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Abstract

We study the entanglement of a multipartite quantum state. An inequality between the bipartite concurrence and the multipartite concurrence is obtained. More effective lower and upper bounds of the multipartite concurrence are obtained. By using the lower bound, the entanglement of more multipartite states are detected.

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As a potential resource for communication and information processing, quantum entanglement has rightly been the subject of much study in recent years [1]. However the boundary between the entangled states and the separable states, states that can be prepared by means of local operations and classical communications [2], is still not well characterized. Entanglement detection turns out to be a rather tantalizing problem. A more general question is to calculate the well defined quantitative measures of quantum entanglement such as entanglement of formation (EOF) [3] and concurrence [4, 5]. A series of excellent results have been obtained recently.

There have been some (necessary) criteria for separability, the Bell inequalities [6], PPT (positive partial transposition) [7] (which is also sufficient for the cases 2×2 and 2×3 bipartite systems [8]), realignment [9–11] and generalized realignment [12], as well as some necessary and sufficient operational criteria for low rank density matrices[13–15]. Further more, separability criteria based on local uncertainty relation [16–19] and the correlation matrix [20, 21] of the Bloch representation for a quantum state have been derived, which are strictly stronger than or independent of the PPT and realignment criteria. The calculation of entanglement of formation or concurrence is complicated except for 2×2 systems [22] or for states with special forms [23]. For general quantum states with higher dimensions or multipartite case, it seems to be a very difficult problem to obtain analytical formulas. However, one can try to find the lower and the upper bounds to estimate the exact values of the concurrence [24–27].

In this paper, we focus on the concurrence. We derive new lower and upper bounds of concurrence for arbitrary quantum states. From the bounds we can detect more entangled states. Detailed examples are given to show that the new bounds of concurrence are better than that have been obtained before.

For a pure N-partite quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $dim\mathcal{H}_i = d_i$, $i = 1, ..., N$, the concurrence of bipartite decomposition between subsystems $12 \cdots M$ and $M+1 \cdots N$ is defined by

$$C_2(|\psi\rangle\langle\psi|) = \sqrt{2(1 - \mathrm{Tr}\{\rho_{12\cdots M}^2\})} \tag{1}$$

where $\rho_{12\cdots M}^2 = \mathrm{Tr}_{M+1\cdots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over subsystems $M + 1 \cdots N$.

On the other hand, the concurrence of $|\psi\rangle$ is defined by [5]

$$C_N(|\psi\rangle\langle\psi|) = 2^{1-\frac{N}{2}} \sqrt{(2^N - 2) - \sum_\alpha \mathrm{Tr}\{\rho_\alpha^2\}}, \tag{2}$$

where α labels all different reduced density matrices.

For a mixed multipartite quantum state, $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding concurrence of (1) and (2) are then given by the convex roof:

$$C_2(\rho) = \min_{\{p_i,|\psi_i\}\rangle} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|), \tag{3}$$

$$C_N(\rho) = \min_{\{p_i,|\psi_i\}\rangle} \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|). \tag{4}$$

We now investigate the relation between the two kinds of concurrences.

Lemma 1: For a bipartite density matrix $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, one has

$$1 - \mathrm{Tr}\{\rho^2\} \leq 1 - \mathrm{Tr}\{\rho_A^2\} + 1 - \mathrm{Tr}\{\rho_B^2\}, \tag{5}$$

where $\rho_{A/B} = \mathrm{Tr}_{B/A}\{\rho\}$ be the reduced density matrices.

Proof: Let $\rho = \sum\limits_{ij} \lambda_{ij}|ij\rangle\langle ij|$ be the spectral decomposition, where $\lambda_{ij} \geq 0, \sum_{ij} \lambda_{ij} = 1$.

Then $\rho_1 = \sum_{ij} \lambda_{ij}|i\rangle\langle i|, \rho_2 = \sum_{ij} \lambda_{ij}|j\rangle\langle j|$. Therefore

$$\begin{aligned}
& 1 - \mathrm{Tr}\{\rho_A^2\} + 1 - \mathrm{Tr}\{\rho_B^2\} - 1 + \mathrm{Tr}\{\rho^2\} = 1 - \mathrm{Tr}\{\rho_A^2\} - \mathrm{Tr}\{\rho_B^2\} + \mathrm{Tr}\{\rho^2\} \\
= \ & (\sum_{ij} \lambda_{ij})^2 - \sum_{i,j,j'} \lambda_{ij}\lambda_{ij'} - \sum_{i,i',j} \lambda_{ij}\lambda_{i'j} + \sum_{ij} \lambda_{ij}^2 \\
= \ & (\sum_{i=i',j=j'} \lambda_{ij}^2 + \sum_{i=i',j\neq j'} \lambda_{ij}\lambda_{ij'} + \sum_{i\neq i',j=j'} \lambda_{ij}\lambda_{i'j} + \sum_{i\neq i',j\neq j'} \lambda_{ij}\lambda_{i'j'}) - (\sum_{i,j=j'} \lambda_{ij}^2 + \sum_{i,j\neq j'} \lambda_{ij}\lambda_{ij'}) \\
& -(\sum_{i=i',j} \lambda_{ij}^2 + \sum_{i\neq i',j} \lambda_{ij}\lambda_{i'j}) + \sum_{i,j} \lambda_{ij}^2 \\
= \ & \sum_{i\neq i',j\neq j'} \lambda_{ij}\lambda_{i'j'} \geq 0.
\end{aligned}$$

□

The same result in this lemma has also been derived in [27, 28] to prove the subadditivity of the linear entropy. Here we just give a simpler proof. In the following we compare the bi- and multi-partite concurrence in (3)(4) by using the lemma.

Theorem 1: For a multipartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ with $N \geq 3$, the following inequality holds,

$$C_N(\rho) \geq \max 2^{\frac{3-N}{2}} C_2(\rho), \tag{6}$$

where the maximum is taken over all kinds of bipartite concurrence.

Proof: Without lose of generality, we suppose that the maximal bipartite concurrence is attained between subsystems $12\cdots M$ and $(M+1)\cdots N$.

For a pure multipartite state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $\mathrm{Tr}\{\rho_{12\cdots M}^2\} = \mathrm{Tr}\{\rho_{(M+1)\cdots N}^2\}$. From (5) we have

$$\begin{aligned}
C_N^2(|\psi\rangle\langle\psi|) &= 2^{2-N}((2^N-2) - \sum_\alpha \mathrm{Tr}\{\rho_\alpha^2\}) \geq 2^{3-N}(N - \sum_{k=1}^N \mathrm{Tr}\{\rho_k^2\}) \\
&\geq 2^{3-N}(1 - \mathrm{Tr}\{\rho_{12\cdots M}^2\} + 1 - \mathrm{Tr}\{\rho_{(M+1)\cdots N}^2\}) \\
&= 2^{3-N} * 2(1 - \mathrm{Tr}\{\rho_{12\cdots M}^2\}) = 2^{3-N} C_2^2(|\psi\rangle\langle\psi|),
\end{aligned}$$

i.e. $C_N(|\psi\rangle\langle\psi|) \geq 2^{\frac{3-N}{2}} C_2(|\psi\rangle\langle\psi|)$.

Let $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ attain the minimal decomposition of the multipartite concurrence. One has

$$\begin{aligned}
C_N(\rho) = \sum_i p_i C_N(|\psi_i\rangle\langle\psi_i|) &\geq 2^{\frac{3-N}{2}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|) \\
&\geq 2^{\frac{3-N}{2}} \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|) = 2^{\frac{3-N}{2}} C_2(\rho).
\end{aligned}$$

$\square$

Corollary For a tripartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, the following inequality hold:

$$C_3(\rho) \geq \max C_2(\rho) \tag{7}$$

where the maximum is taken over all kinds of bipartite concurrence.

In [24] a lower bound for a bipartite state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$, $d_A \leq d_B$, has been obtained,

$$C_2(\rho) \geq \sqrt{\frac{2}{d_A(d_A-1)}}[\max(||\mathcal{T}_A(\rho)||, ||R(\rho)||) - 1]. \tag{8}$$

where $\mathcal{T}_A$, R and $||\cdot||$ stand for the partial transpose, realignment, and the trace norm (i.e., the sum of the singular values), respectively.

In [26, 29], from the separability criteria related to local uncertainty relation, covariance matrix and correlation matrix, the following lower bounds for bipartite concurrence are obtained:

$$C_2(\rho) \geq \frac{2||C(\rho)|| - (1-\mathrm{Tr}\{\rho_A^2\}) - (1-\mathrm{Tr}\{\rho_B^2\})}{\sqrt{2d_A(d_A-1)}} \tag{9}$$

and

$$C_2(\rho) \geq \sqrt{\frac{8}{d_A^3 d_B^2(d_A-1)}}(||T(\rho)|| - \frac{\sqrt{d_A d_B(d_A-1)(d_B-1)}}{2}), \tag{10}$$

where the entries of the matrix C, $C_{ij} = \langle\lambda_i^A \otimes \lambda_j^B\rangle - \langle\lambda_i^A \otimes I_{d_B}\rangle\langle I_{d_A} \otimes \lambda_j^B\rangle$, $T_{ij} = \frac{d_A d_B}{2}\langle\lambda_i^A \otimes \lambda_j^B\rangle$, $\lambda_k^{A/B}$ stands for the normalized generator of $SU(d_A/d_B)$, i.e. $\mathrm{Tr}\{\lambda_k^{A/B}\lambda_l^{A/B}\} = \delta_{kl}$ and $\langle X\rangle = \mathrm{Tr}\{\rho X\}$. It is shown that the lower bounds (9) and (10) are independent of (8).

Now we consider a multipartite quantum state $\rho \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ as a bipartite state belonging to $\mathcal{H}^A \otimes \mathcal{H}^B$ with the dimensions of the subsystems A and B being $d_A = d_{s_1}d_{s_2}\cdots d_{s_m}$ and $d_B = d_{s_{m+1}}d_{s_{m+2}}\cdots d_{s_N}$ respectively. By using the corollary, (8), (9) and (10) we have the following lower bound:

Theorem 2: For any N-partite quantum state ρ, we have:

$$C_N(\rho) \geq 2^{\frac{3-N}{2}} \max\{B1, B2, B3\}, \tag{11}$$

where

$$\begin{aligned}
B1 &= \max_{\{i\}} \sqrt{\frac{2}{M_i(M_i-1)}}\left[\max(||\mathcal{T}_A(\rho^i)||, ||R(\rho^i)||) - 1\right], \\
B2 &= \max_{\{i\}} \frac{2||C(\rho^i)|| - (1-\mathrm{Tr}\{(\rho_A^i)^2\}) - (1-\mathrm{Tr}\{(\rho_B^i)^2\})}{\sqrt{2M_i(M_i-1)}}, \\
B3 &= \max_{\{i\}} \sqrt{\frac{8}{M_i^3 N_i^2(M_i-1)}}(||T(\rho^i)|| - \frac{\sqrt{M_i N_i(M_i-1)(N_i-1)}}{2}),
\end{aligned}$$

ρ^is are all possible bipartite decompositions of ρ, and $M_i = \min\{d_{s_1}d_{s_2}\cdots d_{s_m}, d_{s_{m+1}}d_{s_{m+2}}\cdots d_{s_N}\}$, $N_i = \max\{d_{s_1}d_{s_2}\cdots d_{s_m}, d_{s_{m+1}}d_{s_{m+2}}\cdots d_{s_N}\}$.

In [27, 30, 31], it is shown that the upper and lower bound of multipartite concurrence satisfy

$$\sqrt{(4-2^{3-N})\mathrm{Tr}\{\rho^2\} - 2^{2-N}\sum_\alpha \mathrm{Tr}\{\rho_\alpha^2\}} \leq C_N(\rho) \leq \sqrt{2^{2-N}[(2^N-2) - \sum_\alpha \mathrm{Tr}\{\rho_\alpha^2\}]}. \quad (12)$$

In fact we can obtain a more effective upper bound for multi-partite concurrence. Let $\rho = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, where $|\psi_i\rangle$s are the orthogonal pure states and $\sum_i \lambda_i = 1$. We have

$$C_N(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_N(|\varphi_i\rangle\langle\varphi_i|) \leq \sum_i \lambda_i C_N(|\psi_i\rangle\langle\psi_i|). \quad (13)$$

The right side of (13) gives a new upper bound of $C_N(\rho)$. Since

$$\begin{aligned}\sum_i \lambda_i C_N(|\psi_i\rangle\langle\psi_i|) &= 2^{1-\frac{N}{2}}\sum_i \lambda_i\sqrt{(2^N-2) - \sum_\alpha \mathrm{Tr}\{(\rho_\alpha^i)^2\}} \\ &\leq 2^{1-\frac{N}{2}}\sqrt{(2^N-2) - \sum_\alpha \mathrm{Tr}\{\sum_i \lambda_i(\rho_\alpha^i)^2\}} \\ &\leq 2^{1-\frac{N}{2}}\sqrt{(2^N-2) - \sum_\alpha \mathrm{Tr}\{(\rho_\alpha)^2\}},\end{aligned}$$

the upper bound obtained in (13) is better than that in (12).

The lower and upper bounds can be used to estimate the value of the concurrence. Meanwhile, the lower bound of concurrence can be used to detect entanglement of quantum states. We now show that our upper and lower bounds can be better than that in (12) by several detailed examples.

Example 1: Consider the $2\times2\times2$ Dü r-Cirac-Tarrach states defined by [32]:

$$\rho = \sum_{\sigma=\pm}\lambda_0^\sigma|\Psi_0^\sigma\rangle\langle\Psi_0^\sigma| + \sum_{j=1}^{3}\lambda_j(|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \quad (14)$$

where the orthonormal Greenberger-Horne-Zeilinger (GHZ)-basis $|\Psi_j^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|j\rangle_{12}|0\rangle_3 \pm |(3-j)\rangle_{12}|1\rangle_3)$, $|j\rangle_{12} \equiv |j_1\rangle_1|j_2\rangle_2$ with $j = j_1j_2$ in binary notation. From theorem 2 we have that the lower bound of ρ is $\frac{1}{3}$. If we mix the state with white noise,

$$\rho(x) = \frac{(1-x)}{8}I_8 + x\rho, \quad (15)$$

by direct computation we have, as shown in FIG. 1, the lower bound obtained in (12) is always zero, while the lower bound in (11) is larger than zero for $0.425 \leq x \leq 1$, which

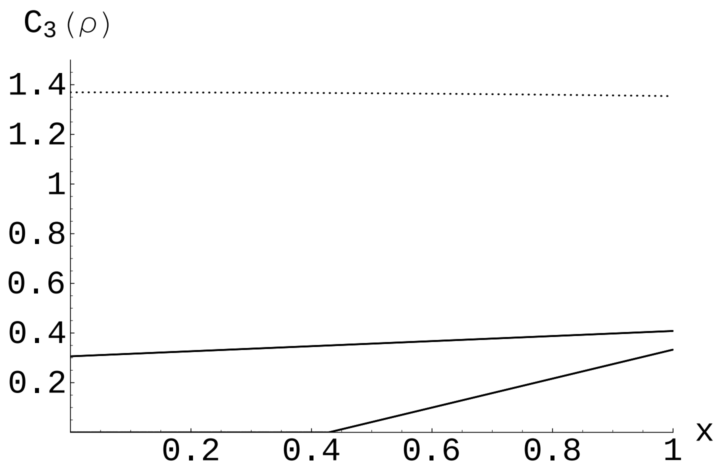

FIG. 1: Our lower and upper bounds of $C_3(\rho)$ from (11)(13)(solid line) and the upper bound obtained in (12)(dot line) while the lower bound in (12) is always zero.

shows that $\rho(x)$ is detected to be entangled at this situation. And the upper bound (dot line) in (12) is much larger than the upper bound we have obtained in (13) (solid line).

Example 2: We consider the depolarized state [32]:

$$\rho = \frac{(1-x)}{8} I_8 + x|\psi^+\rangle\langle\psi^+|, \tag{16}$$

where $0 \leq x \leq 1$ representing the degree of depolarization, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. From FIG. 2 one can obviously seen that our upper bound is tighter. For $0 \leq x \leq 0.7237$ our lower bound is higher than that in (12), i.e. our lower bound is closer to the true concurrence. Moreover for $0.2 \leq x \leq 0.57735$, our lower bound can detect the entanglement of ρ, while the lower bound in (12) not.

We have studied the concurrence for arbitrary multipartite quantum states. We derived new better lower and upper bounds. The lower bound can also be used to detect more multipartite entangled quantum states.

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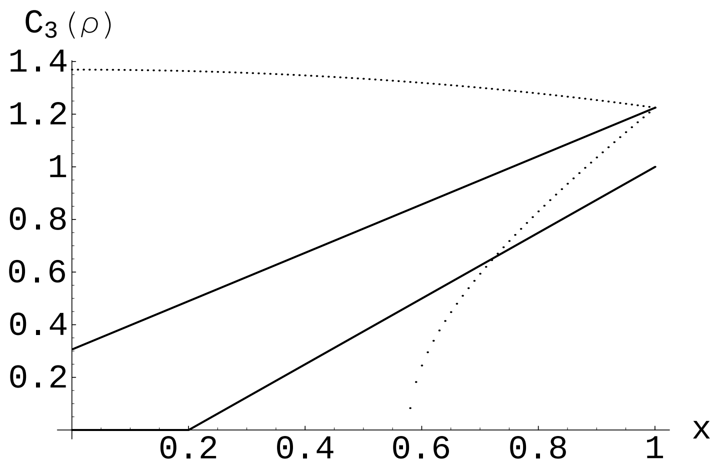

FIG. 2: Our lower and upper bounds of $C_3(\rho)$ from (11)(13) (solid line) and the bounds obtained in (12)(dot line).

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