oscillation controls for harmonic maps. A crucial point is that our estimates will not depend on the energy of the harmonic map to be estimated. Therefore, in particular, we do not need to make any energy minimizing assumption, and when we turn to global issues, we only need to assume the map to have locally finite energy, but not necessarily globally.

Following the scheme of [22], we can therefore apply our a-priori estimates to the Bernstein problem for minimal hypersurfaces in spheres and Euclidean spaces. The connection between such Bernstein problems and harmonic maps into spheres comes from the Ruh-Vilms theorem [33] that says that the Gauss map of a minimal hypersurface is a *harmonic* map (with values in a sphere). Showing that the original minimal hypersurface is trivial (a totally geodesic subsphere or a hyperplane, resp.) then is reduced to showing that the Gauss map is constant. In order to apply our results, we therefore have to show that the Gauss map is constant under the assumption that its image is contained in a compact subset of the complement of a half-equator. In the case of the sphere, where we are in interested in *compact* minimal hypersurfaces (the spherical Bernstein problem introduced by Chern [11]), this is easy: When we compose our harmonic Gauss map with a strictly convex function, we obtain a subharmonic function which on our compact hypersurface then has to be constant, implying that the Gauss map itself is constant, as desired. In fact, these results can also be obtained by the method of Solomon [41]. In the Euclidean case, where we are interested in complete minimal hypersurfaces, this is more difficult. There, we need very precise a-priori estimates that can be translated into a Liouville type theorem by a scaling argument. For that purpose, unfortunately, we need to impose some additional restrictions on the geometry of our minimal hypersurface. In particular, we need a condition on the volume growth of balls as a function of their radii, and we need a Poincaré inequality. Fortunately, however, these assumptions are known to be satisfied in a number of important and interesting cases, but the final answers do not yet seem to be known (we are grateful to Neshan Wickramasekera for some useful information in this regard, including some description of his still unpublished work).

Let us finally try to put our results into the perspective of the Bernstein problem (our survey will be rather incomplete, however; see [44] for a more detailed account). The original result of Bernstein that there is no other entire minimal graph in $\mathbb{R}^3$, i.e., a minimal graph defined on the entire plane $\mathbb{R}^2$, than an affine plane, has been extended by Simons [39] to such entire minimal graphs in $\mathbb{R}^n$ for $n \leq 7$ whereas Bombieri-de Giorgi-Giusti [5] constructed counterexamples in higher dimensions. In fact, the Bernstein problem has been one of the central driving forces of geometric measure theory which is concerned with area (or in higher dimensions, volume) minimizing currents (see [14]). More generally, such Bernstein type results have been obtained for complete stable minimal hypersurfaces, on the basis of curvature estimates by Heinz [20] (in dimension 2), Schoen-Simon-Yau [35], Simon [37, 38], Ecker-Huisken [13], and others. Minimal graphs are automatically stable, and so this approach applies to the original problem. Also, in contrast to the counterexample of [5], Moser [31] had shown that an entire minimal graph in any dimension has to be affinely linear, provided its slope is uniformly bounded. [22] then introduced the method of deriving Bernstein type theorems by showing that the Gauss map of a minimal submanifold of $\mathbb{R}^n$ is constant, as explained above. In particular, this method could generalize Moser's result. See also [40] for a combination of the Gauss map with geometric measure theory constructions. An important advantage of the method of [22] as compared to either the geometric measure theory approach or the strategy of curvature estimates is that it naturally extends to higher codimension, the only difference being that the Gauss map now takes its values in a Grassmann variety whose geometry is somewhat more complicated than the one of a sphere. Nevertheless, the Gauss map is still harmonic by [33], and when one can derive good enough a-priori estimates, one can again deduce a Liouville type theorem and Bernstein type results, see [22, 28]. Therefore, the strategy of the present paper can also be extended to higher codimension, and we shall develop the necessary convex geometry of Grassmannians in a sequel to this paper.

Contents

2. Construction of convex functions and the spherical Bernstein problem

2.1. **Convex supporting sets.** Let (M, g) be a smooth Riemannian manifold. A C^2-function F is said to be strictly **convex** on an open subset U of M if the Hessian form of F is positive definite at every point of U, i.e.,

$$\text{Hess } F(X, X) = \nabla_X \nabla_X F - (\nabla_X X)F > 0 \qquad \text{for every nonzero } X \in TU.$$

(Here ∇ denotes the Levi-Civita connection on M induced by g.) Equivalently, for any arc-length-parametrized geodesic γ lying in U, $F \circ \gamma$ is a strictly convex function in the usual sense.

The notion of a **convex supporting** set was proposed in [18]. A subset U of M is said to be convex supporting if and only if any compact subset of U has an open neighborhood in M on which there is defined a strictly convex function F. For the sequel, it may be helpful to point out that this does not require F be defined on all U, and in fact in the case we shall be interested in below, there will be no strictly convex function on U.

A maximal open convex supporting set is one which is not properly contained in any other open convex supporting set. Take the ordinary 2-sphere equipped with the canonical metric as an example. An open hemisphere is obviously a convex supporting set, but it is not a maximal one. To obtain a maximal open convex supporting domain on S^2, it suffices to remove half of a great circle γ joining north pole and south pole. We will prove this fact and its higher dimensional analogue and construct a maximal open convex supporting set on S^n $(n \geq 2)$ equipped with the canonical metric.

Lemma 2.1. *Let M be a Riemannian manifold, A be a compact domain of M and h be a non-negative C^2-function $|\nabla h| \neq 0$ everywhere on A. If there is a positive constant C such that*

$$\textit{Hess } h(Y, Y) \geq C|Y|^2 \tag{2.1}$$

for any $Y \in TA$ with $dh(Y) = 0$, then there exists a positive constant λ_0, only depending on C, $\sup_A |\textit{Hess } h|$ and $\inf_A |\nabla h|$, such that whenever $\lambda \geq \lambda_0$,

$$\textit{Hess } \big(\lambda^{-1} \exp(\lambda h)\big)(X, X) \geq \frac{C}{2}|X|^2 \tag{2.2}$$

for any $X \in TA$.

Proof. With $\nu := \frac{\nabla h}{|\nabla h|}$, for any unit tangent vector $X \in TA$, there exist $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and a unit tangent vector $Y \in TA$ such that $dh(Y) = 0$ and

$$X = \sin\alpha\ \nu + \cos\alpha\ Y.$$

With $c_0 := \sup_U |\text{Hess } h|$, then

$$\text{Hess } h(\nu, \nu) \geq -c_0 \qquad \text{and } \big|\text{Hess } h(\nu, Y)\big| \leq c_0.$$

Thereby

$$\begin{aligned}\text{Hess } h(X,X) &= \sin^2\alpha\text{Hess } h(\nu,\nu) + 2\sin\alpha\cos\alpha\text{Hess } h(\nu,Y) + \cos^2\alpha\text{Hess } h(Y,Y)\\ &\geq -c_0\sin^2\alpha - 2c_0|\sin\alpha\cos\alpha| + C\cos^2\alpha\\ &\geq -c_0\sin^2\alpha - \frac{C}{2}\cos^2\alpha - 2c_0^2C^{-1}\sin^2\alpha + C\cos^2\alpha\\ &= \frac{C}{2}\cos^2\alpha - (c_0 + 2c_0^2C^{-1})\sin^2\alpha.\end{aligned}$$

Denote $c_1 := \inf_U |\nabla h|$ and take

$$\lambda_0 = c_1^{-2}\Big(\frac{C}{2} + c_0 + 2c_0^2C^{-1}\Big),$$

then

$$\begin{aligned}\text{Hess}\big(\lambda^{-1}\exp(\lambda h)\big)(X,X) &= \exp(\lambda h)(\text{Hess } h + \lambda dh\otimes dh)(X,X)\\ &\geq \text{Hess } h(X,X) + \lambda\big(dh(X)\big)^2\\ &\geq \frac{C}{2}\cos^2\alpha - (c_0 + 2c_0^2C^{-1})\sin^2\alpha + \lambda|\nabla h|^2\sin^2\alpha\\ &\geq \frac{C}{2}\cos^2\alpha - (c_0 + 2c_0^2C^{-1})\sin^2\alpha + \lambda c_1^2\sin^2\alpha\\ &\geq \frac{C}{2}\end{aligned}$$

whenever $\lambda \geq \lambda_0$ and (2.2) follows. □

2.2. **Maximal convex supporting subsets of S^n.** We work on the standard Euclidean sphere $S^n \subset \mathbb{R}^{n+1}$ with its metric g.

We consider a closed half hemisphere of S^n of codimension 1, that is, half an equator,

$$\overline{S}_+^{n-1} := \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : x_1 = 0, x_2 \geq 0\} \tag{2.3}$$

and put

$$\mathbb{V} := S^n \backslash \overline{S}_+^{n-1}. \tag{2.4}$$

$\mathbb{V}$ is open and connected. $\overline{S}_+^{n-1}$ consists of the geodesics joining $x_0 = (0,1,0,\cdots,0) \in \overline{S}_+^{n-1}$ with the points in

$$S^{n-2} = \{(x_1, x_2, \cdots, x_{n+1}) \in S^n : x_1 = x_2 = 0\},$$

which is a totally geodesic submanifold of S^n with codimension 2. When $n = 2$, $S^{n-2} = S^0 = \{(0,0,1),(0,0,-1)\}$ and hence $\overline{S}_+^1$ is simply the shortest geodesic joining $(0,0,1)$(north pole) and $(0,0,-1)$(south pole) passing through $x_0 = (0,1,0)$.

We start with some simple and well-known computations and consider the projection π from S^n onto $\bar{\mathbb{D}}^2$ (2-dimensional closed unit disk):

$$\pi: \ S^n \to \bar{\mathbb{D}}^2 \qquad (x_1, \cdots, x_{n+1}) \mapsto (x_1, x_2).$$

Then $x \in \mathbb{V}$ if and only if $\pi(x)$ is contained in the domain obtained by removing the radius connecting (0,0) and (0,1) from the closed unit disk. Hence for any $x \in \mathbb{V}$, there exist a unique $v \in (0,1]$ and a unique $\varphi \in (0, 2\pi)$ such that

$$\pi(x) = (v \sin\varphi, v\cos\varphi). \tag{2.5}$$

v and φ can be considered as smooth functions on $\mathbb{V}$.

Put $y_0 := (1, 0, \cdots, 0)$ and let ρ be the distance function from y_0, then by spherical geometry, $x_1 = \cos\rho$. It is well-known that

$$\text{Hess } \rho = \cot\rho(g - d\rho\otimes d\rho); \tag{2.6}$$

hence

$$\begin{aligned} \text{Hess } x_1 = \text{Hess}\cos\rho &= -\sin\rho \text{ Hess } \rho - \cos\rho\ d\rho\otimes d\rho \\ &= -\cos\rho(g - d\rho\otimes d\rho) - \cos\rho\ d\rho\otimes d\rho \\ &= -\cos\rho\ g = -x_1\ g. \end{aligned} \tag{2.7}$$

Similarly

$$\text{Hess } x_2 = -x_2\ g. \tag{2.8}$$

By (2.5),

$$v^2 = x_1^2 + x_2^2. \tag{2.9}$$

and

$$\begin{aligned} dx_1 &= \sin\varphi dv + v\cos\varphi d\varphi = \sin\varphi dv + x_2 d\varphi, \\ dx_2 &= \cos\varphi dv - v\sin\varphi d\varphi = \cos\varphi dv - x_1 d\varphi. \end{aligned} \tag{2.10}$$

Combining with (2.9) and (2.10) yields

$$\begin{aligned} 2v\text{Hess } v + 2dv\otimes dv = \text{Hess } v^2 &= \text{Hess}(x_1^2 + x_2^2) \\ &= 2x_1\text{Hess } x_1 + 2x_2\text{Hess } x_2 + 2dx_1\otimes dx_1 + 2dx_2\otimes dx_2 \\ &= -2v^2\ g + 2dv\otimes dv + 2v^2 d\varphi\otimes d\varphi, \end{aligned}$$

which tells us

$$\text{Hess } v = -v\ g + v\ d\varphi\otimes d\varphi. \tag{2.11}$$

Furthermore, (2.7), (2.11) and (2.5) tell us

$$\begin{aligned} -x_1\ g &= \text{Hess } x_1 \\ &= v\cos\varphi\text{Hess}\varphi + \sin\varphi\text{Hess } v - x_1 d\varphi\otimes d\varphi + \cos\varphi(d\varphi\otimes dv + dv\otimes d\varphi) \\ &= v\cos\varphi\text{Hess}\varphi - x_1\ g + x_1 d\varphi\otimes d\varphi - x_1 d\varphi\otimes d\varphi + \cos\varphi(d\varphi\otimes dv + dv\otimes d\varphi) \\ &= v\cos\varphi\text{Hess}\varphi - x_1\ g + \cos\varphi(d\varphi\otimes dv + dv\otimes d\varphi), \end{aligned}$$

i.e.

$$v\cos\varphi\text{Hess } \varphi = -\cos\varphi(d\varphi\otimes dv + dv\otimes d\varphi).$$

Similarly, we have

$$v\sin\varphi\text{Hess } \varphi = -\sin\varphi(d\varphi\otimes dv + dv\otimes d\varphi).$$

We then have

$$\text{Hess }\varphi = -v^{-1}(d\varphi \otimes dv + dv \otimes d\varphi). \tag{2.12}$$

Let K be a compact subset of $\mathbb{V}$. Define a function

$$\phi = \varphi + f(v) \tag{2.13}$$

on K, where f is to be chosen. A straightforward calculation shows that

$$\begin{aligned} \text{Hess }\phi &= \text{Hess }\varphi + f'(v)\text{Hess }v + f''(v)dv \otimes dv \\ &= -vf'(v)g + vf'(v)d\varphi \otimes d\varphi + f''(v)dv \otimes dv \\ &\quad - v^{-1}(d\varphi \otimes dv + dv \otimes d\varphi). \end{aligned} \tag{2.14}$$

Obviously $d\phi = d\varphi + f'(v)dv \neq 0$, and for every X such that $d\phi(X) = 0$, we have $d\varphi(X) = -f'(v)dv(X)$ and furthermore

$$\begin{aligned} \text{Hess }\phi(X,X) &= -vf'(v)\langle X,X\rangle + vf'(v)d\varphi(X)^2 + f''(v)dv(X)^2 - 2v^{-1}d\varphi(X)dv(X) \\ &= -vf'(v)\langle X,X\rangle + \big(vf'(v)^3 + f''(v) + 2v^{-1}f'(v)\big)dv(X)^2. \end{aligned} \tag{2.15}$$

By the compactness of K, there exists a constant $c \in (0,1)$, such that $v > c$ on K. Hence the function

$$f = \arcsin\big(\frac{c}{v}\big) \tag{2.16}$$

is well-defined on K. By a straightforward computation, we obtain

$$\begin{aligned} f'(v) &= -cv^{-1}(v^2 - c^2)^{-\frac{1}{2}} < 0 \\ f''(v) &= cv^{-2}(v^2 - c^2)^{-\frac{1}{2}} + c(v^2 - c^2)^{-\frac{3}{2}} \end{aligned}$$

and moreover

$$\begin{aligned} &vf'(v)^3 + f''(v) + 2v^{-1}f'(v) \\ &\quad = -c^3v^{-2}(v^2-c^2)^{-\frac{3}{2}} + cv^{-2}(v^2-c^2)^{-\frac{1}{2}} + c(v^2-c^2)^{-\frac{3}{2}} \\ &\qquad - 2cv^{-2}(v^2-c^2)^{-\frac{1}{2}} = 0. \end{aligned}$$

Therefore Hess $\phi(X,X) > 0$ for every $d\phi(X) = 0$ and $|X| = 1$. The compactness of K implies that we can find a positive constant C satisfying (2.1). Then by Lemma 2.1 we can find λ large enough so that

$$F = \lambda^{-1}\exp(\lambda\phi) \tag{2.17}$$

is strictly convex on K. Since K is arbitrary, we conclude that $\mathbb{V}$ is a convex supporting subset of S^n.

Theorem 2.1. $\mathbb{V} = S^n\backslash\overline{S}^{n-1}_+$ *is a maximal open convex supporting subset of* S^n.

Proof. It remains to show that $\mathbb{V}$ is maximal. Let $\mathbb{U} \supset \mathbb{V}$ be another open convex supporting subset of S^n. If there exist $\theta \in (0, \frac{\pi}{2}]$ and $y \in S^{n-2}$, such that $(0, \sin\theta, y\cos\theta) \in \mathbb{U}$, then a closed geodesic of S^n defined by

$$\gamma : t \in \mathbb{R} \mapsto (\sin t, \cos t \sin\theta, y\cos t\cos\theta) \tag{2.18}$$

lies in $\mathbb{U}$. (It is easily-seen that $|\dot{\gamma}| = 1$ and

$$d\big(\gamma(t_0), \gamma(t_0+t)\big) = \arccos\big(\gamma(t_0), \gamma(t_0+t)\big) = t$$

whenever $t \in [0, \pi]$, hence γ is a geodesic.) Since $\mathbb{U}$ is convex supporting, there exist an open neighborhood U of $\mathrm{Im}(\gamma)$ and a strictly convex function F on U. Hence

$$\frac{d^2}{dt^2}(F \circ \gamma) = \text{Hess } F(\dot{\gamma}, \dot{\gamma}) > 0.$$

But on the other hand, since $F \circ \gamma$ is periodic, $F \circ \gamma$ takes its maximum at some point $t_0 \in [0, 2\pi]$; at t_0, $\frac{d^2}{dt^2}(F \circ \gamma) \leq 0$; which causes a contradiction. Therefore $(0, \sin\theta, y\cos\theta) \notin \mathbb{U}$ whenever $\theta \in (0, \frac{\pi}{2}]$. The openness of $\mathbb{U}$ yields $(0, 0, y) \notin \mathbb{U}$ whenever $y \in S^{n-2}$. Hence $\mathbb{U} = \mathbb{V}$ and we complete the proof.

□

The following **Remark** may be helpful for the geometric intuition:

- The functions that we construct in general do not have convex level sets. Only the intersections of their level sets with the compact subset K of $\mathbb{V}$ under consideration have to be convex. That is, their level sets may leave K, become concave outside K, then enter K again as convex sets and leave it again as concave sets.

The following two **Observations** will be useful below:

2.1 Let F be a convex function on an arbitrary compact set $K \subset \mathbb{V}$, and T be an isometry of S^n onto itself, then obviously $F \circ T^{-1}$ is a convex function on $T(K)$. Therefore $\mathbb{U} = T(\mathbb{V}) = S^n \backslash T(\overline{S}^{n-1}_+)$ is also a maximal open convex supporting subset of S^n. Here $T(\overline{S}^{n-1}_+)$ can be characterized by

$$T(\overline{S}^{n-1}_+) = \{x \in S^n : (x, e_1) = 0 \text{ and } (x, e_2) \geq 0\},$$

where e_1, e_2 are two orthogonal vectors on S^n. In the sequel, $\overline{S}^{n-1}_+$ will denote an arbitrary codimension 1 closed half hemisphere.

2.2 Denote $\mathbb{D}^m(r) := \{x \in \mathbb{R}^m : |x| < r\}$ and $\mathbb{D}^m := \mathbb{D}^m(1)$. Then we can define

$$\begin{aligned} \chi: \quad (0, 2\pi) \times \mathbb{D}^{n-1} &\to \mathbb{V} \\ (\theta, y) &\mapsto \big(\sqrt{1-|y|^2}\sin\theta, \sqrt{1-|y|^2}\cos\theta, y\big). \end{aligned}$$

It is easy to check that χ is a diffeomorphism. Thus, $\mathbb{V}$ is diffeomorphic to a convex subset of $\mathbb{R}^n$. This fact will be crucial in the estimates of the oscillation of weakly harmonic maps, see Section 5.

2.3. **Liouville type theorems for harmonic maps from compact manifolds.** The following property of harmonic maps is well-known (see e.g. [27], Section 7.2.C).

Lemma 2.2. *Let (M^m, g), (N^n, h) be two Riemannian manifolds (not necessarily complete) and u be a harmonic map from M to N. If on N there exists a strictly convex function F, then $F \circ u$ is a subharmonic function. Moreover, if there exists a positive constant K_0 such that Hess $F \geq K_0\ h$, then*

$$\Delta(F \circ u) \geq K_0 |du|^2. \tag{2.19}$$

When M is compact and F is a strictly convex function on N, then the compactness of $u(M)$ enables us to find a constant $K_0 > 0$ such that Hess $F \geq K_0 h$ on $u(M)$. Therefore (2.19) holds. Integrating both sides of (2.19) yields

$$0 = \int_M \Delta(F \circ u) * 1 \geq K_0 \int_M |du|^2 * 1 = 2K_0 E(u).$$

Hence $E(u) = 0$, i.e. u is a constant map. We arrive at the following Liouville-type theorem.

Proposition 2.1. *(see* [18]*) Let (M, g) be a compact Riemannian manifold, (N, h) be a Riemannian manifold and u be a harmonic map from M to N. If the image of u is contained in a convex supporting set of N, then u has to be a constant map.*

Theorem 2.1, Proposition 2.1 and Observation 2.1 imply:

Theorem 2.2. *Let (M, g) be a compact Riemannian manifold, u be a harmonic map from M to S^n. If $u(M) \subset S^n \backslash \overline{S}^{n-1}_+$, then u has to be a constant map.*

2.4. **A spherical Bernstein theorem.** S. S. Chern [11] has raised the spherical Bernstein conjecture: Is any imbedded minimal $(n-1)$-dimensional sphere in S^n an equator? For $n = 3$, this was affirmatively solved by Almgren [1] and Calabi [8]. The answer is negative in higher dimensions, however, by counterexamples due to Hsiang [23]. Since then, the spherical Bernstein problem is understood as the question under what conditions a compact minimal hypersurface in S^n has to be an equator. An important result of Solomon [40] concerns this problem for compact minimal hypersurfaces with vanishing first Betti number.

We now study this problem for compact minimal hypersurfaces of arbitrary topological type.

Let $M \to S^{m+p} \subset \mathbb{R}^{m+p+1}$ be an m-dimensional submanifold in the sphere. For $x \in M$, by parallel translation in $\mathbb{R}^{m+p+1}$, we can move the normal space $N_x M$ of M in S^{m+p} to the origin of $\mathbb{R}^{m+p+1}$. Thereby we get a p-subspace of $\mathbb{R}^{m+p+1}$. This defines the **normal Gauss map** $\gamma : M \to \mathbf{G}_{p,m+1}$. Here $\mathbf{G}_{p,m+1}$ is the Grassmannian manifold of p-subspaces of $\mathbb{R}^{m+p+1}$. When $p = 1$, $\mathbf{G}_{p,m+1}$ is simply the $(m+1)$-dimensional sphere.

There is a natural isometry η between $\mathbf{G}_{p,m+1}$ and $\mathbf{G}_{m+1,p}$ which maps any p-subspace into its orthogonal complementary $(m+1)$-subspace. The map $\gamma^* = \eta \circ \gamma$ maps any point $x \in M$ into the $(m+1)$-subspace consisting of the tangent space of M at x and the position vector of x.

As pointed out and utilized by J. Simons [39], the properties of the (minimal) submanifold M in the sphere are closely related to those of the cone CM generated by M. This cone is the image under the map from $M \times [0, \infty)$ into $\mathbb{R}^{m+p+1}$ defined by $(x, t) \mapsto tx$, where $t \in [0, \infty)$ and $x \in M$. CM has a singularity $t = 0$. To avoid the singularity at the origin, we consider the truncated cone CM_ε, which is the image of $M \times (\varepsilon, \infty)$ under the same map, for $\varepsilon > 0$. We have

Proposition 2.2. *([43] p.64) CM_ε has parallel mean curvature in $\mathbb{R}^{m+p+1}$ if and only if M is a minimal submanifold in S^{m+p}.*

There is a natural map from $\mathbb{R}^{m+p+1} - \{0\}$ to S^{m+p} defined by

$$\psi(x) = \frac{x}{|x|} \qquad x \in \mathbb{R}^{m+p+1} - \{0\}.$$

Hence for a map f_1 from a submanifold $M \subset S^{m+p}$ into a Riemannian manifold N, we obtain a map f from CM_ε into N defined by $f = f_1 \circ \psi$, which is called the **cone-like map** (see [43] p.66). One computes that f_1 is harmonic if and only if f is harmonic (see [43] p.67).

By the definition, it is clear that the Gauss map $\gamma_c : CM_\varepsilon \to \mathbf{G}_{m+1,p}\ x \mapsto T_x(CM_\varepsilon)$ is a cone-like map. We have already defined the normal Gauss map $\gamma : M \to \mathbf{G}_{p,m+1}$ and $\gamma^* = \eta \circ \gamma : M \to \mathbf{G}_{m+1,p}$, where $\eta : \mathbf{G}_{p,m+1} \to \mathbf{G}_{m+1,p}$ is an isometry. Obviously

$$\gamma_c = \gamma^* \circ \psi. \tag{2.20}$$

The well-known Ruh-Vilms Theorem (see [33]) tells us that CM_ε has parallel mean curvature if and only if the Gauss map γ_c is a harmonic map, which holds if and only if the normal Gauss map γ is a harmonic map. In conjunction with Proposition 2.2 we have

Proposition 2.3. *([10] [24] [43]p.67) M is a minimal submanifold in the sphere if and only if its normal Gauss map $\gamma : M \to \mathbf{G}_{p,m+1}$ is a harmonic map.*

Combining Proposition 2.3 and Theorem 2.1, we obtain the following spherical Bernstein theorem:

Theorem 2.3. *Let M be a compact minimal hypersurface in S^n. If the image under the normal Gauss map omits $\overline{S}^{n-1}_+$, then M has to be an equator.*

Remarks:

- Theorem 2.3 is an improvement of Simons' extrinsic rigidity theorem (see [39]).

- Solomon [40] (see also [41]) showed that under the additional assumption that the first Betti number of M vanishes, such a spherical Bernstein already holds when the Gauss image omits a neighborhood of some totally geodesic S^{n-2}. Without that topological assumption, however, there are easy counterexamples, like the Clifford torus, and its higher dimensional analogues, as described in [40]. In fact, Theorem 2.3 can also be obtained by Solomon's method in [41].

3. Construction of a smooth family of convex functions on S^n

So far, we have constructed and utilized a single convex function on the target of our Gauss maps in the sphere. For the general regularity theory for harmonic maps that we now wish to develop and later utilize for Bernstein type theorems, we need suitable families of convex functions. Therefore, we need to refine and extend our preceding construction.

On $\mathbb{R}^n$, the squared distances from the points $x \in \mathbb{R}^n$ constitute a smooth family of strictly convex functions, or expressed differently, for every $x \in \mathbb{R}^n$, we have a strictly convex function that assumes its minimum at x. In this vein, we now wish to construct a smooth family of strictly convex functions on arbitrary compact set $K \subset \mathbb{V}$, sufficiently many points in K occur as the minimal points of corresponding convex functions. To realize this, we need the following lemmas concerning the relationship between convex hypersurfaces and convex functions.

3.1. **Convex functions and convex hypersurfaces.** For later reference, we recall some elementary facts.

Definition 3.1. *Let N be a hypersurface in the Riemannian manifold (M, g). If there is a unit normal vector field ν on N with $\langle B(X, X), \nu\rangle < 0$ for any nonzero $X \in TN$ (where B denotes the second fundamental form of N), then we call N a **convex hypersurface**, and the direction in which ν points is called the **direction of convexity**.*

The following result is well known and easy to verify:

Lemma 3.1. *Let ϕ be a C^2-function on the Riemannian manifold (M, g) and $N = \{x \in M : \phi(x) = c\}$ a level set of ϕ. If $|\nabla\phi| \neq 0$ on N, then*

$$\text{Hess}\,\phi(X, X) > 0 \qquad \text{for every nonzero } X \in TN$$

if and only if N is a convex hypersurface and the direction of convexity is the direction of increasing ϕ.

Combining Lemma 3.1 with Lemma 2.1, we obtain

Lemma 3.2. *Let A be a compact domain in M. If ϕ is a nonnegative function on A, every level set of ϕ is a convex hypersurface, and the direction of convexity is the direction of increasing ϕ, then there exists $\lambda > 0$ such that $\lambda^{-1}\exp(\lambda\phi)$ is convex on A.*

The following result is again well known and easy to prove:

Lemma 3.3. *For $x_0 \in S^n$ and $c \in (0,1)$, the hypersurface*

$$N_{x_0,c} = \{x \in S^n : (x, x_0) = c\}$$

where $(.,.)$ denotes the Euclidean scalar product, is convex, and the direction of convexity is the direction of decreasing (x, x_0).

3.2. **Refined construction of convex functions.** We shall use the functions v and φ defined in (2.5). Let K be a compact subset of $\mathbb{V} = S^n \backslash \overline{S}^{n-1}_+$, then there exists a constant $c \in (0, \frac{1}{3}]$ with

$$v \geq 3c$$

on K.

$$U := \{x = (x_1, x_2, \cdots, x_{n+1}) \in \mathbb{V} : v = \sqrt{x_1^2 + x_2^2} > 2c\}, \tag{3.1}$$

is an open domain of S^n and $K \subset U$.

Theorem 3.1. *For any compact subset Φ of $(0, 2\pi)$, there exists a smooth family of nonnegative, smooth functions $F(\cdot, \varphi_0)$ $(\varphi_0 \in \Phi)$ on U, such that:*

(i) $F(\cdot, \varphi_0)$ is strictly convex on K;

(ii) $F(x, \varphi_0) = 0$ if and only if $x = x_{\varphi_0} := (\sin\varphi_0, \cos\varphi_0, 0, \cdots, 0)$;

(iii) $F(x, \varphi_0) \leq 1$ (or $F(x, \varphi_0) < 1$) if and only if $(x, x_{\varphi_0}) \geq \frac{3}{2}c$ (or respectively, $(x, x_{\varphi_0}) > \frac{3}{2}c$) and $|\varphi - \varphi_0| \leq \pi$.

Proof. Let f be a smooth function on $[0, \infty)$ satisfying

$$\begin{cases} f(t) = 0, & t \in \left[0, 1 - \dfrac{3}{2}c\right]; \\ f(t) = t - 1 + c, & t \in \left[1 - \dfrac{1}{2}c, \infty\right); \\ \quad 0 \leq f' \leq 1. & \end{cases}$$

Then we can define H on $U \times (0, 2\pi) \times [0, \infty)$ by

$$(x, \varphi_0, t) \mapsto \begin{cases} -v\cos(\varphi - \varphi_0 + f(t)) + f(t) - t + 1 & \text{if } \varphi \leq \varphi_0, \\ -v\cos(\varphi - \varphi_0 - f(t)) + f(t) - t + 1 & \text{if } \varphi > \varphi_0. \end{cases} \tag{3.2}$$

Now we fix $\varphi_0 \in (0, 2\pi)$ and denote $H_{\varphi_0}(x, t) := H(x, \varphi_0, t)$. For arbitrary $x \in U$, put

$$I_x := \big\{t \in [0, \infty) : \max\{0, |\varphi - \varphi_0| - \pi\} \le f(t) \le |\varphi - \varphi_0|\big\} \tag{3.3}$$

then obviously I_x is a closed interval, $I_x =: [m_x, M_x]$. If $\varphi \le \varphi_0$ and $t \in I_x$, then

$$\partial_2 H_{\varphi_0}(x, t) = f'(t)\big(1 + v\sin(\varphi - \varphi_0 + f(t))\big) - 1 \le 0 \tag{3.4}$$

and $\partial_2 H_{\varphi_0}(x, t) = 0$ if and only if $f'(t) = 1$ and $f(t) = |\varphi - \varphi_0| - \pi$ or $|\varphi - \varphi_0|$; which implies $t = m_x$ or M_x. Hence

$$\partial_2 H_{\varphi_0}(x, \,\cdot\,) < 0 \qquad \text{on } (m_x, M_x). \tag{3.5}$$

Similarly (3.5) holds when $\varphi > \varphi_0$.

It is easily seen that when $|\varphi - \varphi_0| \le \pi$,

$$H_{\varphi_0}(x, m_x) = H_{\varphi_0}(x, 0) = -v\cos(\varphi - \varphi_0) + 1 \ge 0. \tag{3.6}$$

Since $f' \le 1$, $f(t) - t$ is a decreasing function. So $f(t) - t \ge \lim_{t\to\infty}\big(f(t) - t\big) = -1 + c$. Moreover, when $|\varphi - \varphi_0| > \pi$ we have $f(m_x) = |\varphi - \varphi_0| - \pi$ and by (3.2)

$$H_{\varphi_0}(x, m_x) = v + f(m_x) - m_x + 1 \ge v + c > 0. \tag{3.7}$$

By the definition of M_x, f cannot be identically zero on any neighborhood of M_x; hence $M_x \ge 1 - \frac{3}{2}c$ and moreover $f(M_x) - M_x \le f(1 - \frac{3}{2}c) - (1 - \frac{3}{2}c) = -1 + \frac{3}{2}c$. Therefore

$$H_{\varphi_0}(x, M_x) = -v + f(M_x) - M_x + 1 < -2c - 1 + \frac{3}{2}c + 1 = -\frac{1}{2}c < 0. \tag{3.8}$$

By (3.5)-(3.8), for each $x \in U$, there exists a unique $\psi = \psi(x, \varphi_0) \in [m_x, M_x)$, such that

$$H(x, \varphi_0, \psi(x, \varphi_0)) = H_{\varphi_0}(x, \psi(x, \varphi_0)) = 0. \tag{3.9}$$

Denote

$$\Omega := \{(x, \varphi_0) \in U \times (0, 2\pi) : \varphi \ne \varphi_0\},$$

then H is obviously smooth on $\Omega \times [0, \infty)$. The implicit function theorem implies that ψ is smooth on Ω. To show the smoothness of ψ, it remains to prove that ψ is smooth on $\{(x, \varphi_0) \in U \times (0, 2\pi) : \varphi = \varphi_0\}$. Denote

$$\Omega_0 := \{(x, \varphi_0) \in U \times (0, 2\pi) : (x, x_{\varphi_0}) \ge \frac{3}{2}c \text{ and } |\varphi - \varphi_0| \le \pi\},$$

then for every $(x, \varphi_0) \in \Omega_0$, $1 - (x, x_{\varphi_0}) = 1 - v\cos(\varphi - \varphi_0) \le 1 - \frac{3}{2}c$ and hence $f\big(1 - (x, x_{\varphi_0})\big) = 0$; which implies $1 - (x, x_{\varphi_0}) \in [m_x, M_x)$ and

$$H_{\varphi_0}(x, 1 - (x, x_{\varphi_0})) = -v\cos(\varphi - \varphi_0) - (1 - (x, x_{\varphi_0})) + 1 = 0.$$

Therefore

$$\psi(x, \varphi_0) = 1 - (x, x_{\varphi_0}) = 1 - \cos\rho(x) \qquad \forall (x, \varphi_0) \in \Omega_0 \tag{3.10}$$

(where ρ denotes the distance from x_{φ_0} on S^n), and ψ is obviously smooth on the interior of Ω_0. From (3.1) it is easily seen that $\{(x, \varphi_0) \in U \times (0, 2\pi) : \varphi = \varphi_0\} \subset \text{int}(\Omega_0)$, which yields the smoothness of ψ.

For $\varphi_0 \in (0, 2\pi)$, put

$$V_{\varphi_0} := \{x \in U : (x, x_{\varphi_0}) \geq \frac{3}{2}c \text{ and } |\varphi - \varphi_0| \leq \pi\}. \tag{3.11}$$

Then by (3.10) and (2.6), on V_{φ_0},

$$\begin{aligned} \text{Hess } \psi(\cdot, \varphi_0) &= \sin\rho \text{ Hess } \rho + \cos\rho \; d\rho \otimes d\rho \\ &= \cos\rho(g - d\rho \otimes d\rho) + \cos\rho \; d\rho \otimes d\rho \\ &= (1 - \psi)g \geq \frac{3}{2}c \; g; \end{aligned} \tag{3.12}$$

i.e. $\psi(\cdot, \varphi_0)$ is strictly convex on V_{φ_0}.

¿From (3.10) it is easily seen that $\psi(\cdot, \varphi_0) \leq 1 - \frac{3}{2}c$ on V_{φ_0}. On the other hand, for arbitrary $x \in U \backslash V_{\varphi_0}$, one of the following two cases must occur: (I) $|\varphi - \varphi_0| > \pi$; (II) $|\varphi - \varphi_0| \leq \pi$ and $(x, x_{\varphi_0}) < \frac{3}{2}c$.

If case (I) holds, then $\psi(x, \varphi_0) \geq m_x > 1 - \frac{3}{2}c$ by $f(t) > 0$; in the second case, since

$$H_{\varphi_0}(x, 1 - \frac{3}{2}c) = -(x, x_{\varphi_0}) - (1 - \frac{3}{2}c) + 1 = -(x, x_{\varphi_0}) + \frac{3}{2}c > 0$$

and by the monotonicity of H_{φ_0} with respect to the t variable (see (3.5)), we also have $\psi(x, \varphi_0) > 1 - \frac{3}{2}c$. Therefore

$$V_{\varphi_0} = \{x \in U : \psi(x, \varphi_0) \leq 1 - \frac{3}{2}c\}. \tag{3.13}$$

Similarly

$$\text{int}(V_{\varphi_0}) = \{x \in U : \psi(x, \varphi_0) < 1 - \frac{3}{2}c\}. \tag{3.14}$$

For each $t_0 \geq 1 - \frac{3}{2}c$ and $x \in U$ satisfying $\varphi < \varphi_0$, $\psi(x, \varphi_0) = t_0$ if and only if

$$0 = H_{\varphi_0}(x, t_0) = -v\cos(\varphi - \varphi_0 + f(t_0)) + f(t_0) - t_0 + 1,$$

i.e.

$$(x, y(t_0)) = f(t_0) - t_0 + 1,$$

where

$$y(t_0) = (\sin(\varphi_0 - f(t_0)), \cos(\varphi_0 - f(t_0)), 0, \cdots, 0).$$

Since $f' \leq 1$, $t \mapsto f(t) - t + 1$ is a decreasing function, hence

$$\frac{3}{2}c \geq f(t_0) - t_0 + 1 \geq \lim_{t \to +\infty}(f(t) - t + 1) = c.$$

By Lemma 3.3,

$$N^-_{t_0, \varphi_0} \stackrel{def.}{=} \{x \in U : \varphi < \varphi_0, \psi(x, \varphi_0) = t_0\}$$

is a convex hypersurface, and the direction of convexity is the direction of decreasing the function $x \to (x, y(t_0))$. By noting that

$$H_{\varphi_0}(x, t_0) = -(x, y(t_0)) + f(t_0) - t_0 + 1,$$

we have

$$\nabla_\nu H_{\varphi_0}(\cdot, t_0) = -\nabla_\nu(\cdot, y(t_0)) > 0,$$

where ν is the unit normal vector field on $N^-_{t_0,\varphi_0}$ pointing in the direction of convexity. Then $\tilde{H}(x) := H_{\varphi_0}(x, \psi(x, \varphi_0))$ satisfies $\tilde{H} \equiv 0$ and at each $x \in N^-_{t_0,\varphi_0}$

$$0 = \nabla_\nu \tilde{H} = \nabla_\nu H_{\varphi_0}(\cdot, t_0) + \partial_2 H_{\varphi_0} \nabla_\nu \psi(\cdot, \varphi_0);$$

which implies $\nabla_\nu \psi(\cdot, \varphi_0) > 0$ (since $\partial_2 H_{\varphi_0}(x, \psi(x, \varphi_0)) < 0$). In other words, $|\nabla\psi(\cdot, \varphi_0)| \neq 0$ on $N^-_{t_0,\varphi_0}$ and the direction of convexity of $N^-_{t_0,\varphi_0}$ is the direction of increasing $\psi(\cdot, \varphi_0)$. By Lemma 3.1, Hess $\psi(\cdot, \varphi_0)(X, X) > 0$ for every nonzero $X \in TN^-_{t_0,\varphi_0}$ such that $d\psi(\cdot, \varphi_0)(X) = 0$.

Similarly, putting

$$N^+_{t_0,\varphi_0} := \{x \in U : \varphi > \varphi_0, \psi(x, \varphi_0) = t_0\},$$

then $|\nabla\psi(\cdot, \varphi_0)| \neq 0$ on $N^+_{t_0,\varphi_0}$ and Hess $\psi(\cdot, \varphi_0)(X, X) > 0$ for every nonzero $X \in TN^+_{t_0,\varphi_0}$ such that $d\psi(\cdot, \varphi_0)(X) = 0$.

By the compactness of K and $\Phi \subset (0, 2\pi)$, there are positive constants c_2, c_3 and c_4, such that for all $\varphi_0 \in \Phi$,

$$\text{Hess } \psi(\cdot, \varphi_0)(X, X) \geq c_2|X|^2$$

for every nonzero $X \in TK$ which is tangential to one of the level sets of $\psi(\cdot, \varphi_0)$, and

$$|\text{Hess } \psi(\cdot, \varphi_0)| \leq c_3, \qquad |\nabla\psi(\cdot, \varphi_0)| \geq c_4$$

on K. Lemma 2.1 then implies that there exists $\lambda_0 > 0$ satisfying

$$\text{Hess}(\lambda_0^{-1} \exp(\lambda_0 \psi(\cdot, \varphi_0)) \geq \frac{1}{2} c_2\, g. \tag{3.15}$$

Now we take

$$F(\cdot, \varphi_0) := \frac{\exp(\lambda_0 \psi(\cdot, \varphi_0)) - 1}{\exp\left(\lambda_0(1 - \frac{3}{2}c)\right) - 1}. \tag{3.16}$$

Then from (3.15), $F(\cdot, \varphi_0)$ is a strictly convex function on K; while conclusions (ii) and (iii) in the Theorem follow from (3.10), (3.13) and (3.14), respectively. □

Remark: The auxiliary function f in the above proof can be easily obtained from the standard bump functions. We choose a nonnegative smooth function h on $\mathbb{R}$, whose supporting set is $[0, 1]$. Let

$$h_1(t) := \frac{\int_0^t h}{\int_0^1 h},$$

then $0 \le h_1 \le 1$, $h_1|_{(-\infty,0]} \equiv 0$ and $h_1|_{[1,+\infty)} \equiv 1$. Define

$$h_2(t) := h_1^\beta(\frac{t}{c}),$$

where $\beta > 0$ to be chosen, then $0 \le h_2 \le 1$, $h_2|_{(-\infty,0]} \equiv 0$, and $h_2|_{[c,+\infty)} \equiv 1$. Note that $\alpha \in (0,+\infty) \mapsto \int_0^c h_1^\alpha(\frac{t}{c})$ is a strictly decreasing function, which converges to 0 as $\alpha \to +\infty$ and converges to c as $\alpha \to 0$. It enables us to find $\beta > 0$, such that

$$\int_0^c h_2 = \frac{1}{2}c;$$

Then

$$f(t) := \int_0^{t-1+\frac{3}{2}c} h_2$$

is the required function.

4. Some properties of weakly harmonic maps

Let (M^m, g) and (N^n, h) be Riemannian manifolds, not necessarily complete. Here and in the sequel, we denote by $\{e_1, \cdots, e_m\}$ a local orthonormal frame field on M and by $\{f_1, \cdots, f_n\}$ a local orthonormal frame field on N. We use the summation convention with the index ranges

$$1 \le \alpha, \beta \le m, \qquad 1 \le i, j \le n.$$

$u \in H^{1,2}_{loc}(M, N)$ is called a **weakly harmonic map** if it is a critical point of the energy functional E; i.e.

$$\frac{d}{dt}\Big|_{t=0} E(\exp_u(t\xi)) = 0. \tag{4.1}$$

for all compactly supported bounded sections ξ of $u^{-1}TN$ of class $H^{1,2}$, where $u^{-1}TN$ denotes the pull-back bundle of TN (see [27] p.452). A straightforward calculation shows

$$\int_M \langle du(e_\alpha), \nabla_{e_\alpha}\xi\rangle * 1 = 0. \tag{4.2}$$

Here ∇ is the connection on $u^{-1}TN$ induced by the Levi-Civita connections of M and N.

Suppose Ω is an open domain of M and K is a compact domain of N, such that $u(\Omega) \subset K$ and there is a smooth and strictly convex function F on K, i.e. there exists a positive constant K_0 such that Hess $F \ge K_0 h$.

Let η be a non-negative smooth function on Ω with compact support. Put

$$\xi(y) := \eta(y)\nabla^N F\big(u(y)\big). \tag{4.3}$$

Then (4.2) tells us

$$
\begin{aligned}
0 &= \int_\Omega \langle du(e_\alpha), \nabla_{e_\alpha}\xi\rangle * 1 \\
&= \int_\Omega \langle du(e_\alpha), (\nabla_{e_\alpha}\eta)\nabla^N F\big(u(y)\big)\rangle * 1 + \int_\Omega \langle du(e_\alpha), \eta\nabla_{e_\alpha}\nabla^N F\big(u(y)\big)\rangle * 1 \\
&= I + II.
\end{aligned}
\tag{4.4}
$$

$f := F\circ u$ then satisfies $\nabla_{e_\alpha} f = \nabla^N_{u_* e_\alpha} F$, hence

$$
I = \int_\Omega \nabla\eta\cdot\nabla f \ \ * 1. \tag{4.5}
$$

Without loss of generality, one can assume $\nabla^N f_i = 0$ for every $1 \le i \le n$ at the considered point, then

$$
\nabla_{e_\alpha}\nabla^N F\big(u(y)\big) = \nabla_{e_\alpha}\big((\nabla^N_{f_i}F)f_i\big) = (\nabla^N_{u_*e_\alpha}\nabla^N_{f_i}F)f_i = \text{Hess } F(u_*e_\alpha, f_i)f_i \tag{4.6}
$$

and moreover

$$
\begin{aligned}
II &= \int_\Omega \eta\langle du(e_\alpha), \nabla_{e_\alpha}\nabla^N F\big(u(y)\big)\rangle * 1 = \int_\Omega \eta \text{Hess } F(u_*e_\alpha, f_i)\langle f_i, u_*e_\alpha\rangle * 1 \\
&= \int_\Omega \eta\text{Hess } F(u_*e_\alpha, u_*e_\alpha) * 1 \ge K_0 \int_\Omega \eta|u_*e_\alpha|^2 * 1 \\
&= K_0 \int_\Omega \eta|du|^2 * 1.
\end{aligned}
\tag{4.7}
$$

Substituting (4.5) and (4.7) into (4.4) yields

$$
K_0 \int_\Omega \eta|du|^2 * 1 \le -\int_\Omega \nabla\eta\cdot\nabla f * 1. \tag{4.8}
$$

It says that $f = F\circ u$ is a subharmonic function in the weak sense.

4.1. **Additional assumptions.** In the following, we shall assume that (M, g) satisfies 3 additional conditions:

(D) There is a distance function d on M (which is not necessary induced from the Riemannian metric on M), and the metric topology induced by d is equivalent to the Riemannian topology of M; moreover, for each $y_1, y_2 \in M$, $d(y_1, y_2) \le r(y_1, y_2)$, where $r(\cdot,\cdot)$ is the distance function of M with respect to the Riemannian metric.

(V) **Doubling property**: Let $B_R(y)$ be the ball centered at y of radius R given by the distance d, denote by $V(y, R)$ the volume of $B_R(y)$, then there are $R_0 \in (0, \infty]$ and a positive constant K_1 independent of y and R, such that

$$
V(y, 2R) \le K_1\, V(y, R) \qquad \text{whenever } R \le R_0. \tag{4.9}
$$

(P) **Neumann-Poincaré inequality**: For arbitrary $y \in M$ and $R > 0$ satisfying $B_R(y) \subset\subset M$, the following inequality holds

$$\int_{B_R(y)} |v - \bar{v}_{B_R(y)}|^2 * 1 \leq K_2 R^2 \int_{B_R(y)} |\nabla v|^2 * 1 \tag{4.10}$$

where

$$\bar{v}_{B_R(y)} := \frac{\int_{B_R(y)} v * 1}{V(y, R)}$$

is the average value of v on $B_R(y)$, and K_2 is a positive constant not depending on y and R.

We say that the manifold M satisfies the DVP-condition if it satisfies these 3 conditions.

Remarks:

4.1 Condition (D) implies $|\nabla d(\cdot, y)| \leq 1$ for each $y \in M$. Hence it is easy to construct a cut-off function η on $B_R(y)$ satisfying

$$0 \leq \eta \leq 1, \ \eta|_{B_r(y)} \equiv 1, \text{and } |\nabla \eta| \leq c_0 (R - r)^{-1}$$

by letting $\eta = \varphi\big(d(\cdot, y)\big)$, where φ is a smooth function on $[0, \infty)$, such that $0 \leq \varphi \leq 1$, $\varphi|_{[0,r]} = 1$, $\varphi|_{[R,\infty)} = 0$, and $|\varphi'| \leq \frac{c_0}{R-r}$.

4.2 Put

$$\nu_0 := \frac{\log K_1}{\log 2}. \tag{4.11}$$

For arbitrary $0 < r < R \leq R_0$, we consider the integer k such that $2^{k-1} < \frac{R}{r} \leq 2^k$; from the doubling property it follows that

$$V(x, R) \leq V(x, 2^k r) \leq K_1^k V(x, r) < K_1 \big(\frac{R}{r}\big)^{\nu_0} V(x, r). \tag{4.12}$$

4.3 It is well-known that the Neumann-Poincaré inequality is closely related to the eigenvalues of the Laplace-Beltrami operator with Neumann boundary values. More precisely, let $\mu_2(\Omega)$ be the second eigenvalue of

$$\begin{aligned} \Delta v + \mu v = 0 \qquad & \text{in } \Omega \\ \frac{\partial v}{\partial n} = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{4.13}$$

where n denotes the outward normal vector field, then $\mu_2(\Omega)$ is characterized by

$$\mu_2(\Omega) = \min_{\int_\Omega v * 1 = 0} \frac{\int_\Omega |\nabla v|^2 * 1}{\int_\Omega |v|^2 * 1}. \tag{4.14}$$

Therefore Condition (P) is equivalent to $\mu_2(B_R(y)) \geq K_2^{-1} R^{-2}$.

The Neumann-Poincaré inequality is also related to Cheeger's [9] isoperimetric constant

$$h_N(\Omega) := \inf_A \frac{\mathrm{Vol}(\partial A \cap \mathrm{int}(\Omega))}{\mathrm{Vol}(A)} \tag{4.15}$$

where A stands for an open subset of Ω satisfying $\mathrm{Vol}(A) \le \frac{1}{2}\mathrm{Vol}(\Omega)$. Cheeger proved

$$\mu_2(\Omega) \ge \frac{1}{4} h_N^2(\Omega). \tag{4.16}$$

4.4 (4.10) is the strong form of the Poincaré inequality; in contrast, the weak form of Poincaré is

$$\int_{B_{\kappa R}(y)} |v - \bar{v}_{B_{\kappa R}(y)}|^2 * 1 \le C R^2 \int_{B_R(y)} |\nabla v|^2 * 1 \tag{4.17}$$

where $\kappa \in (0,1)$ and C is a constant not depending on y and R. It follows from the work of D. Jerison [25] that the doubling property and (4.17) implies (4.10). Hence Condition (P) could be replaced by (4.17).

4.5 ¿From the work of Saloff-Coste [34] and Biroli-Mosco [3], Conditions (V) and (P) imply the following Sobolev-type inequality: for $y \in M$ and $R > 0$ satisfying $B_{2R}(y) \subset\subset M$,

$$\Big(\int_{B_R(y)} |v|^{\frac{2\nu}{\nu-2}} * 1\Big)^{\frac{\nu-2}{2\nu}} \le K_3 R\, V(y,R)^{-\frac{1}{\nu}} \Big(\int_{B_R(y)} |\nabla v|^2 * 1\Big)^{\frac{1}{2}}, \tag{4.18}$$

where $v \in H_0^{1,2}(B_R(y))$, ν is a constant only depending on K_1, which is greater or equal to $\nu_0 = \frac{\log K_1}{\log 2}$ and strictly greater than 2; $R \le R_0$ and K_3 is a positive constant only depending on K_1 and K_2. With

$$⨍_\Omega v := \frac{\int_\Omega v * 1}{\mathrm{Vol}(\Omega)}, \tag{4.19}$$

(4.18) is equivalent to

$$\Big(⨍_{B_R(y)} |v|^{\frac{2\nu}{\nu-2}}\Big)^{\frac{\nu-2}{2\nu}} \le K_3 R \Big(⨍_{B_R(y)} |\nabla v|^2\Big)^{\frac{1}{2}}. \tag{4.20}$$

4.2. **Harnack inequality.** In the sequel, we shall make use of the following abbreviations: Fix a point $y_0 \in M$ and let $B_R = B_R(y_0) \subset\subset M$ with $R \le \frac{1}{2}R_0$, then $V(R) := V(y_0, R)$ and for arbitrary $v \in L^\infty(B_R)$,

$$\begin{aligned} &v_{+,R} := \sup_{B_R} v, \qquad v_{-,R} := \inf_{B_R} v, \qquad \bar{v}_R := ⨍_{B_R} v, \\ &|\bar{v}|_{p,R} := \Big(⨍_{B_R} |v|^p\Big)^{\frac{1}{p}} \qquad p \in (-\infty, +\infty). \end{aligned} \tag{4.21}$$

It is easily seen that $p \mapsto |\bar{v}|_{p,R}$ is an increasing function, $\lim_{p\to+\infty} |\bar{v}|_{p,R} = |v|_{+,R}$ and $\lim_{p\to-\infty} |\bar{v}|_{p,R} = |v|_{-,R}$, if $|\bar{v}|_{p,R}$ is well-defined.

Lemma 4.1. *Let M be an m-dimensional Riemannian manifold satisfying DVP-condition, then for any positive superharmonic function v on B_R satisfying $R \leq \frac{1}{2}R_0$ and $B_{2R} \subset\subset M$, $p \in (0, \frac{\nu}{2-\nu})$ and $\theta \in [\frac{1}{2}, 1)$, we have the estimate*

$$|\bar{v}|_{p,\theta R} \leq \gamma_1 v_{-,\theta R} \tag{4.22}$$

Here γ_1 is a positive constant only depending on K_1, K_2, p and θ.

This Harnack inequality follows from the work of Moser[31], Bombieri-Giusti[6], Saloff-Coste[34] and Biroli-Mosco[4] as we shall now briefly describe. Firstly, the superharmonicity of v implies a reserve Poincaré inequality for v^k for arbitrary $k < \frac{1}{2}$ (see [31] Lemma 4); with the aid of the Sobolev inequality (4.20) and suitable cut-off functions as described in Remark 4.1, we can obtain

$$v_{-,(1-\tau)R} \geq (c_1 \tau^{-c_2})^{-\frac{1}{q}} |\bar{v}|_{-q,R} \tag{4.23}$$

for arbitrary $\tau \in (0, \frac{1}{2}]$ and $q \in (0, \infty)$ by Moser's iteration. Again using Moser's iteration repeatedly, one can get

$$|\bar{v}|_{q,(1-\tau)R} \leq (c_3 \tau^{-c_4})^{\frac{1}{s}-\frac{1}{q}} |\bar{v}|_{s,R}. \tag{4.24}$$

Here $\tau \in (0, \frac{1}{2}]$, $q \in (0, \frac{\nu}{\nu-2})$, $s \in (0, (\frac{2}{q} - \frac{\nu-2}{\nu})^{-1}]$ and c_3, c_4 are positive constants depending only on K_1, K_2 and q. By the Neumann-Poincaré inequality, we arrive at

$$\sup_{\tau \in [\tau_0, \frac{1}{2}]} \inf_{k \in \mathbb{R}} \fint_{B_{(1-\tau)R}} \Big| \log\big(\frac{v}{k}\big) \Big| \leq c_5(\tau_0, K_1, K_2) \tag{4.25}$$

as in [6], where $\tau_0 \in (0, \frac{1}{2})$. Combining with (4.23), (4.24) and (4.25), one can apply an abstract John-Nirenberg inequality ([6], Theorem 4) to obtain the result.

¿From Lemma 4.1, we get analogues of Corollary 1 and Lemma 7 in [26]:

Corollary 4.1. *Let M be an m-dimensional Riemannian manifold satisfying DVP-condition, v be a subharmonic function on B_R satisfying $R \leq \frac{1}{2}R_0$ and $B_{2R} \subset\subset M$. Then there exists a constant $\delta_0 \in (0,1)$, only depending on K_1 and K_2, such that*

$$v_{+,\frac{R}{2}} \leq (1-\delta_0) v_{+,R} + \delta_0 \bar{v}_{\frac{R}{2}}. \tag{4.26}$$

Proof. For arbitrary $\varepsilon > 0$, $v_{+,R} - v + \varepsilon$ is a positive superharmonic function on B_R, then Lemma 4.1 implies

$$|v_{+,R} - v + \varepsilon|_{1,\frac{R}{2}} \leq \gamma_1 (v_{+,R} - v + \varepsilon)_{-,\frac{R}{2}}$$

where γ_1 is a positive constant only depending on K_1 and K_2. This is equivalent to $v_{+,R} - \bar{v}_{\frac{R}{2}} + \varepsilon \leq \gamma_1 (v_{+,R} - v_{+,\frac{R}{2}} + \varepsilon)$; letting $\varepsilon \to 0$ yields

$$v_{+,R} - \bar{v}_{\frac{R}{2}} \leq \gamma_1 (v_{+,R} - v_{+,\frac{R}{2}}).$$

(4.26) follows by putting $\delta_0 = \frac{1}{\gamma_1}$. $\square$

The next result is proved as in [26]:

Corollary 4.2. *Let v be as in Corollary 4.1, and suppose $0 < \varepsilon < \frac{1}{2}$. There exists $k \in \mathbb{N}$, independent of v and ε, such that*

$$v_{+,\varepsilon^k R} \le \varepsilon^2 v_{+,R} + (1-\varepsilon^2)\bar{v}_{R'} \tag{4.27}$$

for some R' with $\varepsilon^k R \le R' \le \frac{R}{2}$ (R' may depend on v and ε).

4.3. **Mollified Green function.** Obviously, $b : H_0^{1,2}(\Omega) \times H_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$(\phi, \psi) \mapsto \int_\Omega \nabla\phi \cdot \nabla\psi * 1$$

is a bounded, positive definite bilinear form. For arbitrary $y \in \Omega$ and $\rho > 0$ such that $B_\rho(y) \subset \Omega$,

$$\phi \in H_0^{1,2}(\Omega) \mapsto \fint_{B_\rho(y)} \phi$$

is a bounded linear functional. By the Lax-Milgram Theorem, there exists a unique function $G^\rho(\cdot, y) \in H_0^{1,2}(\Omega)$, such that for all $\phi \in H_0^{1,2}(\Omega)$,

$$\int_\Omega \nabla G^\rho(\cdot, y) \cdot \nabla\phi * 1 = \fint_{B_\rho(y)} \phi. \tag{4.28}$$

G^ρ is called the **mollified Green function** with respect to the Laplace-Beltrami operator on Ω. We can follow [19] and [4] to obtain estimates on G^ρ, to be used in the next paragraphs. (Riemannian manifolds satisfying the DVP- condition are certain metric spaces (homogeneous spaces in the sense of [12] Ch. III, Section 1) on which a weak version of the Poincaré inequality holds, and $(u, v) \mapsto \int_M \nabla u \cdot \nabla v * 1$ is a Dirichlet form. Hence the results in [4] can be applied.)

Lemma 4.2. *Let M be an m-dimensional Riemannian manifold satisfying DVP-condition, $R \in (0, \frac{1}{3}R_0]$ satisfying $B_{3R} \subset\subset M$. Then the mollified Green function G^ρ on B_R enjoys the following properties:*

$$G^\rho(\cdot, y) \le C_1 \frac{R^2}{V(R)} \qquad \text{on } S_R := B_R - \bar{B}_{\frac{3R}{8}} \tag{4.29}$$

and

$$\int_{T_R} |\nabla G^\rho(\cdot, y)|^2 * 1 \le C_2 \frac{R^2}{V(R)} \qquad \text{on } T_R := B_R - \bar{B}_{\frac{R}{2}} \tag{4.30}$$

for all $y \in B_{\frac{R}{4}}$ and $\rho \le \frac{R}{8}$. Here C_1, C_2 are positive constants depending only on K_1 and K_2.

Proof. $y \in B_{\frac{R}{4}}$ implies $B_R \subset B_{\frac{5R}{4}}(y)$; since $B_{\frac{5R}{2}}(y) \subset B_{\frac{11R}{4}} \subset B_{3R} \subset\subset M$, one can apply (6.13)-(6.15) in [4] to obtain

$$\sup_{\partial B_{\frac{R}{8}}(y)} G^\rho_{B_{\frac{5R}{4}}(y)}(\cdot, y) \le c_6(K_1, K_2) \frac{R^2}{V(y, \frac{R}{8})}. \tag{4.31}$$

Here $G^\rho_{B_{\frac{5R}{4}}(y)}$ denotes the mollified Green function on $B_{\frac{5R}{4}}(y)$, which is harmonic on $B_{\frac{5R}{4}}(y) - B_\rho(y)$. Hence the maximal principle implies for each $z \in S_R \subset B_{\frac{5R}{4}}(y) - B_{\frac{R}{8}}(y)$,

$$G^\rho_{B_{\frac{5R}{4}}(y)}(z,y) \le \sup_{\partial B_{\frac{R}{8}}(y)} G^\rho_{B_{\frac{5R}{4}}(y)}(\cdot,y) \le c_6(K_1,K_2)\frac{R^2}{V(y,\frac{R}{8})} \tag{4.32}$$

Noting that $G^\rho_{B_{\frac{5R}{4}}(y)}$ and G^ρ are all nonnegative (which can be seen by a simple truncation argument, see [19]) and $G^\rho_{B_{\frac{5R}{4}}(y)} - G^\rho$ is harmonic on B_R, again using the maximal principle yields

$$G^\rho(z,y) - G^\rho_{B_{\frac{5R}{4}}(y)}(z,y) \le \sup_{\partial B_R}\left[G^\rho(\cdot,y) - G^\rho_{B_{\frac{5R}{4}}(y)}(\cdot,y)\right] \le 0.$$

Hence (4.29) immediately follows from (4.32) and

$$V(y,\frac{R}{8}) \ge K_1^{-4} V(y,\frac{5R}{4}) \ge K_1^{-4}V(R).$$

As in [19], we choose a cut-off function η satisfying $\eta \equiv 1$ in T_R, $\eta \equiv 0$ in $B_{\frac{3R}{8}}$ and $|\nabla\eta| \le \frac{c_0}{R}$, and insert $G^\rho(\cdot,y)\eta^2$ into (4.28). (4.30) then follows from (4.29).

□

Lemma 4.3. *Let M be an m-dimensional Riemannian manifold satisfying DVP-condition, $R \in (0,\frac{1}{2}R_0]$ satisfying $B_{2R} \subset\subset M$. With*

$$\omega^R := \frac{V(\frac{R}{2})}{R^2} G^{\frac{R}{2}}(\cdot,y_0), \tag{4.33}$$

then

$$\omega^R \le C_3 \qquad \text{on } B_R \tag{4.34}$$

and

$$\omega^R \ge C_4 \qquad \text{on } B_{\frac{R}{2}}, \tag{4.35}$$

where C_3 and C_4 are positive constants depending only on K_1 and K_2, but not depending on R.

Proof. (4.28) and (4.33) imply that

$$\int_{B_R} \nabla\omega^R \cdot \nabla\phi * 1 = \frac{1}{R^2}\int_{B_{\frac{R}{2}}} \phi * 1 = \int_{B_R} \frac{1}{R^2}\mathbf{1}_{B_{\frac{R}{2}}} \cdot \phi * 1 \tag{4.36}$$

holds for every $\phi \in H_0^{1,2}(B_R)$. Then applying Theorem 4.1 in [4] yields

$$\sup_{B_R} \omega^R \le C_3 R^2 \sup_{B_R}\left(\frac{1}{R^2}\mathbf{1}_{B_{\frac{R}{2}}}\right) = C_3. \tag{4.37}$$

By (6.13)-(6.15) in [4],

$$\inf_{\partial B_{\frac{R}{2}}} G^{\frac{R}{2}}(\cdot, y_0) \geq C_4 \frac{R^2}{V(\frac{R}{2})}. \tag{4.38}$$

Since $G^{\frac{R}{2}}(\cdot, y_0)$ is a superharmonic function, it assumes its minimum in $\bar{B}_{\frac{R}{2}}$ at the boundary; therefore, (4.35) immediately follows.

□

4.4. **Telescoping lemma.** Based on (4.8) and Corollary 4.1, we can obtain a version of the telescoping lemma of Giaquinta-Giusti [15] and Giaquinta-Hildebrandt [16] as in [26].

Lemma 4.4. *Let (M^m, g) be a Riemannian manifold satisfying DVP-condition, (N^n, h) be a Riemannian manifold, $u \in H^{1,2}_{loc}(M, N)$ be a weakly harmonic map, K be a compact domain of N, and let there exist a smooth and strictly convex function F on K such that Hess $F \geq K_0\ h$. If there is $R_1 \in (0, \frac{1}{2}R_0]$, such that $B_{2R_1} \subset\subset M$ and $u(B_{R_1}) \subset K$, then there is a positive constant C_5, only depending on K_0, K_1 and K_2, such that for arbitrary $R \leq R_1$*

$$\frac{R^2}{V(\frac{R}{2})} \int_{B_{\frac{R}{2}}} |du|^2 * 1 \leq C_5 (f_{+,R} - f_{+,\frac{R}{2}}). \tag{4.39}$$

Here $f = F \circ u$. Moreover, there exists a positive constant C_6, only depending on K_0, K_1, K_2 and $\sup_K F - \inf_K F$, with the property that for arbitrary $\varepsilon > 0$, we can find $R \in [\exp(-C_6 \varepsilon^{-1}) R_1, R_1]$ such that

$$\frac{R^2}{V(\frac{R}{2})} \int_{B_{\frac{R}{2}}} |du|^2 * 1 \leq \varepsilon. \tag{4.40}$$

The telescoping lemma will be so powerful for our purposes because it does not require an energy bound on our weakly harmonic map. Instead, the energy of u is locally controlled by the oscillation of its composition with the strictly convex function F, essentially via a lower bound on the Hessian of F. (In general, when applying this scheme, one will also need an upper bound for the gradient of the strictly convex function F in order to relate the oscillation of $F \circ u$ to the one of u itself. In the situation of the present paper, this will be implicitely contained in the geometry of the sphere and therefore not come up as an issue.)

Proof. $v := f - f_{+,R}$ satisfies $v \le 0$. Choosing $(\omega^R)^2 \in H_0^{1,2}(B_R)$ as a test function in (4.8) (ω^R is defined in (4.33)), we obtain

$$\begin{aligned} K_0 \int_{B_R} |du|^2 (\omega^R)^2 * 1 &\le -\int_{B_R} \nabla(\omega^R)^2 \cdot \nabla v * 1 = -2\int_{B_R} \nabla\omega^R \cdot \omega^R \nabla v * 1 \\ &= -2\int_{B_R} \nabla\omega^R \cdot \big(\nabla(\omega^R v) - v\nabla\omega^R\big) * 1 \le -2\int_{B_R} \nabla\omega^R \cdot \nabla(\omega^R v) * 1 \\ &= -\frac{2}{R^2}\int_{B_{\frac{R}{2}}} \omega^R v * 1 \le -\frac{2C_3}{R^2}\int_{B_{\frac{R}{2}}} v * 1. \end{aligned}$$

Here we have used (4.36) and the pointwise estimates for ω^R in Lemma 4.3. On the other hand,

$$\int_{B_R} |du|^2 (\omega^R)^2 * 1 \ge C_4^2 \int_{B_{\frac{R}{2}}} |du|^2 * 1.$$

Hence

$$\begin{aligned} \int_{B_{\frac{R}{2}}} |du|^2 * 1 &\le \frac{2C_3}{K_0 C_4^2 R^2}\int_{B_{\frac{R}{2}}} (f_{+,R} - f) * 1 \\ &= c_7(K_0, K_1, K_2)\frac{V(\frac{R}{2})}{R^2}(f_{+,R} - \bar{f}_{\frac{R}{2}}). \end{aligned} \tag{4.41}$$

By Corollary 4.1, $f_{+,R} - \bar{f}_{\frac{R}{2}} \le \delta_0^{-1}(f_{+,R} - f_{+,\frac{R}{2}})$. Substituting it into (4.41) yields (4.39).

For arbitrary $k \in \mathbb{N}$, (4.39) tells us

$$\begin{aligned} \sum_{i=0}^{k} \frac{(2^{-i}R_1)^2}{V(2^{-i-1}R_1)} \int_{B_{2^{-i-1}R_1}} |du|^2 * 1 &\le C_5 \sum_{i=0}^{k} (f_{+,2^{-i}R_1} - f_{+,2^{-i-1}R_1}) \\ &= C_5(f_{+,R_1} - f_{+,2^{-k-1}R_1}) \\ &\le C_5(\sup_K F - \inf_K F) \end{aligned} \tag{4.42}$$

For arbitrary $\varepsilon > 0$, we take

$$k := \big[C_5(\sup_K F - \inf_K F)\varepsilon^{-1}\big]. \tag{4.43}$$

Here and in the sequel, $[x]$ denotes the greatest integer not larger than x. Then we can find j with $0 \le j \le k$, such that

$$\frac{(2^{-j}R_1)^2}{V(2^{-j-1}R_1)} \int_{B_{2^{-j-1}R_1}} |du|^2 * 1 \le \frac{1}{k+1} C_5(\sup_K F - \inf_K F) \le \varepsilon. \tag{4.44}$$

Since $2^{-j} \ge 2^{-k} \ge 2^{-C_5(\sup_K F - \inf_K F)\varepsilon^{-1}} = \exp\big(-(\log 2)C_5(\sup_K F - \inf_K F)\varepsilon^{-1}\big)$, it is sufficient to take $C_6 = C_5 \log 2(\sup_K F - \inf_K F)$. $\square$

5. Regularity of weakly harmonic maps and Liouville type theorems

5.1. **Pointwise estimates.** M, N, u, K and R_0 are as in Lemma 4.4. Now we assume that there exists $R_1 \in (0, \frac{1}{3}R_0]$ with $B_{3R_1} \subset\subset M$ and $u(B_{R_1}) \subset K$. Let H be a smooth function on K, η be a non-negative smooth function on B_{R_1} with compact support and φ be a $H^{1,2}$-function on B_{R_1}. Denoting

$$\xi(y) := \eta(y)\varphi(y)\nabla^N H(u(y)), \tag{5.1}$$

then similar to (4.4)-(4.7), we have

$$\int_{B_{R_1}} \varphi \nabla\eta \cdot \nabla h * 1 + \int_{B_{R_1}} \eta\nabla\varphi \cdot \nabla h * 1 + \int_{B_{R_1}} \eta\varphi\Delta h * 1 = 0 \tag{5.2}$$

where $h = H \circ u$. It implies

$$\begin{aligned} \int_{B_{R_1}} \nabla\varphi \cdot \nabla(\eta h) * 1 = &- \int_{B_{R_1}} \varphi\nabla\eta \cdot \nabla h * 1 - \int_{B_{R_1}} \eta\varphi\Delta h * 1 \\ &+ \int_{B_{R_1}} h\nabla\varphi \cdot \nabla\eta * 1. \end{aligned} \tag{5.3}$$

For arbitrary $R \le \frac{1}{2}R_1$, we can take a cut-off function η with the support in the interior of B_R, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\frac{R}{2}}$ and $|\nabla\eta| \le \frac{c_0}{R}$. For each $\rho \le \frac{R}{8}$, denote by G^ρ the mollified Green function on B_R. Then by inserting $\varphi = G^\rho(\cdot, y)$ into (5.3), where y is an arbitrary point in $B_{\frac{R}{4}}$, we have

$$\begin{aligned} \int_{B_R} \nabla G^\rho(\cdot, y) \cdot \nabla(\eta h) * 1 = &- \int_{B_R} G^\rho(\cdot, y)\nabla\eta \cdot \nabla h * 1 \\ &- \int_{B_R} \eta G^\rho(\cdot, y)\Delta h * 1 \\ &+ \int_{B_R} h\nabla G^\rho(\cdot, y) \cdot \nabla\eta * 1. \end{aligned} \tag{5.4}$$

We write (5.4) as

$$I_\rho = II_\rho + III_\rho + IV_\rho. \tag{5.5}$$

By (4.28), we arrive at

$$I_\rho = \fint_{B_\rho(y)} \eta h = \fint_{B_\rho(y)} h. \tag{5.6}$$

If we choose for h its Lebesgue representative, then we can find a subsequence of the ρs with the property that

$$\lim_{\rho \to 0} I_\rho = h(y). \tag{5.7}$$

Put $T_R := B_R - \bar{B}_{\frac{R}{2}}$. Since $\nabla\eta \equiv 0$ outside T_R,

$$\begin{aligned}|II_\rho| =& \Big|\int_{T_R} G^\rho(\cdot,y)\nabla\eta\cdot\nabla h * 1\Big| \le \int_{T_R} G^\rho(\cdot,y)|\nabla\eta||\nabla h| * 1 \\ \le& c_0 R^{-1}\sup_K |\nabla^N H| \int_{T_R} G^\rho(\cdot,y)|du| * 1\end{aligned} \tag{5.8}$$

By (4.29), $G^\rho(\cdot,y) \le C_1 \frac{R^2}{V(R)}$ on T_R, hence

$$\begin{aligned}|II_\rho| &\le C\sup_K |\nabla^N H| \frac{R}{V(R)}\int_{T_R} |du| * 1 \\ &\le C\sup_K |\nabla^N H| \frac{R}{V(R)}\Big(\int_{T_R}|du|^2 * 1\Big)^{\frac{1}{2}}\mathrm{Vol}(T_R)^{\frac{1}{2}} \\ &\le c_1(K_1,K_2)\sup_K |\nabla^N H|\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2 * 1\Big)^{\frac{1}{2}}\end{aligned} \tag{5.9}$$

Obviously

$$III_\rho = -\int_{T_R} \eta G^\rho(\cdot,y)\Delta h * 1 - \int_{B_{\frac{R}{2}}} G^\rho(\cdot,y)\Delta h * 1. \tag{5.10}$$

According to (4.29) and (4.39) in Lemma 4.4,

$$\begin{aligned}\Big| -\int_{T_R} \eta G^\rho(\cdot,y)\Delta h * 1\Big| &= \Big|\int_{T_R}\eta G^\rho(\cdot,y)\mathrm{Hess}\ H(u_* e_\alpha, u_* e_\alpha) * 1\Big| \\ &\le \sup_K |\mathrm{Hess}\ H| \int_{T_R} G^\rho(\cdot,y)|du|^2 * 1 \\ &\le C_1 \sup_K |\mathrm{Hess}\ H| \frac{R^2}{V(R)}\int_{B_R}|du|^2 * 1 \\ &\le c_2 \sup_K |\mathrm{Hess}\ H|\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2 * 1\Big)^{\frac{1}{2}}.\end{aligned} \tag{5.11}$$

Here c_2 is a positive constant depending on K_0, K_1, K_2 and $\sup_K F - \inf_K F$.

Additionally assume that $K \subset N$ is diffeomorphic to a convex domain V of $\mathbb{R}^n$, χ denotes the diffeomorphism from V to K; and suppose that there exist positive constants K_3 and K_4, such that for arbitrary $X \in TV$,

$$K_3|X| \le |\chi_*(X)| \le K_4|X|. \tag{5.12}$$

Then $u \in L^1(B_R, K)$ can be viewed as an L^1-function from B_R into $V \subset \mathbb{R}^n$. Define the mean value of u on B_R by

$$\bar{u}_R := \chi\Big[\frac{\int_{B_R}(\chi^{-1}\circ u) * 1}{V(R)}\Big]. \tag{5.13}$$

Applying the Neumann-Poincaré inequality yields

$$
\begin{aligned}
\int_{B_R} d_N^2(u,\bar{u}_R)*1 &\le K_4^2\int_{B_R}|\chi^{-1}(u)-\chi^{-1}(\bar{u}_R)|^2*1\\
&\le K_4^2K_2R^2\int_{B_R}|d(\chi^{-1}\circ u)|^2*1 \qquad (5.14)\\
&\le \frac{K_4^2K_2}{K_3^2}R^2\int_{B_R}|du|^2*1.
\end{aligned}
$$

Here d_N denotes the distance function on N induced by the metric. Now we write

$$h=H(u)=H(\bar{u}_R)+\big(H(u)-H(\bar{u}_R)\big),$$

then

$$IV_\rho=H(\bar{u}_R)\int_{B_R}\nabla G^\rho(\cdot,y)\cdot\nabla\eta*1+\int_{T_R}\big(H(u)-H(\bar{u}_R)\big)\nabla G^\rho(\cdot,y)\cdot\nabla\eta*1. \qquad (5.15)$$

Similar to (5.6)-(5.7), the first term can be estimated by

$$\lim_{\rho\to0}H(\bar{u}_R)\int_{B_R}\nabla G^\rho(\cdot,y)\cdot\nabla\eta*1=H(\bar{u}_R). \qquad (5.16)$$

We recall $|\nabla\eta|\le\frac{c_0}{R}$. In conjunction with (5.14) we have

$$
\begin{aligned}
&\Big|\int_{T_R}\big(H(u)-H(\bar{u}_R)\big)\nabla G^\rho(\cdot,y)\cdot\nabla\eta*1\Big|\\
\le& c_0\sup_K|\nabla^NH|R^{-1}\int_{T_R}d_N(u,\bar{u}_R)\big|\nabla G^\rho(\cdot,y)\big|*1 \qquad (5.17)\\
\le& c_0\sup_K|\nabla^NH|R^{-1}\Big(\int_{T_R}d_N^2(u,\bar{u}_R)*1\Big)^{\frac12}\Big(\int_{T_R}\big|\nabla G^\rho(\cdot,y)\big|^2\Big)^{\frac12}\\
\le& c_3(K_2,K_3,K_4)\sup_K|\nabla^NH|\Big(\int_{B_R}|du|^2*1\Big)^{\frac12}\Big(\int_{T_R}\big|\nabla G^\rho(\cdot,y)\big|^2\Big)^{\frac12}.
\end{aligned}
$$

Substituting (4.30) into (5.17) implies

$$\Big|\int_{T_R}\big(H(u)-H(\bar{u}_R)\big)\nabla G^\rho(\cdot,y)\cdot\nabla\eta*1\Big|\le c_4\sup_K|\nabla^NH|\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac12} \qquad (5.18)$$

where c_4 is a positive constant depending on K_1,K_2,K_3 and K_4.

¿From (5.4), (5.6)-(5.7), (5.9), (5.10)-(5.11), (5.15)-(5.16), (5.18), letting $\rho\to0$ we arrive at the following important formula

(5.19)

$$
\begin{aligned}
h(y)=H(u(y))\le& H(\bar{u}_R)+C_7(\sup_K|\nabla^NH|+\sup_K|\text{Hess }H|)\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac12}\\
&-\liminf_{\rho\to0}\int_{B_{\frac{R}{2}}}G^\rho(\cdot,y)\Delta h*1
\end{aligned}
$$

for arbitrary $y \in B_{\frac{R}{4}}$. Here C_7 is a positive constant depending on K_0, K_1, K_2, K_3, K_4 and $\sup_K F - \inf_K F$.

5.2. **Image shrinking property.** Based on the convex functions constructed in Theorem 3.1, with the aid of (5.19) and Lemma 4.4, we can derive an image shrinking property for weakly harmonic maps, that is, when we make the domain smaller, the image also gets smaller in a controlled manner.

Recall that on $\mathbb{V} = S^n \backslash \overline{S}^{n-1}_+$ there are well defined functions v and φ (see (2.5)).

Theorem 5.1. *Let M be a Riemannian manifold satisfying the DVP-condition, K be an arbitrary compact subset of $\mathbb{V} = S^n \backslash \overline{S}^{n-1}_+$, and we put*

$$c := \min\left\{\frac{1}{3}\inf_{x\in K} v, \inf_{x\in K}\varphi, \inf_{x\in K}(2\pi - \varphi)\right\}. \tag{5.20}$$

If $u \in H^{1,2}_{loc}(M, S^n)$ is a weakly harmonic map and there exist $y_0 \in M$ and $R_1 \le \frac{1}{3}R_0$ with $B_{3R_1} \subset\subset M$ and $u(B_{R_1}(y_0)) \subset K$, then there exists $\delta_1 \in (0,1)$, only depending on K_1, K_2 and c, such that $u(B_{\delta_1 R_1}(y_0))$ is contained in a geodesic ball in S^n of radius $\arccos(\frac{3}{2}c) < \frac{\pi}{2}$.

Proof. The definition of $\chi : (0, 2\pi) \times \mathbb{D}^{n-1} \to \mathbb{V}$ is shown in Observation 2.2. From (5.20), it is easily seen that

$$\chi^{-1}(K) \subset [c, 2\pi - c] \times \mathbb{D}^{n-1}(\sqrt{1-(3c)^2}). \tag{5.21}$$

Denote

$$\tilde{K} := \chi\big([c, 2\pi - c] \times \mathbb{D}^{n-1}(\sqrt{1-(3c)^2})\big), \tag{5.22}$$

then $\tilde{K} \supset K$ is diffeomorphic to a compact and convex subset of $\mathbb{R}^n$. Then we can define the mean value of u on B_R (denoted by $\bar{u}_R$) as in (5.13) ; and obviously $\bar{u}_R \in \tilde{K}$. Also note that the constants K_3 and K_4 given in (5.12) depend only on c.

Let F be the convex function on $\tilde{K}$ given in Theorem 3.1, then there exists $K_0 > 0$, such that Hess $F \ge K_0\ h$ on $\tilde{K}$, where h is the canonical metric on S^n. K_0 and $\sup_{\tilde{K}} F - \inf_{\tilde{K}} F$ depend only on c.

If U is defined as in (3.1), then $\tilde{K} \subset U$. Let $F(\cdot, \varphi_0)$ ($\varphi_0 \in [c, 2\pi - c]$) be the smooth family of the smooth functions on U constructed in Theorem 3.1. Put

$$\Sigma := \left\{(x, \varphi_0) \in \tilde{K} \times [c, 2\pi - c] : (x_1, x_2) = \sqrt{x_1^2 + x_2^2}(\sin\varphi_0, \cos\varphi_0)\right\},$$

then for each $(x, \varphi_0) \in \Sigma$, for $x_{\varphi_0} := (\sin\varphi_0, \cos\varphi_0, 0, \cdots, 0)$, then $(x, x_{\varphi_0}) = \sqrt{x_1^2 + x_2^2} \ge 3c$; which implies $F|_\Sigma < 1$. Hence by the compactness of Σ, we can find a positive constant c_5, such that

$$F|_\Sigma \le 1 - c_5. \tag{5.23}$$

Put

$$c_6 := \sup_{\varphi_0\in[c,2\pi-c]}\Big(\sup_{\tilde{K}}\big|\nabla F(\cdot,\varphi_0)\big| + \sup_{\tilde{K}}\big|\text{Hess } F(\cdot,\varphi_0)\big|\Big). \tag{5.24}$$

Then for $\varepsilon = c_5^2 c_6^{-2} C_7^{-2}$, Lemma 4.4 enables us to find $R \in [\frac{1}{2}\exp(-C_6\varepsilon^{-1})R_1, \frac{1}{2}R_1]$, where C_6 is a positive constant only depending on K_1, K_2 and c , such that

$$\frac{(2R)^2}{V(R)}\int_{B_R}|du|^2 * 1 \le \varepsilon. \tag{5.25}$$

Since $\bar{u}_R \in \tilde{K}$, there is $\varphi_0 \in [c, 2\pi - c]$ satisfying $(\bar{u}_R, \varphi_0) \in \Sigma$, and moreover (5.19) yields

$$\begin{aligned} F(u(y),\varphi_0) \le{}& F(\bar{u}_R,\varphi_0) + \frac{1}{2}C_7 c_6 \varepsilon^{\frac{1}{2}} \\ & - \liminf_{\rho\to 0}\int_{B_{\frac{R}{2}}} G^\rho(\cdot,y)\Delta\big(F(\cdot,\varphi_0)\circ u\big) * 1 \\ \le{}& 1 - c_5 + \frac{1}{2}C_7 c_6\varepsilon^{\frac{1}{2}} = 1 - \frac{1}{2}c_5 < 1 \end{aligned} \tag{5.26}$$

for all $y \in B_{\frac{R}{4}}$. Hence if we take $\delta_1 = \frac{1}{8}\exp(-C_6\varepsilon^{-1})$, then for arbitrary $y \in B_{\delta_1 R_1} \subset B_{\frac{R}{4}}$, we have $(u(y), x_{\varphi_0}) > \frac{3}{2}c$; i.e., $u(B_{\delta_1 R_1})$ is contained in the geodesic ball centered at x_{φ_0} and of radius $\arccos(\frac{3}{2}c)$.

□

5.3. **Estimating the oscillation.** Now we put $R^{(0)} := \delta_1 R_1$, $r_0 := \arccos(\frac{3}{2}c)$, $x^{(0)} := x_{\varphi_0}$ and denote by $\mathcal{B}_r(x)$ the geodesic ball of S^n centered at x and of radius r. First of all, note that one can find positive constants c_7, c_8 and c_9, only depending on c, such that

$$\sup_{\tilde{K}}|\nabla\rho(x,\cdot)^2| + \sup_{\tilde{K}}|\text{Hess } \rho(x,\cdot)^2| \le c_7 \tag{5.27}$$

and

$$c_8 \le |d\exp_x| \le c_9 \qquad \text{on } \bar{\mathbb{D}}(r_0) \tag{5.28}$$

for all $x \in S^n$. Here $\rho(x,\cdot)$ is the distance function on S^n from x, and $\exp_x$ denotes the exponential mapping of S^n at x; its restriction on $\bar{\mathbb{D}}(r_0)$ is a diffeomorphism. Since

$$u(B_{R^{(0)}}) \subset \mathcal{B}_{r_0}(x^{(0)}) = \exp_{x^{(0)}}(\mathbb{D}(r_0)),$$

we can define the mean value of u on B_R with $R \le R^{(0)}$ by

$$\bar{u}_R = \exp_{x^{(0)}}\Big[\frac{\int_{B_R}(\exp_{x^{(0)}}^{-1}\circ u) * 1}{V(R)}\Big],$$

and $u(B_{R^{(0)}})\subset\mathcal{B}_{r_0}(x^{(0)})$ implies $\bar{u}_R\in\mathcal{B}_{r_0}(x^{(0)})$. Hence by (5.19), there is a positive constant c_{10}, only depending on K_1,K_2 and c, such that for all $R\le R^{(0)}$, $y\in B_{\frac{R}{4}}$ and $x\in S^n$

$$\begin{aligned}\rho(x,u(y))^2\le&\rho(x,\bar{u}_R)^2+c_{10}c_7\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac{1}{2}}\\&-\liminf_{\rho\to0}\int_{B_{\frac{R}{2}}}G^\rho(\cdot,y)\Delta\big(\rho(x,\cdot)^2\circ u\big)*1.\end{aligned}\tag{5.29}$$

For arbitrary $\varepsilon>0$, Lemma 4.4 enables us to find $R\in[4R^{(1)},R^{(0)}]$, where $R^{(1)}=\delta^{(0)}(\varepsilon,K_1,K_2,c)R^{(0)}$, such that

$$c_{10}c_7\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac{1}{2}}\le\varepsilon.$$

Since $\bar{u}_R\in\mathcal{B}_{r_0}(x^{(0)})$, one can easily find $x^{(1)}\in\mathcal{B}_{r_0}(x^{(0)})$, such that

$$\rho(x^{(0)},x^{(1)})\le\frac{\pi}{2}-r_0\qquad\text{and}\qquad\rho(x^{(1)},\bar{u}_R)\le 2r_0-\frac{\pi}{2}.$$

Thereby $\mathcal{B}_{r_0}(x^{(0)})\subset\mathcal{B}_{\frac{\pi}{2}}(x^{(1)})$ and hence $\rho(x^{(1)},\cdot)^2$ is convex on $u(B_{R^{(0)}})\supset u(B_{\frac{R}{2}})$. Letting $x=x^{(1)}$ in (5.29) yields

$$\rho(x^{(1)},u(y))^2\le(2r_0-\frac{\pi}{2})^2+\varepsilon\qquad\forall y\in B_{R^{(1)}}\subset B_{\frac{R}{4}}.$$

Let

$$\varepsilon=(\frac{3}{2}r_0-\frac{\pi}{4})^2-(2r_0-\frac{\pi}{2})^2,$$

then we arrive at

$$u(B_{R^{(1)}})\subset\mathcal{B}_{r_1}(x^{(1)})\qquad\text{where }r_1=\frac{3}{2}r_0-\frac{\pi}{4}.\tag{5.30}$$

Similarly for each $j\ge1$, if $r_j>\frac{\pi}{4}$, we can find $x^{(j+1)}\in\mathcal{B}_{r_j}(x^{(j)})$ and $R^{(j+1)}=\delta^{(j)}(K_1,K_2,c)R^{(j)}$, such that

$$u(B_{R^{(j+1)}})\subset\mathcal{B}_{r_{j+1}}(x^{(j)})\qquad\text{where }r_{j+1}=\frac{3}{2}r_j-\frac{\pi}{4}.\tag{5.31}$$

Noting that $r_0>r_1>r_2>\cdots$ and $r_j-r_{j+1}=\frac{1}{2}(\frac{\pi}{2}-r_j)\ge\frac{1}{2}(\frac{\pi}{2}-r_0)$, after k steps (k only depending on c) we can arrive at

$$u(B_{R^{(k)}})\subset\mathcal{B}_{r_k}(x^{(k)})\subset\mathcal{B}_{\frac{\pi}{4}}(x^{(k)}).\tag{5.32}$$

This implies that for arbitrary $R\le R^{(k)}$, $\bar{u}_R\subset\mathcal{B}_{\frac{\pi}{4}}(x^{(k)})$ and moreover $u(B_{R^{(k)}})\subset\mathcal{B}_{\frac{\pi}{2}}(\bar{u}_R)$. Hence $\rho(\bar{u}_R,\cdot)^2$ is convex on $u(B_{R^{(k)}})$. Letting $x=\bar{u}_R$ in (5.29) yields

$$\rho(\bar{u}_R,u(y))^2\le c_{10}c_7\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac{1}{2}}.\tag{5.33}$$

for arbitrary $y\in B_{\frac{R}{4}}$. Hence the oscillation of u on $B_{\frac{R}{4}}$ can be controlled by

$$\mathrm{osc}_{B_{\frac{R}{4}}}u\le2(c_{10}c_7)^{\frac{1}{2}}\Big(\frac{R^2}{V(R)}\int_{B_R}|du|^2*1\Big)^{\frac{1}{4}}.\qquad\forall R\le R^{(k)}.\tag{5.34}$$

Again applying Lemma 4.4 we have the following theorem:

Theorem 5.2. *When $M, u, \mathbb{V}, K, c, y_0, R_1$ satisfy the assumptions of Theorem 5.1, then u is continuous at y_0. More precisely, for arbitrary $\varepsilon > 0$, there is $\delta_2 \in (0,1)$, only depending on K_1, K_2, c and ε, such that*

$$osc_{B_{\delta_2 R_1}(y_0)} u \leq \varepsilon. \tag{5.35}$$

5.4. **Hölder estimates.** Now we can proceed as in [26] and Ch. 7.6 in [27] to get the Hölder estimates for weakly harmonic maps.

By Theorem 5.2, there is a constant $\delta_2 \in (0,1)$, depending only on K_1, K_2 and c, such that

$$u(B_{R_2}(y_0)) \subset \mathcal{B}_{\frac{\pi}{8}}(x_0) \qquad \text{where } R_2 = \delta_2 R_1, x_0 = u(y_0).$$

This implies that the function $\rho^2(\cdot, x)$ is strictly convex on $u(B_{R_2})$ for arbitrary x in the convex hull of $u(B_{R_2})$; furthermore, one can find a positive constant c_{11}, independent of the choice of x, such that

$$\text{Hess } \rho^2(\cdot, x) \geq c_{11} h. \tag{5.36}$$

Similar to the above, one can define the mean value of u on B_R with $R \leq R_2$ by

$$\bar{u}_R = \exp_{x_0}\Big[\frac{\int_{B_R}(\exp_{x_0}^{-1} \circ u) * 1}{V(R)}\Big].$$

Then $\bar{u}_R$ lies in the convex hull of $u(B_{R_2})$. The convexity of $\rho^2(\cdot, x)$ implies that the compositions

$$v = \rho^2(\cdot, x_0) \circ u, \qquad \text{and } w = \rho^2(\cdot, \bar{u}_{\frac{R}{2}}) \circ u$$

are both subharmonic functions. Applying Corollary 4.2 yields

$$w_{+,\varepsilon^k R} \leq \varepsilon^2 w_{+,R} + (1-\varepsilon^2)\bar{w}_{R'} \tag{5.37}$$

for some $R' \in [\varepsilon^k R, \frac{R}{2}]$. By (5.14), the doubling property and the Telescoping Lemma 4.4, we arrive at

$$\begin{aligned}
\bar{w}_{R'} &= ⨍_{B_{R'}} \rho^2(u, \bar{u}_{\frac{R}{2}}) \leq \frac{1}{V(R')}\int_{B_{\frac{R}{2}}} \rho^2(u, \bar{u}_{\frac{R}{2}}) * 1 \\
&\leq \frac{CR^2}{V(R')}\int_{B_{\frac{R}{2}}} |du|^2 * 1 \leq \frac{CR^2}{V(\frac{R}{2})}\int_{B_{\frac{R}{2}}} |du|^2 * 1 \\
&\leq c_{12}(v_{+,R} - v_{+,\frac{R}{2}}).
\end{aligned} \tag{5.38}$$

Here c_{12} depends on ε. With the aid of the triangle inequality, it is easily seen that $v_{+,\varepsilon^k R} \leq 4w_{+,\varepsilon^k R}$ and $w_{+,R} \leq 4v_{+,R}$. Substituting (5.38) into (5.37) yields

$$\begin{aligned}
v_{+,\varepsilon^k R} &\leq 4w_{+,\varepsilon^k R} \leq 4\varepsilon^2 w_{+,R} + 4(1-\varepsilon^2)w_{R'} \\
&\leq 16\varepsilon^2 v_{+,R} + 4c_{12}(1-\varepsilon^2)(v_{+,R} - v_{+,\frac{R}{2}}) \\
&\leq 16\varepsilon^2 v_{+,R} + c_{13}(v_{+,R} - v_{+,\varepsilon^k R})
\end{aligned} \tag{5.39}$$

where c_{13} is a positive constant depending on ε. Take $\varepsilon = \frac{1}{8}$, and put $\delta = \varepsilon^k$, then

$$v_{+,\delta R} \leq \frac{\frac{1}{4} + c_{13}}{1 + c_{13}} v_{+,R}. \tag{5.40}$$

By iteration, we arrive at

$$\sup_{y \in B_R(y_0)} \rho\big(u(y), u(y_0)\big) \leq c_{14} \Big(\frac{R}{R_2}\Big)^{\sigma} = c_{14} \delta_2^{-\sigma} R_1^{-\sigma} R^{\sigma} \tag{5.41}$$

for arbitrary $0 < R \leq R_2$, where $c_{14} > 0$, $\sigma \in (0,1)$ are constants depending only on K_1, K_2 and c.

We note that for each $y_1 \in B_{\frac{R_1}{2}}(y_0)$, $u(B_{\frac{R_1}{2}}(y_1)) \subset u(B_{R_1}(y_0)) \subset K$; therefore, (5.41) still holds true for arbitrary $0 < R \leq \frac{R_2}{2}$ when y_0 is replaced by y_1. We can get the following theorem:

Theorem 5.3. *With $M, u, \mathbb{V}, K, c, y_0, R_1$ satisfying the above assumptions, there exist numbers $\sigma \in (0,1)$, $\delta_3 \in (0,1)$ and $C_8 > 0$, depending only on K_1, K_2 and c, such that the σ-Hölder seminorm of u on $\bar{B}_{\delta_3 R_1}(y_0)$ can be estimated by*

$$[u]_{C^{\sigma}(\bar{B}_{\delta_3 R_1})(y_0)} \leq C_8 R_1^{-\sigma}. \tag{5.42}$$

Here and in the sequel,

$$[u]_{C^{\sigma}(S)} := \sup_{y_1, y_2 \in S, y_1 \neq y_2} \frac{\rho\big(u(y_1), u(y_2)\big)}{r(y_1, y_2)^{\sigma}} \tag{5.43}$$

where $r(\cdot,\cdot)$ is the distance function on M induced by the metric g.

We point out that the Hölder bound thus depends only on the geometry of the domain, as incorporated in the volume doubling constant K_1 of (4.9) and the Poincaré inequality constant K_2 of (4.10) and on the convexity condition on the image as reflected in the constant c that controls the Hessian of our convex functions, but not on the map u, and in particular not on its energy. Therefore, in the sequel, in our Liouville theorem, we do not need to require that the map in question have finite energy.

5.5. **A Liouville type theorem.** Letting $R_1 \to +\infty$ in Theorem 5.2 or Theorem 5.3, we obtain the following Liouville-type theorem:

Theorem 5.4. *Let (M, g) be a complete Riemannian manifold satisfying the DVP-condition with $R_0 = +\infty$, and K be an arbitrary compact subset of $\mathbb{V} = S^n \backslash \overline{S}^{n-1}_{+}$. If $u : M \to S^n$ is a weakly harmonic map, and almost every $y \in M$ satisfies $u(y) \in K$, then u is constant.*

Remark: Solomon [41] obtained regularity and Liouville theorems for **energy minimizing** harmonic maps when the image omits a neighborhood of a totally geodesics S^{n-2}. Since we wish to apply our Liouville theorem to the Bernstein problem in the next section, we cannot make the assumption that the harmonic maps under consideration be energy minimizing as in general it is not clear under which conditions Gauss maps are energy minimizing. In any case, his Liouville theorem is only derived for the case where the domain is Euclidean space. And Solomon's result ceases to be true without the energy minimizing assumption.

6. Analytic and geometric conclusions

6.1. **Simple manifolds.** Let $M = \mathbb{D}^m(r_0) \subset \mathbb{R}^m$ with metric $g = g_{\alpha\beta}(y^1, \cdots, y^m)dy^\alpha dy^\beta$. We suppose that there exist two positive constants λ and μ, such that

$$\lambda^2|\xi|^2 \le g_{\alpha\beta}\xi^\alpha\xi^\beta \le \mu^2|\xi|^2. \tag{6.1}$$

Now we define a distance function d on M:

$$(y_1, y_2) \in M \times M \mapsto \lambda|y_1 - y_2|. \tag{6.2}$$

Then obviously

$$r(y_1, y_2) \ge \lambda|y_1 - y_2| = d(y_1, y_2)$$

where $r(\cdot,\cdot)$ denotes the usual distance function induced by the metric g.

Denoting by $\mathbb{D}(y, R)$ the Euclidean disk centered at y and of radius R, then with respect to the distance function d,

$$B_R(y) = \mathbb{D}(y, \lambda^{-1}R) \cap \mathbb{D}(r_0). \tag{6.3}$$

Taking

$$R_0 := \lambda r_0, \tag{6.4}$$

then for arbitrary $R \le R_0$,

$$V(y, 2R) = \mathrm{Vol}\big(\mathbb{D}\big(y, 2\lambda^{-1}R) \cap \mathbb{D}(r_0)\big) \le \Big(\frac{2\mu}{\lambda}\Big)^m R^m\omega_m \tag{6.5}$$

and

$$V(y, R) \ge \mathrm{Vol}\Big(\mathbb{D}\big(y - \frac{1}{2}\lambda^{-1}R\frac{y}{|y|}, \frac{1}{2}\lambda^{-1}R\big)\Big) \ge 2^{-m}R^m\omega_m. \tag{6.6}$$

Here ω_m is the volume of the m-dimensional Euclidean unit disk equipped with the canonical metric. Hence (M, g) satisfies condition (V) with $K_1 = (\frac{4\mu}{\lambda})^m$.

We note that $B_R(y) \subset\subset M$ if and only if $B_R(y) = \mathbb{D}(y, \lambda^{-1}R)$. By [32],

$$\int_{\mathbb{D}(y,\lambda^{-1}R)} |v - \bar{v}_{\lambda^{-1}R}|^2 dy \le 4\pi^{-2}\lambda^{-2}R^2 \int_{\mathbb{D}(y,\lambda^{-1}R)} |Dv|^2 dy \tag{6.7}$$

where

$$\bar{v}_{\lambda^{-1}R} = \frac{\int_{\mathbb{D}(y,\lambda^{-1}R)} v\ dy}{\int_{\mathbb{D}(y,\lambda^{-1}R)} dy}$$

and $|Dv|^2 = \sum_\alpha (\partial^\alpha v)^2$. Noting that $*1 = \sqrt{\det(g_{\alpha\beta})}dy$ and $|\nabla v|^2 = g^{\alpha\beta}\partial^\alpha v \partial^\beta v$, where $\big(g^{\alpha\beta}\big)$ denotes the inverse matrix of $\big(g_{\alpha\beta}\big)$, it is easy for us to arrive at

$$\begin{aligned}\int_{B_R(y)} |v - \bar{v}_{B_R(y)}|^2 * 1 &\le \mu^m \int_{\mathbb{D}(y,\lambda^{-1}R)} |v - \bar{v}_{\lambda^{-1}R}|^2 dy \\ &\le 4\pi^{-2}\lambda^{-2}\mu^m R^2 \int_{\mathbb{D}(y,\lambda^{-1}R)} |Dv|^2 dy \\ &\le 4\pi^{-2}\Big(\frac{\mu}{\lambda}\Big)^{m+2} R^2 \int_{B_R(y)} |\nabla v|^2 * 1\end{aligned} \tag{6.8}$$

which means that M satisfies condition (P) with $K_2 = 4\pi^{-2}\big(\frac{\mu}{\lambda}\big)^{m+2}$.

Therefore, applying Theorem 5.3 yields the following Hölder estimate:

Theorem 6.1. *Let $M = \mathbb{D}^m(r_0)$ with metric $g = g_{\alpha\beta}dy^\alpha dy^\beta$, and suppose that there exist two positive constants λ and μ, such that*

$$\lambda^2|\xi|^2 \le g_{\alpha\beta}\xi^\alpha\xi^\beta \le \mu^2|\xi|^2$$

for arbitrary $\xi \in M$. Suppose $u \in H^{1,2}_{loc}(M, S^n)$ is a weakly harmonic map, and there exists a compact set $K \subset S^n \backslash \overline{S}^{n-1}_+$, such that $u(y) \in K$ for almost every $y \in M$. Then there exist numbers $\sigma_1 \in (0,1)$, $\varepsilon_4 \in (0,1)$, and $C_9 > 0$, depending only on $m, \frac{\mu}{\lambda}$ and K, but not on r_0, such that the σ_1-Hölder seminorm of u on $\bar{B}_{\varepsilon_4 r_0}$ is estimated by

$$[u]_{C^{\sigma_1}(\bar{B}_{\varepsilon_4 r_0})} \le C_9(\lambda r_0)^{-\sigma_1}. \tag{6.9}$$

Remark: Here the definition of Hölder seminorm is the same as (5.43). In many references, e.g. [22], the σ-Hölder seminorm is given by

$$[u]_{C^\sigma(S)} := \sup_{y_1,y_2\in S, y_1\neq y_2} \frac{\rho\big(u(y_1), u(y_2)\big)}{|y_1 - y_2|^\sigma}. \tag{6.10}$$

With that definition, the corresponding estimate would read as

$$[u]_{C^{\sigma_1}(\bar{B}_{\varepsilon_4 r_0})} \le C_9 r_0^{-\sigma_1}. \tag{6.11}$$

If (M, g) is a Riemannian manifold, then every point $y_0 \in M$ has a coordinate patch with induced metric, hence from Theorem 6.1 the following estimate immediately follows:

Theorem 6.2. *Let (M, g) be an m-dimensional Riemannian manifold, $u \in H^{1,2}_{loc}(M, S^n)$ be a weakly harmonic map, and K be a compact subset of $S^n \backslash \overline{S}^{n-1}_+$, with the property that almost every $y \in M$ satisfies $u(y) \in K$. Then for any compact subset S of M, there exist numbers $\sigma_2 \in (0,1)$, and $C_{10} > 0$, depending on m, K, S, and on the metric of M, but not on u, such that the estimate*

$$[u]_{C^{\sigma_2}(S)} \le C_{10} \tag{6.12}$$

holds. Moreover, if M is homogeneously regular (in the sense of Morrey) with constants λ and μ, then σ_2 depends only on m, λ, μ and K, while C_{10} depends, apart from these parameters, also on S.

We use here a slightly altered definition for **homogenously regular** manifolds (cf. [30], p.363): A C^1-manifold M is said to be homogenously regular if there exist positive numbers λ and μ, such that each point y_0 of M is the center of a coordinate patch $\{z : |z| \leq 1\}$ for which

$$\lambda^2|\xi|^2 \leq g_{\alpha\beta}(z)\xi^\alpha\xi^\beta \leq \mu^2|\xi|^2$$

holds for all $\xi \in \mathbb{R}^m$ and each $|z| \leq 1$.

Recall that a Riemannian manifold M is said to be **simple**, if it is described by a single set of coordinates y on $\mathbb{R}^m$ and by a metric

$$g = g_{\alpha\beta}(y)dy^\alpha dy^\beta$$

for which there exist positive numbers λ and μ such that

$$\lambda^2|\xi|^2 \leq g_{\alpha\beta}(y)\xi^\alpha\xi^\beta \leq \mu^2|\xi|^2$$

holds for all $\xi, y \in \mathbb{R}^m$.

Applying Theorem 5.4 yields the following Liouville-type theorem:

Theorem 6.3. *Let u be a weakly harmonic map from a simple Riemannian manifold M to S^n. If $u(M)$ is contained in a compact subset of $S^n\backslash\overline{S}^{n-1}_+$, then u has to be a constant map.*

Remark: Let M be an entire graph given by $f : \mathbb{R}^n \to \mathbb{R}$. If $|\nabla f| \leq \beta < \infty$, then the induced metric on M is simple. Furthermore, the image under its Gauss map lies in a closed subset of an open hemisphere. This is the situation of [31].

For higher codimensional graphs with suitable bounded slope of defining functions we also obtain simple manifolds with convex Gauss image in Grassmannian manifolds. This is the situation of [22].

6.2. **Manifolds with nonnegative Ricci curvature.** Let (M, g) be a Riemannian manifold with Ric $M \geq 0$. For the canonical distance function $d(\cdot,\cdot)$ induced by g, Condition (D) is obviously satisfied. By the classical relative volume comparison theorem, M enjoys the doubling property with constant $K_1 = 2^m$. In [7], P. Buser shows that M satisfies the Neumann-Poincaré inequality with constant $K_2 = K_2(m)$. Therefore, Theorem 5.4 yields the following Liouville-type theorem:

Theorem 6.4. *Let u be a weakly harmonic map from a Riemannian manifold M with nonnegative Ricci curvature to S^n. If $u(M)$ is contained in a compact subset of $S^n\backslash\overline{S}^{n-1}_+$, then u has to be a constant map.*

6.3. **Bernstein type theorems.** Let $M^m \subset \mathbb{R}^{m+1}$ be a complete hypersurface with the induced Riemannian metric. Then we can define $d : M \times M \to \mathbb{R}$

$$(y_1, y_2) \to |y_1 - y_2| \tag{6.13}$$

where $|y_1 - y_2|$ denotes the Euclidean distance from y_1 to y_2. Obviously $d(y_1, y_2) \leq r(y_1, y_2)$. Therefore M satisfied Condition (D) if and only if the inclusion map $i : M \to \mathbb{R}^{m+1}$ is injective, i.e. M is an imbedded hypersurface.

Given $y \in M$ and $R > 0$, the density is defined by

$$\Theta(y, R) = \frac{V(y, R)}{\omega_m R^m}. \tag{6.14}$$

Here ω_m is the volume of $S^m \subset \mathbb{R}^{m+1}$ equipped with the canonical metric. The following monotonicity of the volume is well-known.

Lemma 6.1. *If M^m is any complete minimal submanifold in Euclidean space, then $\Theta(y, R)$ is monotonically nondecreasing in r and $\lim_{R\to 0} \Theta(y, R) = 1$.*

We say that a minimal hypersurface M has Euclidean volume growth if there exist $y_0 \in M$ and a positive constant C, such that

$$\Theta(y_0, R) \leq C \tag{6.15}$$

for arbitrary $R > 0$. For each $y \in M$, if we denote $r = d(y, y_0)$, then

$$\Theta(y, R) = \frac{V(y, R)}{\omega_m R^m} \leq \frac{V(y_0, R + r)}{\omega_m R^m} = \Big(\frac{R + r}{R}\Big)^m \Theta(y_0, R + r)$$

Letting $R \to +\infty$ implies

$$\lim_{R\to+\infty} \Theta(y, R) \leq \lim_{R\to+\infty} \Theta(y_0, R) = C.$$

And moreover

$$\frac{V(y, 2R)}{V(y, R)} = 2^m \frac{\Theta(y, 2R)}{\Theta(y, R)} \leq 2^m C. \tag{6.16}$$

i.e. M satisfies Condition (V) with $R_0 = +\infty$.

The Gauss map $\gamma : M \to S^m$ is defined by

$$\gamma(y) = T_y M \in S^m$$

via the parallel translation in $\mathbb{R}^{m+1}$ for all $y \in M$. Ruh-Vilms [33] proved that the mean curvature vector of M is parallel if and only if its Gauss map is a harmonic map. This fact enables us to apply Theorem 5.4 and obtain a Bernstein type theorem as follows.

Theorem 6.5. *Let $M^m \subset \mathbb{R}^{m+1}$ be a complete minimal embedded hypersurface. Assume M has Euclidean volume growth, and there is a positive constant C, such that for arbitrary $y \in M$ and $R > 0$, the Neumann-Poincaré inequality*

$$\int_{B_R(y)} |v - \bar{v}_{B_R(y)}|^2 * 1 \leq CR^2 \int_{B_R(y)} |\nabla v|^2 * 1$$

holds for all $v \in C^\infty(B_R(y))$, *where*

$$B_R(y) = \{z \in M : |z - y| < R\}.$$

If the image under the Gauss map omits a neighborhood of $\overline{S}^{n-1}_+$, *then* M *has to be an affine linear space.*

This naturally raises the question under which conditions a complete embedded minimal hypersurfaces in Euclidean space satisfies the Neumann-Poincaré inequality. So far, only partial results in this direction seem to be known. When M is an **area-minimizing** hypersurface, the Neumann-Poincaré inequality has been proved by Bombieri-Giusti (see [6]). Hence we have:

Theorem 6.6. *Let* $M^m \subset \mathbb{R}^{m+1}$ *be a complete embedded area-minimizing hypersurface. Assume* M *has Euclidean volume growth. If the image under the Gauss map omits a neighborhood of* $\overline{S}^{n-1}_+$, *then* M *has to be an affine linear space.*

Remarks:

- Solomon [40] proved such a result under a somewhat weaker condition on the Gauss image, but needed the additional topological assumption that the first Betti number of M vanishes. Presumably, Solomon's result ceases to be true without that topological assumption.
- Recently, N.Wickramasekera[42] proved new Poincaré type inequalities for stable minimal hypersurfaces of dimension at most 6 in Euclidean space of controlled volume growth. The dimensional restriction in his results is related to the fact that in higher dimensions, stable minimal hypersurfaces may have singularities.

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Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany.

E-mail address: jost@mis.mpg.de

Institute of Mathematics, Fudan University, Shanghai 200433, China.

E-mail address: ylxin@fudan.edu.cn

Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, 04103 Leipzig, Germany.

E-mail address: lingyang@mis.mpg.de