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Abstract

We study certain quantum states for which the PPT criterion is both sufficient and necessary for separability. We present a class of $(2k+1)\times(2k+1)$, $k\in\mathbb{N}$, bipartite mixed states and derive the conditions of PPT for these states. The separable pure state decompositions of these states are explicitly constructed when they are PPT.

Keywords: PPT, Entanglement, Separability

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Quantum entangled states have become one of the key resources in quantum information processing. The study of quantum teleportation, quantum cryptopraphy, quantum dense coding, quantum error correction and parallel computation [1, 2, 3] has spurred a flurry of activities in the investigation of quantum entanglement. Despite the potential applications of quantum entangled states, the theory of quantum entanglement itself is far from being satisfied. The separability of bipartite and multipartite quantum mixed states is one of the important questions in quantum entanglement.

Let H_1 (resp. H_2) be an M (resp. N)-dimensional complex Hilbert space, with $|i\rangle$, $i=1,...,M$ (resp. $|j\rangle, j=1,...,N$) as an orthonormal normalized basis. A bipartite mixed state is said to be separable if the density matrix can be written as

$$\rho=\sum_i p_i\rho_i^1\otimes\rho_i^2, \tag{1}$$

where $0 < p_i \leq 1$, $\sum_i p_i = 1$, ρ_i^1 and ρ_i^2 are density matrices on H_1 and H_2 respectively. It is a challenge to find a decomposition like (1) or proving that such a decomposition does not exist for a given state ρ. With considerable effort in analyzing the separability, there have been some (necessary) criteria for separability in recent years, for instance, Bell inequalities [4], PPT (positive partial transposition) [5], reduction criterion [6, 7], majorization criterion [8], entanglement witnesses [9, 10], realignment [11, 12] and generalized realignment [13] , as well as some necessary and sufficient criteria for low rank density matrices [14, 15, 16].

The PPT criterion is generally a necessary condition for separability, and it becomes sufficient in the cases 2×2 and 2×3 bipartite states [17]. In [18], It has been shown that a state ρ supported on $m \times n$ $(m \leq n)$ with $rank(\rho) \leq m$ is separable if and only if ρ is PPT. In [19] we have studied a class of 3×3 mixed states ρ with $rank(\rho) = 3$, where their PPT condition was proved sufficient for separability with an explicit decomposition under PPT. In this paper we generalize our results to general $(2k+1) \times (2k+1), k \in \mathbb{N}$ quantum mixed states. We derive the conditions for PPT and construct the separable pure state decompositions when the states are PPT.

We first consider 5×5 quantum mixed states on tensor space $H \otimes H$ with basis $|1\rangle = (1,0,0,0,0)^t$, $|2\rangle = (0,1,0,0,0)^t$, $|3\rangle = (0,0,1,0,0)^t$, $|4\rangle = (0,0,0,1,0)^t$ and $|5\rangle = (0,0,0,0,1)^t$ for each vector space H, where t stands for transposition. The states have the following eigenvector decomposition,

$$\rho = \sum_i \lambda_i |X_i\rangle\langle X_i|, \quad i = 1, ..., 5, \tag{2}$$

where λ_i are the eigenvalues, $\sum_i \lambda_i = 1$, $0 < \lambda_i < 1$, $|X_i\rangle$, $i = 1, ..., 5$, are normalized vectors of the form:

$$|X_i\rangle = (I \otimes P^{i-1})|X(a_i, b_i, c_i, d_i, e_i)\rangle, \quad i = 1, ..., 5,$$

$$|X(a_i, b_i, c_i, d_i, e_i)\rangle = a_i|11\rangle + b_i|22\rangle + c_i|33\rangle + d_i|44\rangle + e_i|55\rangle,$$

$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ is a permutation operator, a_i, b_i, c_i, d_i, d_i, $i = 1, 2, 3, 4, 5$, are non-zero complex numbers.

According to [18], the states (2) are separable if and only if they are PPT. We now derive the condition such that ρ are PPT. Let ρ^{pt} denote the partial transposed matrix of ρ. The non-negativity of ρ^{pt} implies that $\langle\psi|\rho^{pt}|\psi\rangle \geq 0$ for all vector $|\psi\rangle \in H \otimes H$. Let us take $|\psi\rangle = (x_1, y_1, z_1, u_1, v_1, x_2, y_2, z_2, u_2, v_2, x_3, y_3, z_3, u_3, v_3, x_4, y_4, z_4, u_4, v_4, x_5, y_5, z_5, u_5, v_5)^t$, for any $x_i, y_i, z_i, u_i, v_i \in \mathbb{C}$, $i = 1, 2, 3, 4, 5$. Then $\langle\psi|\rho^{pt}|\psi\rangle \geq 0$ is equivalent to the simultaneous non-negativity of the following five inequalities: $\langle x|A|x\rangle \geq 0$, $\langle y|B|y\rangle \geq 0$, $\langle z|C|z\rangle \geq 0$, $\langle u|D|u\rangle \geq 0$, $\langle v|E|v\rangle \geq 0$, where $|x\rangle = (x_1, x_2, x_3, x_4, x_5)^t$, $|y\rangle = (y_1, y_2, y_3, y_4, y_5)^t$, $|z\rangle = (z_1, z_2, z_3, z_4, z_5)^t$, $|u\rangle = (u_1, u_2, u_3, u_4, u_5)^t$, $|v\rangle = (v_1, v_2, v_3, v_4, v_5)^t$, and

$$A = \begin{pmatrix} \lambda_1 a_1 a_1^* & \lambda_5 a_5 b_5^* & \lambda_4 a_4 c_4^* & \lambda_3 a_3 d_3^* & \lambda_2 a_2 e_2^* \\ \lambda_5 b_5 a_5^* & \lambda_4 b_4 b_4^* & \lambda_3 b_3 c_3^* & \lambda_2 b_2 d_2^* & \lambda_1 b_1 e_1^* \\ \lambda_4 c_4 a_4^* & \lambda_3 c_3 b_3^* & \lambda_2 c_2 c_2^* & \lambda_1 c_1 d_1^* & \lambda_5 c_5 e_5^* \\ \lambda_3 d_3 a_3^* & \lambda_2 d_2 b_2^* & \lambda_1 d_1 c_1^* & \lambda_5 d_5 d_5^* & \lambda_4 d_4 e_4^* \\ \lambda_2 e_2 a_2^* & \lambda_1 e_1 b_1^* & \lambda_5 e_5 c_5^* & \lambda_4 e_4 d_4^* & \lambda_3 e_3 e_3^* \end{pmatrix}, \tag{3}$$

$$B = \begin{pmatrix} \lambda_2 a_2 a_2^* & \lambda_1 a_1 b_1^* & \lambda_5 a_5 c_5^* & \lambda_4 a_4 d_4^* & \lambda_3 a_3 e_3^* \\ \lambda_1 b_1 a_1^* & \lambda_5 b_5 b_5^* & \lambda_4 b_4 c_4^* & \lambda_3 b_3 d_3^* & \lambda_2 b_2 e_2^* \\ \lambda_5 c_5 a_5^* & \lambda_4 c_4 b_4^* & \lambda_3 c_3 c_3^* & \lambda_2 c_2 d_2^* & \lambda_1 c_1 e_1^* \\ \lambda_4 d_4 a_4^* & \lambda_3 d_3 b_3^* & \lambda_2 d_2 c_2^* & \lambda_1 d_1 d_1^* & \lambda_5 d_5 e_5^* \\ \lambda_3 e_3 a_3^* & \lambda_2 e_2 b_2^* & \lambda_1 e_1 c_1^* & \lambda_5 e_5 d_5^* & \lambda_4 e_4 e_4^* \end{pmatrix}, \tag{4}$$

$$C = \begin{pmatrix} \lambda_3 a_3 a_3^* & \lambda_2 a_2 b_2^* & \lambda_1 a_1 c_1^* & \lambda_5 a_5 d_5^* & \lambda_4 a_4 e_4^* \\ \lambda_2 b_2 a_2^* & \lambda_1 b_1 b_1^* & \lambda_5 b_5 c_5^* & \lambda_4 b_4 d_4^* & \lambda_3 b_3 e_3^* \\ \lambda_1 c_1 a_1^* & \lambda_5 c_5 b_5^* & \lambda_4 c_4 c_4^* & \lambda_3 c_3 d_3^* & \lambda_2 c_2 e_2^* \\ \lambda_5 d_5 a_5^* & \lambda_4 d_4 b_4^* & \lambda_3 d_3 c_3^* & \lambda_2 d_2 d_2^* & \lambda_1 d_1 e_1^* \\ \lambda_4 e_4 a_4^* & \lambda_3 e_3 b_3^* & \lambda_2 e_2 c_2^* & \lambda_1 e_1 d_1^* & \lambda_5 e_5 e_5^* \end{pmatrix}, \tag{5}$$

$$D = \begin{pmatrix} \lambda_4 a_4 a_4^* & \lambda_3 a_3 b_3^* & \lambda_2 a_2 c_2^* & \lambda_1 a_1 d_1^* & \lambda_5 a_5 e_5^* \\ \lambda_3 b_3 a_3^* & \lambda_2 b_2 b_2^* & \lambda_1 b_1 c_1^* & \lambda_5 b_5 d_5^* & \lambda_4 b_4 e_4^* \\ \lambda_2 c_2 a_2^* & \lambda_1 c_1 b_1^* & \lambda_5 c_5 c_5^* & \lambda_4 c_4 d_4^* & \lambda_3 c_3 e_3^* \\ \lambda_1 d_1 a_1^* & \lambda_5 d_5 b_5^* & \lambda_4 d_4 c_4^* & \lambda_3 d_3 d_3^* & \lambda_2 d_2 e_2^* \\ \lambda_5 e_5 a_5^* & \lambda_4 e_4 b_4^* & \lambda_3 e_3 c_3^* & \lambda_2 e_2 d_2^* & \lambda_1 e_1 e_1^* \end{pmatrix}, \tag{6}$$

$$E = \begin{pmatrix} \lambda_5 a_5 a_5^* & \lambda_4 a_4 b_4^* & \lambda_3 a_3 c_3^* & \lambda_2 a_2 d_2^* & \lambda_1 a_1 e_1^* \\ \lambda_4 b_4 a_4^* & \lambda_3 b_3 b_3^* & \lambda_2 b_2 c_2^* & \lambda_1 b_1 d_1^* & \lambda_5 b_5 e_5^* \\ \lambda_3 c_3 a_3^* & \lambda_2 c_2 b_2^* & \lambda_1 c_1 c_1^* & \lambda_5 c_5 d_5^* & \lambda_4 c_4 e_4^* \\ \lambda_2 d_2 a_2^* & \lambda_1 d_1 b_1^* & \lambda_5 d_5 c_5^* & \lambda_4 d_4 d_4^* & \lambda_3 d_3 e_3^* \\ \lambda_1 e_1 a_1^* & \lambda_5 e_5 b_5^* & \lambda_4 e_4 c_4^* & \lambda_3 e_3 d_3^* & \lambda_2 e_2 e_2^* \end{pmatrix}. \tag{7}$$

Namely all the principal minors of the matrices A, B, C, D and E should be non-negative. In fact it can be shown that all these principal minors should be zero, otherwise one gets contradiction. From the second order principal minors of the five matrices we have the following relations,

$$\begin{array}{l} \frac{\sqrt{\lambda_3\lambda_4}}{\lambda_1} b_4 e_3 = b_1 e_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_3\lambda_4}}{\lambda_1} c_3 e_4 = c_1 e_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_3\lambda_4}}{\lambda_1} a_3 c_4 = c_1 a_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_3\lambda_4}}{\lambda_1} d_3 a_4 = d_1 a_1 e^{i\theta_1}, \\ \frac{\sqrt{\lambda_3\lambda_4}}{\lambda_1} b_3 d_4 = b_1 d_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_2\lambda_5}}{\lambda_1} e_2 a_5 = a_1 e_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_2\lambda_5}}{\lambda_1} d_2 e_5 = d_1 e_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_2\lambda_5}}{\lambda_1} a_2 b_5 = b_1 a_1 e^{i\theta_1}, \\ \frac{\sqrt{\lambda_2\lambda_5}}{\lambda_1} b_2 c_5 = b_1 c_1 e^{i\theta_1},\ \frac{\sqrt{\lambda_2\lambda_5}}{\lambda_1} c_2 d_5 = c_1 d_1 e^{i\theta_1}; \end{array} \tag{8}$$

$$\frac{\sqrt{\lambda_1\lambda_3}}{\lambda_2}a_1e_3 = a_2e_2e^{i\theta_2},\ \frac{\sqrt{\lambda_1\lambda_3}}{\lambda_2}d_1c_3 = c_2d_2e^{i\theta_2},\ \frac{\sqrt{\lambda_1\lambda_3}}{\lambda_2}b_1a_3 = a_2b_2e^{i\theta_2},\ \frac{\sqrt{\lambda_1\lambda_3}}{\lambda_2}c_1b_3 = b_2c_2e^{i\theta_2},$$
$$\frac{\sqrt{\lambda_1\lambda_3}}{\lambda_2}e_1d_3 = e_2d_2e^{i\theta_2},\ \frac{\sqrt{\lambda_4\lambda_5}}{\lambda_2}a_4c_5 = a_2c_2e^{i\theta_2},\ \frac{\sqrt{\lambda_4\lambda_5}}{\lambda_2}d_4a_5 = a_2d_2e^{i\theta_2},\ \frac{\sqrt{\lambda_4\lambda_5}}{\lambda_2}c_4e_5 = c_2e_2e^{i\theta_2},$$
$$\frac{\sqrt{\lambda_4\lambda_5}}{\lambda_2}e_4b_5 = b_2e_2e^{i\theta_2},\ \frac{\sqrt{\lambda_4\lambda_5}}{\lambda_2}b_4d_5 = b_2d_2e^{i\theta_2}; \tag{9}$$

$$\frac{\sqrt{\lambda_2\lambda_4}}{\lambda_3}e_4a_2 = a_3e_3e^{i\theta_3},\ \frac{\sqrt{\lambda_2\lambda_4}}{\lambda_3}b_4c_2 = b_3c_3e^{i\theta_3},\ \frac{\sqrt{\lambda_2\lambda_4}}{\lambda_3}c_4d_2 = d_3c_3e^{i\theta_3},\ \frac{\sqrt{\lambda_2\lambda_4}}{\lambda_3}d_4e_2 = d_3e_3e^{i\theta_3},$$
$$\frac{\sqrt{\lambda_2\lambda_4}}{\lambda_3}a_4b_2 = b_3a_3e^{i\theta_3},\ \frac{\sqrt{\lambda_1\lambda_5}}{\lambda_3}a_1d_5 = a_3d_3e^{i\theta_3},\ \frac{\sqrt{\lambda_1\lambda_5}}{\lambda_3}d_1b_5 = b_3d_3e^{i\theta_3},\ \frac{\sqrt{\lambda_1\lambda_5}}{\lambda_3}c_1a_5 = a_3c_3e^{i\theta_3},$$
$$\frac{\sqrt{\lambda_1\lambda_5}}{\lambda_3}e_1c_5 = e_3c_3e^{i\theta_3},\ \frac{\sqrt{\lambda_1\lambda_5}}{\lambda_3}b_1e_5 = b_3e_3e^{i\theta_3}; \tag{10}$$

$$\frac{\sqrt{\lambda_1\lambda_2}}{\lambda_4}a_1c_2 = a_4c_4e^{i\theta_4},\ \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_4}d_1a_2 = a_4d_4e^{i\theta_4},\ \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_4}b_1d_2 = b_4d_4e^{i\theta_4},\ \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_4}e_1b_2 = e_4b_4e^{i\theta_4},$$
$$\frac{\sqrt{\lambda_1\lambda_2}}{\lambda_4}c_1e_2 = e_4c_4e^{i\theta_4},\ \frac{\sqrt{\lambda_3\lambda_5}}{\lambda_4}d_5e_3 = d_4e_4e^{i\theta_4},\ \frac{\sqrt{\lambda_3\lambda_5}}{\lambda_4}b_5c_3 = b_4c_4e^{i\theta_4},\ \frac{\sqrt{\lambda_3\lambda_5}}{\lambda_4}e_5a_3 = a_4e_4e^{i\theta_4}$$
$$\frac{\sqrt{\lambda_3\lambda_5}}{\lambda_4}c_5d_3 = d_4c_4e^{i\theta_4},\ \frac{\sqrt{\lambda_3\lambda_5}}{\lambda_4}a_5b_3 = a_4b_4e^{i\theta_4}; \tag{11}$$

$$\frac{\sqrt{\lambda_1\lambda_4}}{\lambda_5}a_1b_4 = a_5b_5e^{i\theta_5},\ \frac{\sqrt{\lambda_1\lambda_4}}{\lambda_5}d_1e_4 = d_5e_5e^{i\theta_5},\ \frac{\sqrt{\lambda_1\lambda_4}}{\lambda_5}b_1c_4 = c_5b_5e^{i\theta_5},\ \frac{\sqrt{\lambda_1\lambda_4}}{\lambda_5}e_1a_4 = a_5e_5e^{i\theta_5},$$
$$\frac{\sqrt{\lambda_1\lambda_4}}{\lambda_5}c_1d_4 = c_5d_5e^{i\theta_5},\ \frac{\sqrt{\lambda_2\lambda_3}}{\lambda_5}c_2e_3 = c_5e_5e^{i\theta_5},\ \frac{\sqrt{\lambda_2\lambda_3}}{\lambda_5}a_2c_3 = c_5a_5e^{i\theta_5},\ \frac{\sqrt{\lambda_2\lambda_3}}{\lambda_5}d_2a_3 = d_5a_5e^{i\theta_5}$$
$$\frac{\sqrt{\lambda_2\lambda_3}}{\lambda_5}e_2b_3 = b_5e_5e^{i\theta_5},\ \frac{\sqrt{\lambda_2\lambda_3}}{\lambda_5}b_2d_3 = b_5d_5e^{i\theta_5}, \tag{12}$$

where $\theta_i \in [-\pi, \pi]$, $i = 1, ..., 5$.

All the rest relations from the high order principal minors are not independent and can be deduced from the second order principal minors. For instance from (8)-(12) we have

$$a_1b_4c_2d_5e_3 = b_5c_3d_1e_4a_2 = c_4d_2e_5a_3b_1 = d_3e_1a_4b_2c_5 = e_2a_5b_3c_1d_4. \tag{13}$$

Moreover one can also get following relations:

$$\theta_2 + \theta_5 = \theta_3 + \theta_4,\ \theta_2 + \theta_4 = \theta_1 + \theta_5,\ \theta_1 + \theta_2 = \theta_3 + \theta_5,\ \theta_1 + \theta_4 = \theta_2 + \theta_3. \tag{14}$$

Therefore we have

Theorem State ρ is separable if and only if the conditions (8)-(12) are satisfied.

In the following, we construct separable pure state decompositions of ρ when relations (8)-(12) are satisfied. Suppose

$$\rho = \sum_{n=1}^{n=5} |\psi_n\rangle\langle\psi_n| \tag{15}$$

is a separable pure state decomposition, i.e. $|\psi_n\rangle$, $n = 1, 2, 3, 4, 5$, are separable. $|\psi_n\rangle$ can be generally expressed as

$$|\psi_n\rangle = \sum_{m}^{5} u_{mn}|Y_m\rangle = \sum_{ij}^{5} a_{ij}^n |ij\rangle, \tag{16}$$

for some $a_{ij}^n \in \mathbb{C}$, where u_{mn} are the entries of a 5×5 unitary matrix U and

$$|Y_i\rangle = \sqrt{\lambda_i}|X_i\rangle, \quad i = 1,2,3,4,5.$$

Suppose the unitary matrix U has the form with entries $u_{mn} = C_{mn}e^{i\theta_m}$, where $C_{mn} = e^{i\varphi_{mn}}/\sqrt{5}$. Denote A_n the 5×5 matrix with entries a_{ij}^n, then from (16) one has

$$A_n = \begin{pmatrix} u_{1n}a_1\sqrt{\lambda_1} & u_{2n}a_2\sqrt{\lambda_2} & u_{3n}a_3\sqrt{\lambda_3} & u_{4n}a_4\sqrt{\lambda_4} & u_{5n}a_5\sqrt{\lambda_5} \\ u_{5n}b_5\sqrt{\lambda_5} & u_{1n}b_1\sqrt{\lambda_1} & u_{2n}b_2\sqrt{\lambda_2} & u_{3n}b_3\sqrt{\lambda_3} & u_{4n}b_4\sqrt{\lambda_4} \\ u_{4n}c_4\sqrt{\lambda_4} & u_{5n}c_5\sqrt{\lambda_5} & u_{1n}c_1\sqrt{\lambda_1} & u_{2n}c_2\sqrt{\lambda_2} & u_{3n}c_3\sqrt{\lambda_3} \\ u_{3n}d_3\sqrt{\lambda_3} & u_{4n}d_4\sqrt{\lambda_4} & u_{5n}d_5\sqrt{\lambda_5} & u_{1n}d_1\sqrt{\lambda_1} & u_{2n}d_2\sqrt{\lambda_2} \\ u_{2n}e_2\sqrt{\lambda_2} & u_{3n}e_3\sqrt{\lambda_3} & u_{4n}e_4\sqrt{\lambda_4} & u_{5n}e_5\sqrt{\lambda_5} & u_{1n}e_1\sqrt{\lambda_1} \end{pmatrix}. \tag{17}$$

The state $|\psi_n\rangle$ is separable if the rank of $A_n A_n^\dagger$ is one. Consider the characteristic polynomial $|A_n - xI|$, where I is the 5×5 identity matrix. The coefficient of x^3 in $|A_n - xI|$ is given by

$$\begin{aligned} Q &= -\lambda_1 u_{1n}^2(a_1b_1 + a_1c_1 + b_1c_1 + a_1d_1 + b_1d_1 + c_1d_1 + a_1e_1 + b_1e_1 + c_1e_1 + d_1e_1) \\ &\quad +(a_3c_4 + a_4d_3 + b_3d_4 + b_4e_3 + c_3e_4)\sqrt{\lambda_3\lambda_4}u_{3n}u_{4n} \\ &\quad +(a_2b_5 + b_2c_5 + c_2d_5 + a_5e_2 + d_2e_5)\sqrt{\lambda_2\lambda_5}u_{2n}u_{5n}. \end{aligned} \tag{18}$$

It can be shown that if Q is set to be zero, then the coefficients of x^2, x and the constant term in $|A_n - xI|$ are also zero. Thus A_n has only one non-zero eigenvalue and the rank of A_n is then one. Applying similar analysis to $|PA_nP^{-1} - xI|$, $|(P^2)A_n(P^2)^{-1} - xI|$, ..., $|(P^4)A_n(P^4)^{-1} - xI|$, where P is the permutation matrix, and taking into account the conditions (8)-(12), it is straightforward to verify that the matrix A_n has rank one if

$$\begin{aligned} C_{1n}^4 e^{i2\theta_1} &= C_{2n}C_{3n}C_{4n}C_{5n}e^{i(\theta_2+\theta_3+\theta_4+\theta_5)}, \\ C_{2n}^4 e^{i2\theta_2} &= C_{1n}C_{3n}C_{4n}C_{5n}e^{i(\theta_1+\theta_3+\theta_5+\theta_4)}, \\ C_{3n}^4 e^{i2\theta_3} &= C_{2n}C_{4n}C_{1n}C_{5n}e^{i(\theta_2+\theta_4+\theta_1+\theta_5)}, \\ C_{4n}^4 e^{i2\theta_4} &= C_{1n}C_{2n}C_{3n}C_{5n}e^{i(\theta_1+\theta_2+\theta_3+\theta_5)}, \\ C_{5n}^4 e^{i2\theta_5} &= C_{1n}C_{3n}C_{2n}C_{4n}e^{i(\theta_1+\theta_3+\theta_2+\theta_4)}. \end{aligned} \tag{19}$$

As $0 < rank(A_nA_n^+) \le rank(A_n)rank(A_n^+)$, if the rank of A_n is one, matrix $A_nA_n^+$ has also rank one and $|\psi_n\rangle$ is separable. Therefore if one can find $C_{1n}, C_{2n}, C_{3n}, C_{4n}, C_{5n}$,

$n=1,...,5$, satisfying the unitary condition of U and the rank one condition, then (15) is a separable pure state decomposition of ρ.

Let us assume

$$\begin{array}{l}
\varphi_{11}-\varphi_{21}=\xi_1,\ \varphi_{12}-\varphi_{22}=\xi_1+\frac{2\pi}{5},\ \varphi_{13}-\varphi_{23}=\xi_1+\frac{4\pi}{5},\\
\varphi_{14}-\varphi_{24}=\xi_1+\frac{6\pi}{5},\ \varphi_{15}-\varphi_{25}=\xi_1+\frac{8\pi}{5};\\
\varphi_{11}-\varphi_{31}=\xi_2,\ \varphi_{12}-\varphi_{32}=\xi_2+\frac{4\pi}{5},\ \varphi_{13}-\varphi_{33}=\xi_2+\frac{8\pi}{5},\\
\varphi_{14}-\varphi_{34}=\xi_2+\frac{2\pi}{5},\ \varphi_{15}-\varphi_{35}=\xi_2+\frac{6\pi}{5};\\
\varphi_{11}-\varphi_{41}=\xi_3,\ \varphi_{12}-\varphi_{42}=\xi_3+\frac{6\pi}{5},\ \varphi_{13}-\varphi_{43}=\xi_3+\frac{2\pi}{5},\\
\varphi_{14}-\varphi_{44}=\xi_3+\frac{8\pi}{5},\ \varphi_{15}-\varphi_{45}=\xi_3+\frac{4\pi}{5};\\
\varphi_{11}-\varphi_{51}=\xi_4,\ \varphi_{12}-\varphi_{52}=\xi_4+\frac{8\pi}{5},\ \varphi_{13}-\varphi_{53}=\xi_4+\frac{6\pi}{5},\\
\varphi_{14}-\varphi_{54}=\xi_4+\frac{4\pi}{5},\ \varphi_{15}-\varphi_{55}=\xi_4+\frac{2\pi}{5},
\end{array} \tag{20}$$

for some real numbers ξ_1,ξ_2,ξ_3,ξ_4. Then the unitary condition of U is satisfied.

The rank-one conditions (8)-(12) require that:

$$\begin{array}{l}
4\varphi_{1n}-(\varphi_{2n}+\varphi_{3n}+\varphi_{4n}+\varphi_{5n})=-2\theta_1+(\theta_2+\theta_3+\theta_4+\theta_5),\\
4\varphi_{2n}-(\varphi_{1n}+\varphi_{3n}+\varphi_{4n}+\varphi_{5n})=-2\theta_2+(\theta_1+\theta_3+\theta_4+\theta_5),\\
4\varphi_{3n}-(\varphi_{1n}+\varphi_{2n}+\varphi_{4n}+\varphi_{5n})=-2\theta_3+(\theta_1+\theta_2+\theta_4+\theta_5),\\
4\varphi_{4n}-(\varphi_{1n}+\varphi_{2n}+\varphi_{3n}+\varphi_{5n})=-2\theta_4+(\theta_1+\theta_2+\theta_3+\theta_5),\\
4\varphi_{5n}-(\varphi_{1n}+\varphi_{2n}+\varphi_{3n}+\varphi_{4n})=-2\theta_5+(\theta_1+\theta_2+\theta_3+\theta_4),
\end{array} \tag{21}$$

which can be realized by simply choosing

$$\xi_i=\frac{3}{5}(\theta_{i+1}-\theta_1),\quad i=1,...,4. \tag{22}$$

For simplicity we can further take $\varphi_{1i}=0$, $i=1,2,...,5$, then

$$U=\frac{1}{\sqrt{5}}\begin{pmatrix}
e^{i\theta_1} & e^{i\theta_1} & e^{i\theta_1} & e^{i\theta_1} & e^{i\theta_1}\\
e^{\frac{i}{5}(3\theta_1+2\theta_2)} & e^{\frac{i}{5}(3\theta_1+2\theta_2-2\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_2-4\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_2-6\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_2-8\pi)}\\
e^{\frac{i}{5}(3\theta_1+2\theta_3)} & e^{\frac{i}{5}(3\theta_1+2\theta_3-4\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_3-8\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_3-2\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_3-6\pi)}\\
e^{\frac{i}{5}(3\theta_1+2\theta_4)} & e^{\frac{i}{5}(3\theta_1+2\theta_4-6\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_4-2\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_4-8\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_4-4\pi)}\\
e^{\frac{i}{5}(3\theta_1+2\theta_5)} & e^{\frac{i}{5}(3\theta_1+2\theta_5-8\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_5-6\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_5-4\pi)} & e^{\frac{i}{5}(3\theta_1+2\theta_5-2\pi)}
\end{pmatrix}, \tag{23}$$

and $|\psi_i\rangle$ have the corresponding separable expressions:

$$
\begin{aligned}
|\psi_1\rangle &= (1, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_5)}b_5\sqrt{\lambda_5}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_4)}c_4\sqrt{\lambda_4}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_3)}d_3\sqrt{\lambda_3}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_2)}e_2\sqrt{\lambda_2}}{e^{i\theta_1}a_1\sqrt{\lambda_1}})^t \\
&\otimes\frac{1}{\sqrt{5}}(e^{i\theta_1}a_1\sqrt{\lambda_1},\ e^{\frac{i}{5}(3\theta_1+2\theta_2)}a_2\sqrt{\lambda_2},\ e^{\frac{i}{5}(3\theta_1+2\theta_3)}a_3\sqrt{\lambda_3},\ e^{\frac{i}{5}(3\theta_1+2\theta_4)}a_4\sqrt{\lambda_4}, \\
&e^{\frac{i}{5}(3\theta_1+2\theta_5)}a_5\sqrt{\lambda_5})^t; \\
|\psi_2\rangle &= (1, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_5-8\pi)}b_5\sqrt{\lambda_5}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_4-6\pi)}c_4\sqrt{\lambda_4}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_3-4\pi)}d_3\sqrt{\lambda_3}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_2-2\pi)}e_2\sqrt{\lambda_2}}{e^{i\theta_1}a_1\sqrt{\lambda_1}})^t \\
&\otimes\frac{1}{\sqrt{5}}(e^{i\theta_1}a_1\sqrt{\lambda_1},\ e^{\frac{i}{5}(3\theta_1+2\theta_2-2\pi)}a_2\sqrt{\lambda_2},\ e^{\frac{i}{5}(3\theta_1+2\theta_3-4\pi)}a_3\sqrt{\lambda_3}, \\
&e^{\frac{i}{5}(3\theta_1+2\theta_4-6\pi)}a_4\sqrt{\lambda_4},\ e^{\frac{i}{5}(3\theta_1+2\theta_5-8\pi)}a_5\sqrt{\lambda_5})^t; \\
|\psi_3\rangle &= (1, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_5-6\pi)}b_5\sqrt{\lambda_5}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_4-2\pi)}c_4\sqrt{\lambda_4}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_3-8\pi)}d_3\sqrt{\lambda_3}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_2-4\pi)}e_2\sqrt{\lambda_2}}{e^{i\theta_1}a_1\sqrt{\lambda_1}})^t \\
&\otimes\frac{1}{\sqrt{5}}(e^{i\theta_1}a_1\sqrt{\lambda_1},\ e^{\frac{i}{5}(3\theta_1+2\theta_2-4\pi)}a_2\sqrt{\lambda_2},\ e^{\frac{i}{5}(3\theta_1+2\theta_3-8\pi)}a_3\sqrt{\lambda_3}, \\
&e^{\frac{i}{5}(3\theta_1+2\theta_4-2\pi)}a_4\sqrt{\lambda_4},\ e^{\frac{i}{5}(3\theta_1+2\theta_5-6\pi)}a_5\sqrt{\lambda_5})^t; \\
|\psi_4\rangle &= (1, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_5-4\pi)}b_5\sqrt{\lambda_5}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_4-8\pi)}c_4\sqrt{\lambda_4}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_3-2\pi)}d_3\sqrt{\lambda_3}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_2-6\pi)}e_2\sqrt{\lambda_2}}{e^{i\theta_1}a_1\sqrt{\lambda_1}})^t \\
&\otimes\frac{1}{\sqrt{5}}(e^{i\theta_1}a_1\sqrt{\lambda_1},\ e^{\frac{i}{5}(3\theta_1+2\theta_2-6\pi)}a_2\sqrt{\lambda_2},\ e^{\frac{i}{5}(3\theta_1+2\theta_3-2\pi)}a_3\sqrt{\lambda_3}, \\
&e^{\frac{i}{5}(3\theta_1+2\theta_4-8\pi)}a_4\sqrt{\lambda_4},\ e^{\frac{i}{5}(3\theta_1+2\theta_5-4\pi)}a_5\sqrt{\lambda_5})^t; \\
|\psi_5\rangle &= (1, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_5-2\pi)}b_5\sqrt{\lambda_5}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_4-4\pi)}c_4\sqrt{\lambda_4}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_3-6\pi)}d_3\sqrt{\lambda_3}}{e^{i\theta_1}a_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{5}(3\theta_1+2\theta_2-8\pi)}e_2\sqrt{\lambda_2}}{e^{i\theta_1}a_1\sqrt{\lambda_1}})^t \\
&\otimes\frac{1}{\sqrt{5}}(e^{i\theta_1}a_1\sqrt{\lambda_1},\ e^{\frac{i}{5}(3\theta_1+2\theta_2-8\pi)}a_2\sqrt{\lambda_2},\ e^{\frac{i}{5}(3\theta_1+2\theta_3-6\pi)}a_3\sqrt{\lambda_3}, \\
&e^{\frac{i}{5}(3\theta_1+2\theta_4-4\pi)}a_4\sqrt{\lambda_4},\ e^{\frac{i}{5}(3\theta_1+2\theta_5-2\pi)}a_5\sqrt{\lambda_5})^t.
\end{aligned}
$$

Remark Due to the fact that there is still freedom in choosing the parameters φ_{mn}, $m, n = 1, 2, 3, 4, 5$, there exist many other separable pure state decompositions.

We now consider $(2k+1)\times(2k+1)$ quantum mixed states on the tensor space $H\otimes H$ with basis $|1\rangle = (1, 0, ..., 0)^t$, $|2\rangle = (0, 1, ..., 0)^t$, ..., $|2k+1\rangle = (0, 0, ..., 1)^t$ on each vector space H,

$$\rho = \sum_{i=1}^{2k+1} \lambda_i |X_i\rangle\langle X_i|, \quad k \in \mathbb{N}, \tag{24}$$

where $\sum_{i=1}^{2k+1}\lambda_i = 1$, $0 < \lambda_i < 1$, and $|X_i\rangle$, $i = 1, 2, ..., 2k+1$, are normalized vectors of the form:

$$|X_i\rangle = (I\otimes P^{i-1})|X(a_1^i, a_2^i, ..., a_{2k+1}^i)\rangle, \tag{25}$$

where $|X(a_1^i,a_2^i,...,a_{2k+1}^i)\rangle=\sum_{j=1}^{2k+1}a_j^i|jj\rangle$, $0\neq a_i^j\in\mathbb{C}$, $1\leq i,j\leq 2k+1$, $k\in\mathbb{N}$, and

$$P=\begin{pmatrix}0&0&\dots&0&1\\1&0&\dots&0&0\\0&1&\dots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\dots&1&0\end{pmatrix}$$

is a permutation operator.

ρ can be generally written as

$$\rho=\sum_{n=1}^{2k+1}|\psi_n\rangle\langle\psi_n|, \tag{26}$$

where the pure states $|\psi_n\rangle$ are of the form

$$|\psi_n\rangle=\sum_{m}^{2k+1}u_{mn}|Y_m\rangle, \tag{27}$$

u_{mn} are the entries of a $(2k+1)\times(2k+1)$ unitary matrix U and

$$|Y_i\rangle=\sqrt{\lambda_i}|X_i\rangle,\quad i=1,\dots,2k+1.$$

Similar to the case of 5×5 systems, the PPT condition of ρ gives rise to $2k+1$ independent linear inequality sets, and hence equations like (8)-(12). Under the PPT condition of ρ, the unitary matrix U has the form with entries $u_{mn}=e^{i(\varphi_{mn}+\theta_m)}/\sqrt{2k+1}$, and φ_{mn} satisfying the following relations:

$$\varphi_{1n}-\varphi_{jn}=\xi_{j-1}+\frac{2(j-1)(n-1)\pi}{2k+1},\quad n,j=1,2,...,2k+1, \tag{28}$$

where

$$\xi_i=\frac{k+1}{2k+1}(\theta_{i+1}-\theta_1),\quad i=0,...,2k. \tag{29}$$

Then all $|\psi_n\rangle$ in (26) are product states.

For example if we simply take $\varphi_{1i}=0$, $i=1,2,...,2k+1$, then we have

$$U=\frac{1}{\sqrt{2k+1}}\begin{pmatrix}e^{i\theta_1}&e^{i\theta_1}&\dots&e^{i\theta_1}\\e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2)}&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2-2\pi)}&\dots&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2-4k\pi)}\\e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_3)}&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_3-4\pi)}&\dots&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_3-(4k-2)\pi)}\\\vdots&\vdots&\vdots&\vdots\\e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1})}&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1}-4k\pi)}&\dots&e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1}-2\pi)}\end{pmatrix}.$$

$|\psi_i\rangle$, $i=1,...,2k+1$, have the separable expressions:

$$
\begin{aligned}
|\psi_1\rangle &= (1, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k+1}]}a^2_{2k+1}\sqrt{\lambda_{2k+1}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k}]}a^3_{2k}\sqrt{\lambda_{2k}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, ..., \\
&\quad \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_2]}a^{2k+1}_2\sqrt{\lambda_2}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}) \otimes \frac{1}{\sqrt{2k+1}}(e^{i\theta_1}a^1_1\sqrt{\lambda_1}, e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2)}a^1_2\sqrt{\lambda_2}, ..., \\
&\quad e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1})}a^1_{2k+1}\sqrt{\lambda_{2k+1}}), \\
|\psi_2\rangle &= (1, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k+1}-4k\pi]}a^2_{2k+1}\sqrt{\lambda_{2k+1}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k}-(4k-2)\pi]}a^3_{2k}\sqrt{\lambda_{2k}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, ..., \\
&\quad \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_2-2\pi]}a^{2k+1}_2\sqrt{\lambda_2}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}) \otimes \frac{1}{\sqrt{2k+1}}(e^{i\theta_1}a^1_1\sqrt{\lambda_1}, e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2-2\pi)}a^1_2\sqrt{\lambda_2}, \\
&\quad ..., e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1}-4k\pi)}a^1_{2k+1}\sqrt{\lambda_{2k+1}}), \\
&\cdots \\
|\psi_{2k+1}\rangle &= (1, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k+1}-2\pi]}a^2_{2k+1}\sqrt{\lambda_{2k+1}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_{2k}-4\pi]}a^3_{2k}\sqrt{\lambda_{2k}}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}, ..., \\
&\quad \frac{e^{\frac{i}{2k+1}[(k+1)\theta_1+k\theta_2-4k\pi]}a^{2k+1}_2\sqrt{\lambda_2}}{e^{i\theta_1}a^1_1\sqrt{\lambda_1}}) \otimes \frac{1}{\sqrt{2k+1}}(e^{i\theta_1}a^1_1\sqrt{\lambda_1}, e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_2-4k\pi)}a^1_2\sqrt{\lambda_2}, \\
&\quad ..., e^{\frac{i}{2k+1}((k+1)\theta_1+k\theta_{2k+1}-2\pi)}a^1_{2k+1}\sqrt{\lambda_{2k+1}}).
\end{aligned}
$$

We have investigated a class of $(2k+1)\times(2k+1)$, $k\in\mathbb{N}$, bipartite mixed states for which the PPT criterion is both sufficient and necessary for separability. The PPT conditions for these states are derived. We have presented a general approach to find the separable pure state decompositions of this class, and the separable pure state decompositions have been explicitly constructed. Our method can be applied to separability analysis and separable pure state decomposition for other class quantum mixed states.

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