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An Operational Haar Wavelet Method for Solving Fractional Volterra Integral Equations

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Abstract

In this work, the Haar wavelet operational matrix of fractional integration is first obtained. Haar wavelet approximating method is then utilized to reduce the fractional Volterra integral equations (which are also called the weakly-singular linear Volterra integral equations) and in particular the Abel integral equations, to a system of algebraic equations. An error bound is estimated and some numerical examples are included to demonstrate the validity and applicability of the method.

Keywords: Volterra Integral Equation, Fractional Calculus, Haar Wavelet Method, Operational Matrices.

1 Introduction

The conception of the fractional derivatives was introduced for the first time in the middle of the 19th century by Riemann and Liouville. After that the number of researches and studies about the fractional calculus has rapidly increased, because some physical processes such as anomalous diffusion [5], complex viscoelasticity [14], behavior of mechatronic and biological systems [12], rheology [15] and etc. can't be described by classical models. Fractional differential equations have been discussed in many papers and in most of them, they are transformed into fractional volterra integral equations. Also different techniques have been used for solving them like Fourier and Laplace transforms [1], power spectral density [20], Adomian decomposition method [11], Path integration[6], etc. But in comparison to the above methods, the wavelet method has not been much considered yet, specially for solving Volterra integral equations. We found only these papers [3, 10]. The solutions are often quite complicated, so we are looking for simplifications. For this reason we use Haar wavelets in present paper, which are the most simple wavelets, to approximate the solution of such equations [8, 9, 13, 7].
This paper is organized as follows: In section 2 we state the preliminaries. In section 3 We introduce the function approximation via Haar wavelets. In section 4 the generalized operational matrix of fractional integration is obtained and in section 5 this matrix is utilized for solving the fractional Volterra integral equation and Abel integral equation. Finally in section 6 an error bound for approximation and the error function are estimated and in section 7 the numerical results are shown in figures and tables.

2 Preliminaries

2.1 Riemann-Liouville fractional integration

The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of the function $f(x)$ is defined as [17, 16]:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \qquad \alpha > 0,\ x > 0, \tag{2.1}$$

where $\Gamma(.)$ is Gamma function with this property: $\Gamma(x+1) = x\Gamma(x),\ x \in \mathbb{R}$.
Some properties of the operator I^α can be found in [17], we mention only the following. For $\alpha,\ \beta \geqslant 0$ and $\gamma > -1$ we have:

$$I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{2.2}$$

2.2 Haar Wavelets

The orthogonal set of the Haar wavelets $h_n(x)$ is a group of square waves defined as follows:

$$h_0(x) = \begin{cases} 1, & 0 \leq \mathrm{x} \leq 1; \\ 0, & \text{elsewhere}. \end{cases}, \quad h_1(x) = \begin{cases} 1, & 0 \leq \mathrm{x} < \frac{1}{2}; \\ -1, & \frac{1}{2} \leq \mathrm{x} < 1; \\ 0, & \text{elsewhere}. \end{cases} \tag{2.3}$$
$$h_n(x) = h_1(2^j x - k), \quad n = 2^j + k, \quad j, k \in \mathbb{N} \cup \{0\};\ 0 \leq k < 2^j,$$

such that:

$$\int_0^1 h_n(x) h_m(x) dx = 2^{-j} \delta_{nm}, \tag{2.4}$$

where δ_{nm} is the Kronecker delta. For more details see [2, 19, 18].

2.3 Laplace transforms

The Laplace transform $\mathcal{L}\{f\}$ of a function $f(x)$ is defined as:

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t) dt. \tag{2.5}$$

The inverse $\mathcal{L}^{-1}\{F\}$ is:

$$\mathcal{L}^{-1}\{F\} = f(x). \tag{2.6}$$

This transformation is linear. In the following, some of the basic properties of the Laplace transform are presented:

$$\mathcal{L}\{x^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha \in \mathbb{R}, \tag{2.7}$$

$$\mathcal{L}\{\int_0^x f(x-t) g(t) dt\} = \mathcal{L}\{f\}\mathcal{L}\{g\}, \tag{2.8}$$

$$\mathcal{L}\{u(x-\tau)\} = \frac{e^{-\tau s}}{s}, \quad \tau \in \mathbb{R}, \tag{2.9}$$

$$\mathcal{L}\{(x-\tau)^\alpha e^{-\beta(x-\tau)} u(x-\tau)\} = \frac{\Gamma(\alpha+1)e^{-\tau s}}{(s+\beta)^{\alpha+1}}, \quad \alpha,\ \tau \in \mathbb{R}, \tag{2.10}$$

where $u(x)$ is the Heaviside step function which is defined as:

$$u(x) = \begin{cases} 1, & \text{x}\geq 0; \\ 0, & \text{x}<0. \end{cases} \tag{2.11}$$

A useful property of the Heaviside step function is:

$$u(x-a)u(x-b) = u(x - max\{a,b\}),\ a,\ b \in \mathbb{R}. \tag{2.12}$$

Note that we can write Eq. (2.3) by using the Heaviside step function as the following:

$$\begin{gathered} h_0(x) = u(x) - u(x-1), \\ h_n(x) = u(x - \tfrac{k}{2^j}) - 2u(x - \tfrac{k+1/2}{2^j}) + u(x - \tfrac{k+1}{2^j}), \\ n = 2^j + k,\ j,k \in \mathbb{N}\cup\{0\};\ 0 \leq k < 2^j. \end{gathered} \tag{2.13}$$

2.4 Fractional volterra integral equation and Abel integral equation

A fractional Volterra integral equation has the form:

$$f(x) - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}k(x,t)f(t)dt = g(x),\quad 0 \leq x \leq 1. \tag{2.14}$$

The kernel $k(x,t)$ and the right-hand-side function $g(x)$ are given, $\alpha > 0$ is a real number. This equation is also called the weakly-singular linear Volterra integral equation. The value $\alpha = 1$ corresponds to the ordinary (non-fractional) Volterra integral equation.
Particularly, if $k(x,t) = 1$ and $0 < \alpha < 1$ in equation (2.14), we have an Abel integral equation in the form:

$$f(x) - \lambda\int_0^x \frac{f(t)}{(x-t)^\beta}dt = g(x),\quad 0 < \beta < 1. \tag{2.15}$$

Here $\lambda = \frac{1}{\Gamma(\alpha)}$ and $\beta = 1 - \alpha$.

3 Function approximation

A square integrable function $f(x)$ in the interval $[0,1]$ can be expanded into a Haar series of infinite terms:

$$f(x) = c_0h_0(x) + \sum_{j=0}^{\infty}\sum_{k=0}^{2^j-1} c_{2^j+k}h_{2^j+k}(x),\quad x \in [0,\ 1], \tag{3.1}$$

where the Haar coefficients are determined as:

$$c_i = 2^j\int_0^1 f(x)h_i(x)dx,\ i = 0,\ 2^j+k,\ j,k \in \mathbb{N}\cup\{0\};\ 0 \leq k < 2^j, \tag{3.2}$$

such that the following integral square error ϵ_m is minimized:

$$\epsilon_m = \int_0^1 [f(x) - \sum_{i=0}^{m-1} c_ih_i(x)]^2dx,\quad m = 2^{J+1},\ J \in \mathbb{N}\cup\{0\}. \tag{3.3}$$

By using Eq. (2.13), the above Haar coefficients can be rewritten as:

$$c_i = 2^j[\int_{\frac{k}{2^j}}^{\frac{k+1/2}{2^j}} f(x)dx - \int_{\frac{k+1/2}{2^j}}^{\frac{k+1}{2^j}} f(x)dx],\ i = 2^j + k,\ j,k \in \mathbb{N}\cup\{0\};\ 0 \le k < 2^j. \tag{3.4}$$

If $f(x)$ is a piecewise constant or may be approximated as a piecewise constant during each subinterval, the series sum in Eq. (3.1) can be truncated after m terms ($m = 2^{J+1}$, $J \geq 0$ is a resolution, level of wavelet), that is:

$$f(x) \cong c_0h_0(x) + \sum_{j=0}^{J}\sum_{k=0}^{2^j-1} c_{2^j+k}h_{2^j+k}(x) = \mathbf{c}^T\mathbf{h}(x) = f_m(x), \quad x \in [0,\ 1], \tag{3.5}$$

where $\mathbf{c} = \mathbf{c}_{m\times 1} = [c_0,\ c_1, \ldots, c_{m-1}]^T$, $\mathbf{h}(x) = \mathbf{h}_{m\times 1}(x) = [h_0(x),\ h_1(x), \ldots, h_{m-1}(x)]^T$.

3.1 Operational Matrix of Integration

The integration of $\mathbf{h}(x)$ can be expanded into Haar series with Haar coefficient matrix $\mathbf{P}_m$ [4] as:

$$\int_0^x \mathbf{h}(x)dx \cong \mathbf{P}_m\mathbf{h}(x), \tag{3.6}$$

the $m \times m$ square matrix $\mathbf{P}_m$ is called the operational matrix of integration and is given in [7] as:

$$\mathbf{P}_m = \frac{1}{2m}\left[\begin{array}{cc} 2m\mathbf{P}_{m/2} & -\mathbf{H}_{m/2\times m/2} \\ \mathbf{H}^{-1}_{m/2\times m/2} & \mathbf{0} \end{array}\right], \tag{3.7}$$

where $\mathbf{H}_{1\times 1} = [1]$, $\mathbf{P}_1 = [1/2]$ and $\mathbf{H}_{m\times m} = [\mathbf{h}(\frac{1}{2m}), \mathbf{h}(\frac{3}{2m}), \ldots, \mathbf{h}(\frac{2m-1}{2m})]$.

3.2 Product Operational Matrix

Three basic multiplication properties of Haar wavelets are as follows [7]:

- (i) $h_n(x)h_0(x) = h_n(x)$ for any $n \in \mathbb{N}\cup\{0\}$.

- (ii) For any two Haar wavelets $h_n(x)$ and $h_l(x)$ with $n < l$:

$$h_n(x)h_l(x) = \rho_{nl}h_l(x), \tag{3.8}$$

$$\rho_{nl} = h_n(2^{-i}(q+1/2)) = \begin{cases} 1, & 2^{i-j}k \le q < 2^{i-j}(k+1/2); \\ -1, & 2^{i-j}(k+1/2) \le q < 2^{i-j}(k+1); \\ 0, & \text{O.W}, \end{cases} \tag{3.9}$$

  where $n = 2^j + k, \quad j \geq 0, \quad 0 \le k < 2^j$ and $l = 2^i + q, \quad i \geq 0, \quad 0 \le q < 2^i$.

- (iii) The square of any Haar wavelet is a block pulse with magnitude of 1 during both positive and negative half waves.

The product of $\mathbf{h}(x)$, $\mathbf{h}^T(x)$ and $\mathbf{c}$ can be expanded into Haar series with Haar coefficient matrix $\mathbf{M}_m$ as follows:

$$\mathbf{h}(x)\mathbf{h}^T(x)\mathbf{c} = \mathbf{M}_m\mathbf{h}(x), \tag{3.10}$$

where $\mathbf{M}_m$ is an $m \times m$ matrix and is called the product operational matrix and given by [7]:

$$\mathbf{M}_m = \begin{bmatrix} \mathbf{M}_{m/2} & \mathbf{H}_{m/2}diag(\widetilde{\mathbf{c}}_b) \\ diag(\widetilde{\mathbf{c}}_b)\mathbf{H}_{m/2}^{-1} & diag(\widetilde{\mathbf{c}}_a^T.\mathbf{H}_{m/2}) \end{bmatrix}, \tag{3.11}$$

such that $\mathbf{M}_1 = c_0$ and $\widetilde{\mathbf{c}}_a = [c_0, \ldots, c_{m/2-1}]^T,\ \widetilde{\mathbf{c}}_b = [c_{m/2}, \ldots, c_{m-1}]^T$.

4 Operational Matrix of Fractional Integration

In this section we want to obtain the operational matrix of fractional integration for Haar wavelets, which is the generalized form of $\mathbf{P}_m$ in (3.7). The fractional integration of order α of $\mathbf{h}(x)$ can be expanded into Haar series with Haar coefficient matrix $\mathbf{P}_m^\alpha$ as follows:

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\mathbf{h}(t)dt = \mathbf{P}_m^\alpha\mathbf{h}(x). \tag{4.1}$$

We call this $m \times m$ square matrix $\mathbf{P}_m^\alpha$ the (generalized) operational matrix of fractional integration. If $f(x)$ is expanded into Haar wavelet series, as shown in Eq. (3.5), the Riemann-Liouville fractional integral of $f(x)$ becomes:

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt \cong \mathbf{c}^T\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\mathbf{h}(t)dt. \tag{4.2}$$

Thus for expanding the Riemann-Liouville integral, it is enough to expand:

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_n(t)dt, \tag{4.3}$$

for $n = 0,\ 1, \ldots, m-1$, in Haar series. We know:

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_n(t)dt = \frac{1}{\Gamma(\alpha)}\{x^{\alpha-1} * h_n(x)\}, \tag{4.4}$$

where $*$ is the convolution operator of two functions. By taking the Laplace transform of the above equation and using Eq. (2.8) we have:

$$\mathcal{L}\{\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_n(t)dt\} = \frac{1}{\Gamma(\alpha)}\mathcal{L}\{x^{\alpha-1}\}\mathcal{L}\{h_n(x)\}, \tag{4.5}$$

where:

$$\begin{aligned} \mathcal{L}\{x^{\alpha-1}\} &= \frac{\Gamma(\alpha)}{s^\alpha}, \\ \mathcal{L}\{h_n(x)\} &= \mathcal{L}\{u(x-\frac{k}{2^j}) - 2u(x-\frac{k+1/2}{2^j}) + u(x-\frac{k+1}{2^j})\} \\ &= \frac{1}{s}\{e^{-\frac{k}{2^j}s} - 2e^{-\frac{k+1/2}{2^j}s} + e^{-\frac{k+1}{2^j}s}\}, \end{aligned}$$

the last two equalities are obtained by Eqs. (2.13), (2.9). Therefore (4.5) can be rewritten as:

$$\mathcal{L}\{\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_n(t)dt\} = \frac{1}{\Gamma(\alpha+1)}.\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}\{e^{-\frac{k}{2^j}s} - 2e^{-\frac{k+1/2}{2^j}s} + e^{-\frac{k+1}{2^j}s}\}. \tag{4.6}$$

Now taking the inverse Laplace transform of the above equation and using (2.10) yields:

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_n(t)dt =$$

$$\frac{1}{\Gamma(\alpha+1)}\{\underbrace{(x-\frac{k}{2^j})^\alpha u(x-\frac{k}{2^j})}_{X(x)} - 2\underbrace{(x-\frac{k+1/2}{2^j})^\alpha u(x-\frac{k+1/2}{2^j})}_{Y(x)} + \underbrace{(x-\frac{k+1}{2^j})^\alpha u(x-\frac{k+1}{2^j})}_{Z(x)}\}$$

$$= \frac{1}{\Gamma(\alpha+1)}\{X(x) - 2Y(x) + Z(x)\}. \tag{4.7}$$

Specially for $n = 0$ we have:

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}h_0(t)dt &= \frac{1}{\Gamma(\alpha+1)}\{\underbrace{x^\alpha u(x) - (x-1)^\alpha u(x-1)}_{W(x)}\} \\ &= \frac{1}{\Gamma(\alpha+1)}W(x).\end{aligned} \tag{4.8}$$

The Eqs. (4.7) and (4.8) can be expanded into Haar wavelets as:

$$I^\alpha h_n(x) = c_{n0}h_0(x) + \sum_{p=0}^{J}\sum_{q=0}^{2^p-1} c_{n2^p+q}h_{2^p+q}(x),\ n = 0,\ 1, \dots, m-1. \tag{4.9}$$

Now we want to obtain the coefficients, $c_{nl},\ n,l = 0,\ 1, ..., m-1$ in the above equation. According to (2.12), (2.13) we have:

$$c_{00} = \frac{1}{\Gamma(\alpha+1)}\int_0^1 W(t)h_0(t)dt = \frac{1}{\Gamma(\alpha+2)}, \tag{4.10}$$

$$\begin{aligned} c_{02^p+q} &= \frac{2^p}{\Gamma(\alpha+1)}\int_0^1 W(t)h_{2^p+q}(t)dt \\ &= \frac{2^p}{\Gamma(\alpha+1)}\int_0^1 W(t)\{u(t-\frac{q}{2^p}) - 2u(t-\frac{q+1/2}{2^p}) + u(t-\frac{q+1}{2^p})\}dt \\ &= \frac{2^p}{\Gamma(\alpha+1)}[\int_{\frac{q}{2^p}}^1 t^\alpha dt - 2\int_{\frac{q+1/2}{2^p}}^1 t^\alpha dt + \int_{\frac{q+1}{2^p}}^1 t^\alpha dt]. \end{aligned}$$

Thus:

$$c_{02^p+q} = -\frac{2^p}{\Gamma(\alpha+2)}[(\frac{q}{2^p})^{\alpha+1} - 2(\frac{q+1/2}{2^p})^{\alpha+1} + (\frac{q+1}{2^p})^{\alpha+1}], \tag{4.11}$$

where $p = 0, 1, \dots, J$, $q = 0, 1, \dots, 2^p - 1$.

Similarly to calculate c_{n0} and c_{n2^p+q} for $n = 1, 2, \dots, m-1$, $p = 0, 1, \dots, J$ and $q = 0, 1, \dots, 2^p - 1$ (in Eq. (4.9)), we have:

$$\begin{aligned}
c_{n0} &= \frac{1}{\Gamma(\alpha+1)} \int_0^1 [X(t) - 2Y(t) + Z(t)] h_0(t) dt \\
&= \frac{1}{\Gamma(\alpha+1)} \int_0^1 [X(t) - 2Y(t) + Z(t)][u(t) - u(t-1)] dt \\
&= \frac{1}{\Gamma(\alpha+1)} [\int_{\frac{k}{2^j}}^1 (t - \frac{k}{2^j})^\alpha dt - 2 \int_{\frac{k+1/2}{2^j}}^1 (t - \frac{k+1/2}{2^j})^\alpha dt + \int_{\frac{k+1}{2^j}}^1 (t - \frac{k+1}{2^j})^\alpha dt],
\end{aligned}$$

so:

$$c_{n0} = \frac{1}{\Gamma(\alpha+2)} [(1 - \frac{k}{2^j})^{\alpha+1} - 2(1 - \frac{k+1/2}{2^j})^{\alpha+1} + (1 - \frac{k+1}{2^j})^{\alpha+1}]. \tag{4.12}$$

and:

$$\begin{aligned}
c_{n2^p+q} &= \frac{2^p}{\Gamma(\alpha+1)} \int_0^1 [X(t) - 2Y(t) + Z(t)] h_{2^p+q}(t) dt \\
&= \frac{2^p}{\Gamma(\alpha+1)} \int_0^1 [X(t) - 2Y(t) + Z(t)][u(t - \frac{q}{2^p}) - 2u(t - \frac{q+1/2}{2^p}) + u(t - \frac{q+1}{2^p})] dt \\
&= \frac{2^p}{\Gamma(\alpha+1)} [\int_{\eta_0}^1 (t - \frac{k}{2^j})^\alpha dt - 2 \int_{\eta_{1/2}}^1 (t - \frac{k}{2^j})^\alpha dt + \int_{\eta_1}^1 (t - \frac{k}{2^j})^\alpha dt] \\
&\quad -2. \frac{2^p}{\Gamma(\alpha+1)} [\int_{\theta_0}^1 (t - \frac{k+1/2}{2^j})^\alpha dt - 2 \int_{\theta_{1/2}}^1 (t - \frac{k+1/2}{2^j})^\alpha dt + \int_{\theta_1}^1 (t - \frac{k+1/2}{2^j})^\alpha dt \\
&\quad + \frac{2^p}{\Gamma(\alpha+1)} [\int_{\xi_0}^1 (t - \frac{k+1}{2^j})^\alpha dt - 2 \int_{\xi_{1/2}}^1 (t - \frac{k+1}{2^j})^\alpha dt + \int_{\xi_1}^1 (t - \frac{k+1}{2^j})^\alpha dt,
\end{aligned}$$

where:

$$\begin{aligned}
\eta_i &= max\{\frac{k}{2^j},\ \frac{q+i}{2^p}\},\ i = 0,\ 1/2,\ 1, \\
\theta_i &= max\{\frac{k+1/2}{2^j},\ \frac{q+i}{2^p}\},\ i = 0,\ 1/2,\ 1, \\
\xi_i &= max\{\frac{k+1/2}{2^j},\ \frac{q+i}{2^p}\},\ i = 0,\ 1/2,\ 1.
\end{aligned}$$

Therefore:

$$\begin{aligned}
c_{n2^p+q} &= -\frac{2^p}{\Gamma(\alpha+2)} [(\eta_0 - \frac{k}{2^j})^{\alpha+1} - 2(\eta_{1/2} - \frac{k}{2^j})^{\alpha+1} + (\eta_1 - \frac{k}{2^j})^{\alpha+1}] \\
&\quad +2. \frac{2^p}{\Gamma(\alpha+2)} [(\theta_0 - \frac{k+1/2}{2^j})^{\alpha+1} - 2(\theta_{1/2} - \frac{k+1/2}{2^j})^{\alpha+1} + (\theta_1 - \frac{k+1/2}{2^j})^{\alpha+1}] \\
&\quad - \frac{2^p}{\Gamma(\alpha+2)} [(\xi_0 - \frac{k+1}{2^j})^{\alpha+1} - 2(\xi_{1/2} - \frac{k+1}{2^j})^{\alpha+1} + (\xi_1 - \frac{k+1}{2^j})^{\alpha+1}],
\end{aligned} \tag{4.13}$$

for $n = 1, 2, \dots, m-1$, $p = 0, 1, \dots, J$ and $q = 0, 1, \dots, 2^p - 1$.
Thus, we can write the operational matrix of fractional integration, which is introduced in (4.1) as:

$$\mathbf{P}_m^\alpha = \begin{bmatrix} \mathbf{P}_{m/2}^\alpha & \mathbf{R}_{m/2\times m/2} \\ \mathbf{S}_{m/2\times m/2} & \mathbf{U}_{m/2\times m/2} \end{bmatrix}, \tag{4.14}$$

where:

$$\mathbf{R}_{m/2\times m/2}=[c_{n2^J+q}] \quad n=0,1,\ldots,m/2-1,\ q=0,1,\ldots,2^J-1$$

and:

$$\mathbf{S}_{m/2\times m/2}=[c_{nl}], \quad n=m/2,\ldots,m-1,\ l=0,1,\ldots,m/2-1$$

are calculated easily by (4.11), (4.13) and (4.12), (4.13), respectively, and $\mathbf{U}_{m/2\times m/2}$ is an upper triangular matrix which is in the form:

$$\mathbf{U}_{m/2\times m/2}=u_1\mathbf{I}+u_2\mu+u_3\mu^2+\ldots+u_{m/2}\mu^{m/2-1}, \tag{4.15}$$

such that:

$$u_i=\begin{cases}\frac{2^J}{\Gamma(\alpha+2)}\cdot(\frac{1}{2^J})^{\alpha+1}[4(1/2)^{\alpha+1}-1], & \text{i=1;}\\ \frac{2^J}{\Gamma(\alpha+2)}\cdot(\frac{1}{2^J})^{\alpha+1}[-i^{\alpha+1}+4(i-1/2)^{\alpha+1}-6(i-1)^{\alpha+1}+4(i-3/2)^{\alpha+1}-(i-2)^{\alpha+1}], & \text{i=2,...,}m/2,\end{cases}$$

$\mathbf{I}_{m/2}$ is the identity matrix and:

$$\mu_{m/2}=\begin{bmatrix}0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ & & & & 1\\ 0 & & & \ldots & 0\end{bmatrix}. \tag{4.16}$$

It is considerable that for $\alpha=1$, the fractional integration (2.1) is the ordinary integration and the generalized operational matrix of fractional integration $\mathbf{P}_m^\alpha$ is the same as $\mathbf{P}_m$, which is introduced in (3.7). Here we present $\mathbf{P}_m^{1/3}$ for $J=0,\ 1,\ 2$:

$$\mathbf{P}_2^{1/3}=\begin{bmatrix}0.8399 & -0.1733\\ 0.1733 & 0.4933\end{bmatrix},\quad \mathbf{P}_4^{1/3}=\begin{bmatrix}0.8399 & -0.1733 & -0.1375 & -0.0571\\ 0.1733 & 0.4933 & -0.1375 & 0.2179\\ 0.0286 & 0.1090 & 0.3916 & -0.0428\\ 0.0688 & -0.0688 & 0 & 0.3916\end{bmatrix},$$

$$\mathbf{P}_8^{1/3}=\begin{bmatrix}0.8399 & -0.1733 & -0.1375 & -0.0571 & -0.1092 & -0.0453 & -0.0320 & -0.0256\\ 0.1733 & 0.4933 & -0.1375 & 0.2179 & -0.1092 & -0.0453 & 0.1863 & 0.0651\\ 0.0286 & 0.1090 & 0.3916 & -0.0428 & -0.1092 & 0.1730 & -0.0505 & -0.0068\\ 0.0688 & -0.0688 & 0 & 0.3916 & 0 & 0 & -0.1092 & 0.1730\\ 0.0064 & 0.0163 & 0.0865 & -0.0034 & 0.3108 & -0.0339 & -0.0029 & -0.0009\\ 0.0080 & 0.0466 & -0.0546 & -0.0253 & 0 & 0.3108 & -0.0339 & -0.0029\\ 0.0113 & -0.0113 & 0 & 0.0865 & 0 & 0 & 0.3108 & -0.0339\\ 0.0273 & -0.0273 & 0 & -0.0546 & 0 & 0 & 0 & 0.3108\end{bmatrix}.$$

5 Approximation of the Fractional Volterra Integral Equation and Abel Integral Equation Via Haar Waveles

5.1 Fractional Volterra Integral Equation

Now we consider the fractional Volterra integral equation (2.14). According to section (3), the right-hand-side of the mentioned equation is approximated as:

$$g(x)\cong g_0h_0(x)+\sum_{j=0}^{J}\sum_{k=0}^{2^j-1}g_{2^j+k}h_{2^j+k}(x)=\mathbf{g}^T\mathbf{h}(x). \tag{5.1}$$

Similarly, $K(x,t) \in L^2([0,\ 1) \times [0,\ 1))$ can be approximated as:

$$k(x,\ t) \cong \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} k_{ij} h_i(x) h_j(t), \tag{5.2}$$

or in the matrix form:

$$k(x,\ t) \cong \mathbf{h}^T(x) \mathbf{K} \mathbf{h}(t), \tag{5.3}$$

where $\mathbf{K} = [k_{ij}]_{m \times m}$ such that:

$$k_{ij} = 2^{i+j} \int_0^1 \int_0^1 k(x,t) h_i(x) h_j(t) dt dx,\ i,j = 0, 1, \ldots, m-1.$$

Also the fractional integral part of (2.14) is written as the following:

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} k(x,t) f(t) dt \cong \widetilde{\mathbf{v}} \mathbf{h}(x), \tag{5.4}$$

for calculating $\widetilde{\mathbf{v}}$ we need some preliminaries, then we will introduce it in subsection (5.3). By substituting the approximations (3.5), (5.4) and (5.1) into (2.14) we obtain:

$$\mathbf{h}^T(x) \mathbf{c} - \mathbf{h}^T(x) \widetilde{\mathbf{v}} = \mathbf{h}^T(x) \mathbf{g}, \tag{5.5}$$

therefore:

$$\mathbf{c} - \widetilde{\mathbf{v}} = \mathbf{g}. \tag{5.6}$$

Eq. (5.6) is a system of linear equations and can be easily solved for the unknown vector $\mathbf{c}$. Note that the entries of the vector $\widetilde{\mathbf{v}}$ are related to the entries of $\mathbf{c}$.

5.2 Abel Integral Equation

similarly, the Abel integral equation (2.15) as a particular kind of the fractional Volterra integral equation can be written as:

$$\mathbf{h}^T(x) \mathbf{c} - \mathbf{h}^T(x) (\mathbf{P}_m^\alpha)^T \mathbf{c} = \mathbf{h}^T(x) \mathbf{g}, \tag{5.7}$$

therefore:

$$\mathbf{c} - (\mathbf{P}_m^\alpha)^T \mathbf{c} = \mathbf{g}. \tag{5.8}$$

Eq. (5.8) is a system of linear equations and can be easily solved for the unknown vector $\mathbf{c}$, as:

$$\mathbf{c} = (I - (\mathbf{P}_m^\alpha)^T)^{-1} \mathbf{g}. \tag{5.9}$$

5.3 Evaluating $\widetilde{\mathbf{v}}$

As we mentioned before, the fractional integral part of (2.14) can be written via Haar wavelets as (5.4), where $\widetilde{\mathbf{v}} = [\widetilde{v}_0,\ \widetilde{v}_1, \ldots, \widetilde{v}_{m-1}]^T$ and according to Eq. (3.2), we have:

$$\widetilde{v}_i = 2^j \int_0^1 [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} k(x,t) f(t) dt] h_i(x) dx,\ i = 2^j + k, \tag{5.10}$$

by substituting $f(t) \cong \mathbf{h}^T(x)\mathbf{c}$ and (5.3) into Eq. (5.10) we get:

$$\begin{aligned}
\widetilde{v}_i &\cong 2^j \int_0^1 [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\ \mathbf{h}^T(x)\mathbf{K}\mathbf{h}(t)\ \mathbf{h}^T(t)\mathbf{c}dt]\ h_i(x)dx \\
&= 2^j \int_0^1\ \mathbf{h}^T(x)\mathbf{K}\ \mathbf{M}\ [\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\mathbf{h}(t)dt]\ h_i(x)dx,
\end{aligned}$$

the $m \times m$ matrix $\mathbf{M}$ is introduced in (3.11). Now by using (4.1), we have:

$$\begin{aligned}
\widetilde{v}_i &\cong 2^j \int_0^1\ \mathbf{h}^T(x)\mathbf{K}\ \mathbf{M}\ \mathbf{P}_m^\alpha \mathbf{h}(x)\ h_i(x)dx \\
&= 2^j \int_0^1\ \mathbf{h}^T(x)\mathbf{A}\mathbf{h}(x)\ h_i(x)dx,
\end{aligned}$$

where $\mathbf{A} = \mathbf{K}\ \mathbf{M}\ \mathbf{P}_m^\alpha = [a_{ij}]_{m\times m}$. It is obvious that $\mathbf{h}^T(x)\mathbf{A}\mathbf{h}(x)$ is a 1×1 matrix, and:

$$\begin{aligned}
\mathbf{h}^T(x)\mathbf{A}\mathbf{h}(x)h_i(x) &= \sum_{m_1=2}^{m}\sum_{n_1=m_1}^{m}(a_{n_1(m_1-1)} + a_{(m_1-1)n_1})h_{n_1-1}(x)h_{m_1-2}(x) + \sum_{i_1=1}^{m} a_{i_1i_1}h_{i_1-1}^2(x) \\
&= \sum_{n_1=2}^{m}(a_{1n_1} + a_{n_11})h_{n_1-1}(x)h_0(x) \\
&\quad + \sum_{m_1=3}^{m}\sum_{n_1=m_1}^{m}(a_{n_1(m_1-1)} + a_{(m_1-1)n_1})h_{n_1-1}(x)h_{m_1-2}(x) + \sum_{i_1=1}^{m} a_{i_1i_1}h_{i_1-1}^2(x) \\
&= \sum_{n_1=2}^{m}(a_{1n_1} + a_{n_11})h_{n_1-1}(x) \\
&\quad + \sum_{m_1=3}^{m}\sum_{n_1=m_1}^{m}(a_{n_1(m_1-1)} + a_{(m_1-1)n_1})\rho_{n_1-1m_1-2}h_{n_1-1}(x) + \sum_{i_1=1}^{m} a_{i_1i_1}h_{i_1-1}^2(x),
\end{aligned}$$

where $\rho_{n_1m_1}$ is defined in (3.9). Therefore:

$$\begin{aligned}
\widetilde{v}_i &\cong 2^j[\sum_{n_1=2}^{m}(a_{1n_1} + a_{n_11})\int_0^1 h_{n_1-1}(x)h_i(x)dx \\
&\quad + \sum_{m_1=3}^{m}\sum_{n_1=m_1}^{m}(a_{n_1(m_1-1)} + a_{(m_1-1)n_1})\rho_{n_1-1m_1-2}\int_0^1 h_{n_1-1}(x)h_i(x)dx \\
&\quad + \sum_{i_1=1}^{m} a_{i_1i_1}\int_0^1 h_{i_1-1}^2(x)h_i(x)dx] \\
&= (a_{1(i+1)} + a_{(i+1)1}) + \sum_{m_1=2}^{i}(a_{m_1(i+1)} + a_{(i+1)m_1})\rho_{i(m_1-1)} + 2^j\sum_{i_1=i+2}^{m} a_{i_1i_1}\ \rho_{i_1i}\cdot\frac{1}{2^l},
\end{aligned}$$

where $i_1 = 2^l + w, l, w \in \{0\} \cup \mathbb{N};\ 0 \le w < 2^l$. Specially $\widetilde{v}_0 = a_{11} + \sum_{i_1=2}^{m} a_{i_1i_1}\cdot\frac{1}{2^{l_1}}$, for $i_1 - 1 = 2^{l_1} + z,\ \ l_1, z \in \{0\} \cup \mathbb{N};\ 0 \le z < 2^{l_1}$.

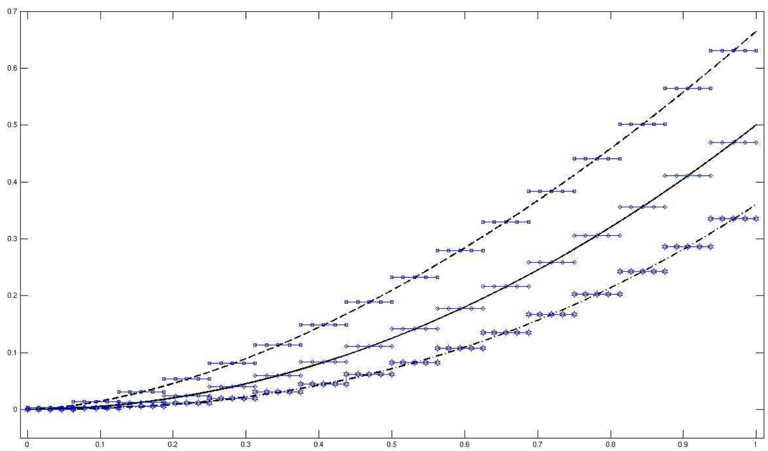

Figure 1. Fractional integration of $f(x) = x$, $(--,\ -,\ -.-)$ and its approximation $(I^{\alpha} f_{16}(x))$ $(\square, \diamond, \star)$ for $\alpha = 2/3, 1, 4/3$.

6 Error Approximation

Let $f(x)$ be the exact solution of (2.14) , $f(x) \cong \mathbf{c}^T\mathbf{h}(x) = f_m(x)$ and $e_m(x) = f(x) - f_m(x)$. By (3.1) and (3.5) we have:

$$e_m(x) = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k} h_{2^j+k}(x), \tag{6.1}$$

where $e_m(x)$ is called the error function of Haar wavelet method. Suppose that $f(x)$ satisfies in the Lipschitz condition on $[0,\ 1]$, that is:

$$\exists M > 0\ ;\ \forall x,\ y \in [0,\ 1]:\ |f(x) - f(y)| \leq M|x - y|. \tag{6.2}$$

In this section we first obtain a bound for $\|e_m(x)\|_2$, then a method for estimating error function, (6.1), is introduced.

6.1 A Bound for $\|e_m(x)\|_2$

$$\begin{aligned}
\|e_m(x)\|_2^2 &= \int_0^1 \Big(\sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k} h_{2^j+k}(x)\Big)^2 dx \\
&= \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k}^2 \int_0^1 h_{2^j+k}^2(x)dx \\
&\quad + \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{p=J+1}^{\infty} \sum_{q=0,\ q\neq k}^{2^p-1} c_{2^j+k} c_{2^p+q} \int_0^1 h_{2^j+k}(x) h_{2^p+q}(x)dx \\
&= \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k}^2 \Big(\frac{1}{2^j}\Big).
\end{aligned}$$

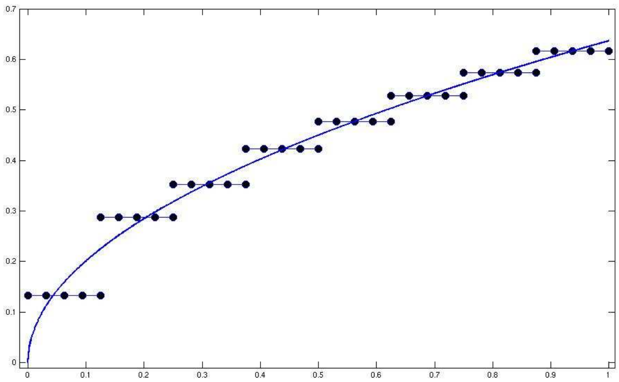

Figure 2. Exact solution (-) and Numerical solution (•) of the example (7.2) for $J = 2$.

Since $c_{2^j+k} = 2^j \int_0^1 f(x)h_i(x)dx$, by (3.4) and using mean value theorem we have:

$$\exists \; x_1^{jk} \in [\frac{k}{2^j}, \; \frac{k+1/2}{2^j}], \; x_2^{jk} \in [\frac{k+1/2}{2^j}, \; \frac{k+1}{2^j}],$$

such that:

$$\begin{aligned} c_{2^j+k} &= 2^j[(\frac{k+1/2}{2^j} - \frac{k}{2^j})f(x_1^{jk}) - (\frac{k+1}{2^j} - \frac{k+1/2}{2^j})f(x_2^{jk})] \\ &= \frac{1}{2}[f(x_1^{jk}) - f(x_2^{jk})] \\ &\leq \frac{1}{2}M(x_1^{jk} - x_2^{jk}) \\ &\leq \frac{1}{2}M\frac{1}{2^j} \\ &= M\frac{1}{2^{j+1}}, \end{aligned}$$

the first inequality is obtained by (6.2). Therefore $c_{2^j+k}^2 \leq M^2 \frac{1}{2^{2j+2}}$ and:

$$\begin{aligned} \|e_m(x)\|_2^2 &= \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{2^j+k}^2 (\frac{1}{2^j}) \\ &\leq \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} M^2 \frac{1}{2^{2j+2}} (\frac{1}{2^j}) \\ &= \frac{M^2}{4} \sum_{j=J+1}^{\infty} 2^j \frac{1}{2^{3j}} \\ &= \frac{M^2}{3} (\frac{1}{2^{J+1}})^2. \end{aligned}$$

Since $m = 2^{J+1}$, we have $\|e_m(x)\|_2^2 \leq \frac{M^2}{3}(\frac{1}{m})^2$ or in other words: $\|e_m(x)\|_2 = O(\frac{1}{m})$.

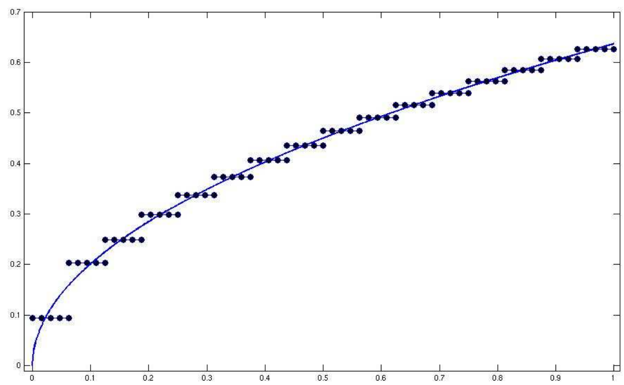

Figure 3. Exact solution (-) and Numerical solution (•) of the example (7.2) for $J = 3$.

7 Numerical Examples

Example 7.1. *Let $f(x) = x$, here we want to approximate $I^\alpha f(x)$ by the proposed $\mathbf{P}_m^\alpha$ for $\alpha = 2/3, 1, 4/3$ and compare it to the exact fractional integration of the function $f(x) = x$, which is easily obtained by (2.2). If we put $x \cong \mathbf{c}^T\mathbf{h}(x)$, we will have $I^\alpha x \cong \mathbf{c}^T I^\alpha \mathbf{h}(x) \cong \mathbf{c}^T\mathbf{P}_m^\alpha\mathbf{h}(x)$. Numerical results for $m = 16$ are shown in Figure 1.*

Example 7.2. *Consider the following Abel integral equation:*

$$\int_0^x \frac{f(t)}{\sqrt{x-t}}dt = x,$$

with the exact solution $f(x) = \frac{2}{\pi}\sqrt{x}$. Here $\alpha = 1/2$. Numerical results are shown in Table(1) and Figures (2, 3), for $J = 2,\ J = 3$, respectively.

x_i	$\lvert f(x_i) - f_8(x_i)\rvert$	$\lvert f(x_i) - f_{16}(x_i)\rvert$
0.0312	0.0200	0.0188
0.0937	0.0623	0.0087
0.1562	0.0363	0.0021
0.2187	0.0098	0.0011
0.2812	0.0153	1.0127e-4
0.3437	0.0204	2.2067e-4
0.4062	0.0169	7.0290e-5
0.4687	0.0132	9.0791e-5
0.5312	0.0133	6.3717e-5
0.5937	0.0132	5.7721e-5
0.6562	0.0124	4.8337e-5
0.7187	0.0116	4.2550e-5
0.7812	0.0112	3.7400e-5
0.8437	0.0108	3.3342e-5
0.9062	0.0105	2.9912e-5
0.9687	0.0101	2.7052e-5

Table 1.

Example 7.3. *Consider the following fractional Volterra integral equation:*

$$f(x) + \int_0^1 \frac{xt}{\sqrt{x-t}} f(t)dt = g(x), \quad 0 \le x \le 1, \tag{7.1}$$

where $g(x) = x(1-x) + \frac{16}{105}x^{\frac{7}{2}}(7-6x)$ *and* $f(x) = x(1-x)$ *is the exact solution. The numerical results are shown in Table(2) and Figures (4, 5), for* $J = 2,\ J = 3$*, respectively.*

x_i	$\lvert f(x_i) - f_8(x_i)\rvert$	$\lvert f(x_i) - f_{16}(x_i)\rvert$
0.0312	0.0289	0.0020
0.0937	0.0258	0.0014
0.1562	0.0208	0.0011
0.2187	0.0182	0.0014
0.2812	0.0152	0.0040
0.3437	0.0078	0.0038
0.4062	0.0076	0.0035
0.4687	0.0002	0.0034
0.5312	0.0057	0.0025
0.5937	0.0021	0.0022
0.6562	0.0129	0.0020
0.7187	0.0105	0.0018
0.7812	0.0214	0.0030
0.8437	0.0177	0.0026
0.9062	0.0283	0.0023
0.9687	0.0263	0.0020

Table 2.

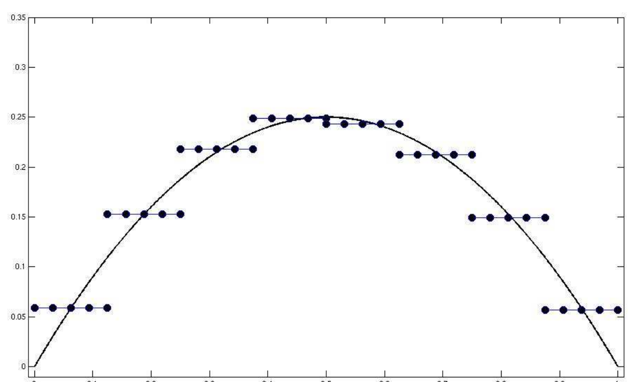

Figure 4. Exact solution (-) and Numerical solution (•) of the example (7.3) for $J = 3$.

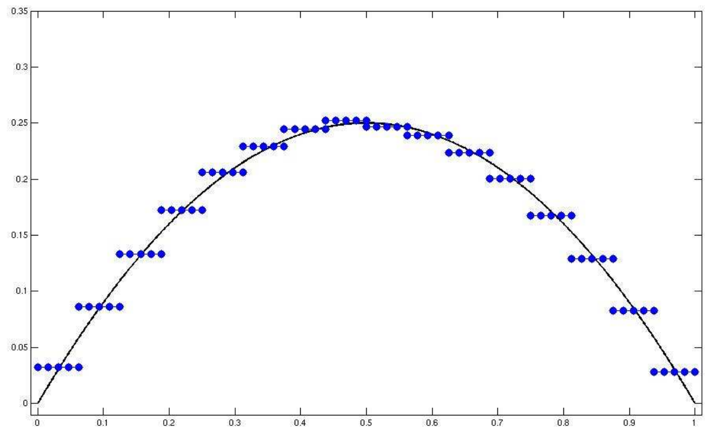

Figure 5. Exact solution (-) and Numerical solution (•) of the example (7.3) for $J = 2$.

8 Conclusion

Haar wavelets have been used before for solving integral equations, in this paper we use them for solving fractional Volterra integral equations by introducing a new fractional operational matrix which is the generalized form of the operational matrix of integration. The error analysis shows that, the larger resolution J is used, the more accurate results are obtained.

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