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ENERGY IDENTITIES AND BLOW UP ANALYSIS FOR SOLUTIONS OF THE SUPER LIOUVILLE EQUATION

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Abstract. In this paper, we study the super Liouville equations, a natural generalization of the Liouville equation. We establish energy identities and a precise blow-up analysis for solutions of the super Liouville equations.

1. Introduction

In [JWZ], we have introduced the **super Liouville functional**, a conformally invariant functional that couples a real-valued function and a spinor ψ on a closed Riemann surface M with conformal metric g and a spin structure,

$$E(u,\psi)=\int_M\{\frac{1}{2}|\nabla u|^2+K_gu+\langle(\not{D}+e^u)\psi,\psi\rangle-e^{2u}\}dv. \tag{1}$$

K_g is the Gaussian curvature of M. The Dirac operator $\not{D}$ is defined by $\not{D}\psi:=\sum_{\alpha=1}^{2}e_\alpha\cdot\nabla_{e_\alpha}\psi$, where $\{e_1,e_2\}$ is an orthonormal basis on TM, ∇ is the Levi-Civita connection on M with respect to g and $\cdot$ denotes Clifford multiplication in the spinor bundle ΣM of M. Finally, $\langle\cdot,\cdot\rangle$ is the natural Hermitian metric on ΣM induced by g. For the geometric background, see [LM] or [Jo].
The Euler-Lagrange system for $E(u,\psi)$ is

$$\begin{cases} -\Delta u &= 2e^{2u}-e^u\langle\psi,\psi\rangle-K_g \\ \not{D}\psi &= -e^u\psi \end{cases}\qquad \text{in } M. \tag{2}$$

where Δ is the Laplacian with respect to g. These equations are called the **super Liouville equations**.

It is clear that, when ψ vanishes, we obtain the Liouville equation

$$-\Delta u=2e^{2u}-K_g \qquad \text{in } M. \tag{3}$$

Liouville [Liou] studied this equation in the plane, that is, for $K_g=0$. The Liouville equation is a basic equation for the complex analysis and differential geometry of Riemann surfaces; in particular it shows up in the prescribing curvature problem. It also occurs naturally in string theory as discovered by Polyakov [Po], from the gauge anomaly in quantizing the string action. There then also is a natural supersymmetric version of the Liouville functional and equation, coupling the bosonic scalar field to a fermionic spinor field. This is the motivation behind the functional (1). Note, however, that we consider ordinary instead of fermionic spinor fields in the super-Liouville functional. An essential feature of the Liouville action is its conformal invariance. For results by physicists about super-Liouville equations, we refer to [CC] and [Pr].

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The conformal invariance of the super Liouville functional suggests that the space of solutions is not compact, but that sequences of solutions may blow up at isolated points, with a quantized loss of "energy". In this paper, we wish to probe into this blow up behavior, and in particular to relate the number of blow up points to the genus of the underlying Riemann surface M. It turns out that for this analysis the precise coupling between the "bosonic" u and the "fermionic" ψ is essential.

In technical terms, we shall be able to build upon [JWZ], where we have provided an analytic foundation for system (2). We have established the small energy regularity theorem, proved a removable singularity theorem, and developed the fundamental blow up analysis of solutions. The key analytical points are that singularities in solutions (u_n, ψ_n) of (2) on closed surfaces, or more generally with bounded energy $\int e^{2u_n} + |\psi_n|^4$, can form only at isolated points x where the limit $\max\{u_n(x), |\psi_n(x)|\}$ tends to infinity. Away from those singularities $u_n(x)$ remains either uniformly bounded or converge to $-\infty$. The precise results are contained in:

Theorem 1.1. *(see [JWZ]) Assume that (u_n, ψ_n) satisfy*

$$\begin{cases} -\Delta u_n &= 2e^{2u_n} - e^{u_n}\langle \psi_n, \psi_n\rangle - K_g, \\ \not{D}\psi_n &= -e^{u_n}\psi_n, \end{cases} \tag{4}$$

in M with the energy condition

$$\int_M e^{2u_n} dv < C, \text{ and } \int_M |\psi_n|^4 \, dv < C. \tag{5}$$

for some positive constant C.

Define the blow up set of (u_n, ψ_n)

$$\begin{aligned} \Sigma_1 &= \{x \in M, \text{ there is a sequence } y_n \to x \text{ such that } u_n(y_n) \to +\infty\} \\ \Sigma_2 &= \{x \in M, \text{ there is a sequence } y_n \to x \text{ such that } |\psi_n(y_n)| \to +\infty\}. \end{aligned}$$

Then $\Sigma_2 \subset \Sigma_1$ and (u_n, ψ_n) admits a subsequence, still denoted by (u_n, ψ_n), satisfying one of the following cases:

i) *u_n is bounded in $L^\infty(M)$.*
ii) *$u_n \to -\infty$ uniformly on M.*
iii) *Σ_1 is finite, nonempty and either*

$$u_n \text{ is bounded in } L^\infty_{loc}(M\backslash\Sigma_1)$$

or

$$u_n \to -\infty \text{ uniformly on compact subsets of } M\backslash\Sigma_1.$$

The challenge is to extend the full blow up theory for the Liouville equation, see [BCL], [CL1], [CL2], [CL3], [LSh], [Ly] and [JLW] and the references therein. In particular, this should contain the energy identity for solutions, the blow up values at the blow up points, and the profile of solutions near the blow up point. Here, we investigate these problems. What makes the analysis really interesting is that the finer aspects of the blow up are revealed by analyzing the behavior of the spinor part ψ_n which, in fact, turns out to be similar to two-dimensional harmonic maps.

We can show

Theorem 1.2. *Let M be a closed Riemann surface with a fixed spin structure, and suppose (u_n, ψ_n) is a sequence of smooth solutions of (4) and (5), with $\Sigma_1 = \{x_1, x_2, \cdots, x_l\}$. Then there are finitely many solutions of (2) on S^2: $(u^{i,k}, \psi^{i,k})$, $i = 1, 2, \cdots, l; k = 1, 2, \cdots, L_i$, such that, after selection of a subsequence, ψ_n converges in C^∞_{loc} to ψ on $M \backslash \Sigma_1$ and we have the energy identity:*

$$\lim_{n\to\infty} \int_M |\psi_n|^4 dv = \int_M |\psi|^4 dv + \sum_{i=1}^{l} \sum_{k=1}^{L_i} \int_{S^2} |\psi^{i,k}|^4 dv. \tag{6}$$

The key point behind the energy identity in (6) is that the neck energy of spinors ψ_n is zero. Therefore, as an application of the energy identity for spinors ψ_n, we rule out the first case in (iii) in Theorem 1.1. This completes the qualitative picture of the blow up process of u_n. The remaining quantitative aspects, i.e., the energy identity for u_n and the profile of u_n at the blow up point, will be considered in a later paper. Thus, we can state our Theorem as:

Theorem 1.3. *Assume that (u_n, ψ_n) is a sequence of solutions to (4) and (5), and the blow up set $\Sigma_1 \neq \emptyset$. Then we have*

$$u_n \to -\infty \quad \text{uniformly on compact subset of } M \backslash \Sigma_1,$$

and

$$2e^{2u_n} - e^{u_n}|\psi_n|^2 \rightharpoonup \sum_{x_i \in \Sigma_1} \alpha_i \delta_{x_i},$$

in the distribution sense and with $\alpha_i \geq 4\pi$.

Remark 1.4. In [JWZ], we have obtained this result under the condition $\Sigma_1 \backslash \Sigma_2 \neq \emptyset$.

Further exploring the energy identity for spinors ψ_n, we will compute the blow up value at the blow up point. Assuming that $p \in \Sigma_1$, we define the blow up value at p as

$$m(p) = \lim_{r\to 0} \lim_{n\to\infty} \int_{B_r(p)} (2e^{2u_n} - e^{u_n}|\psi_n|^2) dv.$$

To calculate the value of $m(p)$, we need a Pohozaev type identity for smooth solutions of (2). This will be established in the second section. With this Pohozaev type identity and the asymptotic behavior of (u_n, ψ_n) at a blow up point obtained in Theorem 1.3, we can show:

Theorem 1.5. *If $p \in \Sigma_1$, then we have $m(p) = 4\pi$.*

Furthermore, from (4) and the Gauss-Bonnet formula, we deduce

$$\int_M (2e^{2u_n} - e^{u_n}|\psi_n|^2) dv = 4\pi(1 - g_M)$$

where g_M is the genus of M. Therefore we have the following Theorem:

Theorem 1.6. *For the blow up set Σ_1 we have*
(1) If M is a closed surface with $g_M = 0$, then the blow up set Σ_1 contains at most one point.
(2) If M is a closed surface with $g_M \geq 1$, then the blow set $\Sigma_1 = \emptyset$. Thus, there is no blow-up in this case.

We remark that in the case $g_M = 0$, ie, M is a sphere, the solution space is not compact.

2. Basic analytic properties for solutions

In this section, we first recall some basic analytic properties for solutions of super Liouville equations obtained in [JWZ], which will be the key tools for the blow up analysis. Then we prove a removability result for local singularities. At the end of this section, we shall derive the Pohozaev identity for solutions.

Proposition 2.1. [JWZ] *The functional $E(u,\psi)$ is conformally invariant. Namely, for any conformal diffeomorphism $\varphi : M \to M$, set*

$$\begin{aligned} \widetilde{u} &= u \circ \varphi - \ln \lambda \\ \widetilde{\psi} &= \lambda^{-\frac{1}{2}} \psi \circ \varphi \end{aligned}$$

where λ is the conformal factor of the conformal map φ, i.e., $\varphi^*(g) = \lambda^2 g$. Then $E(u,\psi) = E(\widetilde{u},\widetilde{\psi})$. In particular, if (u,ψ) is a solution of (2), so is $(\tilde{u},\tilde{\psi})$.

We say that (u,ψ) is a weak solution of (2) if $u \in W^{1,2}(M)$ and $\psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ satisfy

$$\begin{aligned} \int_M \nabla u \nabla \phi dv &= \int_M (2e^{2u} - e^u|\psi|^2 - K_g)\phi dv \\ \int_M \langle \psi, \not{D}\xi \rangle dv &= -\int_M e^u \langle \psi, \xi \rangle dv \end{aligned}$$

for any smooth function ϕ and any smooth spinor ξ. It is clear that $(u,\psi) \in W^{1,2}(M) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ is a weak solution if and only if (u,ψ) is a critical point of E in $W^{1,2}(M) \times W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$. A weak solution is a classical solution by the following

Proposition 2.2. [JWZ] *Any weak solution (u,ψ) to (2) on M with $\int_M e^{2u} + |\psi|^4 dv < \infty$ is smooth.*

Lemma 2.3. [JWZ] *(ε_0-regularity) Let $\varepsilon_0 < \pi$ be a constant. For any sequence of solutions (u_n,ψ_n) with*

$$\int_{B_r} e^{2u_n} dx < \varepsilon_0, \qquad \int_{B_r} |\psi_n|^4 \, dx < C$$

for some fixed constant $C > 0$ we have that $\|u_n^+\|_{L^\infty(B_{\frac{r}{4}})}$ is uniformly bounded.

It follows from Lemma 2.3 that the blow-up set Σ_1 can also be defined by

$$\Sigma_1 = \bigcap_{r>0} \{x \in M | \lim_{n\to\infty} \inf \int_{B(x,r)} e^{2u_n} dx \geq \varepsilon_0\}.$$

Lemma 2.4. ([JWZ]) *There is an $0 < \varepsilon_0 < \pi$ if (u,ψ) is a smooth solution to (2) on $B_1 \backslash \{0\}$ with energy $\int_{|x|\leq 1} e^{2u} dx < \varepsilon_0$, and $\int_{|x|\leq 1} |\psi|^4 dx < C$, then for any $x \in B_{\frac{1}{2}}$ we have*

$$|\psi(x)||x|^{\frac{1}{2}} + |\nabla\psi(x)||x|^{\frac{3}{2}} \leq C(\int_{B_{2|x|}} |\psi|^4 dx)^{\frac{1}{4}}.$$

Furthermore, if we assume that $e^{2u} = O(\frac{1}{|x|^{2-\varepsilon}})$*, then, for any* $x \in B_{\frac{1}{2}}$*, we have*

$$|\psi(x)||x|^{\frac{1}{2}} + |\nabla\psi(x)||x|^{\frac{3}{2}} \leq C|x|^{\frac{1}{4C}}(\int_{B_1} |\psi|^4 dx)^{\frac{1}{4}},$$

for some positive constant C*. Here* ε *is any sufficiently small positive number.*

Proposition 2.5. [JWZ] *(Removability of a global singularity) Let* (u, ψ) *be a smooth solution of (2) in* $\mathbb{R}^2$ *with* $\int_{\mathbb{R}^2} e^{2u} + |\psi|^4 dx < \infty$*. Then* (u, ψ) *extends to a smooth solution on* $\mathbb{S}^2$*. Moreover we have*

$$\int_{\mathbb{R}^2} 2e^{2u} - e^u|\psi|^2 dx = 4\pi.$$

In general, a local singularity of (u, ψ) is not removable. For example, if we set

$$u(x) = \log \frac{(2+2\beta)|x|^{\beta}}{1+2|x|^{2+2\beta}},$$

then u is a solution of

$$-\Delta u = 2e^{2u}, \quad \text{in } \mathbb{R}^2\backslash\{0\}$$

where $\beta > -1$. Therefore $(u, 0)$ is also a solution of (2) with finite energy in $\mathbb{R}^2\backslash\{0\}$. It is clear that $x = 0$ is a local singularity which isn't removable when $\beta \neq 0$.

Let $z = x + iy$ be a local isothermal parameter of M with $g = ds^2 = \rho|dz^2|$. Define the quadratic differential for (2) by

$$T(z)dz^2 = \{(\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle\psi, dz \cdot \partial_{\bar{z}}\psi\rangle + \frac{1}{4}\langle d\bar{z} \cdot \partial_z\psi, \psi\rangle\}dz^2.$$

From Proposition 3.3 in [JWZ], we know that $\partial_{\bar{z}}T(z) = -\frac{1}{4}\partial_z K_g$. Hence $T(z)$ is holomorphic if the curvature of the surface is constant. On the other hand, it is clear that $\int_{B_r(0)} |T(z)|dz = +\infty$ for $(u, 0)$ in the above example.

A simple, but crucial observation for the removability of local singularities is:

Proposition 2.6. *(Removability of a local singularity) Let* (u, ψ) *be a smooth solution in* $B_1\backslash\{0\}$ *of*

$$\begin{cases} -\Delta u &= 2e^{2u} - e^u \langle\psi, \psi\rangle \\ \not{D}\psi &= -e^u\psi \end{cases} \tag{7}$$

with $\int_{B_1} e^{2u} + |\psi^4|dx < C$*. If the quadratic differential* $T(z)dz^2$ *satisfies*

$$\int_{B_1} |T(z)|dz \leq C,$$

then the singularity of (u, ψ) *is removable.*

Proof. Since $\int_{B_1} e^{2u}dx$ is conformally invariant, we assume for convenience that $\int_{B_1} e^{2u}dx < \varepsilon_0$, where ε_0 is as in Lemma 2.4. Since u is a smooth solution of

$$-\Delta u = 2e^{2u} - e^u|\psi|^2$$

in $B_1\backslash\{0\}$ with $\int_{B_1} e^{2u}+|\psi|^4dx<\infty$. By the standard potential analysis it follows that there is a constant γ such that

$$\lim_{|x|\to 0}\frac{u}{-\log|x|}=\gamma.$$

By $\int_{B_1} e^{2u}+|\psi|^4dx<\infty$ we obtain that $\gamma\leq 1$. Furthermore, by the argument of Proposition 6.3 of [JLW], we can improve this inequality to $\gamma<1$.

Define $v(x)$ by

$$v(x)=-\frac{1}{2\pi}\int_{B_1}\log|x-y|(2e^{2u}-e^u|\psi|^2)dx$$

and set $w=u-v$. It is clear that $-\Delta v=2e^{2u}-e^u|\psi^2|$ in B_1 and $\Delta w=0$ in $B_1\backslash\{0\}$. One can check that

$$\lim_{|x|\to 0}\frac{v(x)}{-\log|x|}=0$$

which implies that

$$\lim_{|x|\to 0}\frac{w(x)}{-\log|x|}=\lim_{|x|\to 0}\frac{u-v}{-\log|x|}=\gamma.$$

Since w is harmonic in $B_1\backslash\{0\}$ we have

$$w=-\gamma\log|x|+w_0$$

with a smooth harmonic function w_0 in D. Therefore we have

$$u=-\gamma\log|x|+w_0+v\qquad\text{near }0$$

Then by Lemma 2.4 we have

$$T(z)=\frac{\gamma^2-2\gamma}{4z^2}+o(\frac{1}{z^2}).$$

Since $\int_{B_1}|T(z)|dz\leq C$ we have $\gamma(\gamma-2)=0$, consequently $\gamma=0$. Then the standard elliptic theory implies that (u,ψ) is smooth in B_1. $\square$

We now come to the Pohozaev type identity for smooth solutions of super Liouville equations.

Proposition 2.7. *Let (u,ψ) be a smooth solution of (2). Then, for every geodesic ball $B_R\subseteq M$,*

$$\begin{aligned}&R\int_{\partial B_R}|\frac{\partial u}{\partial\nu}|^2-\frac{1}{2}|\nabla u|^2d\sigma\\ =&\int_{B_R}2e^{2u}-e^u|\psi|^2dv-R\int_{\partial B_R}e^{2u}d\sigma\\ &+\int_{B_R}K_gx\cdot\nabla udv+\frac{1}{2}\int_{\partial B_R}\langle\frac{\partial\psi}{\partial\nu},x\cdot\psi\rangle+\langle x\cdot\psi,\frac{\partial\psi}{\partial\nu}\rangle d\sigma\end{aligned}$$

where ν is the outward normal vector to ∂B_R.

Proof. We choose a local orthonormal basis e_1,e_2 on M such that $\nabla_{e_\alpha}e_\beta=0$ at a considered point. Denote $x=x_1e_1+x_2e_2$. As usual in deriving Pohozaev type identities, we multiply the first equation of (2) by $x \cdot \nabla u$ and integrate over B_R to obtain

$$-\int_{B_R} \Delta u x \cdot \nabla u dv = \int_{B_R} 2e^{2u} x \cdot \nabla u dv - \int_{B_R} e^u |\psi|^2 x \cdot \nabla u dv - \int_{B_R} K_g x \cdot \nabla u dv.$$

By a direct computation we have

$$\int_{B_R} \Delta u x \cdot \nabla u dv = R \int_{\partial B_R} |\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2 d\sigma,$$

$$\int_{B_R} 2e^{2u} x \cdot \nabla u dv = R \int_{\partial B_R} e^{2u} d\sigma - \int_{B_R} 2e^{2u} dv,$$

and

$$\int_{B_R} e^u |\psi|^2 x \cdot \nabla u dv = R \int_{\partial B_R} e^u |\psi|^2 d\sigma - \int_{B_R} e^u x \cdot \nabla(|\psi|^2) dv - 2\int_{B_R} e^u |\psi|^2 dv.$$

Therefore we have

$$\begin{aligned} & R \int_{\partial B_R} |\frac{\partial u}{\partial \nu}|^2 - \frac{1}{2}|\nabla u|^2 d\sigma \\ = \quad & -R \int_{\partial B_R} e^{2u} d\sigma + \int_{B_R} 2e^{2u} dv + R \int_{\partial B_R} e^u |\psi|^2 d\sigma \\ & - \int_{B_R} e^u x \cdot \nabla(|\psi|^2) dv - 2\int_{B_R} e^u |\psi|^2 dv + \int_{B_R} K_g x \cdot \nabla u dv. \qquad (8) \end{aligned}$$

The local orthonormal basis $\{e_1, e_2\}$ on M satisfies the Clifford multiplication relation

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij}, \text{ for } 1 \le i, j \le 2$$

and

$$\langle \psi, \varphi \rangle = \langle e_i \cdot \psi, e_i \cdot \varphi \rangle$$

for any spinors $\psi, \varphi \in \Gamma(\Sigma M)$. It is clear that

$$\langle \psi, e_i \cdot \psi \rangle + \langle e_i \cdot \psi, \psi \rangle = 0 \qquad (9)$$

for any $i = 1, 2$. Using the Schrödinger-Lichnerowicz formula $\not{D}^2 = -\Delta + \frac{1}{2}K_g$, we have from the second equation of (2)

$$\Delta \psi = \sum_{\alpha=1}^{2} \nabla_{e_\alpha}(e^u) e_\alpha \cdot \psi - e^{2u}\psi + \frac{1}{2}K_g \psi \qquad (10)$$

Then we multiply (10) by $x \cdot \psi$ (where $\cdot$ denotes the Clifford multiplication) and integrate over B_R to obtain

$$\int_{B_R} \langle \Delta \psi, x \cdot \psi \rangle dv = \int_{B_R} \sum_{\alpha,\beta=1}^{2} \langle \nabla_{e_\alpha}(e^u) e_\alpha \cdot \psi, e_\beta \cdot \psi \rangle x_\beta dv - \int_{B_R} (e^{2u} - \frac{1}{2}K_g) \langle \psi, x \cdot \psi \rangle dv,$$

and

$$\int_{B_R} \langle x \cdot \psi, \Delta \psi \rangle dv = \int_{B_R} \sum_{\alpha,\beta=1}^{2} \langle e_\beta \cdot \psi, \nabla_{e_\alpha}(e^u) e_\alpha \cdot \psi \rangle x_\beta dv - \int_{B_R} (e^{2u} - \frac{1}{2}K_g) \langle x \cdot \psi, \psi \rangle dv.$$

On the other hand, by partial integration,

$$\begin{aligned}
&\int_{B_R}\langle\Delta\psi, x\cdot\psi\rangle dv\\
=\;& \int_{B_R} div\langle\nabla\psi, x\cdot\psi\rangle dv-\int_{B_R}\sum_{\alpha=1}^{2}\langle\nabla_{e_\alpha}\psi, e_\alpha\cdot\psi\rangle dv-\int_{B_R}\langle\nabla\psi, x\cdot\nabla\psi\rangle\\
=\;& \int_{\partial B_R}\langle\frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle d\sigma+\int_{B_R}\langle\not{D}\psi,\psi\rangle dv-\int_{B_R}\langle\nabla\psi, x\cdot\nabla\psi\rangle\\
=\;& \int_{\partial B_R}\langle\frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle d\sigma-\int_{B_R}e^u|\psi|^2dv-\int_{B_R}\langle\nabla\psi, x\cdot\nabla\psi\rangle,
\end{aligned}$$

and similarly

$$\int_{B_R}\langle x\cdot\psi,\Delta\psi\rangle=\int_{\partial B_R}\langle x\cdot\psi,\frac{\partial\psi}{\partial\nu}\rangle d\sigma-\int_{B_R}e^u|\psi|^2dv-\int_{B_R}\langle x\cdot\nabla\psi,\nabla\psi\rangle.$$

Furthermore we also have

$$\begin{aligned}
&\int_{B_R}\sum_{\alpha,\beta=1}^{2}\langle\nabla_{e_\alpha}(e^u)e_\alpha\cdot\psi, e_\beta\cdot\psi\rangle x_\beta dv+\int_{B_R}\sum_{\alpha,\beta=1}^{2}\langle e_\beta\cdot\psi,\nabla_{e_\alpha}(e^u)e_\alpha\cdot\psi\rangle x_\beta dv\\
=\;& 2\int_{B_R}\sum_{\alpha=1}^{2}\langle\nabla_{e_\alpha}(e^u)e_\alpha\cdot\psi, e_\alpha\cdot\psi\rangle x_\alpha dv\\
=\;& 2\int_{B_R}x\cdot\nabla(e^u)|\psi|^2dv\\
=\;& -2\int_{B_R}e^u x\cdot\nabla(|\psi|^2)dv-4\int_{B_R}e^u|\psi|^2dv+2R\int_{\partial B_R}e^u|\psi|^2dv.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&R\int_{\partial B_R}e^u|\psi|^2d\sigma-\int_{B_R}e^u x\cdot\nabla(|\psi|^2)dv\\
=\;& \frac{1}{2}\int_{\partial B_R}\langle\frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle d\sigma+\frac{1}{2}\int_{\partial B_R}\langle x\cdot\psi,\frac{\partial\psi}{\partial\nu}\rangle d\sigma+\int_{B_R}e^u|\psi|^2dv. \qquad (11)
\end{aligned}$$

Putting (8) and (11) together, we obtain our Pohozaev type identity

$$\begin{aligned}
&R\int_{\partial B_R}|\frac{\partial u}{\partial\nu}|^2-\frac{1}{2}|\nabla u|^2d\sigma\\
=\;& \int_{B_R}2e^{2u}-e^u|\psi|^2dv-R\int_{\partial B_R}e^{2u}d\sigma\\
&+\int_{B_R}K_g x\cdot\nabla u dv+\frac{1}{2}\int_{\partial B_R}\langle\frac{\partial\psi}{\partial\nu}, x\cdot\psi\rangle+\langle x\cdot\psi,\frac{\partial\psi}{\partial\nu}\rangle d\sigma.
\end{aligned}$$

□

3. Energy identity for spinors

In this section, we shall prove the energy identity in Theorem 1.2. For harmonic maps in dimension two and holomorphic curves as well as for Dirac harmonic maps and solutions of certain nonlinear Dirac equations, similar results are derived in [DT], [PW], [CJLW], [Z2] and the references therein. Firstly, we derive a local estimate:

Lemma 3.1. *Let (u,ψ) be a smooth solution of (2) on the annulus $A_{r_1,r_2}=\{x\in\mathbb{R}^2|r_1\le|x|\le r_2\}$, where $0<r_1<2r_1<\frac{r_2}{2}<r_2<1$. Then we have*

$$\begin{aligned}&(\int_{A_{2r_1,\frac{r_2}{2}}}|D\psi|^{\frac{4}{3}})^{\frac{3}{4}}+(\int_{A_{2r_1,\frac{r_2}{2}}}|\psi|^4)^{\frac{1}{4}} \\ \le\ &\Lambda(\int_{A_{r_1,r_2}}e^{2u})^{\frac{1}{2}}(\int_{A_{r_1,r_2}}|\psi|^4)^{\frac{1}{4}}+C(\int_{A_{r_1,2r_1}}|\psi|^4)^{\frac{1}{4}}+C(\int_{A_{\frac{r_2}{2},r_2}}|\psi|^4)^{\frac{1}{4}}\end{aligned}\tag{12}$$

for a positive constant Λ and some universal positive constant C.

Proof. Let D be the unit disk. Choose a cut-off function $\eta\in[0,1]$ on D satisfying

$$\eta\in C_0^\infty(A_{r_1,r_2});\quad \eta\equiv1\text{ in }A_{2r_1,\frac{r_2}{2}}$$
$$|\nabla\eta|\le\frac{4}{r_1}\text{ in }A_{r_1,2r_1};\quad |\nabla\eta|\le\frac{4}{r_2}\text{ in }A_{\frac{r_2}{2},r_2}.$$

By L^p boundary estimates for Dirac operators, see Lemma 2.2. in [CJW], we have

$$\begin{aligned}(\int_D|D(\eta\psi)|^{\frac{4}{3}})^{\frac{3}{4}}&\le C(\int_D|\not{D}(\eta\psi)|^{\frac{4}{3}})^{\frac{3}{4}}\\ &\le C(\int_D|\eta\not{D}\psi|^{\frac{4}{3}})^{\frac{3}{4}}+C(\int_D(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}\\ &= C(\int_D|\eta e^u\psi|^{\frac{4}{3}})^{\frac{3}{4}}+C(\int_D(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}\\ &\le C(\int_{A_{r_1,r_2}}e^{2u})^{\frac{1}{2}}(\int_D|\eta\psi|^4)^{\frac{1}{4}}+C(\int_D(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}\end{aligned}$$

and

$$\begin{aligned}(\int_D(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}&\le(\int_{A_{r_1,2r_1}}(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}+(\int_{A_{\frac{r_2}{2},r_2}}(|\nabla\eta||\psi|)^{\frac{4}{3}})^{\frac{3}{4}}\\ &\le\frac{4}{r_1}(\int_{A_{r_1,2r_1}}|\psi|^{\frac{4}{3}})^{\frac{3}{4}}+\frac{4}{r_2}(\int_{A_{\frac{r_2}{2},r_2}}|\psi|^{\frac{4}{3}})^{\frac{3}{4}}\\ &\le C(\int_{A_{r_1,2r_1}}|\psi|^4)^{\frac{1}{4}}+C(\int_{A_{\frac{r_2}{2},r_2}}|\psi|^4)^{\frac{1}{4}}.\end{aligned}$$

Therefore we have

$$\begin{aligned}&(\int_{A_{2r_1,\frac{r_2}{2}}}|D\psi|^{\frac{4}{3}})^{\frac{3}{4}}+(\int_{A_{2r_1,\frac{r_2}{2}}}|\psi|^4)^{\frac{1}{4}}\\ \le\ &(\int_D|D(\eta\psi)|^{\frac{4}{3}})^{\frac{3}{4}}+(\int_D|\eta\psi|^4)^{\frac{1}{4}}\le C(\int_D|D(\eta\psi)|^{\frac{4}{3}})^{\frac{3}{4}}\\ \le\ &\Lambda(\int_{A_{r_1,r_2}}e^{2u})^{\frac{1}{2}}(\int_{A_{r_1,r_2}}|\psi|^4)^{\frac{1}{4}}+C(\int_{A_{r_1,2r_1}}|\psi|^4)^{\frac{1}{4}}+C(\int_{A_{\frac{r_2}{2},r_2}}|\psi|^4)^{\frac{1}{4}}.\end{aligned}$$

□

Now we apply Lemma 2.3 and the analytic properties in the second section to prove Theorem 1.2.

Proof of Theorem 1.2. We will follow closely the argument for the energy identity of harmonic maps, see [DT] and [CJLW]. Since the blow up set Σ_1 is finite, we can find small disks D_{δ_i} for each blow-up point x_i such that $D_{\delta_i}\bigcap D_{\delta_j} = \emptyset$ for $i \neq j, i, j = 1, 2, \cdots, l$, and on $M\setminus\bigcup_{i=1}^{l} D_{\delta_i}$, ψ_n strongly converges to ψ in L^4. So, we need to prove that there are $(u^{i,k}, \xi^{i,k})$, which are solutions of (2) on S^2, $i = 1, 2, \cdots, l; k = 1, 2, \cdots, L_i$, such that

$$\sum_{i=1}^{l} \lim_{\delta_i\to 0} \lim_{n\to\infty} \int_{D_{\delta_i}} |\psi_n|^4 dv = \sum_{i=1}^{l}\sum_{k=1}^{L_i} \int_{S^2} |\xi^{i,k}|^4 dv,$$

or

$$\lim_{\delta_i\to 0} \lim_{n\to\infty} \int_{D_{\delta_i}} |\psi_n|^4 dv = \sum_{k=1}^{L_i} \int_{S^2} |\xi^{i,k}|^4 dv.$$

Without loss of generality, we assume that there is only one bubble at each blow up point p. Then what we need to prove is that there exists a bubble (u, ξ), such that

$$\lim_{\delta\to 0} \lim_{n\to\infty} \int_{D_{\delta}} |\psi_n|^4 dv = \int_{S^2} |\xi|^4 dv, \tag{13}$$

where D_δ is a small neighborhood of the blow up point p.

Each (u_n, ψ_n) is then rescaled at the blow up point p. Choose $x_n \in D_\delta$ such that $u_n(x_n) = \max_{\overline{D}_\delta} u_n(x)$. Then we have $x_n \to p$ and $u_n(x_n) \to +\infty$. Let $\lambda_n = e^{-u_n(x_n)} \to 0$. Denote

$$\begin{cases} \widetilde{u}_n(x) &= u_n(\lambda_n x + x_n) + \ln\lambda_n \\ \widetilde{\psi}_n(x) &= \lambda_n^{\frac{1}{2}} \psi_n(\lambda_n x + x_n) \end{cases}$$

for any $x \in B_{\frac{\delta}{2\lambda_n}}(0)$. Then $(\widetilde{u}_n(x), \widetilde{\psi}_n(x))$ satisfies

$$\begin{cases} -\Delta\widetilde{u}_n(x) &= 2e^{2\widetilde{u}_n(x)} - e^{\widetilde{u}_n(x)}|\widetilde{\psi}_n(x)|^2 - \lambda_n^2 K_g \\ \not{D}\widetilde{\psi}_n(x) &= -e^{\widetilde{u}_n(x)}\widetilde{\psi}_n(x) \end{cases}$$

with the energy conditions

$$\int_{B_{\frac{\delta}{2\lambda_n}}(0)} e^{2\widetilde{u}_n(x)} + |\widetilde{\psi}_n(x)|^4 dv < C.$$

Since $\widetilde{u}_n(0) = 0$ and $\widetilde{u}_n(x) \leq 0$, it follows from Theorem 1.1 that only alternative (i) may occur for $\widetilde{u}_n(x)$. Therefore, we have for any $R > 0$

$$\widetilde{u}_n \text{ is bounded in } L^\infty_{loc}(B_R(0)),$$

$$\widetilde{\psi}_n \text{ is bounded in } L^\infty_{loc}(B_R(0)),$$

and by standard elliptic estimates then also in $C^{1,\alpha}_{loc}(B_R(0))$. Finally, we pass to a subsequence (which we will still denote by $(\widetilde{u}_n, \widetilde{\psi}_n)$) converging in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$ to $\widetilde{u}$ and $\widetilde{\psi}$, which satisfy

$$\begin{cases} -\Delta\widetilde{u} &= 2e^{2\widetilde{u}} - e^{\widetilde{u}}|\psi|^2 \\ \not{D}\widetilde{\psi} &= -e^{\widetilde{u}}\widetilde{\psi} \end{cases} \tag{14}$$

with the energy condition $\int_{\mathbb{R}^2} e^{2\widetilde{u}} + |\widetilde{\psi}|^4 dx < \infty$. Therefore it follows from Proposition 2.5 that

$$\int_{\mathbb{R}^2} 2e^{2\widetilde{u}} - e^{\widetilde{u}}|\widetilde{\psi}|^2 dx = 4\pi.$$

Furthermore, also by the removable singularity proposition 2.5, we get a nonconstant solution $(\widetilde{u}, \widetilde{\psi})$ of (2) on S^2. Thus we get the first bubble at the blow-up point p.

So in order to prove (13) we need to estimate the energy of ψ_n in the neck domain. Let

$$A_{\delta,R,n} = \{x \in \mathbb{R}^2 | \lambda_n R \leq |x - x_n| \leq \delta\}.$$

We call $A_{\delta,R,n}$ the neck domain, and the image of (u_n, ψ_n) is called the neck. Then to prove (13) is equivalent to prove the following

$$\lim_{R\to 0}\lim_{\delta\to 0}\lim_{n\to\infty}\int_{A_{\delta,R,n}} |\psi_n|^4 dv = 0. \tag{15}$$

For convenience and simplicity, we use a polar coordinate system as in [DT] and [CJLW]. Let (r, θ) be the polar coordinates of $\mathbb{R}^2$ centered at 0 and $h = dr^2 + r^2 d\theta^2$ be the Eucliden metric on $\mathbb{R}^2$. Equip the cylinder $\mathbb{R}^1 \times S^1$ with the metric $ds^2 = dt^2 + d\theta^2$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then the following map $f : \mathbb{R}^1 \times S^1 \to \mathbb{R}^2$

$$r = e^{-t}, \theta = \theta, (t, \theta) \in \mathbb{R}^1 \times S^1$$

is a conformal transformation. One can verify that $f^* h = e^{-2t} ds^2$. Given $r_1 > r_2$, then, the annulus $A_{r_1,r_2} = \{re^{i\theta} | r_2 \leq r \leq r_1\}$ is mapped to the cylinder $P_{t_1,t_2} = [t_1, t_2] \times S^1$, where $t_i = -\log r_i, i = 1, 2$.

Denote $T_0 = |\log\delta|$ and $T_n = |\log\lambda_n R|$, then the neck domain changes to a cylinder $P_{\delta,R,n} = [T_0, T_n] \times S^1$. Let

$$\begin{cases} v_n &= f^* u_n + \log e^{-t} \\ \varphi_n &= e^{-\frac{t}{2}} f^* \psi_n \end{cases}$$

Then (v_n, φ_n) satisfies

$$\begin{cases} -\Delta v_n &= 2e^{2v_n} - e^{v_n}|\varphi_n|^2, \\ \not{D}\varphi_n &= -e^{v_n}\varphi_n, \end{cases} \qquad \text{on } P_{\delta,R,n} \tag{16}$$

and with the condition $\int_{P_{\delta,R,n}} e^{2v_n} + |\varphi_n|^4 \leq C$. Therefore to prove (15), it is sufficient to show

$$\lim_{R\to 0}\lim_{\delta\to 0}\lim_{n\to\infty}\int_{P_{\delta,R,n}} |\varphi_n|^4 dv = 0. \tag{17}$$

Next we want to show two claims.

Claim 1 For any ε, there is an $N > 0$ such that for any $n \geq N$, we have

$$\int_{[t,t+1]\times S^1} e^{2v_n} + \int_{[t,t+1]\times S^1} |\varphi_n|^4 < \varepsilon; \quad \forall t \in [T_0, T_n - 1].$$

To prove this claim, we note two facts. The first fact is: for any $T > 0$, set $P_T = [T_0, T_0 + T] \times S^1$, there exists some $N(T)$ such that for any $n \geq N(T)$ we have

$$\int_{P_T} e^{2v_n} + |\varphi_n|^4 < \varepsilon.$$

Actually, from Theorem 1.1, since (u_n, ψ_n) has no blow up point in $D_\delta \backslash \{p\}$, then $|\psi_n|$ is uniformally bounded in $\overline{D_\delta \backslash D_{\delta e^{-T}}}$, and u_n will either be uniformly bounded in $\overline{D_\delta \backslash D_{\delta e^{-T}}}$ or uniformly tend to $-\infty$ in $\overline{D_\delta \backslash D_{\delta e^{-T}}}$. So if u_n uniformly tends to $-\infty$ in $\overline{D_\delta \backslash D_{\delta e^{-T}}}$, it is clear that, for any given $T > 0$, we have an $N(T)$ big enough such that when $n \geq N(T)$

$$\int_{P_T} e^{2v_n} = \int_{D_\delta \backslash D_{\delta e^{-T}}} e^{2u_n} < \frac{\varepsilon}{2}.$$

Moreover, since ψ_n converges to ψ in $L^4_{loc}(M \setminus \{p\})$ and hence φ_n converges to $f^*\psi = \varphi$ in L^4 on P_T, namely,

$$\int_{P_T} |\varphi_n|^4 \to \int_{P_T} |\varphi|^4.$$

For any small $\varepsilon > 0$, we may choose $\delta > 0$ small enough such that $\int_{D_\delta} |\psi|^4 < \frac{\varepsilon}{4}$, then for any given $T > 0$, we have an $N(T)$ big enough such that when $n \geq N(T)$

$$\int_{P_T} |\varphi_n|^4 < \frac{\varepsilon}{2}.$$

Consequently, we have

$$\int_{P_T} e^{2v_n} + |\varphi_n|^4 < \varepsilon.$$

If (u_n, ψ_n) is uniformly bounded in $\overline{D_\delta \backslash D_{\delta e^{-T}}}$, then we know (u_n, ψ_n) converges to a weak solution (u, ψ) of (2) strongly on compact sets of $D_\delta \setminus \{p\}$ and hence (v_n, φ_n) converges to $(f^*u + \log e^{-t}, e^{-\frac{t}{2}} f^*\psi) = (v, \varphi)$ strongly on P_T, and

$$\int_{P_T} e^{2v_n} + |\varphi_n|^4 \to \int_{P_T} e^{2v} + |\varphi|^4.$$

Again, we choose $\delta > 0$ small enough such that $\int_{D_\delta} e^{2u} + |\psi|^4 < \frac{\varepsilon}{2}$, then for any given $T > 0$, we have an $N(T)$ big enough such that when $n \geq N(T)$

$$\int_{P_T} e^{2v_n} + |\varphi_n|^4 < \varepsilon.$$

The second fact is: For any small $\varepsilon > 0$, and $T > 0$, we may choose an $N(T)$ such that when $n \geq N(T)$

$$\int_{Q_T} e^{2v_n} + |\varphi_n|^4 < \varepsilon, \qquad Q_T = [T_n - T, T_n] \times S^1.$$

This fact follows from the following equality:

$$\int_{Q_T} e^{2v_n} + |\varphi_n|^4 = \int_{D_{\lambda_n R e^T} \backslash D_{\lambda_n R}} e^{2u_n} + |\psi_n|^4 = \int_{D_{Re^T} \backslash D_R} e^{2\widetilde{u}_n} + |\widetilde{\psi}_n|^4 < \varepsilon$$

if R is big enough.

Now we can prove the claim. We argue by contradiction by using the above two facts. If there exists $\varepsilon_0 > 0$ and a sequence t_n such that

$$\int_{[t_n,t_n+1]\times S^1} e^{2v_n} + |\varphi_n|^4 \geq \varepsilon_0,$$

then, by the above two facts, we know that $t_n - T_0$ and $T_n - t_n$ tend to infinity as n tends to infinity.

Translating t to $t - t_n$, we get some $(\bar{v}_n, \bar{\varphi}_n)$, and for all n and for all $R > 0$, we have

$$\int_{[0,1]\times S^1} e^{2\bar{v}_n} + |\overline{\varphi}_n|^4 \geq \varepsilon_0$$

and $(\bar{v}_n, \bar{\varphi}_n)$ satisfying

$$\begin{cases} -\Delta \bar{v}_n &= 2e^{2\bar{v}_n} - e^{\bar{v}_n}|\bar{\varphi}_n|^2 \\ \not{D}\bar{\varphi}_n &= -e^{\bar{v}_n}\bar{\varphi}_n \end{cases} \quad \text{in } [-R,R]\times S^1.$$

From Theorem 1.1, there are three possible cases:

(1). There exists some $R > 0$, some $q \in [-R,R]\times S^1$ and energy concentration near the point q, namely along some subsequence we have

$$\lim_{n\to\infty} \int_{D_r(q)} e^{2\bar{v}_n} + |\overline{\varphi}_n|^4 \geq \varepsilon_0 > 0$$

for any small $r > 0$. In such a case, we still obtain a second "bubble" by the rescaling argument. Thus we get a contradiction.

(2). For any $R > 0$, there is no blow up point in $[-R,R]\times S^1$ and $\bar{v}_n \to -\infty$ uniformly in $[-R,R]\times S^1$. Then, it is clear that $\overline{\varphi}_n$ converges to a harmonic spinor $\overline{\varphi}$ (namely, $\not{D}\bar{\varphi} = 0$) in $L^4_{loc}(\mathbb{R}^1\times S^1)$. Note that harmonic spinors on surfaces are special Dirac-harmonic maps studied in [CJLW] and hence $\bar{\varphi}$ conformally extends to a harmonic spinor on S^2. By the well know fact that there is no nontrivial harmonic spinor on S^2, we have that $\bar{\varphi} \equiv 0$ and hence $\overline{\varphi}_n$ converges to 0 in $L^4_{loc}(\mathbb{R}^1\times S^1)$. This will contradict

$$\int_{[0,1]\times S^1} e^{2\bar{v}_n} + |\overline{\varphi}_n|^4 \geq \varepsilon_0.$$

(3). For any $R > 0$, there is no blow up point in $[-R,R]\times S^1$ and $(\bar{v}_n, \bar{\varphi}_n)$ is uniformly bounded in $[-R,R]\times S^1$. In such a case $(\bar{v}_n, \bar{\varphi}_n)$ will converge to (v,φ) strongly on $[-R,R]\times S^1$ and (v,φ) satisfying

$$\begin{cases} -\Delta v &= 2e^{2v} - e^{v}|\varphi|^2 \\ \not{D}\varphi &= -e^{v}\varphi \end{cases} \quad \text{in } [-R,R]\times S^1$$

with finite energy. In this case it is clear that $(v,\varphi) \in C^\infty(\mathbb{R}^1\times S^1)$. Furthermore (v,φ) satisfies that $\int_{\mathbb{R}^1\times S^1} T(z)dz \leq C$, where $T(z)$ is the quadratic differential

$$T(z)dz^2 = \{(\partial_z v)^2 - \partial_z^2 v + \frac{1}{4}\langle \varphi, dz\cdot\partial_{\bar{z}}\varphi\rangle + \frac{1}{4}\langle d\bar{z}\cdot\partial_z\varphi, \varphi\rangle\}dz^2.$$

Indeed, this property inherits from (u_n, ψ_n). Set

$$T_n(z) = (\partial_z u_n)^2 - \partial_z^2 u_n + \frac{1}{4}\langle \psi_n, dz\cdot\partial_{\bar{z}}\psi_n\rangle + \frac{1}{4}\langle d\bar{z}\cdot\partial_z\psi_n, \psi_n\rangle$$

It follows from Proposition 3.3 in [JWZ] that $\partial_{\bar{z}} T_n(z) = -\frac{1}{4}\partial_z K_g$. On the other hand we can write $T_n(z)$ as

$$T_n(z) = \frac{1}{2\pi i\rho}\int_{\partial D_\rho} \frac{T_n(\theta)}{\theta - z} d\theta - \int_{D_\rho} \frac{\bar{\partial} T_n(\xi)}{\xi - z} d\xi,$$

where ρ can be any number in $(0,1]$. Then it follows that

$$\int_{D_\delta} |T_n(z)| dz \leq C.$$

By the L^1-norm of the quadratic differential is conformally invariant and $(\bar{v}_n, \bar{\varphi}_n)$ converges to (v, φ) strongly, we conclude that

$$\int_{\mathbb{R}^1 \times S^1} |T(z)| dz \leq C.$$

Note that $\mathbb{R}^1 \times S^1$ is conformal to $S^2 \backslash \{N, S\}$. By Proposition 2.6 for the removability of the local singularities, we get another bubble on S^2. Thus we get a contradiction to the assumption that $m = 1$.

Thus we have shown that: for any ε, there is an $N > 0$ such that for any $n \geq N$, we have

$$\int_{[t,t+1]\times S^1} e^{2v_n} + |\varphi_n|^4 < \varepsilon.$$

Thus we finish to prove the claim.

Claim 2 We can separate $P_{\delta,R,n}$ into finitely many parts

$$P_{\delta,R,n} = \bigcup_{k=1}^{N_k} P^k, \quad P^k = [T^{k-1}, T^k] \times S^1, T^0 = T_0, T^{N_k} = T_n$$

such that $N_k \leq N_0$, where N_0 is a uniform integer for all n large enough, and on each part

$$\int_{P^k} e^{2v_n} \leq \frac{1}{4\Lambda^2}, \quad k = 1, 2, \cdots, N_k.$$

where Λ is a constant as in Lemma 3.1.

The proof of this claim is very similar to those in [Zh] and [Z1]. The details are as follows. Without loss of generality, we assume that $T_n = T_0 + m_n$, where m_n is an integer and $\lim_{n\to\infty} m_n = \infty$.

By Claim 1, for any $\varepsilon \leq \frac{1}{8\Lambda^2}$, we can find N such that for any $n \geq N$ we have

$$\int_{[t,t+1]\times S^1} e^{2v_n} < \varepsilon \leq \frac{1}{8\Lambda^2}, \quad \forall t \in [T_0, T_n - 1].$$

Then for any $n \geq N$, if

$$\int_{[T_0,T_n]\times S^1} e^{2v_n} \leq \frac{1}{4\Lambda^2},$$

we take $T^1 = T_n$ and denote $P^1 = [T^0, T^1] \times S^1 = [T_0, T_n] \times S^1$. Otherwise, if

$$\int_{[T_0,T_n]\times S^1} e^{2v_n} > \frac{1}{4\Lambda^2},$$

we can choose an integer m_n^1 such that

$$\frac{1}{8\Lambda^2} < \int_{P^1} e^{2v_n} \le \frac{1}{4\Lambda^2}, \quad \text{and} \int_{[T^0,T^1+1]\times S^1} e^{2v_n} > \frac{1}{4\Lambda^2},$$

where $T^1 = T^0 + m_n^1$, $P^1 = [T^0, T^1] \times S^1$ and $1 \le m_n^1 \le m_n^1 - 1$. This is the first step of the division.

Inductively, suppose that $P^l = [T^{l-1}, T^l] \times S^1$ is chosen such that $\int_{P^l} e^{2v_n} \le \frac{1}{4\Lambda^2}$. If

$$\int_{[T^l,T_n]\times S^1} e^{2v_n} \le \frac{1}{4\Lambda^2},$$

Then we take $T^{l+1} = T_n$ and denote $P^{l+1} = [T^l, T^{l+1}] \times S^1$. On the other hand, if

$$\int_{[T^l,T_n]\times S^1} e^{2v_n} > \frac{1}{4\Lambda^2},$$

then similar to the first step, we can find $T^{l+1} = T^l + m_n^{l+1}$, $P^{l+1} = [T^l, T^{l+1}] \times S^1$ such that

$$\frac{1}{8\Lambda^2} < \int_{P^{l+1}} e^{2v_n} \le \frac{1}{4\Lambda^2}, \quad \text{and} \int_{[T^l,T^{l+1}+1]\times S^1} e^{2v_n} > \frac{1}{4\Lambda^2},$$

where $m_n^l + 1 \le m_n^{l+1} \le m_n - 1$. Thus we can get one more part P^{l+1} satisfying $\int_{P^{l+1}} e^{2v_n} \le \frac{1}{4\Lambda^2}$. Since $\int_{P_{\delta,R,n}} e^{2v_k} \le C$ for some positive constant C, we will finish our division after at most $N_0 = [8\Lambda^2 C]$ steps. So we have proved the claim.

Now from claim 1 and claim 2, we can show (17). Let $\varepsilon > 0$ be small, and let δ be small enough, and let R and n be big enough. We apply Lemma 3.1 to each part P^l to obtain

$$\begin{aligned}
(\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} &\le \Lambda(\int_{[T^{l-1}-1,T^l+1]\times S^1} e^{2v_n})^{\frac{1}{2}} (\int_{[T^{l-1}-1,T^l+1]\times S^1} |\varphi_n|^4)^{\frac{1}{4}} \\
&+ C(\int_{[T^{l-1}-1,T^{l-1}]\times S^1} |\varphi_n|^4)^{\frac{1}{4}} + C(\int_{[T^l,T^l+1]\times S^1} |\varphi_n|^4)^{\frac{1}{4}} \\
&\le \Lambda((\int_{P^l} e^{2v_n})^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}})((\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{4}}) + C\varepsilon^{\frac{1}{4}} \\
&\le \Lambda(\int_{P^l} e^{2v_n})^{\frac{1}{2}} (\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} + C(\varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{4}}) \\
&\le \frac{1}{2}(\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} + C(\varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{4}}).
\end{aligned}$$

Therefore we have

$$(\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} \le C(\varepsilon^{\frac{1}{4}} + \varepsilon^{\frac{1}{2}} + \varepsilon^{\frac{3}{4}}).$$

Since ε is small, we may assume $\varepsilon \le 1$. Then we get

$$(\int_{P^l} |\varphi_n|^4)^{\frac{1}{4}} \le C\varepsilon^{\frac{1}{4}}. \tag{18}$$

With similar arguments, and using (18), we have

$$(\int_{P^l} |\nabla\varphi_n|^{\frac{4}{3}})^{\frac{3}{4}} \le C\varepsilon^{\frac{1}{4}}. \tag{19}$$

Summing up (18) and (19) on P^l we get

$$\int_{P_{\delta,R,n}} |\varphi_n|^4 + \int_{P_{\delta,R,n}} |\nabla\varphi_n|^{\frac{4}{3}} = \sum_{l=1}^{N_0} \int_{P^l} |\varphi_n|^4 + |\nabla\varphi_n|^{\frac{4}{3}} \le C\varepsilon^{\frac{1}{3}}. \tag{20}$$

Thus we have shown (17). The proof of the Theorem is complete. □

4. BLOW UP BEHAVIOR

In this section, we will prove Theorem 1.1, which is an application of the energy identity of spinors. The method is motivated by [BM].

Proof of Theorem 1.1. We argue by contradiction. If the theorem is false, then we can assume that u_n is uniformly bounded in $L^\infty_{loc}(M\backslash\Sigma_1)$ by Theorem 1.1. Let $x_0 \in \Sigma_1$ and $R > 0$ small so that x_0 is the only point of Σ_1 in $\overline{B}_R(x_0)$. Since u_n is uniformly bounded in $L^\infty_{loc}(M\backslash\Sigma_1)$, u_n is uniformly bounded in $L^\infty(\partial B_R(x_0))$ and similarly for $|\psi_n|$. Let z_n satisfy

$$\begin{cases} -\Delta z_n = e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g, & \text{in } B_R(x_0), \\ z_n = -C, & \text{on } \partial B_R(x_0). \end{cases}$$

Then by the maximum principle we have $u_n \ge z_n$ in $B_R(x_0)$ and in particular

$$\int_{B_R(x_0)} e^{2z_n} \le \int_{B_R(x_0)} e^{2u_n} \le C. \tag{21}$$

On the other hand, similar to the arguments in [BM], we know that $z_n \to z$ a.e. (even uniformly on compact subsets of $B_R(x_0)\backslash\{x_0\}$) where z is the solution of

$$\begin{cases} -\Delta z = \mu & \text{in } B_R(x_0), \\ z = -C & \text{on } \partial B_R(x_0). \end{cases}$$

Now we choose $x_n \in B_R(x_0)$ with $u(x_n) = \max_{B_{x_0}} u_n$ and set $\lambda_n = e^{-u(x_n)}$. Let R be small enough. Since

$$\begin{aligned} &\int_{B_R(x_0)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g \\ =\ &\int_{B_{\lambda_n R}(x_n)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 + \int_{B_R x_0 \backslash B_{\lambda_n R}(x_n)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 - \int_{B_R x_0} K_g \\ \ge\ &\int_{B_{\lambda_n R}(x_n)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 - \int_{B_R x_0 \backslash B_{\lambda_n R}(x_n)} e^{u_n}|\psi_n|^2 - \int_{B_R x_0} K_g. \end{aligned}$$

Note that the neck energy of the spinor field ψ_n is zero from Theorem 1.2. Let $n \to \infty$, we have

$$\lim_{n\to\infty} \int_{B_R(x_0)} 2e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g \ge 4\pi + o_R(1)$$

where $o_R(1)$ will tend to 0 when $R \to 0$. This imply $\mu(\{x_0\}) \ge 4\pi$ and $\mu \ge 2\pi\delta_{x_0}$. Therefore we have

$$z(x) \ge \log\frac{1}{|x - x_0|} + O(1), \quad \text{as } x \to x_0.$$

Thus we have $e^{2z} \ge \frac{C}{|x-x_0|^2}$ with $C > 0$. Hence $\int_{B_R(x_0)} e^{2z} = \infty$.

On the other hand, by (21) and Fatou's lemma we find that $\int_{B_R(x_0)} e^{2z} \le C$. Thus we get a contradiction.

Consequently, u_n converges to $-\infty$ uniformly on compact subsets of $M\backslash\Sigma_1$. It follows that

$$2e^{2u_n} - e^{u_n}|\psi_n|^2 \rightharpoonup \sum_{x_i\in\Sigma_1} \alpha_i\delta_{x_i},$$

in the distribution sense and with $\alpha_i \ge 4\pi$. Thus we finish the proof of Theorem 1.1. $\square$

5. Blow up value

In this section, we want to characterize the blow up value at blow up points in Σ_1. For $p \in \Sigma_1$, let us define

$$m(p) = \lim_{r\to 0}\lim_{n\to\infty}\int_{B_r(p)} 2e^{2u_n} - e^{u_n}|\psi_n|^2.$$

It is easy to see that $m(p) = 0$ implies that $p \in M$ is a regular point and hence $p \notin \Sigma_1$. Furthermore, we have that $m(p) \neq 0$ if and only if $p \in \Sigma_1$. Actually, it is clear from the previous section that $m(p) \ge 4\pi$ when p is a blow up point. In this section, we want to show that $m(p) = 4\pi$ when the domain M is a closed Riemann surface.

Lemma 5.1. *There exists $G \in W^{1,q}(M)\cap C^2_{loc}(M\backslash\Sigma_1)$ with $\int_M G = 0$ for $1 < q < 2$ such that*

$$u_n - \frac{1}{|M|}\int_M u_n \to G$$

in $C^2_{loc}(M\backslash\Sigma_1)$ and weakly in $W^{1,q}(M)$. Moreover, in $\Sigma_1 = \{p_1, p_2, \cdots, p_l\}$, then for $R > 0$ small such that $B_R(p_k)\cap\Sigma_1 = \{p_k\}$, $k = 1, 2, \cdots, l$, we have

$$G = \frac{1}{2\pi}m(p_k)\log\frac{1}{|x-p_k|} + g(x)$$

for $x \in B_R(p_k)\backslash\{p_k\}$ with $g \in C^2(B_R(p_k))$.

Proof. Let $p = \frac{q}{q-1} > 2$. We have

$$||\nabla u_n||_{L^q(M)} \le \sup\{|\int_M \nabla u_n\nabla\varphi dv||\varphi \in W^{1,p}(M), \int_M \varphi dv = 0, ||\varphi||_{W^{1,p}(M)} = 1\}.$$

By the Sobolev embedding theorem, we get

$$||\varphi||_{L^\infty(M)\le C}.$$

It is clear that

$$|\int_M \nabla u_n\nabla\varphi dv| = |\int_M \Delta u_n\varphi dv| \le \int_M (2e^{2u_n} + e^{u_n}|\phi_n|^2)|\varphi|dv \le C.$$

Therefore, $u - \overline{u}_n$ is uniformly bounded in $W^{1,q}(M)$.

Next, we define the Green function G by

$$\begin{cases} \Delta G = \sum_{p\in\Sigma_1} m(p)\delta_p - K_g, \\ \int_M G = 0. \end{cases}$$

We have for any $\varphi \in C^\infty(M)$

$$\begin{aligned}
&\int_M \nabla(u_n - G)\nabla\varphi dv = \int_M \Delta(u_n - G)\varphi dv \\
= &\int_M (2e^{2u_n} - e^{u_n}|\psi_n|^2 - \sum_{p\in\Sigma_1} m(p)\delta_p)\varphi dv \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Combining the fact that the $u_n - \overline{u}_n$ are uniformly bounded in $W^{1,q}(M)$, we get the conclusion of the lemma. □

Now we can compute the blow up value by using the Pohozaev identity and Lemma 5.1.

Proof of Theorem 1.5 Without loss of generality, we assume that $p = 0$. For sufficiently small $R > 0$, 0 then is the only blow up point in $\overline{B}_{2R}(0) \in M$.By Proposition 2.7, Pohozaev identity for solutions (u_n, ψ_n) is

$$\begin{aligned}
&R\int_{\partial B_R} |\frac{\partial u_n}{\partial \nu}|^2 - \frac{1}{2}|\nabla u_n|^2 d\sigma \\
= &\int_{B_R} 2e^{2u_n} - e^{u_n}|\psi_n|^2 dv - R\int_{\partial B_R} e^{2u_n} d\sigma \\
&+ \int_{B_R} K_g x\cdot\nabla u_n dv + \frac{1}{2}\int_{\partial B_R} \langle\frac{\partial\psi_n}{\partial\nu}, x\cdot\psi_n\rangle + \langle x\cdot\psi_n, \frac{\partial\psi_n}{\partial\nu}\rangle d\sigma. \qquad (22)
\end{aligned}$$

By Lemma 5.1, we have

$$\lim_{R\to 0}\lim_{n\to\infty} R\int_{\partial B_R} |\frac{\partial u_n}{\partial\nu}|^2 - \frac{1}{2}|\nabla u_n|^2 d\sigma = \lim_{R\to 0} R\int_{\partial B_R} |\frac{\partial G}{\partial\nu}|^2 - \frac{1}{2}|\nabla G|^2 d\sigma = \frac{1}{4\pi}m^2(0).$$

Since $u_n \to -\infty$ uniformly on $\partial B_R(0)$ and $u_n - \overline{u}_n$ is uniformly bounded in $W^{1,q}(M)$ for $1 < q < 2$, we have

$$\lim_{R\to 0}\lim_{n\to\infty} R\int_{\partial B_R} e^{2u_n} d\sigma = 0,$$

and

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{B_R} K_g x\cdot\nabla u_n dv = 0.$$

Furthermore, by use the Schrödinger-Lichnerowicz formula $\not{D}^2 = -\Delta + \frac{1}{2}K_g$, we have

$$\Delta\psi_n = e^{u_n} du_n\cdot\psi_n - e^{2u_n}\psi_n + \frac{1}{2}K_g\psi_n \quad \text{ in } B_{2R}(0)\backslash B_{\frac{R}{4}}(0).$$

By $u_n \to -\infty$ uniformly in $B_{2R}(0)\backslash B_{\frac{R}{4}}(0)$, $u_n - \overline{u}_n$ is uniformly bounded in $W^{1,q}(M)$ for $1 < q < 2$ and $|\psi_n|$ is uniformly bounded in $B_{2R}(0)\backslash B_{\frac{R}{4}}(0)$, we know by the standard elliptic estimates that ψ_n is uniformly bounded in $W^{2,q}(B_{\frac{3}{2}R}(0)\backslash B_{\frac{R}{2}}(0))$ for $1 < q < 2$. Then by the trace imbedding Theorem we obtain

$$\lim_{R\to 0}\lim_{n\to\infty} \int_{\partial B_R} |\psi_n||x\cdot\nabla\psi_n| d\sigma = 0.$$

Let $R \to 0$ and $n \to \infty$ in (22), we get that

$$\frac{1}{4\pi} m^2(0) = m(0).$$

It follows that $m(0) = 4\pi$. Thus we finish the proof of Theorem 1.5. □

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