Max-Planck-Institut
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Transversality problems in symplectic field theory and a new Fredholm theory

by

Oliver Fabert

TRANSVERSALITY PROBLEMS IN SYMPLECTIC FIELD THEORY AND A NEW FREDHOLM THEORY

OLIVER FABERT

ABSTRACT. This survey wants to give a short introduction to the transversality problem in symplectic field theory and motivate to approach it using the new Fredholm theory by Hofer, Wysocki and Zehnder. With this it should serve as a lead-in to the user's guide to polyfolds, which is appearing soon and the result of a working group organized by J. Fish, R. Golovko and the author at MSRI Berkeley in fall 2009.

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SUMMARY

Many problems in symplectic topology, like Gromov's pseudoholomorphic curves and Floer homology, are based on the study of solution sets of nonlinear elliptic PDE. These moduli spaces exhibit lack of compactness, but on the other hand lead to very elaborate compactifications, which are the source for interesting algebraic invariants. In many cases, including symplectic field theory (SFT), the algebraic structures of interest are precisely those created by the violent analytic behavior.

In order to provide the right framework in order to deal with all these problems, H. Hofer, K. Wysocki and E. Zehnder have introduced a new class of spaces, called polyfolds, which serve as new ambient space containing the compactified moduli space as the zero set of the nonlinear Cauchy-Riemann operator, for which they provide the usual Fredholm package including an implicit function theorem and abstract perturbation scheme, see [HWZ] and [H] and the references therein.

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As a result of the working group on polyfolds which the author organized together with Joel Fish and Roman Golovko at the MSRI in Berkeley in fall 2009, the three organizers use their own lecture notes in order to produce a user's guide to polyfolds, whose main goal is to equip symplectic geometers (and others) with a relatively short guide to show them how to apply the polyfold theory to their own Fredholm problem.

This survey paper wants to give a short introduction to the transversality problem in SFT and hence serve as a lead-in for the user's guide which will appear soon.

After a short introduction to symplectic field theory in section 1, which will serve as our main example in this introduction, we will first recall in section 2 the classical approach to equip moduli spaces and their compactifications with nice manifold structures (with boundaries and corners) using infinite-dimensional Banach space bundles over Banach manifolds. Since everything relies on an infinite-dimensional version of the classical implicit function theorem, it turns out that the crucial step is to prove a transversality result for the Cauchy-Riemann operator. While it is well-known that transversality holds for a generic choice of almost complex structure as long as all holomorphic curves are simple, that is, not multiply-covered, we will discuss in section 3 how severe the transversality problem with multiply-covered curves actually is, where it will turn out that the biggest problems arise from branched covers of orbit cylinders, also called orbit curves. Instead of discussing other approaches like Kuranishi structures and virtual fundamental cycle techniqes which claim to prove the transversality problem for holomorphic curves in full generality, we will shortly discuss in section 4 special approaches to the transversality problem like automatic transversality, obstruction bundles and domain-dependent almost complex structures, which are applicable in special cases and turn out to be useful when one is interested to perform actual computations. After seeing the limitations in all existing approaches (except the ones not discussed), we hope that the reader is finally motivated enough to approach the transversality problem using a completely new Fredholm theory based on new geometric objects like sc-manifolds and polyfolds, where will outline the main ideas and new concepts in the last section 5.

Since the goal of this survey is just to show how severe the transversality problem in SFT actually is and thereby motivate symplectic geometers to get in touch with the results of the great polyfold project, the author wants to apologize right away for all the appearing simplifications, inaccuracies and missing literature. On the other hand, comments, remarks and suggestions are very welcome.

1. A short introduction to symplectic field theory (SFT)

Symplectic field theory (SFT), introduced by H. Hofer, A. Givental and Y. Eliashberg in 2000 ([EGH]), is a very large project and can be viewed as a topological quantum field theory approach to Gromov-Witten theory. Besides providing a unified view on established pseudoholomorphic curve theories like symplectic Floer homology, contact homology and Gromov-Witten theory, it leads to numerous new applications and opens new routes yet to be explored.

In this survey we will restrict our attention to the transversality problem in SFT. Apart from the fact that SFT aims at being a grand unified theory of J-holomorphic curves in symplectic and contact topology, the main motivation comes from the observation that, in contrast to Gromov-Witten theory and symplectic Floer homology, there do not exist reasonable assumptions on the target manifolds like monotonicity or, more generally, semipositivity, for which transversality (for all relevant moduli spaces) can be proven for generic choices of almost complex structures. Furthermore, while in Gromov-Witten theory K. Cieliebak and K. Mohnke were able to prove in [CM] a transversality result for general symplectic manifolds using domain-dependent almost complex structures and employing Donaldson's construction of symplectic hypersurfaces, a corresponding result in SFT was so far only established in the so-called Floer case by the author in [F1] and currently seems out of reach for the general case, as speculated by K. Cieliebak in his talk in the "broken dreams" seminar at MSRI in fall 2009.

We start with briefly recalling the geometric setup of SFT for closed contact manifolds $(V, \xi = \{\lambda = 0\})$. For the moduli spaces and the functional analytic setup we will for simplicity only consider the case of genus zero.

Recall that a contact one-form λ defines a vector field R on V by $R \in \ker d\lambda$ and $\lambda(R) = 1$, which is called the Reeb vector field. We assume that the contact form is Morse in the sense that all closed orbits of the Reeb vector field are nondegenerate in the sense of [BEHWZ]; in particular, the set of closed Reeb orbits is discrete.

The SFT invariants are defined by counting $\underline{J}$-holomorphic curves in $\mathbb{R} \times V$ which are asymptotically cylindrical over chosen collections of Reeb orbits $\Gamma^\pm = \{\gamma_1^\pm, ..., \gamma_{n^\pm}^\pm\}$ as the $\mathbb{R}$-factor tends to $\pm\infty$, see [BEHWZ]. The almost complex structure $\underline{J}$ on the cylindrical manifold $\mathbb{R} \times V$ is required to be cylindrical in the sense that it is $\mathbb{R}$-independent, links the two natural vector fields on $\mathbb{R} \times V$, namely the Reeb vector field R and the $\mathbb{R}$-direction ∂_s, by $\underline{J}\partial_s = R$, and turns the distribution ξ on V into a complex subbundle of TV, $\xi = TV \cap \underline{J}TV$.

Then the moduli space $\mathcal{M}^0 = \mathcal{M}^0(\Gamma^+, \Gamma^-)$ of parametrized curves consists of tuples (u, j) where j is a complex structure on the punctured Riemann sphere $\dot{S}$ with $s^+ = \#\Gamma^+$ positive and $s^- = \#\Gamma^-$ negative punctures and $u : (\dot{S}, j) \to (\mathbb{R} \times V, \underline{J})$ is a $\underline{J}$-holomorphic map in the sense that it satisfies the Cauchy-Riemann equation

$$\bar{\partial}_{\underline{J}}(u) = du + \underline{J}(u) \cdot du \cdot j = 0$$

and is asymptotically cylindrical over the Reeb orbits in $\Gamma^\pm$ near the positive/negative punctures, which is equivalent to finiteness of the Hofer energy, $E(u) < \infty$ ([BEHWZ],[EGH]).

It will become crucial in our discussion that, before we can look for an appropriate compactification, we first need to divide out the all the natural symmetries of the

domain and the target. While the natural $\mathbb{R}$-action on the target $(\mathbb{R} \times V, \underline{J})$ leads to a natural $\mathbb{R}$-action on every moduli space $\mathcal{M}^0$, in the case when the sphere carries less three punctures we further need to divide out the automorphism group $\mathrm{Aut}(\dot{S}, j)$ of the domain given by

$$\varphi.(u, j) = (u \circ \varphi, \varphi^* j), \ \varphi \in \mathrm{Aut}(\dot{S}, j),$$

which is trivial when there are at least three punctures. Note that in the latter case the holomorphic curve is called domain-stable. Dividing out these obvious symmetries we obtain the desired moduli space of *unparametrized* curves,

$$\mathcal{M} = \mathcal{M}(\Gamma^+, \Gamma^-) = \mathcal{M}^0(\Gamma^+, \Gamma^-)/(\mathrm{Aut}(\dot{S}, j) \times \mathbb{R}).$$

In [BEHWZ] it is shown that the moduli space $\mathcal{M}(\Gamma^+, \Gamma^-)$ can be compactified by adding nodal curves as in Gromov-Witten theory as well as multi-level curves as in Floer theory. In order to define invariants, it remains to prove that the compactified moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}(\Gamma^+, \Gamma^-)$ can be equipped with a nice structure, where it will turn out that the best we can hope for is a branched-labelled manifold with boundaries and corners.

2. Classical Banach space bundle setup

In this section we shortly recall the main points of the classical approach which is used (in the hope) to prove nice results about the compactified moduli space. Instead of looking at the moduli space $\mathcal{M}$ of unparametrized curves or even the compactified moduli space $\overline{\mathcal{M}}$ directly, in the classical approach one first tries to prove a nice result about the moduli space $\mathcal{M}^0$ of parametrized curves.

Here the well-known idea is to use an infinite-dimensional version of the implicit function theorem to equip $\mathcal{M}^0$ with a nice manifold (or orbifold) structure. In order to be able to apply the implicit function theorem, we need to find an infinite-dimensional (Banach) manifold $\mathcal{B}^0$ of maps as well as a function, more generally, a section s in an appropriate infinite-dimensional (Banach space) bundle $\mathcal{E}^0$ over this manifold, whose zero set is the moduli space of parametrized curves, $\mathcal{M}^0 = s^{-1}(0)$. For manifolds of maps and their construction we refer to [Sch] and the foundational paper [E] by Eliasson.

Following [BM] (see also [F1], [F2] and [W]) an appropriate Banach manifold $\mathcal{B}^0$ is given by the product

$$\mathcal{B}^0 = H^{1,p,\delta}(\Gamma^+, \Gamma^-) \times \mathcal{M}_{0,s^++s^-},$$

where $\mathcal{M}_{0,s^++s^-}$ is the moduli space of complex structures on the s^++s^- punctured sphere and $H^{1,p,\delta}(\Gamma^+, \Gamma^-) \subset H^{1,p}(\dot{S}, \mathbb{R} \times V)$ is a space of $H^{1,p}$-maps which again satisfy appropriate asymptotic convergence to the chosen Reeb orbits in $\Gamma^\pm$ near the positive/negative punctures. Note that $\mathcal{M}_{0,s^++s^-}$ just consists of a point when $s^+ + s^- = 3$, where we define that this continues to hold when $s^+ + s^- < 3$, i.e., when the curve is domain-unstable. Furthermore it can be shown that there exists a smooth infinite-dimensional Banach space bundle $\mathcal{E}^0$ over $\mathcal{B}^0$ where the fibre

$$\mathcal{E}^0_{u,j} = L^{p,\delta}(\Lambda^{0,1} \otimes_{j,\underline{J}} u^* T(\mathbb{R} \times V)), \ (u, j) \in \mathcal{B}^0$$

consists of $(0, 1)$-forms (with respect to the complex structures j on $T\dot{S}$ and $\underline{J}$ on $u^* T(\mathbb{R} \times V)$) with values in the pullback bundle $u^* T(\mathbb{R} \times V)$. For more details we refer to the papers above.

It can further be shown that the Cauchy-Riemann operator $\bar{\partial}_{\underline{J}}$ defines a smooth section in the Banach space bundle $\mathcal{E}^0 \to \mathcal{B}^0$ by $\bar{\partial}_{\underline{J}}(u, j) = du + \underline{J}(u) \cdot du \cdot j \in \mathcal{E}^0_{u,j}$

for $(u,j) \in \mathcal{B}^0$. Furthermore it can be shown that, if the asymptotic convergence conditions for $H^{1,p,\delta}(\Gamma^+, \Gamma^-)$ are chosen appropriately, the moduli space $\mathcal{M}^0 = \mathcal{M}^0(\Gamma^+, \Gamma^-)$ agrees with the zero set of the section $\bar{\partial}_{\underline{J}}$ in $\mathcal{E}^0 \to \mathcal{B}^0$. In order to show that the moduli space $\mathcal{M}^0$ can be equipped with a nice manifold structure, in the very same way as for the usual implicit function theorem it hence suffices to prove that zero is a regular value, that is, that the Cauchy-Riemann section meets the zero section transversally.

After equipping the moduli space $\mathcal{M}^0$ of parametrized curves with a nice manifold structure, we deduce from the smoothness of the action of the automorphism group on $\mathcal{M}^0$ that also the moduli space $\mathcal{M}$ of unparametrized curves carries a nice structure. Finally, it follows from the compactness theorem in the Gromov-Hofer topology proven in [BEHWZ], together with a gluing theorem which again relies on the above transversality assumption, that the compactified moduli space carries a manifold with boundary and corner structure, which in turn can be used to deduce the desired algebraic statements.

3. Problems with multiply-covered curves

Up to small modifications it can be shown as in [MDSa] for Gromov-Witten theory that for a generic choice of cylindrical almost complex structure the required transversality assumption is true at every *simple* $\underline{J}$-holomorphic curve $(u,j) \in \mathcal{M}^0$. In other words, assuming that the moduli space $\mathcal{M}^0$ consists only of curves (u,j), for which there does *not* exist a covering map $\varphi : (\dot{S}, j) \to (\dot{S}', j')$ of punctured Riemann surfaces such that $u = v \circ \varphi$, then one can prove that the linearization of the Cauchy-Riemann operator $\bar{\partial}_{\underline{J}}$ is surjective at every $(u,j) \in \mathcal{M}^0$ for a generic choice of $\underline{J}$. In particular, it follows that the moduli space $\mathcal{M}^0$ of parametrized *simple* $\underline{J}$-holomorphic curves carries a nice manifold structure.

In order to prove this fundamental transversality result for simple curves one shows that the space of almost complex structures is indeed large enough to ensure that the Cauchy-Riemann operator is transversal to the zero section when one additionally allows the almost complex structure to vary. Forgetting the (simple) holomorphic curve in the resulting universal moduli space but only remembering the almost complex structure, one can prove that for every almost complex structure which is regular for this forgetful map the resulting linearized Cauchy-Riemann operator is surjective, so that the result follows from Sard's lemma.

While it is not directly clear from this transversality proof for simple $\underline{J}$-holomorphic curves that the transversality result is neccessarily false for multiply-covered curves, it is however not very hard to find examples of multiply-covered curves where transversality can definitively not be satisfied.

Indeed, let $u = v \circ \varphi : (\dot{S}, j) \to (\mathbb{R} \times V, \underline{J})$ be a multiply-covered curve with underlying simple curve $v : (\dot{S}', j') \to (\mathbb{R} \times V, \underline{J})$ and branched covering map $\varphi : (\dot{S}, j) \to (\dot{S}', j')$ and assume that for the Fredholm indices for u and v we have

$$\operatorname{ind}(u) < \operatorname{ind}(v) + 2 \cdot \# \operatorname{Crit}(\varphi),$$

where $\operatorname{Crit}(\varphi)$ is the set of branch points of the branching map φ. Since the kernel of the linearized operator D_u contains at least the infinitesimal variations of u as a

multiply-covered curve, it follows that

$$\dim \ker D_u \geq \dim \ker D_v + 2 \cdot \# \operatorname{Crit}(\varphi) \geq \operatorname{ind}(v) + 2 \cdot \# \operatorname{Crit}(\varphi),$$

and hence $\dim \ker D_u > \operatorname{ind}(u)$, while we need to get equality when transversality would be satisfied.

On the other hand, it is not very hard to find examples of multiply-covered curves for which the above inequality for the Fredholm indices of the multiple cover and the underlying simple curve is satisfied. Indeed, as shown in [F2], the above problem already occurs for the basic examples of $\underline{J}$-holomorphic curves in $\mathbb{R} \times V$, namely the branched covers of orbit cylinders $\mathbb{R} \times \gamma$ over closed Reeb orbits.

While the underlying simple curve, the orbit cylinder, always has Fredholm index zero, it can be shown using standard estimates for the Conley-Zehnder index for multiply-covered orbits that for every $\underline{J}$-holomorphic curve u which is a branched cover of an orbit cylinder, we always have $\operatorname{ind}(u) \leq 2 \cdot \# \operatorname{Crit}(\varphi)$. While it can be shown that the right hand side agrees with $\dim \ker D_u$ and is independent of the underlying closed Reeb orbit, the Fredholm index $\operatorname{ind}(u)$ crucially relies on the underlying orbit, that is, the Conley-Zehnder index of γ and its iterates. It can be seen that in general we have to expect to get a strict inequality, which in turn shows that transversality fails in general even for the basic examples of holomorphic curves studied in SFT.

As a concrete example, choose a hyperbolic orbit in a three-dimensional contact manifold and consider a holomorphic curve with two positive and one negative puncture (pair-of-pants) branching over the corresponding orbit cylinder such that it is asymptotically cylindrical over γ near the two positive punctures and over the double-covered orbit γ^2 near the negative puncture: while $\dim \ker D_u = 2 \cdot \# \operatorname{Crit}(\varphi) = 2$, an easy index computation yields $\operatorname{ind}(u) = 1$.

Note that these orbit curves actually cause trouble in two ways. First, it can be shown using an area estimate, that when the chosen collections of closed Reeb orbits $\Gamma^\pm$ only involve iterates γ^k of the same closed Reeb orbit γ, the corresponding moduli spaces $\mathcal{M}(\Gamma^+, \Gamma^-)$ entirely consist of orbit curves over the corresponding orbit cylinder. To be more precise, it is shown in [F2] that there exists a natural action filtration on SFT such that in the corresponding spectral sequence the first differential only counts orbit curves.

While it was shown by the author in [F2] (and [F3] after introducing gravitational descendants) how one has to deal with these moduli spaces of orbit curves, the orbit curves further cause trouble in another way. Indeed, while it follows from the compactness theorem in [BEHWZ] that an appropriate subsequence of a sequence of $\underline{J}$-holomorphic curves in $\mathcal{M}(\Gamma^+, \Gamma^-)$ converges to a $\underline{J}$-holomorphic curve with multiple levels and nodes, there exists no estimate to exclude that some levels consist of orbit curves. While it is natural to expect this to happen for a general moduli space of $\underline{J}$-holomorphic curves, it is one of the key problems in the embedded contact homology (ECH) of Taubes and Hutchings that orbit curves even appear in the compactification of the nice moduli spaces of embedded holomorphic curves studied in this subtheory of SFT.

Indeed, the compactness statement proven by Hutchings and Taubes (based on [BEHWZ]) shows that a sequence of embedded curves with index two converge to a holomorphic curve with many levels, where only the first and the last level consist

of embedded curves of index one, while all the other levels in between consist of orbit curves of index zero. It follows that, in order to prove that $\partial \circ \partial = 0$ in ECH, the authors had to prove in [HT1] and [HT2] a generalized gluing formula for holomorphic curves in SFT, where curves are glued after possibly inserting additional levels of orbit curves of index zero.

While at first glance this seems to contradict the compactness statement used to prove the algebraic statements in SFT, the difference just reflects the fact that transversality can not be established for all moduli spaces using generic choices of almost complex structures, which is however crucial for positivity of intersections needed in the definition of ECH.

4. Special approaches to the transversality problem

After illustrating how severe the transversality problem in SFT actually is, we want to recall in this section three ways to achieve transversality in special cases. Note that none of these approaches solves the transversality problem in SFT in full generality. On the other hand, we decided not to discuss other approaches like virtual fundamental cycle constructions and Kuranishi structures in this note which claim to prove the transversality problem for holomorphic curves in full generality.

4.1. **Automatic transversality in dimension four.** It is well-known that holomorphic curve techniques are particularly powerful for four-dimensional symplectic manifolds and, in the same way, for three-dimensional contact manifolds. The reason is that, since in this case the holomorphic curves have half the dimension of the surrounding symplectic manifold, one can use intersection theory to prove stronger results. Following [MDSa] the central observation is that holomorphic curves always intersect positively with respect to the induced orientations, so that the number of intersection points and singularities can be bounded from above by purely topological quantities.

The intersection theory is used in two ways: apart from the fact that it excludes that sequences of 'sufficiently nice' curves converge to 'bad' multi-level curves, say, with orbit curve components, it implies automatic transversality results which even hold for multiply-covered curves and non-generic choices of almost complex structures.

To be more precise, based on previous work by Hofer-Lizan-Sikorav, R. Siefrings and V. Shevishisin, C. Wendl has proven in [W] an automatic transversality result for SFT in dimension four, which he also used to prove that sequences of so-called nicely-embedded curves, which are the building blocks of finite energy foliations in SFT in dimension four (see [W] and the reference therein), converge to multi-level curves without orbit curve components. In particular, while none of the definitions is stronger than the other, it follows that the embedded curves studied by Wendl (which, roughly speaking, have embedded image in the contact manifold V) satisfy a stronger compactness result than the embedded curves studied in embedded contact homology by Hutchings and Taubes (which have, again roughly speaking, embedded image in $\mathbb{R} \times V$). On the other hand, apart from the fact that his results only apply when the underlying contact manifold is three-dimensional, it only establishes transversality for very special classes of holomorphic curves, since problems occur when the curve has genus, singularities or approaches Reeb orbits with even Conley-Zehnder index.

4.2. **Obstruction bundles.** As we have seen in the last section it is easy to find examples of multiply-covered curves where transversality can never be satisfied. In other words, it follows from index computations that at such curves the linearized Cauchy-Riemann operator must always have a nonzero cokernel. In these cases, it follows that the best one can hope for is that the moduli space of multiply-covered curves is a manifold of the wrong dimension and that the cokernels fit together to give finite-dimensional bundle over this moduli space, where the rank of the bundle is given by the difference between the wrong dimension of the moduli space and the right dimension expected by the Fredholm index. Following C. Taubes, see also [MDSa] and [LP], one can then use sections of this *obstruction bundle* to perturb the Cauchy-Riemann operator in such a way that the perturbed Cauchy-Riemann is transversal to the zero section in the Banach space bundle whenever the section of the obstruction bundle is transversal to the zero section and the zero set of the obstruction bundle section can be identified with the regular moduli space defined by the perturbed Cauchy-Riemann operator.

Let $\mathcal{M}'$ denote a moduli space of simple $\underline{J}$-holomorphic curves $v : (\dot{S}', j') \to (\mathbb{R} \times V, \underline{J})$ and consider the corresponding moduli space $\mathcal{M}$ of multiple covers $u = v \circ \varphi : (\dot{S}, j) \to (\dot{S}', j') \to (\mathbb{R} \times V, \underline{J})$, where $v : (\dot{S}', j') \to (\mathbb{R} \times V, \underline{J})$ is a simple curve in $\mathcal{M}'$ and $\varphi : (\dot{S}, j) \to (\dot{S}', j')$ is any branched covering map between punctured Riemann surfaces. For generic choice of almost complex structure $\underline{J}$ it easily follows from the transversality result for simple curves that not only $\mathcal{M}'$ but also the moduli space $\mathcal{M}$ of multiple covers $u = v \circ \varphi$ carries a nice manifold structure of (local) real dimension $\mathrm{ind}(v) + 2\# \mathrm{Crit}(\varphi)$. To be more precise, for the tangent spaces we have

$$T_u \mathcal{M} = T_v \mathcal{M}' \oplus \mathbb{C}^{\# \mathrm{Crit}(\varphi)},$$

where the second summand keeps track of the deformations of $u = v \circ \varphi$ as a multiple cover.

In order to show that the cokernels of the linearized operators D_u fit together to give a nice vector bundle of the right rank, it suffices by $\mathrm{ind}(u) = \dim \ker D_u - \dim \mathrm{coker} D_u$ to prove that $\dim T_u \mathcal{M} = \dim \ker D_u$, which by the obvious inclusion $T_u \mathcal{M} \subset \ker D_u$ is indeed equivalent to $T_u \mathcal{M} = \ker D_u$. In other words, it remains to show that multiple covers are sufficiently isolated in the sense that every infinitesimal deformation of a multiple cover as a holomorphic curve is still a multiply-covered curve, which however need not be true and is in general very hard to establish.

However, in [F2] it is shown that this is true for moduli space of orbit curves, the basic examples of holomorphic curves in SFT, using an infinitesimal version of the statement in [BEHWZ] that every curve with zero contact area is an orbit curve. Apart from the fact that in [F2] (and [F3] with gravitational descendants) this obstruction bundle method is employed to actually compute the contribution of orbit curves to the SFT invariants, in an ongoing project with C. Wendl the author is using the above automatic transversality result in dimension four to show that the obstruction bundle technique can be used to study multiple covers of nicely-embedded curves. Together with the invariants for closed Reeb orbits obtained in [F2], [F3] this is used to define a local version of SFT, where the objects are closed Reeb orbits instead of contact manifolds, while the nicely-embedded curves replace the symplectic cobordisms as morphisms of the topological quantum field theory.

4.3. **Domain-dependent almost complex structures.** The last special approach to the transversality problem we want to discuss briefly are domain-dependent almost complex structures. While this is the special approach which is most promising to prove transversality in SFT in full generality, so far it could only be successfully applied to the transversality problem in Gromov-Witten theory by K. Cieliebak and K. Mohnke in [CM] and to SFT in the Floer case in [F1], while the general case currently seems out of reach.

Roughly speaking, the main idea of this approach is to 'correct' the transversality proof for simple curves in such a way that it also proves transversality for multiply-covered curves. Recall that for the transversality result for simple curves one shows that the space of almost complex structures is indeed large enough to ensure that the Cauchy-Riemann operator is transversal to the zero section when one additionally allows the almost complex structure to vary. While the latter is in general not true for multiply-covered curves, it can be shown, see [Sch], that transversality indeed holds also for multiply-covered curves when one enlarges the dimension of the base by allowing the almost complex structure to additionally depend on points on the underlying punctured Riemann surface.

While this approach can be used to show that all moduli spaces $\mathcal{M}^0$ of parametrized curves can be transversally cut out of the Banach manifold $\mathcal{B}^0$ for generic choices of domain-dependent almost complex structures, we would be able to solve the transversality problem in SFT when we could ensure that the automorphism groups $\mathrm{Aut}(\dot{S}, j)$ still act on $\mathcal{M}^0$ so that we can still define the desired moduli space $\mathcal{M}$ of unparametrized curves as quotient, and that all the choices of domain-dependent almost complex structures can be made coherent so that everything extends nicely to the compactified moduli space $\overline{\mathcal{M}}$.

It follows from the first requirement that the chosen domain-dependent almost complex structures must be covariant with respect to the action of the automorphism group. While for domain-stable curves, that is, curves with three or more punctures, this automorphism group is trivial, for domain-independent curves like holomorphic spheres, planes or cylinders (without additional marked points) one can show that the allowed domain-dependent almost complex structure are no longer allowed to depend on points on the domain, i.e., must be almost complex structures in the usual sense. It follows that the approach still cannot establish transversality for curves which are not only multiply-covered but also domain-unstable.

In [CM] the authors solve this problem for Gromov-Witten theory by employing Donaldson's construction of symplectic hypersurfaces in closed symplectic manifolds. Indeed, while holomorphic spheres without marked points are domain-unstable, they use these hypersurfaces to introduce enough marked points on the spheres so that they become domain-stable. In order to guarantee that in the end one counts the same objects as in usual Gromov-Witten theory, it is important that the number of marked points is determined a priori by the topological quantity of energy of the holomorphic sphere, since the hypersurface represents a (large multiple of) the homology class Poincare-dual to the cohomology class given by the symplectic form.

At first sight the situation in SFT seems to be significantly easier, since almost all holomorphic curves studied in SFT are already domain-stable since they carry

enough positive and negative punctures. However, if not explicitly excluded, domain-unstable curves like spheres, planes or cylinders still exist and need to be stabilized in order to prove transversality using domain-dependent almost complex structures.

In [F1] it is shown that in the Floer case with symplectically aspherical symplectic manifold M one can establish transversality using domain-dependent almost complex structures which are symmetric with respect to the natural S^1-symmetry on the underlying trivial mapping torus $V = S^1 \times M$, where the symmetry is used to compute the SFT-invariants for this important special case. While holomorphic spheres and planes do not exist, it is crucial for the proof that for holomorphic cylinders, which agree with Floer cylinders, transversality can be established with S^1-symmetric almost complex structures as long as the Hamiltonian is sufficiently small in the C^2-norm such that all Floer cylinders are indeed Morse trajectories.

Apart from the fact that for the general case it is very hard to exclude the existence of holomorphic planes, there already exists a problem with holomorphic cylinders in the general case, since as in usual Floer homology transversality for holomorphic cylinders can only be established for cylindrical almost complex structures which additionally depend on the natural S^1-coordinate on the cylinder. It follows that one needs to find a way to 'coherently' fix special points on all closed Reeb orbits (not only the simple ones) which in turn can be used to fix unique S^1-coordinates on all holomorphic cylinders, which does not exist to date.

5. A new Fredholm theory

While these special approaches are very useful for computations, seeing the limitations in all existing approaches the reader is hopefully motivated enough to approach the transversality problem for holomorphic curves using a completely new Fredholm theory. Apart from the fact that in this new Fredholm theory transversality for the Cauchy-Riemann operator can be achieved by simply perturbing it into general position as well-known for finite-dimensional vector bundles, it further also allows to prove that all computations using different special approaches lead to the same result as it contains all the perturbations for the Cauchy-Riemann operator obtained from the special approaches as a special case.

5.1. **Transversality using abstract perturbations.** It is well-known from classical differential topology that in finite-dimensional vector bundles over finite-dimensional manifolds every section can be made transversal to the zero section by slightly perturbing it into general position. Instead of using special approaches to achieve transversality like domain-dependent almost complex structures, it is clear that the most natural solution to the transversality problem would consist of such an abstract transversality result using generic perturbations.

Recall that in the classical approach one has to show that the moduli space $\mathcal{M}^0$ of parametrized curves is transversally cut out by the Cauchy-Riemann operator as a section in the infinite-dimensional Banach space bundle $\mathcal{E}^0 \to \mathcal{B}^0$. On the other hand, this moduli space can only be compactified in geometric way after dividing out the the action of the automorphism group $\mathrm{Aut}(\dot{S}, j)$ of the domain (and the natural $\mathbb{R}$-action on the target $\mathbb{R} \times V$) to give the moduli space $\mathcal{M}$ of unparametrized curves, while the compactified moduli space $\overline{\mathcal{M}}$ is equipped with a nice manifold structure with boundary and corners using a gluing result, which however relies on transversality.

Using an abstract transversality result for sections in the Banach space bundle $\mathcal{E}^0 \to \mathcal{B}^0$ as in the finite-dimensional case to obtain a perturbed Cauchy-Riemann operator $\bar{\partial}_{\underline{J}}^{\nu} = \bar{\partial}_{\underline{J}} + \nu$, it would follow that the transversality problem is solved as long as it is guaranteed that

(1) the automorphism group $\mathrm{Aut}(\dot{S}, j)$ of the domain (and the $\mathbb{R}$-action on the target) still acts on the resulting perturbed moduli space

$$(\mathcal{M}^0)^{\nu} = (\bar{\partial}_{\underline{J}}^{\nu})^{-1}(0) \subset \mathcal{B}^0$$

of parametrized curves to define the perturbed moduli space $\mathcal{M}^{\nu}$ of unparametrized curves and

(2) the different abstract perturbations ν for the different moduli spaces appearing the compactification of the moduli space can be chosen coherently so that the perturbed moduli spaces can be glued together to give a new perturbed compactified moduli space $\overline{\mathcal{M}}^{\nu}$.

First, while it is easy to see that (1) is not satisfied for a general abstract perturbation of the Cauchy-Riemann operator in $\mathcal{E}^0 \to \mathcal{B}^0$, recall from the last sectio that it was precisely this symmetry property which caused the problems with domain-unstable curves in the transversality approach using domain-dependent almost complex structures. On the other hand, assuming for the moment that this first problem could be solved, it is not clear on the level of abstract perturbations, which are still sections in infinite-dimensional Banach space bundles, in which sense they "smoothly fit together" such that (2) holds, that is, the resulting perturbed moduli spaces can be glued together to give a perturbed compactified moduli space with a nice manifold structure with boundaries and corners.

While the problems (1) and (2) are much easier to handle for the special transversality approaches like obstruction bundles and domain-dependent almost complex structures, it is not clear how to solve them within the classical approach using Banach space bundles over Banach manifolds. In the end this is due to the fact that in both special approaches the abstract perturbations actually depend on finite-dimensional spaces. Indeed, while in the obstruction bundle and the domain-dependent almost complex structures approaches the resulting perturbations of the Cauchy-Riemann operator depend on points on the underlying nonregular moduli space of multiple covers or moduli space of punctured Riemann surfaces, respectively, the general abstract perturbations using the abstract perturbation approach depend on points on the infinite-dimensional Banach manifold.

Before discussing the problems appearing with infinite-dimensional manifolds, let us first sketch our dreamland.

First, in order to find perturbations for the Cauchy-Riemann operator using an abstract perturbation scheme solving (1), i.e., which are in addition symmetric with respect to the action of the automorphism group of the domain (and the $\mathbb{R}$-symmetry of the target), we would hope to find a new "infinite-dimensional bundle" $\mathcal{E} \to \mathcal{B}$, which contains the Cauchy-Riemann operator as a "smooth" section, such that the moduli space $\mathcal{M}$ of *unparametrized* curves (not $\mathcal{M}^0$) is the zero set of this Cauchy-Riemann section. Indeed, perturbing the Cauchy-Riemann operator in this new bundle setup into general position, the resulting abstract perturbation automatically has the desired property.

On the other hand, proceeding in the same way, we can further solve the problem (2) to find abstract perturbations which coherently fit together "in a smooth way", if we can construct an even more general "infinite-dimensional bundle" $\overline{\mathcal{E}} \to \overline{\mathcal{B}}$ which again contains the Cauchy-Riemann operator as "smooth" section, but whose zero set is now already the compactified moduli space $\overline{\mathcal{M}}$. Again, perturbing the Cauchy-Riemann operator in this new bundle setup into general position, we find a "smooth" abstract perturbation which restricts to smooth abstract perturbations for the noncompact moduli space and the moduli spaces appearing in the boundary such that the resulting perturbed moduli spaces can be glued to a perturbed compactified moduli space with the desired manifold structure.

5.2. **Motivation for sc-structures and sc-manifolds.** In the same way as the moduli space $\mathcal{M}$ was obtained from the moduli space $\mathcal{M}^0$ by dividing out the symmetries of the domain (and the target), $\mathcal{M} = \mathcal{M}^0 / \operatorname{Aut}(\dot{S}, j)(\times \mathbb{R})$, the canonical way to obtain the infinite-dimensional bundle $\mathcal{E} \to \mathcal{B}$ solving (1) is to make use of the fact that the action extends to the Banach manifold of maps $\mathcal{B}^0$ and also lifts to an action on the Banach space bundle $\mathcal{E}^0$ and define

$$\mathcal{E} = \mathcal{E}^0 / \operatorname{Aut} \to \mathcal{B} = \mathcal{B}^0 / \operatorname{Aut}.$$

Provided that the symmetry group $\operatorname{Aut}(\dot{S}, j)$ still acts smoothly with respect to the Banach manifold topology on $\mathcal{E}^0 \to \mathcal{B}^0$, it would follow that the new bundle $\mathcal{E} \to \mathcal{B}$ is still a Banach space bundle over a Banach manifold and (1) could be proved using the same abstract transversality result as used for the classical Banach space bundle $\mathcal{E}^0 \to \mathcal{B}^0$.

In order to solve (1) it hence only remains to verify that the action of $\operatorname{Aut}(\dot{S}, j)$ is indeed smooth with respect to the classical Banach manifold topology. As toy model for symmetries of the domain, we restrict for the moment our attention to classical Morse theory. Here the Banach manifold of maps is the space of paths from one fixed (critical) point on the underlying manifold to another fixed (critical) point, where for simplicity we assume that the manifold is just the linear space and the paths start and end at $0 \in \mathbb{R}^n$.

In this case the Banach manifold of paths is just the linear Banach space $H^{k,p}(\mathbb{R}, \mathbb{R}^n)$ with $k \geq 1, p \geq 2$ and we consider the action by translations,

$$\tau : \mathbb{R} \times H^{k,p}(\mathbb{R}, \mathbb{R}^n) \to H^{k,p}(\mathbb{R}, \mathbb{R}^n), \ (s, u) \mapsto u(\cdot + s).$$

In order to check whether this map is smooth, we first need to check that it is even differentiable. Assuming for the moment that u is actually an element in the dense subspace $C_0^\infty(\mathbb{R}, \mathbb{R}^n) \subset H^{k,p}(\mathbb{R}, \mathbb{R}^n)$ of smooth functions (with appropriate decay to zero near $\pm\infty$), basic calculus gives us the result

$$D\tau : \mathbb{R} \times H^{k,p}(\mathbb{R}, \mathbb{R}^n) \to H^{k,p}(\mathbb{R}, \mathbb{R}^n), \ D\tau(0, u) \cdot (1, \xi) = u' + \xi, \ u' = \frac{\partial u}{\partial s}.$$

Since the formula for the differential of the shift map involves the first derivative of the map $u \in H^{k,p}(\mathbb{R}, \mathbb{R}^n)$ it follows that, in order to define a map to $H^{k,p}(\mathbb{R}, \mathbb{R}^n)$, *the differential is only defined at maps u in the subspace $H^{k+1,p}(\mathbb{R}, \mathbb{R}^n)$!*

In the same way as in the Morse theory example, it follows that the differential of the action of the symmetries of the domain is only defined at elements in the dense subset $\mathcal{B}_1^0 \subset \mathcal{B}^0$ with

$$\mathcal{B}_1^0 = H^{k+1,p,\delta}(\Gamma^+, \Gamma^-) \times \mathcal{M}_{0,s}.$$

Proceeding further, it can be shown that the higher derivatives can only be computed at points in the dense subsets $\mathcal{B}_\ell^0 = H^{k+\ell,p,\delta}(\Gamma^+,\Gamma^-) \times \mathcal{M}_{0,s}$, while all derivatives only exist at maps (u,j) in the dense subset $\mathcal{B}_\infty^0 = C_0^\infty(\Gamma^+,\Gamma^-) \times \mathcal{M}_{0,s}$ of smooth maps, where

$$C_0^\infty(\Gamma^+,\Gamma^-) := \bigcap_{\ell=0}^\infty H^{k+\ell,p,\delta}(\Gamma^+,\Gamma^-).$$

It follows that in the classical Banach manifold topology on $\mathcal{B}^0$ the action of the symmetries is *not* smooth, *so that the naive approach for equipping the bundle $\mathcal{E} = \mathcal{E}^0 / \mathrm{Aut} \to \mathcal{B} = \mathcal{B}^0 / \mathrm{Aut}$ with the structure of a smooth infinite-dimensional bundle to solve (1) does not work !*

On the other hand, in order to define a new (infinite-dimensional) differential topology on $\mathcal{B}^0$ (and $\mathcal{E}^0$), it seems natural to equip the underlying Banach spaces E with a scale structure

$$... \subset E_2 \subset E_1 \subset E_0 = E$$

as it is well-known in wavelet theory (also called multi-resolution analysis) in numerical mathematics. In order to ensure that the derivative always exists, the tangent bundle $T\mathcal{B}^0$ to the manifold of maps $\mathcal{B}^0$ should only sit over the dense subset $\mathcal{B}_1^0$,

$$T\mathcal{B}^0 \to \mathcal{B}_1^0,$$

so that differentiability only needs to be checked at curves with higher regularity.

While all this sounds rather distressing, in the very same way it should come as a great surprise that Hofer, Wysocki and Zehnder were actually able to build a new (infinite-dimensional) differential topology on the afore-mentioned *scale-calculus* (sc-calculus) (see [HWZ] and [H] and the references therein), which is on the one hand

- *weak enough* such that the action of the symmetries on the infinite-dimensional manifolds (in the new sense) is smooth, but on the other hand also
- *strong enough* such that the zero set of the Cauchy-Riemann operator is, after perturbing the latter into general position, a finite-dimensional manifold in the usual sense.

In order to see that this new scale differential topology really reduces to the usual differential topology in finite dimensions, it should be mentioned that the inclusions in the sequences of subspaces $... \subset E_2 \subset E_1 \subset E_0 = E$ are actually required to be compact, so that neccessarily $E = E_0 = E_1 = E_2 = ...$ if E is finite-dimensional. In particular, recall that for the finite-dimensional moduli spaces there is no problem with smoothness since all the holomorphic curves are automatically smooth by elliptic regularity.

5.3. **Motivation for retracts and polyfolds.** After this short digression on the problem of equipping the bundle $\mathcal{E} \to \mathcal{B}$ with a nice smooth structure in order to solve (1) and on the main ideas how Hofer and his collaborateurs were able to solve it using their sc-calculus, it remains the problem to equip the bundle $\overline{\mathcal{E}} \to \overline{\mathcal{B}}$ with an even more general smooth structure in order to also solve (2) to obtain a general transversality result for the Cauchy-Riemann operator, which would in turn solve the transversality problem in SFT in full generality.

Assume that a sequence of holomorphic curves (u_n, j_n) in $\mathcal{M}$ converges to a broken or nodal curve (u_∞, j_∞), where $u_\infty : (\dot{S}_\infty, j_\infty) \to (\mathbb{R} \times V, \underline{J})$ starts from

the nodal Riemann surface with punctures $(\dot{S}_\infty, j_\infty)$. We want to describe a neighborhood of (u_∞, j_∞) in the space of maps $\overline{\mathcal{B}}$ containing the compactified moduli space $\overline{\mathcal{M}}$.

For this we first need to glue the nodal Riemann surface $(\dot{S}_\infty, j_\infty)$ to a smooth Riemann surface $(\dot{S}_r, j_r)$ depending on the gluing parameter $r \in [0, \infty)$ (for r sufficiently large). Since the gluing should be defined as usual by choosing cylindrical coordinates $[0, \infty) \times S^1$ and $(-\infty, 0] \times S^1$ near the double-points, removing the half-infinite cylinders $[r, \infty) \times S^1$ and $(-\infty, -r) \times S^1$ and gluing the resulting Riemann surface with boundary in the obvious way, it follows that we can think of the glued smooth Riemann surface $(\dot{S}_r, j_r)$ as a (non-connected !) subset of $(\dot{S}_\infty, j_\infty)$. Neglecting the use of cut-off function for smoothness for the moment, it follows that there exists a natural (pre-)gluing map to obtain from (u_∞, j_∞) a family of maps (u_r, j_r) starting from the smooth Riemann surface $\dot{S}_r$ by defining $u_r = u_\infty|_{\dot{S}_r}$ as $\dot{S}_r \subset \dot{S}_\infty$.

Forgetting about the variation of the complex structure on the domain, recall that the tangent space at (u_∞, j_∞) to the manifold of maps starting from the nodal Riemann surface $\dot{S}_\infty$ should be given by

$$H^{k,p,\delta}(u_\infty^* T(\mathbb{R} \times V)) = H^{k,p,\delta}(\dot{S}_\infty, \mathbb{C}^n),$$

where the identity follows by choosing a unitary trivialization of the pull-back bundle $u_\infty^* T(\mathbb{R} \times V)$. On the other hand, for the (pre-)glued curve (u_r, j_r) the space of infinitesimal deformations as a map starting from the glued Riemann surface $\dot{S}_r$ is given by

$$H^{k,p,\delta}(u_r^* T(\mathbb{R} \times V)) = H^{k,p,\delta}(\dot{S}_r, \mathbb{C}^n),$$

where we explicitly assume that the unitary trivialization of $u_r^* T(\mathbb{R} \times V)$ is given by the unitary trivialization of $u_\infty^* T(\mathbb{R} \times V)$ using that $u_r = u_\infty|_{\dot{S}_r}$.

Viewing $\dot{S}_r$ as a subset of $\dot{S}_\infty$ it hence follows that the space of infinitesimal variations, i.e., the tangent space to the manifold of maps, of the nodal map (u_∞, j_∞) contains the corresponding space of infinitesimal variations of each (pre-)glued map (u_r, j_r) as a subset. Moreover, there indeed exists a natural projection

$$\pi_r : H^{k,p,\delta}(\dot{S}_\infty, \mathbb{C}^n) \to H^{k,p,\delta}(\dot{S}_r, \mathbb{C}^n),$$

which is just given by restricting. In order to be more precise, one actually has to employ cut-off function as for the definition for the pre-glued map and parallel transport in order to identify the unitary trivializations, which we however want to ignore for simplicity. In particular, observe that, while the moduli spaces which one has to add in order to compactify a moduli space always have a smaller dimension than the original moduli space, at least as expected by the Fredholm index, the space of variations as a (not neccessarily holomorphic) map is always larger for the curves in the boundary.

It follows that the desired manifold of maps $\overline{\mathcal{B}}$, containing the compactified moduli space $\overline{\mathcal{M}}$ as a subset, should locally near the nodal curve (u_∞, j_∞) be given (forgetting the variation of the complex structure for simplicity) by

$$
\begin{aligned}
O &= \bigcup_{r \in R} \{r\} \times H^{k,p,\delta}(\dot{S}_r, \mathbb{C}^n) \\
&= \{(r, \pi_r(u)) : (r, u) \in R \times H^{k,p,\delta}(\dot{S}_\infty, \mathbb{C}^n)\} \\
&= \rho(U),
\end{aligned}
$$

when we choose

$$U = R \times H^{k,p,\delta}(\dot{S}_\infty, \mathbb{C}^n), \ \rho : U \to U, \ \rho(r,u) = (r, \pi_r(u)),$$

so that, in particular, $\rho \circ \rho = \rho$.

But with our reasoning we directly arrived at the definition of a *retract* by Hofer, Wysocki and Zehnder, which is the local model of a M-polyfold. Using their scale-calculus which we motivated before, they show in their papers that these new geometric objects can be equipped with an infinite-dimensional differentiable topology. In particular, it turns out that their scale differential topology is again indeed

- *weak enough* to allow dimension jumps, in particular, deal with the fact that the dimension goes up at the strata which contain the boundary of the moduli space, but still
- *strong enough* to detect the different strata, in particular, distinguish between interior and boundary strata.

5.4. **The Fredholm package.** Besides defining a new infinite-dimensional differential topology for sc-manifolds and, more generally, (M-)polyfolds, Hofer, Wysocki and Zehnder show in their papers that one can generalize the usual Fredholm package used in the classical approach to these new categories of geometric objects. In particular, apart from defining (non-linear) Fredholm sections in bundles, they prove an implicit function theorem and an abstract perturbation scheme leading to the desired abstract transversality result for the Cauchy-Riemann operator.

While in the sc-category the Fredholm property is still very natural and leads to an elegant proof of elliptic regularity by making use of the sc-structure, for M-polyfolds one is faced with the problem that their local model is no longer a linear space but a retract which can be viewed as a parametrized family of subspaces. Since these subspaces have locally varying dimensions, more precisely, are obtained from a fixed infinite-dimensional linear space by removing varying infinite-dimensional subspaces, it does no longer suffice to call the linearization Fredholm when the linear operator is an isomorphism up to a finite-dimensional kernel and cokernel. Due to this problem with the jumping of dimensions, Hofer and his collaborateurs introduced the notion of a "filler" and a "filled section", which solve the problem and lead to a generalized definition of Fredholm.

Finally, while we have already mentioned and used that the new differential topology is weaker than the classical differential topology, this has the draw-back that for the new implicit function theorem (IFT) for sc-manifolds and polyfolds it does not only suffice to study the linearization of the section, but one more generally needs to study the germ of the section (naturally defined using the sc-structure) near the zero. The resulting "germ IFT" (not "german IFT") however suffices to equip the zero set of the Cauchy-Riemann operator with a nice manifold structure with boundary and corners, of course, only after possibly perturbing it into general position using the abstract perturbation scheme.

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