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Resurgent analysis of the Witten Laplacian in one dimension II.

by

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Abstract

The Witten Laplacian in one dimension is studied further by methods of resurgent analysis in order to approach Fukaya's conjectures relating WKB asymptotics and disc instantons. In this paper more precise connection formulae are presented, which allows the calculation of a subdominant exponential term in the hyperasymptotic expansion of a low-lying eigenvalue. Calculation of eigenfunctions corresponding to low-lying eigenvalues is presented in two examples.

1 Introduction

We are continuing the project started in [G08], where we proposed to study the Witten Laplacian by methods of resurgent analysis in order to prove conjectures by Fukaya [F05, §5.2] relating WKB asymptotics and disc instantons. The reader is referred to the introductory section of [G08] for philosophy and motivation, as well as for a brief review of resurgent analysis.

In [G08] we have shown, modulo standard black boxes in resurgent analysis, that for a generic enough real trigonometric polynomial $f(q) \in \mathbb{R}[\sin 2\pi q, \cos 2\pi q]$ with n real local minima and n real local maxima on $[0, 1)$, the Witten Laplacian

$$-h^2\partial_q^2 + (f')^2 - hf'', \qquad h \to 0+ \tag{1}$$

has n exponentially small resurgent eigenvalues $E_k(h)$ and that the corresponding eigenfunctions $\phi_k(q, h)$ are resurgent with respect to h for $q \notin (f')^{-1}(0)$. We have also presented a method of setting up a quantization condition and solving it by means of an iterative procedure involving a Newton polygon.

In this paper we perform calculations more explicitly. Firstly, we obtain asymptotic expansion of the connection coefficients and of various formal monodromies to one more order in h, which allows calculation of the first subdominant exponential in the hyperasymptotic expansion of eigenvalues, see formula (27) in section 5.

Secondly, we show how our methods allow us to calculate (hyper)asymptotic expansions of an eigenfunction of the Witten Laplacian (1) corresponding to a nonzero low-lying eigenvalue. If $q_1, q_2 ..., q_{2n}, q_{2n+1} = q_1 + 1, ...$ denote consecutive real zeros of $f'(q)$, note that for one and the same eigenfunction these expansions will change discontinuously from one of the intervals (q_j, q_{j+1}) to the next, due to the Stokes phenomenon. We calculate these hyperasymptotic expansions for two examples of a function $f(q)$ (sections 5 and 6). In remarks 5.2, 6.1 we put our finger on the specific algebraic reason why methods of resurgent analysis are essential for such a calculation. This information about asymptotic expansions of eigenfunctions is much more specific than the information about quasimodes available through C^∞ methods (see, e.g., [HKN04]).

The ability to perform explicit calculations developed in this paper will be needed in our future work towards Fukaya's conjecture. Remarks 5.2, 6.1 and computations leading to them may be of independent pedagogical interest.

The structure of this paper, that is a continuation of [G08] and uses its material freely, is as follows. In the section 2 we recall the notation and calculate various monodromies of formal solution of the Wittle Laplacian (2). In the section 3 we perform a more precise calculation of the connection coefficients and of the tunneling cycle monodromies τ_j than we did in [G08]. A general procedure of calculationg asymptotic expansions of eigenfunctions is recalled in section 4 and applied to two examples in sections 5 and 6. In addition, for the example of 5, we have calculated the first subdominant exponential in the hyperasymptotic expansion of the nonzero low-lying eigenvalue. The paper concludes with an appendix containing a list of elementary formulae used in this text.

2 Formal WKB solutions and formal monodromies

2.1 Notation, cuts, signs, and branches.

Let us recall the notation of [G08]. Let $f(q)$ be a real polynomial in $\sin 2\pi q$ and $\cos 2\pi q$, with n real local minima $q_1, ..., q_{2n-1}$ and n real local maxima $q_2, .., q_{2n}$ on the period, where $0 < q_1 < q_2 < ... < q_{2n-1} < q_{2n} < 1$. We require $f''(q_j) \neq 0$.

In this section we will discuss formal WKB solutions of

$$P\psi \;:=\; \left[-h^2\partial_q^2 + (f')^2 - hf''\right]\psi \;=\; E\psi, \tag{2}$$

where E is a complex number.

For $E \neq 0$ and $|E|$ sufficiently small, the classical momentum $p(q) = \sqrt{E - (f'(q))^2}$ is defined on a two sheeted cover of the complex plane of q. For $E = 0$, the two determinations of $p(q)$ are $\pm f'(q)$, and one can think of the Riemann surface of $p(q)$ as of two separate sheets having contact at points q_j where $f'(q_j) = 0$.

The formulas related to formal solutions of the equation (2) can be established, for definiteness, for $E > 0$, and then analytically continued to other values of E, whenever appropriate.

In the standard terminology of the WKB method points q satisfying $[f'(q)]^2 = E$ are called *turning points* of the equation (2).

When $E > 0$ and $|E|$ is sufficiently small, the double turning points q_j on the real axis for $E = 0$ split into pairs $q_j^- < q_j < q_j^+ (< q_{j+1}^-)$ of simple turning points still on the real axis. The Riemann surface of $\sqrt{E-(f')^2}$ will be described as the plane with cut connecting q_j^- to q_j^+ and going a little below the real axis. To specify the determination of $p(q,E)$ on the first sheet, we define Arg $(E-(f')^2)$ for real values of q on figure 1. As $E \to 0$, on the first sheet $ip(q,E) \to f'(q)$.

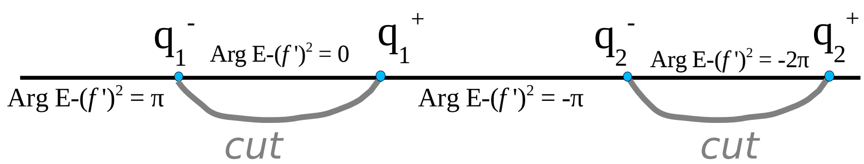

Figure 1: Choice of Arg $E-(f')^2$

A monodromy of an elementary formal WKB solution along some path $\rho(t)$ on a universal cover $\tilde{\mathbb{C}}$ of $\mathbb{C}\backslash\{\text{turning pts}\}$ is defined as $\psi(\rho(1))/\psi(\rho(0))$ and will be denoted $\exp[2\pi i s_\rho]$.

2.2 Formal solutions.

In order to find a formal WKB solution of (2), we will be looking for a series

$$y(h,q) = y_0(q) + hy_1(q) + h^2 y_2(q) + \ldots$$

solving

$$(P-E)\left\{\exp\left\{\int^q \frac{i}{h}\sqrt{E-(f')^2} + y(q')dq'\right\}\right\} = 0.$$

One can calculate y_j's recursively:

$$y_0(q) = \frac{f'f''}{2(E-(f')^2)} - \frac{f''}{2i\sqrt{E-(f')^2}},$$

$$y_1(q) = -\frac{5(f')^2(f'')^2}{8i(E-(f')^2)^{5/2}} - \frac{f'(f'')^2}{2(E-(f')^2)^2} - \frac{(f'')^2}{8i(E-(f')^2)^{3/2}} - \frac{f'f^{(3)}}{4i(E-(f')^2)^{3/2}} - \frac{f^{(3)}}{4(E-(f')^2)},$$

etc.

Let ϕ_+, ϕ_- be the formal resurgent solutons of

$$(-h^2\partial_x^2 + [f']^2 - hf'')\phi = hE_r\phi \tag{3}$$

corresponding to the first and second sheet of the Riemann surface, normalized in such a way that $\psi_+(q_0) = \psi_-(q_0) = 1$ and defined on the domains (complex plane with vertical cuts starting at q_j) shown on fig.2. The point q_0 can be an arbitrary point such that $f'(q_0) \neq 0$, e.g. $q_0 = 0$,

In terms of ϕ_+, ϕ_- the transfer matrix and the quantization condition will be written, in the same way as in [G08, section VIII].

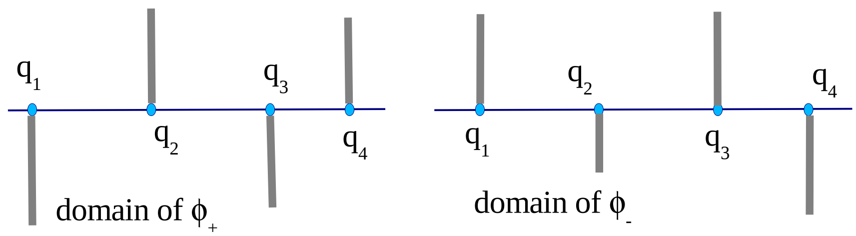

Figure 2: Domains of ϕ_+ and ϕ_-

2.3 Assorted Taylor series

Let $q_\ell \in \mathbb{R}$ and $f'(q_\ell) = 0$. We will need to calculate various integrals along paths passing around or near the point q_ℓ, and in this subsection we will set up the appropriate notation. For q near q_ℓ the substitution $u = -f'(q)$ is one-to-one, and thus u can be taken as a local coordinate near q_ℓ.

Let us introduce the numbers $a_j(q_\ell)$ (sometimes written as a_j if the index ℓ is clear) by

$$-(q - q_\ell) = a_0(q_\ell)f'(q) + \frac{1}{2}a_1(q_\ell)(f')^2 + \frac{1}{3}a_2(q_\ell)(f')^3 + ...$$

In particular,

$$a_0 = -\frac{1}{f''(q_\ell)}, \quad a_1 = \frac{f^{(3)}(q_\ell)}{[f''(q_\ell)]^3}, \quad a_2 = \frac{f^{(4)}(q_\ell)}{2[f''(q_\ell)]^4} - \frac{3[f^{(3)}(q_\ell)]^2}{2[f''(q_\ell)]^5}, ...$$

It follows then that

$$-\frac{1}{f''(q)} = a_0(q_\ell) + a_1(q_\ell)(f') + a_2(q_\ell)(f')^2 + ... = \sum_{j=0}^{\infty} a_j(-1)^j u^j.$$

and

$$f''(q) - f''(q_\ell) = f''(q_\ell)\sum_{j=1}^{\infty} a_j(f')^j f'' \;\; = \;\; f''(q_\ell)\sum_{j=1}^{\infty} a_j(-1)^j u^j f''.$$

Introduce similarly the numbers $b_j = b_j(q_\ell)$ by the requirement that

$$-f'' = b_0(q_\ell) + b_1(q_\ell)u + b_2(q_\ell)u^2 + ...$$

should hold near q_ℓ. In particular,

$$b_0 = -f''(q_\ell); \quad b_1 = \frac{f^{(3)}(q_\ell)}{f''(q_\ell)}; \quad b_2 = \frac{[f^{(3)}(q_\ell)]^2 - f''(q_\ell)f^{(4)}(q_\ell)}{2[f''(q_\ell)]^3}.$$

We obtain by differentiation

$$-f^{(3)} = \sum_{j=0}^{\infty} j b_j(-f')^{j-1}(-f'').$$

We will also use that for ε small enough and $A = -f'(q_\ell - \varepsilon)$,

$$f(q_\ell) - f(q_\ell - \varepsilon) = \int_{q_\ell - \varepsilon}^{q_\ell} f'(q)dq = \int_0^A u[\sum_{j=0}^{\infty}(-1)^j a_j u^j]du = \sum_{j=0}^{\infty}(-1)^j a_j \frac{A^{j+2}}{j+2}. \tag{4}$$

2.4 Calculation of $\int_{\sigma_j} y_0(q)dq$.

The path σ_j is defined for $E > 0$ as starting at $q_j - \varepsilon$ on the first sheet, going under the cut between q_j^- and q_j^+, and ending at $q_j - \varepsilon$ on the second sheet; the path σ_j' is obtained from σ_j by interchanging the sheets, figure 3. It was shown in [G08] that

$$e^{-2\pi i s_{\sigma_1}} = -e^{2\pi i s_{\sigma_1'}}$$

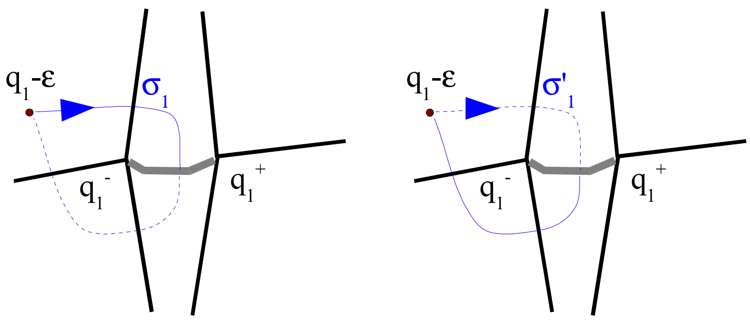

Figure 3: Paths σ_1 and σ_1'.

Recall that

$$y_0(q) = \frac{f'f''}{2(E-(f')^2)} - \frac{f''}{2i\sqrt{E-(f')^2}}$$

Lemma 2.1 *We have*

$$\int_{\sigma_j} y_0(q)dq = \operatorname{arccosh} \frac{(-f'(q_j-\varepsilon))}{\sqrt{E}} - \frac{\pi i}{2}, \quad j \text{ odd};$$

$$\int_{\sigma_j'} y_0(q)dq = -\operatorname{arccosh} \frac{f'(q_j-\varepsilon)}{\sqrt{E}} - \frac{\pi i}{2}, \quad j \text{ even},$$

where the branch of arccosh *is chosen so as to coincide with the principal real value of of* arccosh *for* $E > 0$ *and* $q_j - \varepsilon$ *on the real axis immediately to the left of* q_j^-.

Proof. Let us do the case of j odd. Integrating the first summand in y_0, we have

$$\int_{\sigma_1} \frac{f'f''}{2(E-(f')^2)}dq = -\frac{1}{4}\text{Ln}\,(E-(f')^2)\Big|_{\partial\sigma_1} = -\frac{1}{4}\cdot 2\pi i = -\frac{\pi i}{2}.$$

To integrate the second summand, use a substitution $u = -f'(q)$ and $A = -f'(q)$:

$$\int_{\sigma_1} \frac{f''(q)}{2i\sqrt{E-(f')^2}} = -\int_{\sigma_1} \frac{f''(q)}{2\sqrt{(f')^2-E}}dq = -\int_A^{\sqrt{E}} \frac{(-du)}{\sqrt{u^2-E}} = -\operatorname{arccosh} \frac{A}{\sqrt{E}}.$$

(In the second term of this line the arithmetic square root is meant when $E > 0$ and when q is real immediately to the left of q_j^-.) Subtracting the latter value from the former, obtain the statement. The case of even j is treated similarly. □

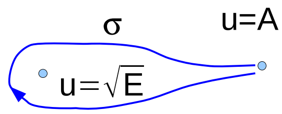

Figure 4:

2.5 Calculation of $\int_{\sigma_j} y_1(q)dq$

We begin by calculation $\int_{\sigma_1} y_1(q)dq$.

Recall that

$$y_1(q) = -\frac{5(f')^2(f'')^2}{8i(E-(f')^2)^{5/2}} - \frac{f'(f'')^2}{2(E-(f')^2)^2} - \frac{(f'')^2}{8i(E-(f')^2)^{3/2}} - \frac{f'f^{(3)}}{4i(E-(f')^2)^{3/2}} - \frac{f^{(3)}}{4(E-(f')^2)}. \quad (5)$$

In the integral $\int_{\sigma_j} y_1(q)dq$ let us make a substitution $u=-f'$, write $-f''=b_0+b_1u+b_2u^2+...$, and put $A=-f'(q_1-\varepsilon)$. Then

$$\int_{\sigma_1} y_1(q)dq = \int_\sigma \left\{ -\frac{5u^2\sum_{j=0}^\infty b_ju^j}{8i(E-u^2)^{5/2}} + \frac{u\sum_{j=0}^\infty b_ju^j}{2(E-u^2)^2} - \frac{\sum_{j=0}^\infty b_ju^j}{8i(E-u^2)^{3/2}} \right.$$
$$\left. -\frac{u\sum_{j=0}^\infty (j+1)b_{j+1}u^j}{4i(E-u^2)^{3/2}} + \frac{\sum_{j=0}^\infty (j+1)b_{j+1}u^j}{4(E-u^2)} \right\} du.$$

Since $\frac{u\sum_{j=0}^\infty b_ju^j}{2(E-u^2)^2} + \frac{\sum_{j=0}^\infty (j+1)b_{j+1}u^j}{4(E-u^2)}$ is a full differential of a function univalued in u, this part of the integrand can be dropped, and so

$$\int_{\sigma_1} y_1(q)dq = \int_\sigma \left\{ -\frac{5u^2\sum_{j=0}^\infty b_ju^j}{8i(E-u^2)^{5/2}} - \frac{\sum_{j=0}^\infty b_ju^j}{8i(E-u^2)^{3/2}} - \frac{\sum_{j=1}^\infty jb_ju^j}{4i(E-u^2)^{3/2}} \right\} du.$$

We prefer to rewrite the denominator in terms of $(u^2-E)^{1/2}$ which is positive for u real, $u>\sqrt{E}$ and close to the beginning of the path σ, and negative close for u close to the end of σ:

$$\int_{\sigma_1} y_1(q)dq = \int_\sigma \left\{ \frac{5u^2\sum_{j=0}^\infty b_ju^j}{8(u^2-E)^{5/2}} - \frac{\sum_{j=0}^\infty (1+2j)b_ju^j}{8(u^2-E)^{3/2}} \right\}.$$

Integrating by parts twice using formulae (39)-(42), obtain

$$\int_{\sigma_1} y_1(q)dq = -\left.\frac{\sum_{j=0}^\infty 5b_ju^{j+1}}{24(u^2-E)^{3/2}}\right|_{\partial\sigma} - \left.\frac{b_0}{12E}\frac{u}{\sqrt{u^2-E}}\right|_{\partial\sigma} -$$
$$-\sum_{j=1}^\infty \frac{(2-j)b_j}{24}\left.\frac{b_ju^{j-1}}{(u^2-E)^{1/2}}\right|_{\partial\sigma} + \int_\sigma \frac{\sum_{j=2}^\infty (2-j)(j-1)b_ju^{j-2}}{24(u^2-E)^{1/2}}du.$$

The first three summands give

$$\frac{5\sum_{j=0}^{\infty} b_j A^{j+1}}{12(A^2-E)^{3/2}} + \frac{b_0}{6E}\frac{A}{(A^2-E)^{1/2}} + \frac{\sum_{j=1}^{\infty}(2-j)b_j A^{j-1}}{12(A^2-E)^{1/2}}.$$

Performing the change of variables in the fourth summand, $u = \sqrt{E}\cosh t$ and using (37),

$$\int_\sigma \frac{\sum_{j=0}^{\infty}(j+1)[-jb_{j+2}]u^j}{24\sqrt{u^2-E}}dq \;=\; \frac{1}{12}\sum_{j=1}^{\infty}(j+1)b_{j+2}A^j \;+\; o(E^0)$$

Thus,

$$\int_{\sigma_1} y_1(q)dq \;=\; \frac{5\sum_{j=0}^{\infty} b_j A^{j+1}}{12(A^2-E)^{3/2}} + \frac{b_0}{6E}\frac{A}{(A^2-E)^{1/2}} + \frac{\sum_{j=1}^{\infty}(2-j)b_j A^{j-1}}{12(A^2-E)^{1/2}} + \frac{1}{12}\sum_{j=1}^{\infty}(j+1)b_{j+2}A^j + o(E^0)$$

$$= -\frac{1}{12}b_2 + \frac{b_0}{6E} + \frac{1}{2}\sum_{j=0}^{\infty} b_j A^{j-2} + o(E^0).$$

Note that the error term $o(E^0)$ in the previous formula cannot be simply replaced by $O(E^1)$, as terms of order $E\ln E$ can also be present.

Analogously,

$$\int_{\sigma_2'} y_1(q)dq \;=\; \frac{1}{12}b_2(q_2) - \frac{b_0(q_2)}{6E} - \frac{1}{2}\sum_{j=0}^{\infty} b_j(q_2)A^{j-2} \;+\; o(E^0).$$

2.6 Monodromy around the turning point

Let γ_k (resp., γ_k') be a counterclockwise loop enclosing both points q_k^- and q_k^+ on the first (resp., second) sheet of the Riemann surface of the momentum (figure 5), and denote $e^{2\pi i s_{\gamma_k}}$ and $e^{2\pi i s_{\gamma_k'}}$ the corresponding monodromies of formal WKB solutions along these loops. From [G08, Lemma V.2] we know that

$$s_{\gamma_k} \;+\; s_{\gamma_k'} \;=\; -1. \tag{6}$$

Proposition 2.2 *We have*

$$s_{\gamma_k} = \frac{1}{2\pi h}\left[-\frac{\pi E}{f''(q_k)} + \frac{\pi a_2(q_k)E^2}{4} + o(E^2)\right] - 1 + O(h),$$

where

$$a_2(q_k) = \frac{f^{(4)}(q_k)}{2[f''(q_k)]^4} - \frac{3}{2}\frac{[f^{(3)}(q_k)]^2}{[f''(q_k)]^5}.$$

Proof. For concreteness, we will prove the statement for $k=1$. With the notation for the formal solution introduced in section 2.2,

$$2\pi i s_{\gamma_1} \;=\; \int_{\gamma_1}\left[\frac{i}{h}\sqrt{E-(f')^2} + y_0(q') + hy_1(q')\right]dq' + O(h^2).$$

Figure 5: Paths γ_k and γ'_k.

We will now separately compute the integrals of the three terms involved.

Step 1: $\omega_{\gamma_1} := \int_{\gamma_1} \sqrt{E-(f')^2}dq = -\frac{\pi}{f''(q_1)} \cdot E + O(E^2)$.

Write $f'(q) = f''(q_1)(q-q_1) + r(q-q_1)$, where $r(q-q_1) = O\left((q-q_1)^2\right)$.

So, using the formula

$$\oint u^k \sqrt{E-u^2}du \;=\; \begin{cases} 0, & k \text{ odd} \\ -2\frac{(k-1)!!}{(k+2)!!}E^{\frac{k}{2}+1}\pi, & k \text{ even} \end{cases},$$

obtain (under the usual substitution $u = -f'$, $-\frac{1}{f''} = \sum_{k=0}^{\infty} a_k(-1)^k u^k$ and using that $f'(q_1^+) = \sqrt{E}$, so $q = q_1^+ \leftrightarrow u = -\sqrt{E}$, $q = q_1^- \leftrightarrow u = \sqrt{E}$.)

$$\oint_{\gamma_1} \sqrt{E-(f')^2}dq = 2\int_{-\sqrt{E}}^{\sqrt{E}} \sqrt{E-u^2}(a_0 - a_1 u + a_2 u^2 - ...)du \;=$$

$$= \sum_{k \text{ even}} 2\frac{(k-1)!!}{(k+2)!!}E^{\frac{k}{2}+1}\pi a_k \;=\; 2\frac{1}{2}E\pi a_0 + 2\frac{1}{2\cdot 4}E^2\pi a_2 \;+\; o(E^2) \;=\; -\frac{\pi E}{f''(q_1)} + \frac{\pi a_2 E^2}{4} \;+\; o(E^2). \quad (7)$$

Step 2: A calculation showing that

$$\oint_{\gamma_1} y_0 dq \;=\; \oint_{\gamma_1} dq \left\{ \frac{f'f''}{2(E-(f')^2)} - \frac{f''}{2i\sqrt{E-(f')^2}} \right\} \;=\; -2\pi i$$

is completely elementary after the substitution $u = -f'(q)$ in the integral.

Step 3. Let us calculate the integral of y_1 around γ_1. Make a change of variables $u = -f'$ and proceed analogously to section 2.5.

$$\int_{\gamma_1} y_1(q)dq \;=\; \int_\sigma \left\{ \frac{5u^2 \sum_{j=0}^{\infty} b_j u^j}{8(u^2-E)^{5/2}} - \frac{\sum_{j=0}^{\infty}(1+2j)b_j u^j}{8(u^2-E)^{3/2}} \right\} \;=$$

$$= \int_\gamma \frac{\sum_{j=2}^{\infty}(2-j)(j-1)b_j u^{j-2}}{3\cdot 8(u^2-E)^{1/2}}du \;=$$

$$= \;2\pi \sum_{2k=j>2} b_j E^{\frac{j-2}{2}} \frac{(j-2)(j-1)}{3\cdot 8}\frac{(j-3)!!}{(j-2)!!} \;=\; = \;\frac{\pi}{12}\sum_{k\geq 2}^{\infty} b_{2k}E^{k-1}\frac{(2k-1)!!}{(2k-4)!!} \;=\; o(E^0)$$

Here use the formula that $\oint_\gamma \frac{u^k}{\sqrt{E-u^2}}du = -2\pi E^{k/2}\frac{(k-1)!!}{k!!}$ for even k.

Adding up the three integrals concludes the proof. □

In the sequel we will also need the following microfunctions μ_j that will be by abuse of language identified with their asymptotic expansions with respect to h:

$$\mu_j \;=\; 1-e^{2\pi i s_{\gamma_j}},\; j \text{ odd}; \quad \mu_j \;=\; 1+e^{-2\pi i s_{\gamma_j}}, j \text{ even}.$$

Using Proposition 2.2 and performing routine simplifications, we obtain that for odd j,

$$\mu_j|_{E=hE_r} \;=\; \frac{E_r\pi i}{f''(q_j)}(1-\frac{a_2(q_j)f''(q_j)}{4}hE_r-\frac{\pi i}{2f''(q_j)}E_r)(1+O(E_r^2)+O(h^2)), \tag{8}$$

and for even j

$$\mu_j|_{E=hE_r} \;=\; \frac{E_r\pi i}{|f''(q_j)|}(1+\frac{a_2(q_j)|f''(q_j)|}{4}hE_r-\frac{\pi i}{2|f''(q_j)|}E_r)(1+O(E_r^2)+O(h^2)). \tag{9}$$

2.7 An application of the Stirling formula

The calculation in this subsection is done for fixed $E>0$.

We have calculated earlier that $s_{\gamma_1}=\frac{\omega_{\gamma_1}}{2\pi h}-1+\beta_1 h$, where $\beta_1=\beta_{1,1}+O(h)$ and $\beta_{1,1}=O(E)$. Since $-s_{\gamma_1}$ has a positive real part which goes to infinity as $h\to 0+$, we can apply the Stirling formula to $\Gamma(-s_{\gamma_1})$ to get

$$\frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})} \;=\; \frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{(-s_{\gamma_1})^{-s_{\gamma_1}-\frac{1}{2}}\cdot e^{s_{\gamma_1}}\cdot\sqrt{2\pi}\cdot(1-\frac{1}{12}s_{\gamma_1}^{-1}+O(h^2))}.$$

A few routine steps of simplification bring us to

$$\frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})} \;=\; \frac{\exp\{-\frac{\omega_{\gamma_1}}{2\pi h}\}}{(-\frac{\omega_{\gamma_1}}{2\pi})^{-\frac{\omega_{\gamma_1}}{2\pi h}+\frac{1}{2}-\beta_1 h}(1-\frac{\pi h}{6\omega_{\gamma_1}})}(1+O(h^2)), \tag{10}$$

where β_1 is $O(E)$ and therefore

$$\frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})} \;=\; \frac{\exp\{-\frac{\omega_{\gamma_1}}{2\pi h}\}}{(-\frac{\omega_{\gamma_1}}{2\pi})^{-\frac{\omega_{\gamma_1}}{2\pi h}+\frac{1}{2}}(1-\frac{\pi h}{6\omega_{\gamma_1}})}(1+O_{E=fix}(h^2)+O(h)O(E\ln E)). \tag{11}$$

2.8 Monodromies from $q_j-\varepsilon$ to $q_{j+1}-\varepsilon$.

Define M_j, M_j' to be the monodromies of the formal WKB solutions along the paths shown on figures 6 and 7, where M_j are taken on the first sheet of the Riemann surface of the classical momentum and M_j' on the second. Note that in [G08] we denoted M_j, M_j' by A_j, A_j'.

Lemma 2.3 *We have:*

$$M_j \;=\; -\exp\left\{\frac{[f(q_{j+1}-\varepsilon)-f(q_j-\varepsilon)]}{h}-\frac{E}{2h}\int_{(q_j-\varepsilon)_I}^{(q_{j+1}-\varepsilon)_I}\frac{dq}{f'(q)}\right\}\sqrt{\frac{|f'(q_j-\varepsilon)|}{|f'(q_{j+1}-\varepsilon)|}}\sqrt[4]{\frac{[f'(q_j-\varepsilon)]^2-E}{[f'(q_{j+1}-\varepsilon)]^2-E}}\times$$

$$\times\exp\left[\frac{E}{8[f'(q_{j+1}-\varepsilon)]^2}-\frac{E}{8[f'(q_j-\varepsilon)]^2}+h(\frac{f''(q_{j+1}-\varepsilon)}{2(f'(q_{j+1}-\varepsilon))^2}-\frac{f''(q_j-\varepsilon)}{2(f'(q_j-\varepsilon))^2})\right](1+O(E^2/h)+O(E^2)+O(Eh)+O(h^2)),$$

where $\int_{(q_j-\varepsilon)_I}^{(q_{j+1}-\varepsilon)_I}$ *means that the integration path lies on the first sheet.*

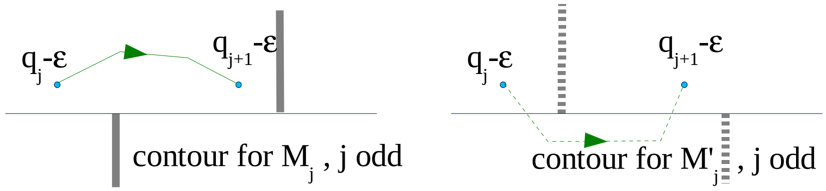

Figure 6: Integration contours defining M_j and M_j', j odd

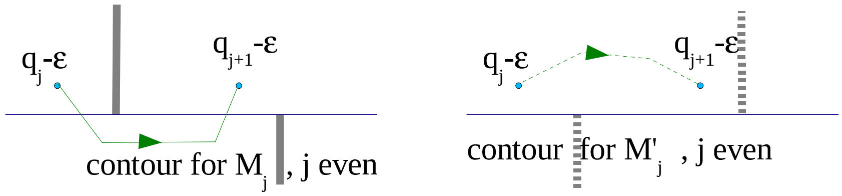

Figure 7: Integration contours defining M_j and M_j', j even

Proof. We will present the argument for M_1, it will be the same for all odd j and very analogous for even j.

We have

$$M_1 = \exp\left\{\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}\left[\frac{i}{h}\sqrt{E-(f')^2}+y_0(q)+hy_1(q)\right]dq+O(h^2)\right\},$$

where $\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}$ means that the path of integration is chosen within the domain of definition of ϕ_+ and determinations of the square root are taken as on the first sheet of the Riemann surface of the classical momentum. We have:

$$\frac{i}{h}\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}\sqrt{E-(f')^2}dq = \frac{i}{h}\int_{q_1-\varepsilon}^{q_2-\varepsilon} f'(q)(1-\frac{1}{2}\frac{E}{(f'(q))^2})dq+O(E^2)$$

$$= \frac{i}{h}\left(f(q_2-\varepsilon)-f(q_1-\varepsilon)-\frac{E}{2}\int_{q_1-\varepsilon}^{q_2-\varepsilon}\frac{dq}{f'(q)}+O(E^2)\right).$$

In the integral

$$\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I} y_0(q)dq = \int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}\left[\frac{f'f''}{2(E-(f')^2)}-\frac{f''}{2i\sqrt{E-(f')^2}}\right]dq$$

the first summand yields

$$\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}\frac{f'f''}{2(E-(f')^2)}dq = -\frac{1}{4}\mathrm{Ln}\,\frac{E-[f'(q_2-\varepsilon)]^2}{E-[f'(q_1-\varepsilon)]^2}$$

and the Ln is analytically continued in the domain with a cut located as on the first sheet.

$$= -\frac{1}{4}\left(\ln\left\{\frac{E-[f'(q_2-\varepsilon)]^2}{E-[f'(q_1-\varepsilon)]^2}\right\}-2\pi i\right) = \frac{\pi i}{2}-\frac{1}{4}\ln\left\{\frac{E-[f'(q_2-\varepsilon)]^2}{E-[f'(q_1-\varepsilon)]^2}\right\},$$

and the second summand

$$\int_{q_1-\varepsilon}^{q_2-\varepsilon} \frac{f''}{2i\sqrt{E-(f')^2}} dq \;=\; \int_{q_1-\varepsilon}^{q_2-\varepsilon} \frac{f''}{2f'}\left(1+\frac{1}{2}\frac{E}{(f')^2}\right) dq \;+\; O(E^2) \;=$$

$$= \;\frac{1}{2}\mathrm{Ln}\;\left(\frac{f'(q_2-\varepsilon)}{f'(q_1-\varepsilon)}\right) \;-\; \frac{E}{8[f'(q)]^2}\Big|_{q=q_1-\varepsilon}^{q=q_2-\varepsilon} \;+\; O(E^2)$$

$$= \;\frac{1}{2}\ln\frac{f'(q_2-\varepsilon)}{[-f'(q_1-\varepsilon)]} - \frac{\pi i}{2} - \frac{E}{8[f'(q_2-\varepsilon)]^2} + \frac{E}{8[f'(q_1-\varepsilon)]^2} + O(E^2).$$

Using (5) and replacing each summand by its limit for $E \to 0$,

$$\int_{(q=q_1-\varepsilon)_I}^{(q=q_2-\varepsilon)_I} y_1(q)dq \;=\; \int_{q=q_1-\varepsilon}^{q=q_2-\varepsilon}\left\{-\frac{5(f')^2(f'')^2}{8(f')^5} - \frac{f'(f'')^2}{2(f')^4} - \frac{(f'')^2}{8(-(f')^3)} - \frac{f' f^{(3)}}{4(-(f')^3)} - \frac{f^{(3)}}{4(-(f')^2)}\right\} dq + O(E) \;=$$

$$= \;\int_{q=q_1-\varepsilon}^{q=q_2-\varepsilon}\left\{-\frac{(f'')^2}{(f')^3} + \frac{f^{(3)}}{2(f')^2}\right\} dq + O(E) \;=\; \frac{f''}{2(f')^2}\Big|_{q=q_1-\varepsilon}^{q=q_2-\varepsilon} + O(E).$$

Hence

$$M_1 \;=\; -e^{\frac{[f(q_2-\varepsilon)-f(q_1-\varepsilon)]}{h} - \frac{E}{2h}\int_{q_1-\varepsilon}^{q_2-\varepsilon}\frac{dq}{f'(q)}} \sqrt{\frac{[-f'(q_1-\varepsilon)]}{f'(q_2-\varepsilon)}}\sqrt[4]{\frac{[f'(q_1-\varepsilon)]^2-E}{[f'(q_2-\varepsilon)]^2-E}}\times$$

$$\times \exp\left[\frac{E}{8[f'(q_2-\varepsilon)]^2} - \frac{E}{8[f'(q_1-\varepsilon)]^2} + h\left(\frac{f''(q_2-\varepsilon)}{2(f'(q_2-\varepsilon))^2} - \frac{f''(q_1-\varepsilon)}{2(f'(q_1-\varepsilon))^2}\right)\right](1+O(E^2/h)+O(E^2)+O(Eh)+O(h^2)),$$

which completes the proof for $j=1$. □

Lemma 2.4 *We have:*

$$M'_j \;=\; \exp\left\{-\frac{[f(q_{j+1}-\varepsilon)-f(q_j-\varepsilon)]}{h} + \frac{E}{2h}\int_{(q_j-\varepsilon)_{II}}^{(q_{j+1}-\varepsilon)_{II}}\frac{dq}{f'(q)}\right\}\sqrt{\frac{|f'(q_{j+1}-\varepsilon)|}{|f'(q_j-\varepsilon)|}}\sqrt[4]{\frac{[f'(q_j-\varepsilon)]^2-E}{[f'(q_{j+1}-\varepsilon)]^2-E}}\times$$

$$\times\exp\left[\frac{E}{8[f'(q_j-\varepsilon)]^2} - \frac{E}{8[f'(q_{j+1}-\varepsilon)]^2}\right](1+O(E^2/h)+O(E^2)+O(Eh)+O(h^2)).$$

where $\int_{(q_j-\varepsilon)_{II}}^{(q_{j+1}-\varepsilon)_{II}}$ *means that the integration path lies within the domain of definition of* ϕ_-*, fig.2, right.*

Two neat formulae easily follow from the above calculation:

Corollary 2.5 *We have*

$$M'_1 M'_2 ... M'_{2n} \;=\; \exp\left(\frac{E}{2h}\oint_{2nd\ sheet}\frac{dq}{f'(q)}\right)(1+O(E^2/h)+O(E^2)+O(Eh)+O(h^2)),$$

or

$$M'_1 M'_2 ... M'_{2n}|_{E=hE_r} \;=\; \exp\left(\frac{E_r}{2}\oint_{2nd\ sheet}\frac{dq}{f'(q)}\right)(1+O(E_r^2 h)+O(h^2)),$$

where $\oint_{2nd\ sheet}$ *means that the integral is taken along a loop from some* q_0 *to* q_0+1 *along a loop lying within the domain of definition of* ϕ_-*.*

Corollary 2.6 *We have*

$$\prod_{j=1}^{2n} M_j^{-1} M_j' \Big|_{E=hE_r} = \exp\left\{ \sum_{j=1}^{2n} \frac{\pi i E_r}{|f''(q_j)|} + E_r \oint_{1st\ sheet} \frac{dq}{f'(q)} \right\} (1 + O(E_r^2 h) + O(h^2)),$$

where $\oint_{1st\ sheet}$ *means that the integral is taken along a loop from some* q_0 *to* $q_0 + 1$ *along a loop lying within the domain of definition of* ϕ_+.

3 Connection coefficient across the double turning point.

We will repeat here the formal calculation of the connection coefficients across the double turning points by the exact matching method done in [G08], while pushing it further to one more order in h.

We will keep the notation the same as in [G08, section VII], except that the connection coefficient called c_1 in [G08, section VII] will now be denoted c_1', consistently with the notation of [G08, sectin VIII]. With this minor difference, we refer to [G08] for notation, terminology, explanations, and literature references relevant to this calculation. We will solve the connection problem in two representative cases – for the double turning point q_1 where $f(q)$ has a local minimum, and for the double turning point q_2 where $f(q)$ has a local maximum; similar results will hold for other real local extrema of $f(q)$.

3.1 Exact matching method around q_1.

Let us consider two formal solutions $\psi_+(q,h)$ and $\psi_-(q,h)$ of (2) corresponding to the first and to the second sheets of the Riemann surface of the classical momentum and normalized in such a way that $\psi_+(q_1 - \varepsilon) = \psi_-(q_1 - \varepsilon) = 1$, where ε is a small positive number.

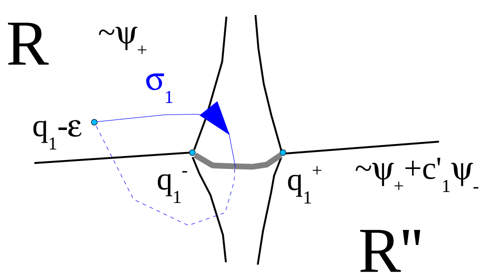

Figure 8: Notation in the exact matching method around q_1

For E a positive real number, the actual solution of (2) represented by ψ_+ in R is represented by $\psi_+ + c_1'\psi_-$ in R''. We know that $c_1'\psi_-$ corresponds to the analytic continuation of ψ_+ along a loop σ_1 with base point $q_1 - \varepsilon$ around the simple turning point $q_1^-(E)$, figure 8, i.e. the Stokes phenomenon transforms ψ_+ into

$$\psi_+(q,E) \ + \ c_1'(E)\psi_-(q,E) \ =$$

$$= \ \psi_+(q,E) \ + \ \sqrt{2\pi}\frac{h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})}(c_1')^{red}(E)\psi_-(q,E),$$

where c_1' is the monodromy of the formal solution along σ_1:

$$c_1' = \exp\left[\int_{\sigma_1} dq\left\{\frac{i}{h}S(q,E) + y_0(q) + hy_1(q) + h^2y_2(q) + ...\right\}\right].$$

and where we have denoted

$$(c_1')^{red} = \frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})}c_1'.$$

For $E = hE_r$ we will obtain

$$\psi_+(q, hE_r) \ + \ \sqrt{2\pi}\left.\frac{h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})}\right|_{E=hE_r}(c_1')^{red}(hE_r)\psi_-(q, hE_r).$$

Remark 3.1 This passage to the limit and replacing $E > 0$ by hE_r has been used in two papers [DDP97] and [DP99], but ideally it would need a more solid mathematical justification. The first issue is purely algebraic: one has to show that the coefficients in the asymptotic expansion of $(c_1')^{red}$ with respect to h are analytic functions of E near the origin, or, equivalently, that all infinitely many coefficients Θ_j, $j \in \mathbb{Z}$, $j \geq -1$, from (12) are analytic in E near the origin. This problem is known in the literature as the Sato's conjecture. For the Schrödinger equation with the harmonic oscillator potential the similar result has been shown in [SS06], and the case of a general potential may perhaps be proven using reduction of an arbitrary potential well to a harmonic oscillator using methods of [AKT09] and references therein. [1] The other issue is analytical: we are not dealing here simply with analytic continuation as h is not a number but a formal parameter, and the functions are not honest analytic functions of h but equivalence classes of those modulo adding a function of h of subexponential decay for $h \to 0$. We hope that these technicalities will be resolved in due time.

Denote

$$\Delta S(E) \ = \ i\int_{\sigma_1}\sqrt{E-(f')^2}dq', \quad \Delta y_k(E) \ = \ \int_{\sigma_1} y_k(q)dq,$$

thus

$$c_1'(E) \ = \ \exp\left\{\frac{1}{h}\Delta S(E) + \Delta y_0(E) + h\Delta y_1(E) + O(h^2)\right\}.$$

Denote further

$$(c_1')^{red}(E) \ = \ \exp\{\sum_{j=-1}^{\infty} h^j\Theta_j(E)\} \ = \ \exp\{\frac{1}{h}\Theta_{-1} + \Theta_0 + h\Theta_1 + O(h^2)\}. \tag{12}$$

Recall that the Stirling formula (10) gives:

$$\frac{\sqrt{2\pi}h^{s_{\gamma_1}+\frac{1}{2}}}{\Gamma(-s_{\gamma_1})} \sim \exp\left\{\frac{1}{h}\left[-\frac{\omega_{\gamma_1}}{2\pi} + \frac{\omega_{\gamma_1}}{2\pi}\text{Ln}\ \left(-\frac{\omega_{\gamma_1}}{2\pi}\right)\right] - \frac{1}{2}\text{Ln}\ \left(-\frac{\omega_{\gamma_1}}{2\pi}\right) + h\frac{\pi}{6\omega_{\gamma_1}} + O_{E=fix}(h^2) + O(E\text{ln}\ E)h\right\},$$

where the notation $O_{E=fix}$ means that the estimate is valid for every fixed $E > 0$. In the above formula the LHS is a true function, and the RHS its hyperasymptotic expansion valid for $E > 0$ and for h in a small sectorial neighborhood of zero in the positive real direction.

[1] We thank Shingo Kamimoto for pointing out to us both of these articles. We thank professors Aoki, Kawai, and Takei for explaining the result of [AKT09] and its significance.

Let us now calculate Θ_{-1}, Θ_0, and Θ_1. Begin with Θ_{-1}.

$$\Theta_{-1} = \Delta S + \frac{\omega_{\gamma_1}}{2\pi} - \frac{\omega_{\gamma_1}}{2\pi}\text{Ln }\left[-\frac{\omega_{\gamma_1}}{2\pi}\right].$$

We have

$$\Delta S = \int_{\sigma_1} i\sqrt{E-(f')^2}dq = 2\int_{q_1-\varepsilon}^{q_1^-}[-\sqrt{(f')^2-E}]dq =$$

(change of variables $u=-f'$, $u=\sqrt{E}\cosh t$, $A=-f'(q_1-\varepsilon)$; $a_j = a_j(q_1)$ were defined in section 2.3)

$$= \ -2\sum_{j=0}^{\infty}(-1)^j a_j\int_A^{\sqrt{E}} u^j\sqrt{u^2-E}du \ = \ -2\sum_{j=0}^{\infty}(-1)^j a_j\int_{\text{arccosh }(A/\sqrt{E})}^{0} E^{\frac{j}{2}+1}\cosh^j t\sinh^2 t dt \ =$$

(use (38), (4) and (36))

$$= \ 2[f(q_1)-f(q_1-\varepsilon)] + a_0(-\frac{E}{2} - \frac{E^2}{8A^2} - E(\text{Ln }\frac{2A}{\sqrt{E}} - \frac{E}{4A^2})) \ + \ a_2(\frac{A^2E}{2} - \frac{E^2}{16} + \frac{E^2}{4}\text{Ln }(\frac{2A}{\sqrt{E}}))+$$

$$+ \sum_{j\,=\,1\text{ or }\geq 3}(-1)^j a_j\left(-\frac{A^jE}{j} - \frac{A^{j-2}E^2}{4(j-2)}\right) + o(E^2,)$$

and hence

$$\Theta_{-1} = 2[f(q_1)-f(q_1-\varepsilon)] - a_0E\text{Ln }(2A) + a_0\frac{E^2}{8A^2} - a_2\frac{A^2}{2}E - \sum_{j=1\text{ or }j\geq 3} a_j(-1)^j\left\{\frac{1}{j}A^jE + \frac{1}{4(j-2)}E^2A^{j-2}\right\} +$$

$$-\ \frac{E}{2f''(q_1)}\text{Ln }(2f''(q_1)) \ + \ \frac{a_2E^2}{16} \ + \ \frac{a_2E^2}{8}\text{Ln }(2f''(q_1)) \ - \ a_2\frac{E^2}{4}\text{Ln }2A + o(E^2).$$

The expressions for Θ_0 and Θ_1 are easier using formulas from sections 2.4 and 2.5, we obtain:

$$\Theta_0 \ = \ \text{arccosh }\frac{A}{\sqrt{E}} \ - \ \frac{\pi i}{2} \ + \ \frac{1}{2}\text{Ln }\left(-\frac{\omega_{\gamma_1}}{2\pi}\right) \ =$$

$$= \ \text{Ln }2A - \frac{E}{4A^2} \ - \ \frac{\pi i}{2} + \frac{1}{2}\left[\text{Ln }(\frac{1}{2f''(q_1)}) - \frac{a_2f''(q_1)E}{4}\right] + o(E),$$

and

$$\Theta_1 \ = \ \int_{\sigma_1} y_1(q)dq - \frac{\pi}{6\omega_{\gamma_1}} + o(E^0) \ = \ -\frac{1}{12}b_2 - \frac{1}{2}\frac{f''(q_1-\varepsilon)}{(f'(q_1-\varepsilon))^2} \ + \ \frac{a_2[f''(q_1)]^2}{24} + o(E^0).$$

By lemma 3.6, this $o(E^0)$ is actually $O(E)$.

As we pointed out in Remark 3.1, there must be a conceptual way of proving analyticity of $\Theta_j(E)$ near $E=0$ for all $j\geq -1$; for now we will prove analyticity of $\Theta_{-1},\Theta_0,\Theta_1$ directly in section 3.3.

Note that the infinite sums appearing in the expression for Θ_{-1} for specific $f(q)$ can be evaluated by integration:

$$\sum_{j=1}^{\infty} a_j(-1)^j \frac{1}{j} A^j \;=\; \int_{q_1}^{q_1-\varepsilon} (-\frac{1}{f'(q)} - a_0 \frac{f''(q)}{f'(q)}) dq, \tag{13}$$

$$\sum_{j=3}^{\infty} a_j(-1)^j \frac{1}{j-2} A^{j-2} \;=\; \int_{q_1}^{q_1-\varepsilon} \left[-\frac{1}{[f'(q)]^3} - \frac{a_0 f''(q)}{[f'(q)]^3} - \frac{a_1 f''(q)}{[f'(q)]^2} - \frac{a_2 f''(q)}{f'(q)} \right] dq. \tag{14}$$

Substituting $E_r h$ for E in the results that we have just obtained and using

$$s_{\gamma_1}|_{E=E_r h} + \frac{1}{2} = -\frac{E_r}{2f''(q_1)} - \frac{1}{2} + O(h^2) + O(E_r h),$$

we get:

$$c_1'|_{E=hE_r} \;=\; \frac{\sqrt{2\pi} h^{-\frac{E_r}{2f''(q_1)} - \frac{1}{2} + O(E_r h) + O(h^2)}}{\Gamma(\frac{E_r}{2f''(q)} + 1 + O(hE_r) + O(h^2))} \exp\left(\frac{1}{h}\Theta_{-1} + \Theta_0 + h\Theta_1 + O(h^2) \right)_{E=hE_r} \;=$$

$$= \;\frac{\sqrt{2\pi} h^{-\frac{E_r}{2f''(q_1)} - \frac{1}{2}}}{\Gamma(\frac{E_r}{2f''(q)} + 1)} \exp\left(\frac{1}{h}\Theta_{-1} + \Theta_0 + h\Theta_1 \right)_{E=hE_r} (1 + O(h^2) + O(E_r h \ln h) + O(E_r^2 h) + O(E_r^2))$$

Finally, we will use the well known formula

$$\frac{1}{\Gamma(1+t)} \;=\; \frac{1}{t\Gamma(t)} \;=\; \frac{1}{t}(t + \underline{\gamma} t^2 + O(t^3)) = 1 + \underline{\gamma} t + o(t), \quad \text{as} \quad t \to 0,$$

where $\underline{\gamma} = 0.5772...$ is the Euler-Mascheroni constant, and obtain

Proposition 3.2 *For the differential equation* (2) *with* $E = hE_r$ *the connection coefficient* c_1' *equals*

$$c_1' = -i\sqrt{2\pi} h^{-\frac{E_r}{2f''(q_1)} - \frac{1}{2}} (1 + \underline{\gamma} \frac{E_r}{2f''(q_1)}) \times$$

$$\times \exp\left[\frac{2[f(q_1) - f(q_1 - \varepsilon)]}{h} + (1 + \frac{E_r}{f''(q_1)}) \text{Ln} \frac{2[-f'(q_1 - \varepsilon)]}{\sqrt{2f''(q_1)}} - \sum_{j=1}^{\infty} a_j \frac{1}{j} [f'(q_1 - \varepsilon)]^j E_r + \right.$$

$$\left. + h \left(-\frac{b_2}{12} - \frac{1}{2} \frac{f''(q_1 - \varepsilon)}{(f'(q_1 - \varepsilon))^2} + \frac{a_2 (f''(q_1))^2}{24} \right) \right]$$

$$\times (1 + O(h^2) + O(E_r h \ln h) + O(E_r^2 h) + O(E_r^2)),$$

where $a_j = a_j(q_1)$, $b_2 = b_2(q_1)$ *were defined in section 2.3.*

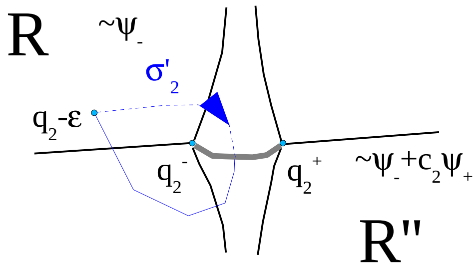

Figure 9: Notation in the exact matching method around q_2

3.2 Exact matching method around q_2.

In this subsection, two formal solutions $\psi_+(q,h)$ and $\psi_-(q,h)$ of (2) correspond to the first and to the second sheets of the Riemann surface of the classical momentum and are normalized in such a way that $\psi_+(q_2-\varepsilon)=\psi_-(q_2-\varepsilon)=1$, where ε is a small positive number.

For E a positive real number, the actual solution represented by ψ_- in R is represented by $\psi_-+c_2\psi_+$ in R'', where $c_2\psi_+$ corresponds to the analytic continuation of ψ_- along a contour σ_2' with base point $q_2-\varepsilon$ around the simple turning point $q_2^-(E)$, figure 9. Thus for $E=hE_r$ the connection coefficient c_2 is the limit of the formal monodromy along the path σ_2'. In other words, the Stokes phenomenon transforms $\psi_-(q,E)$ into

$$c_2(E)\psi_+(q,E) \;+\; \psi_-(q,E) \;=\; \sqrt{2\pi}\frac{h^{s_{\gamma_2'}+\frac{1}{2}}}{\Gamma(-s_{\gamma_2'})}c_2^{red}(E)\psi_+(q,E) \;+\; \psi_-(q,E),$$

where

$$c_2=\exp\left[\int_{\sigma_2'}dq\left\{\frac{1}{h}S(q,E)+y_0(q)+hy_1(q)+O(h^2)\right\}\right]$$

and where c_2^{red} analytically and regularly depends on E. With the same caveat as in the previous subsection, we will substitute E_rh for E and obtain

$$c_2(hE_r)\psi_+(q,hE_r) \;+\; \psi_-(q,hE_r) \;=\; \sqrt{2\pi}\left.\frac{h^{s_{\gamma_2'}+\frac{1}{2}}}{\Gamma(-s_{\gamma_2'})}\right|_{E=hE_r} c_2^{red}(hE_r)\psi_+(q,hE_r) \;+\; \psi_-(q,hE_r).$$

Together with calculations of the monodromy of the formal solution along the contour σ_2', we will use the following asymptotic expansion valid for a fixed $E>0$ and derived from the Stirling formula: The Stirling formula (10) together with (6) give:

$$\frac{\sqrt{2\pi}h^{s_{\gamma_2'}+\frac{1}{2}}}{\Gamma(-s_{\gamma_2'})}\sim\exp\left\{\frac{1}{h}\left[\frac{\omega_{\gamma_2}}{2\pi}-\frac{\omega_{\gamma_2}}{2\pi}\mathrm{Ln}\left(\frac{\omega_{\gamma_2}}{2\pi}\right)\right]-\frac{1}{2}\mathrm{Ln}\left(\frac{\omega_{\gamma_2}}{2\pi}\right)-h\frac{\pi}{6\omega_{\gamma_2}}+O_{E=fix}(h^2)+O(E\ln E)h\right\}.$$

Introduce Θ_j's for $j\geq -1$ by formula

$$c_2^{red} \;=\; \exp\{\frac{1}{h}\Theta_{-1}+\Theta_0+h\Theta_1+...\}$$

and so, similarly to the previous subsection, with $A=-f'(q_2-\varepsilon)$, $a_j=a_j(q_2)$, $b_j=b_j(q_2)$, we have

$$\Theta_{-1} \;=\; i\int_{\sigma_2'} p(q,E)dq \;-\; \frac{\omega_{\gamma_2}}{2\pi}+\frac{\omega_{\gamma_2}}{2\pi}\mathrm{Ln}\;\left(\frac{\omega_{\gamma_2}}{2\pi}\right)$$

$$=\; 2S_{crit}(q_2)+Ea_0\ln\,(-2A)-a_0\frac{E^2}{8A^2}+\left(\frac{Ea_0}{2}+\frac{E^2a_2}{8}\right)\mathrm{Ln}\;\frac{a_0}{2}$$

$$+a_2\frac{A^2E}{2}-a_2\frac{E^2}{16}+a_2\frac{E^2}{4}\mathrm{Ln}\;(-2A)+E\sum_{j\geq 1,\;j\neq 2}^{\infty}a_j(-1)^j\left(\frac{1}{j}A^j+\frac{1}{4(j-2)}A^{j-2}E\right)+o(E^2);$$

$$\Theta_0 \;=\; \int_{\sigma_2'} y_0(q,E)dq \;-\; \frac{1}{2}\mathrm{Ln}\;\left(\frac{\omega_{\gamma_2}}{2\pi}\right) \;=$$

$$=\; -\mathrm{Ln}\;i-\mathrm{Ln}\;\frac{(-2A)}{\sqrt{2|f''(q_2)|}}+\frac{E}{4A^2}+\frac{a_2|f''(q_2)|\cdot E}{8}+o(E);$$

$$\Theta_1 \;=\; \int_{\sigma_2'} y_1(q,E)dq \;+\; \frac{\pi}{6\omega_{\gamma_2}}+o(E^0),$$

$$\Theta_1 \;=\; \frac{1}{12}b_2-\frac{1}{2}\sum_{j=0}^{\infty}b_jA^{j-2} \;-\; \frac{a_2[f''(q_2)]^2}{24}+O(E^0).$$

Now we are almost ready combine these formulae and calculate

$$c_2(hE_r) \;=\; \sqrt{2\pi}\left.\frac{h^{s_{\gamma_2'}+\frac{1}{2}}}{\Gamma(-s_{\gamma_2'})}\right|_{E=hE_r} c_2^{red}(hE_r).$$

Notice that when $E_r=0$, then $e^{-f(q)/h}$ is the formal solution of (2) corresponding to the second sheet determination of the classical momentum, and also the actual solution of (2), and therefore for $E_r=0$ the connection coefficient c_2 must vanish. This implies that $1/\Gamma(-s_{\gamma_2'})$ is divisible by E_r. Hence, using

$$s_{\gamma_2'}+\frac{1}{2}=\frac{E_r}{2f''(q_2)}+\frac{1}{2}+O(E_r^2)+O(hE_r),$$

we can write

$$\frac{1}{\Gamma(-s_{\gamma_2'})} \;=\; -\frac{E_r}{2f''(q_2)}+\underline{\gamma}\frac{E_r^2}{[2f''(q_2)]^2}+O(E_r^3)+h\times O(E_r).$$

Finally, we have obtained:

Proposition 3.3 *For the differential equation* (2) *the connection coefficient* c_2 *equals*

$$c_2=\sqrt{2\pi}h^{\frac{E_r}{2f''(q_2)}+\frac{1}{2}}\left(-\frac{E_r}{2f''(q_2)}+\underline{\gamma}\frac{E_r^2}{[2f''(q_2)]^2}+O(E_r^3)+O(h^2)O(E_r)\right)\times$$

$$\times\exp[2\frac{-f(q_2)+f(q_2-\varepsilon)}{h}-(1+\frac{E_r}{f''(q_2)})\ln\;\frac{2f'(q_2-\varepsilon)}{\sqrt{-2f''(q_2)}}+E_r\sum_{j=1}^{\infty}a_j\frac{1}{j}[f'(q_2-\varepsilon)]^j-\mathrm{Ln}\;i]\times$$

$$\times\exp\left[h\left(\frac{1}{12}b_2-\frac{1}{2}\sum_{j=0}^{\infty}b_j[-f'(q_2-\varepsilon)]^{j-2} \;-\; \frac{a_2[f''(q_2)]^2}{24}+o(E^0)\right)+O(h^2)\right],$$

where $a_j=a_j(q_2)$, $b_2=b_2(q_2)$ *were defined in section 2.3.*

3.3 A partial proof of analyticity of $(c_1')_{red}$

The purpose of this section is not to give a full justification of the methods, but rather to prove some results confirming that our approach is consistent and makes sense.

Lemma 3.4 *The quantity*

$$\Theta_{-1} \;=\; \Delta S + \frac{\omega_{\gamma_1}}{2\pi} - \frac{\omega_{\gamma_1}}{2\pi}\ln\,[-\frac{\omega_{\gamma_1}}{2\pi}]$$

is analytic with respect to E in the neighborhood of zero.

PROOF. We need to show that the term containing $\ln E$ in ΔS cancels $\frac{\omega_{\gamma_1}}{2\pi}\mathrm{Ln}\,E$. Indeed, we have seen that

$$\Delta S = -2\sum_{k=0}^{\infty}(-1)^k a_k \int_A^{\sqrt{E}} u^k\sqrt{u^2-E}du = -2\sum_{k=0}^{\infty}(-1)^k a_k \int_{\mathrm{arccosh}\;A/\sqrt{E}}^{0} E^{\frac{k}{2}+1}\cosh^k t\sinh^2 t dt$$

$$= \;-2\sum_{k=0}^{\infty}(-1)^k a_k \int_{\mathrm{arccosh}\;A/\sqrt{E}}^{0} 2^{-k-2}E^{\frac{k}{2}+1}(e^t+e^{-t})^k(e^t-e^{-t})^2dt$$

Writing the integrand as the sum of exponents and integrating, we realize that only the summand $e^{0t}dt$ will eventually give rise to a logarithmic singularity for $E\to 0$. Writing $reg(E)$ for an arbitrary function that is analytic with respect to E near the origin, we have:

$$\Theta_{-1} \;=\; reg(E) \;-\; 2\sum_{k=2j=0}^{\infty} a_{2j}\int_{\mathrm{arccosh}\;A/\sqrt{E}}^{0} 2^{-2j-2}E^{j+1}(C_{2j}^{j-1}-2C_{2j}^{j}+C_{2j}^{j+1})dt$$

$$=\; reg(E) \;-\; \sum_{k=2j=0}^{\infty} a_{2j}2^{-2j}E^{j+1}\frac{(2j)!}{j!j!}(1-\frac{j}{j+1})\mathrm{arccosh}\;(\frac{A}{\sqrt{E}}) \;=$$

$$=\; reg(E) \;+\; \sum_{k=2j=0}^{\infty} a_{2j}\frac{(2j-1)!!}{(2j+2)!!}E^{j+1}\mathrm{Ln}\;E.$$

The singularity that comes out of $\frac{\omega_{\gamma_1}}{2\pi}\mathrm{Ln}\;E$ is the same by formula (7). □

Lemma 3.5 Θ_0 *is analytic for E around* 0.

PROOF is obvious from (36) and proposition 2.2 □

Lemma 3.6 Θ_1 *is analytic for E around* 0

PROOF. The fact that Θ_1 has no pole (i.e. $\frac{1}{E}$) singularity has been demonstrated in section 3.1. Now let us check that all logarithmic singularities $E^k\mathrm{Ln}\;E$ are absent in Θ_1 as well. The question reduced to identifying the logarithmic singularity in the integral along the contour σ, fig.4:

$$\int_\sigma \frac{\sum_{j=0}^{\infty}(j+1)[-jb_{j+2}]u^j}{24\sqrt{u^2-E}}dq \;=$$

$$= \frac{1}{12}\sum_{j=0}^{\infty} E^{\frac{j}{2}}(j+1)[-jb_{j+2}]\int_{\mathrm{arccosh}\ (A/\sqrt{E})}^{0}\cosh^j(t)dt =$$

$$reg(E) + \frac{1}{12}\sum_{j=2k\geq 0} E^{\frac{j}{2}}(j+1)[-jb_{j+2}]2^{-j}C_j^{(j/2}(-\mathrm{arccosh}\ (A/\sqrt{E}))$$

$$= reg(E) - \frac{1}{24}\sum_{k\geq 0} E^k b_{2k+2}\frac{(2k+1)!!}{(2k-2)!!}\mathrm{Ln}\ E.$$

The logarithmic singularity in the h^1 term of Ln $\frac{\sqrt{2\pi}h^{s_\gamma+\frac{1}{2}}}{\Gamma(-s_\gamma)}$ comes from $-\beta\mathrm{Ln}\ \left(-\frac{\omega_{\gamma_1}}{2\pi}\right)$, i.e. from $-\beta\mathrm{Ln}\ E$, where

$$\beta = \frac{1}{2\pi}\int_{\gamma_1} y_1 dq = \frac{1}{24}\sum_{k\geq 2}^{\infty} b_{2k}E^{k-1}\frac{(2k-1)!!}{(2k-4)!!}.$$

That means that $E^k\mathrm{Ln}\ E$ terms in Θ_1 cancel for all k. □

3.4 Calculations of τs

We define

$$\text{for odd } j: \quad \tau_j = c'_j c_{j+1} M_j^{-1} M'_j,$$

$$\text{for even } j: \quad \tau_j = c_j c'_{j+1} M_j (M'_j)^{-1}.$$

Calculation of τ_1. Using lemmas 2.3, 2.4 and propositions 3.2, 3.3, and inserting $E = hE_r$ into the corresponding formulas, obtain after routine simplification:

$$\tau_1 = e^{2\frac{f(q_1)-f(q_2)}{h}}\frac{E_r\pi}{\sqrt{|f''(q_2)|\cdot f''(q_1)}}h^{-\frac{E_r}{2f''(q_1)}+\frac{E_r}{2f''(q_2)}}(1+\underline{\gamma}\frac{E_r}{2f''(q_1)}+\underline{\gamma}\frac{E_r}{2|f''(q_2)|})\times$$

$$\times\exp\left[\frac{E_r}{f''(q_1)}\ln\frac{2(-f'(q_1-\varepsilon))}{\sqrt{2f''(q_1)}} - E_r\sum_{j=1}^{\infty}a_j(q_1)\frac{1}{j}(f'(q_1-\varepsilon))^j + h\left(-\frac{b_2(q_1)}{12}+\frac{a_2(q_1)(f''(q_1))^2}{24}\right)\right]$$

$$\times\exp\left[-\frac{E_r}{f''(q_2)}\ln\frac{2f'(q_2-\varepsilon)}{\sqrt{-2f''(q_2)}} + E_r\sum_{j=1}^{\infty}a_j(q_2)\frac{1}{j}(f'(q_2-\varepsilon))^j + h\left(\frac{b_2(q_2)}{12}-\frac{a_2(q_2)[f''(q_2)]^2}{24}\right)\right]\times$$

$$\times\exp\left[E_r\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}\frac{dq}{f'(q)} + \frac{\pi i E_r}{f''(q_1)}\right](1+O(E_r^2)+O(E_r h\ln\ h)+O(h^2)).$$

Here $\int_{(q_1-\varepsilon)_I}^{(q_2-\varepsilon)_I}$ denotes the integral along a path lying within the domain of definition of ϕ_+, fig.2. Formulae for other τ_j wih odd j are analogous.

Calculation of τ_2. Analogously,

$$\tau_2 = e^{2\frac{f(q_3)-f(q_2)}{h}}\frac{E_r\pi}{\sqrt{|f''(q_2)|\cdot f''(q_3)}}h^{\frac{E_r}{2f''(q_2)}-\frac{E_r}{2f''(q_3)}}(1+\underline{\gamma}\frac{E_r}{2|f''(q_2)|}+\underline{\gamma}\frac{E_r}{2f''(q_3)})\times$$

$$\times \exp\left[-\frac{E_r}{f''(q_2)}\ln\,\frac{2f'(q_2-\varepsilon)}{\sqrt{-2f''(q_2)}}+E_r\sum_{j=1}^{\infty}a_j(q_2)\frac{1}{j}(f'(q_2-\varepsilon))^j+h\left(\frac{1}{12}b_2(q_2)\;-\;\frac{a_2(q_2)[f''(q_2)]^2}{24}\right)\right]\times$$

$$\exp\left[\frac{E_r}{f''(q_3)}\ln\,\frac{-2f'(q_3-\varepsilon)}{\sqrt{2f''(q_3)}}-E_r\sum_{j=1}^{\infty}a_j(q_3)\frac{1}{j}(f'(q_3-\varepsilon))^j+h\left(-\frac{b_2(q_3)}{12}+\frac{a_2(q_3)(f''(q_3))^2}{24}\right)\right]$$

$$\times \exp\left\{-E_r\int_{(q_2-\varepsilon)_I}^{(q_3-\varepsilon)_I}\frac{dq}{f'(q)}\;-\;\frac{\pi i E_r}{f''(q_2)}\right\}(1+O(E_r^2)+O(E_r h\ln\,h)+O(h^2)),$$

and similarly for other τ_j wih even j.

In a calculation of these monodromies for a specific $f(q)$ we can use formula (13).

4 A procedure for calculating eingenfunctions

In [G08], section VIII we have introduced a transfer matrix $F(E_r)$ and a related matrix G_0 and we wrote the quantization condition as

$$\ker(G_0-\frac{1}{1+E_r k}Id)\;\neq\{0\}. \tag{15}$$

In the case of f having two local minima and two local maxima on the period, the matrix G_0 has the following explicit form:

$$G_0\;=\;\begin{pmatrix}\tau_4 & \\ & 1\end{pmatrix}\begin{pmatrix}\tau_3^{-1}+1 & \mu_3\tau_3^{-1}+1\\ \mu_4\tau_3^{-1}+1 & \mu_3\mu_4\tau_3^{-1}+1\end{pmatrix}\begin{pmatrix}\tau_2 & \\ & 1\end{pmatrix}\begin{pmatrix}\tau_1^{-1}+1 & \mu_1\tau_1^{-1}+1\\ \mu_2\tau_1^{-1}+1 & \mu_1\mu_2\tau_1^{-1}+1\end{pmatrix}\;=$$

$$=\begin{pmatrix}\tau_4(\tau_3^{-1}+1)\tau_2(\tau_1^{-1}+1)+\tau_4(\mu_3\tau_3^{-1}+1)(\mu_2\tau_1^{-1}+1) & \tau_4(\tau_3^{-1}+1)\tau_2(\mu_1\tau_1^{-1}+1)+\tau_4(\mu_3\tau_3^{-1}+1)(\mu_1\mu_2\tau_1^{-1}+1)\\ (\mu_4\tau_3^{-1}+1)\tau_2(\tau_1^{-1}+1)+(\mu_3\mu_4\tau_3^{-1}+1)(\mu_2\tau_1^{-1}+1) & (\mu_4\tau_3^{-1}+1)\tau_2(\mu_1\tau_1^{-1}+1)+(\mu_3\mu_4\tau_3^{-1}+1)(\mu_1\mu_2\tau_1^{-1}+1)\end{pmatrix} \tag{16}$$

Assume hE_r is a low-lying resurgent eigenvalue of our Witten Laplacian.

Suppose a vector of resurgent symbols $(Z_+^{(0)},Z_-^{(0)})^T$ belongs to the kernel (15). Then the vector

$$\begin{pmatrix}D_+^{(0)}\\ D_-^{(0)}\end{pmatrix}\;=\;\begin{pmatrix}B_0'B_0^{-1}(c_1')^{-1}Z_+^{(0)}\\ Z_-^{(0)}\end{pmatrix}$$

belongs to $\ker(F(E_r)-Id)=0$ and thus the eigenfunction corresponding to the resurgent eigenvalue hE_r will be representable, for $q\in(0,q_1)$, by a hyperasymptotic expansion

$$D_+^{(0)}\phi_+ + D_-^{(0)}\phi_-,$$

where ϕ_+ and ϕ_- are formal solutions of the Witten Laplacian normalized by $\phi_+(q_0)=\phi_-(q_0)=1$ for some $q_0\in\mathbb{C}$, $f'(q_0)\neq 0$.

Now we will write down the expressions for the coefficients $D_+^{(j)}, D_-^{(j)}$, for $j=1,2,3,4$, such that

$$D_+^{(j)}\phi_+ + D_-^{(j)}\phi_-$$

represents the same eigenfunction on the interval (q_j, q_{j+1}). These coefficients will be written in terms of auxiliary coefficients $Z_+^{(j)}, Z_-^{(j)}$ and the calculation will consist in successive application of connection matrices across the turning points q_j given in [G08], section VIII.

We have:

$$\begin{pmatrix} D_+^{(1)} \\ D_-^{(1)} \end{pmatrix} = B_0' \begin{pmatrix} B_0^{-1}(c_1')^{-1} Z_+^{(1)} \\ (B_0')^{-1} Z_-^{(1)} \end{pmatrix}; \quad \begin{pmatrix} Z_+^{(1)} \\ Z_-^{(1)} \end{pmatrix} = \begin{pmatrix} Z_+^{(0)} + \mu_1 Z_-^{(0)} \\ Z_+^{(0)} + Z_-^{(0)} \end{pmatrix}; \tag{17}$$

$$\begin{pmatrix} D_+^{(2)} \\ D_-^{(2)} \end{pmatrix}; = B_0' M_1' \begin{pmatrix} B_0 M_1 & 0 \\ 0 & B_0' M_1' \end{pmatrix}^{-1} \begin{pmatrix} c_2 Z_+^{(2)} \\ Z_-^{(2)} \end{pmatrix}; \quad \begin{pmatrix} Z_+^{(2)} \\ Z_-^{(2)} \end{pmatrix} = \begin{pmatrix} \tau_1^{-1} Z_+^{(1)} + Z_-^{(1)} \\ \mu_2 \tau_1^{-1} Z_+^{(1)} + Z_-^{(1)} \end{pmatrix}; \tag{18}$$

$$\begin{pmatrix} D_+^{(3)} \\ D_-^{(3)} \end{pmatrix} = B_0' M_1' M_2' \begin{pmatrix} B_0 M_1 M_2 & 0 \\ 0 & B_0' M_1' M_2' \end{pmatrix}^{-1} \begin{pmatrix} (c_3')^{-1} Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix}; \quad \begin{pmatrix} Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix} = \begin{pmatrix} \tau_2 Z_+^{(2)} + \mu_3 Z_-^{(2)} \\ \tau_2 Z_+^{(2)} + Z_-^{(2)} \end{pmatrix}; \tag{19}$$

$$\begin{pmatrix} D_+^{(4)} \\ D_-^{(4)} \end{pmatrix} = B_0' M_1' M_2' M_3' \begin{pmatrix} B_0 M_1 M_2 M_3 & 0 \\ 0 & B_0' M_1' M_2' M_3' \end{pmatrix}^{-1} \begin{pmatrix} c_4 Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix}; \quad \begin{pmatrix} Z_+^{(4)} \\ Z_-^{(4)} \end{pmatrix} = \begin{pmatrix} \tau_3^{-1} Z_+^{(3)} + Z_-^{(3)} \\ \mu_4 \tau_3^{-1} Z_+^{(3)} + Z_-^{(3)} \end{pmatrix}. \tag{20}$$

Remark also that if E_r is an eigenvalue of the Witten Laplacian and if for $j = 0, ..., 4$ the hyperasymptotic expansions $D_+^{(j)} \phi_+(q) + D_-^{(j)} \phi_-(q)$ define its eigenfunction $\psi(q)$ satisfying $\psi(q) = \psi(q+1)$, then we must have

$$[M_1 M_2 M_3 M_4]^{-1} D_+^{(0)} = D_+^{(4)}; \quad [M_1' M_2' M_3' M_4']^{-1} D_-^{(0)} = D_-^{(4)}.$$

Rewriting this condition in terms of $Z_\pm^{(j)}$, we arrive at

$$Z_+^{(0)} = (1 + E_r k) c_1' c_4 (M_4')^{-1} M_4 Z_+^{(4)}; \quad Z_-^{(0)} = (1 + E_r k) Z_-^{(4)}. \tag{21}$$

If $Z_\pm^{(j)}$ are calculated without algebraic mistakes, they must satisfy the formulae (21).

5 Quantization condition with subdominant terms – Example 1.

Notation. For a resurgent symbol ϕ we will write $\phi \tilde{\sim} e^{\frac{\alpha}{h}}$ if $-\alpha$ is the location of the left-most nonzero microfunction in the decomposition of ϕ, or, informally, if $e^{\frac{\alpha}{h}}$ is the leading exponential in ϕ. We will denote by $\mathcal{E}^a$ the set of resurgent symbols or corresponding resurgent functions of exponential type $\leq a$ in h, i.e. of those whose majors have no singularities left of the vertical line Re $\xi = -a$.

In order to solve the quantization condition

$$\left(\frac{1}{1 + E_r k} \right)^2 - \mathrm{Tr} G_0 \frac{1}{1 + E_r k} + \det G_0 = 0 \tag{22}$$

for the rescaled energy E_r, it is important to understand the determinant and the trace of the matrix G_0.

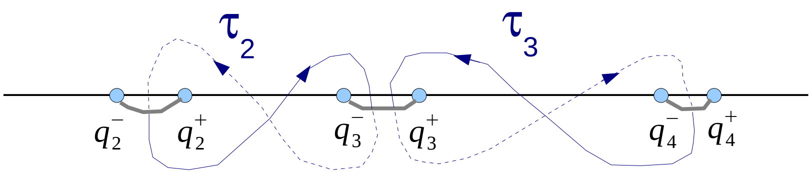

Figure 10: Contours defining τs for $E > 0$.

Take as the superpotential

$$f = \frac{1}{2\pi}\left[\sin 2\pi(q+\frac{1}{8}) + \cos 4\pi(q+\frac{1}{8})\right], \tag{23}$$

$$f' = \cos 2\pi(q+\frac{1}{8}) - 2\sin 4\pi(q+\frac{1}{8}).$$

The critical points of f are all real in this case:

$$\begin{array}{ccc} q_1 = \frac{1}{8} & f(q_1) = 0 & f''(q_1) = 6\pi, \\ q_2 = \frac{3}{8} - \frac{1}{2\pi}\arcsin\frac{1}{4} & f(q_2) = \frac{9}{16\pi} & f''(q_2) = -7.5\pi, \\ q_3 = \frac{5}{8} & f(q_3) = -\frac{1}{\pi} & f''(q_3) = 10\pi, \\ q_4 = \frac{7}{8} + \frac{1}{2\pi}\arcsin\frac{1}{4} & f(q_4) = \frac{9}{16\pi} & f''(q_4) = -7.5\pi. \end{array}$$

Now we are going to exploit the symmetry of the superpotential (23).

Lemma 5.1 *Suppose f has two local minima q_1, q_3 and two local maxima q_2, q_4 and satisfies $f(q) = f(2q_3 - q)$. Then*

$$\tau_1 = \tau_4, \quad \tau_2 = \tau_3, \quad \mu_2 = \mu_4.$$

Proof. Observe that for E real, the equation $P\psi = E\psi$ has two real solutions, i.e. those satisfying

$$\psi(\bar{q}) = \overline{\psi(q)}, \tag{24}$$

therefore the same equality must be satisfied by any solution of this equation. Observe, furher, that reflecting a contour δ on the Riemann surface of the classical momentum with respect to the real line while keeping it on the same sheet of the Riemann surface changes the monodromy of a formal solution satisfying (24) by a complex conjugation. (the monodromy changes from $\frac{\psi(\delta(1))}{\psi(\delta(0))}$ to $\frac{\psi(\overline{\delta(1)})}{\psi(\overline{\delta(0)})} = \overline{\frac{\psi(\delta(1))}{\psi(\delta(0))}}$, where $\delta(0)$ and $\delta(1)$ are the endpoint of δ.

When f is a real trigonometric polynomial, $E > 0$, $h > 0$, then τ_j's are also real, and so flipping the contours (see fig. 10) defining them with respect to the real axis will give rise to the same formal mondromies.

Now if we reflect an integration contour δ with respect to the point q_3, we obtain an integration path that we will denote $2q_3 - \delta$. Notice that if $\psi(q)$ is a formal WKB solution of $P\psi = E\psi$, then so is $\psi(2q_3 - q)$, and both solutions correspond to the same sheet of the Riemann surface of the classical momentum (because both are either exponentially growing or exponentially decreasing in the direction away from q_3 along the real line). Therefore, if we reflect a contour δ with respect to q_3 while keeping it on the same sheet, the formal monodromies along that contour will remain the same.

The lemma follows from the fact that the contour defining τ_3 can be obtained from that defining τ_2 by reflecting it with respect to q_3 and then reflecting it with respect to the real axis. Analogously for τ_1 and τ_4.

Similarly, $s_{\gamma_2} = s_{\gamma_4}$ and hence $\mu_2 = \mu_4$ if $f(q) = f(2q_3 - q)$. $\square$

Recall that

$$G_0 = \begin{pmatrix} \tau_4(\tau_3^{-1}+1) & \tau_4(\mu_3\tau_3^{-1}+1) \\ \mu_4\tau_3^{-1}+1 & \mu_3\mu_4\tau_3^{-1} \end{pmatrix} \begin{pmatrix} \tau_2(\tau_1^{-1}+1) & \tau_2(\mu_1\tau_1^{-1}+1) \\ \mu_2\tau_1^{-1}+1 & \mu_1\mu_2\tau_1^{-1} \end{pmatrix}$$

and

$$\begin{array}{rcrrrr} TrG_0 & = & \tau_4\tau_3^{-1}\tau_2\tau_1^{-1} & +\tau_4\tau_2\tau_3^{-1} & +\tau_4\tau_2\tau_1^{-1} & +\tau_2\tau_4 \\ & & +\mu_2\mu_3\tau_4\tau_3^{-1}\tau_1^{-1} & +\mu_3\tau_4\tau_3^{-1} & +\mu_2\tau_4\tau_1^{-1} & +\tau_4 \\ & & +\mu_1\mu_4\tau_2\tau_3^{-1}\tau_1^{-1} & +\mu_4\tau_2\tau_3^{-1} & +\mu_1\tau_2\tau_1^{-1} & +\tau_2 \\ & & +\mu_1\mu_2\mu_3\mu_4\tau_1^{-1}\tau_3^{-1} & +\mu_1\mu_2\tau_1^{-1} & +\mu_3\mu_4\tau_3^{-1} & +1. \end{array} \tag{25}$$

Using the formulae for τ_k,

$$\tau_1 = \tau_4 \hat{\sim} e^{-\frac{9}{8\pi h}} E_r, \quad \tau_2 = \tau_3 \hat{\sim} e^{-\frac{25}{8\pi h}} E_r,$$

and therefore we can write, loosely,

$$\begin{array}{rcrrrr} TrG_0 & \hat{\sim} & 1 & +E_r e^{-\frac{9}{8\pi h}} & +E_r e^{-\frac{25}{8\pi h}} & +E_r^2 e^{-\frac{34}{8\pi h}} \\ & & +E_r e^{\frac{25}{8\pi h}} & +E_r e^{\frac{16}{8\pi h}} & +E_r & +E_r e^{-\frac{9}{8\pi h}} \\ & & +E_r e^{\frac{9}{8\pi h}} & +E_r & +E_r e^{-\frac{16}{8\pi h}} & +E_r e^{-\frac{25}{8\pi h}} \\ & & +E_r^2 e^{\frac{34}{8\pi h}} & +E_r e^{\frac{9}{8\pi h}} & +E_r e^{\frac{25}{8\pi h}} & +1, \end{array}$$

by which we mean that, e.g., the exponential type of the summand $\tau_4\tau_2\tau_3^{-1}$ is the same as the exponential type of $E_r e^{-\frac{9}{8\pi h}}$ and that, therefore, this summand contributes to the points corresponding to $E_r^k e^{-\frac{9}{8\pi h}}$, $k \geq 1$, in the Newton polygon of the quantization conditon (26).

5.1 Solving the quantization condition with subdominant terms

Since f has only two local minima and two local maxima on its period, the Witten Laplacian will have only two resurgent exponentially small eigenvalues, one of them 0 and the other one will be denoted $hE_{r,*}$. In this subsection we are going to calculate the beginning of the hyperasymptotic expansion of $E_{r,*}$ using methods of [G08].

Rewrite the quantization condition (22) as

$$-\frac{1}{1+E_r k} + \mathrm{Tr}\ G_0 - (1+E_r k)\det G_0 \ = \ 0. \tag{26}$$

Represent the l.h.s. of (26) as a sum of powers of E_r and $e^{\frac{1}{h}}$, namely,

$$-\frac{1}{1+E_r k} + \mathrm{Tr}\ G_0 - (1+E_r k)\det G_0 \ = \ \sum_{j,\omega} a_{j\omega} E_r^j e^{\frac{\omega}{8\pi h}},$$

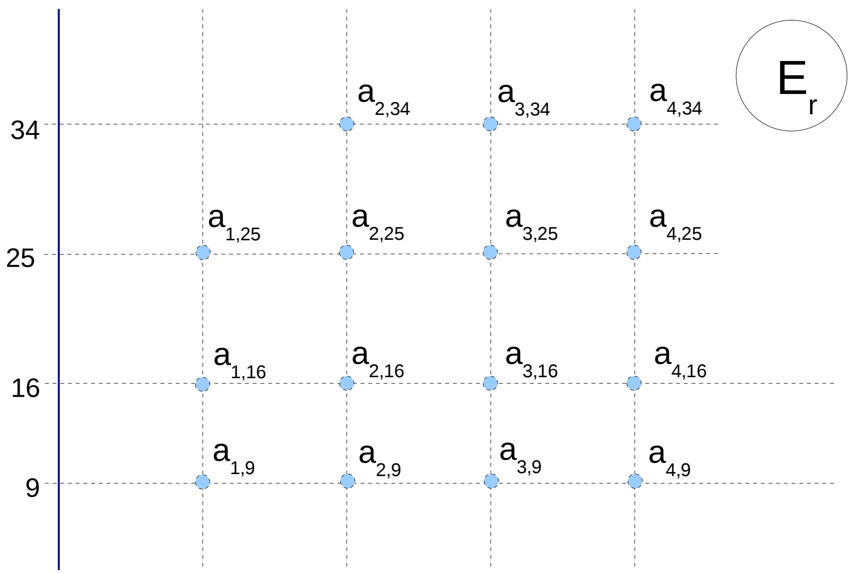

Figure 11: Terms in the quantizaton condition (26) for the Example 1.

and draw the Newton polygon of (26) on figure 11 as explained in [G08]. For the calculation that we are going to carry out, only terms of exponential order $\geq \frac{8}{9\pi h}$ will be important; in particular contributions from the first and the third term on the l.h.s. of (26) are of exponential order ≤ 0 and therefore need not be considered in detail.

In [G08] we explained that the leading exponential summand of (the hyperasymptotic expansion of) $E_{r,*}$ is obtained by looking the the north-west edges of the Newton polygon, in our case that means – by solving

$$a_{1,25} E_r e^{\frac{25}{8\pi h}} + a_{2,34} E_r^2 e^{\frac{34}{8\pi h}} \;=\; 0.$$

Thus, up to subdominant exponentials, $E_{r,*} \;\approx\; -\frac{a_{1,25}}{a_{2,34}} e^{-\frac{9}{8\pi h}}$. To find exponentially subdominant corrections, make a substitution

$$E_r = \left(-\frac{a_{1,25}}{a_{2,34}} + E_1 \right) e^{-\frac{9}{8\pi h}}$$

in the quantization condition and solve it for E_1 under additional requirement that E_1 should be exponentially small.

We re-express the quantization condition in terms of E_1,

$$\sum_{j,\omega} b_{j,\omega} E_1^j e^{\frac{\omega}{8\pi h}} \;=\; e^{-\frac{16}{8\pi h}} \sum_{j,\omega} a_{j,\omega} E_r^j e^{\frac{\omega}{8\pi h}}$$

and plot the summands on the figure 12, where:

$$b_{1,0} = -a_{1,25}, \quad b_{2,0} = a_{2,34},$$

$$b_{0,-9} = -\frac{a_{1,25} a_{1,16}}{a_{2,34}} + a_{2,25} (\frac{a_{1,25}}{a_{2,34}})^2 - a_{3,34} (\frac{a_{1,25}}{a_{2,34}})^3.$$

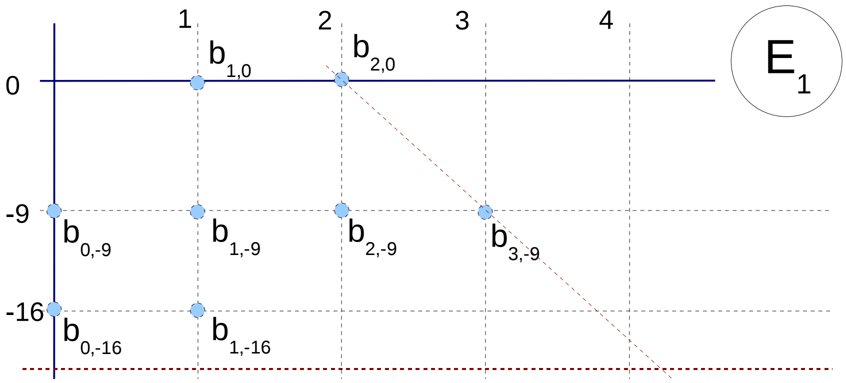

Figure 12: Newton polygon for E_1.

From the Newton polygon on fig.12 we infer

$$E_1 \;=\; e^{-\frac{9}{8\pi h}}\left(-\frac{b_{0,-9}}{b_{1,0}}+E_2\right)$$

where E_2 has to be exponentially small. The next step of this procedure and a Newton polygon for E_2 (which we will not draw here) yields $E_2 \in \mathcal{E}^{-\frac{7}{8\pi}}$.

Thus we need to calculate elements $a_{2,34}, a_{3,34}, a_{1,25}, a_{2,25}, a_{1,16}$ of the Newton polygon, and there the following four summands in Tr G_0 that contribute to these elements, namely:

$$\mu_1\mu_2\mu_3\mu_4\tau_1^{-1}\tau_3^{-1} \;=\; e^{\frac{34}{8h}}\frac{E_r^2\pi^2}{\sqrt{f''(q_1)\cdot|f''(q_2)|\cdot f''(q_3)\cdot|f''(q_4)|}}\times$$

$$\times\left(1+\frac{E_r\ln\, h}{2}\left[\frac{1}{f''(q_1)}+\frac{1}{|f''(q_2)|}+\frac{1}{f''(q_3)}+\frac{1}{|f''(q_4)|}\right]\right)(1+O(h)+O(E_rh^0)+O_{h=fix}(E_r^2));$$

$$\mu_2\mu_3\tau_4\tau_3^{-1}\tau_1^{-1} \;=\; -e^{\frac{2[f(q_2)-f(q_3)]}{h}}\frac{E_r\pi}{\sqrt{f''(q_3)|f''(q_2)|}}\left(1+\frac{E_r\ln\, h}{2}(\frac{1}{|f''(q_2)|}+\frac{1}{f''(q_3)})\right)(1+O(h)+O(E_r)h^0+O_{h=fix}(E_r^2));$$

$$\mu_2\mu_3\tau_2\tau_3^{-1}\tau_1^{-1} \;=\; -e^{\frac{2[-f(q_3)+f(q_4)]}{h}}\frac{E_r\pi}{\sqrt{f''(q_3)|f''(q_4)|}}\times$$

$$\times\left(1+\frac{E_r\ln\, h}{2}(\frac{1}{f''(q_3)}+\frac{1}{|f''(q_4)|})\right)(1+O(h)+O(E_rh^0)+O_{h=fix}(E_r^2));$$

$$\mu_3\tau_4\tau_3^{-1} \;=\; \frac{E_r\pi i}{\sqrt{f''(q_1)f''(q_3)}}\exp\left\{\frac{2[f(q_1)-f(q_3)]}{h}\right\}(1+O_{h=fix}(E_r)+O(h)).$$

The notation $O_{h=fix}(E_r)$ means terms that contain factors of degree ≥ 1 with respect to E_r, regardless of their degree with respect to h or ln h.

We have:

$$a_{2,34} = \frac{\pi^2}{\sqrt{f''(q_1)\cdot|f''(q_2)|\cdot f''(q_3)\cdot|f''(q_4)|}}(1+O(h)) \;=\; \frac{1}{15\sqrt{15}};$$

$$a_{3,34} = \frac{\pi^2}{\sqrt{f''(q_1)\cdot|f''(q_2)|\cdot f''(q_3)\cdot|f''(q_4)|}}\left[\frac{1}{f''(q_1)}+\frac{1}{|f''(q_2)|}+\frac{1}{f''(q_3)}+\frac{1}{|f''(q_4)|}\right]\frac{\ln\, h+O(h^0)}{2} =$$

$$= \frac{1}{15\sqrt{15}}\left[\frac{1}{6\pi}+\frac{2}{7.5\pi}+\frac{1}{10\pi}\right]\frac{\ln\, h+O(h^0)}{2} = \frac{4}{15^2\sqrt{15}\pi}\ln\, h+O(h^0);$$

$$a_{1,25} = -\frac{\pi}{\sqrt{f''(q_3)|f''(q_2)|}} - \frac{\pi}{\sqrt{f''(q_3)|f''(q_4)|}} + O(h) = -\frac{2}{\sqrt{75}}+O(h);$$

$$a_{2,25} = \left\{-\frac{\pi}{\sqrt{f''(q_3)|f''(q_2)|}}\left(\frac{1}{f''(q_3)}+\frac{1}{|f''(q_2)|}\right) - \frac{\pi}{\sqrt{f''(q_3)|f''(q_4)|}}\left(\frac{1}{f''(q_3)}+\frac{1}{|f''(q_4)|}\right)\right\}\frac{\ln\, h+O(h^0)}{2}$$

$$= -\frac{2}{\sqrt{75}}[\frac{1}{10\pi}+\frac{2}{15\pi}]\frac{\ln\, h+O(h^0)}{2} = -\frac{1}{5\sqrt{3}}\cdot\frac{7}{30\pi}\ln\, h+O(h^0) = -\frac{7}{150\sqrt{3}\pi}\ln\, h+O(h^0);$$

$$a_{1,16} = \frac{\pi i}{\sqrt{f''(q_1)f''(q_3)}}+O(h) = \frac{i}{\sqrt{60}}+O(h).$$

Proceed with the calculation:

$$r = -\frac{a_{1,25}}{a_{2,34}} = \frac{1}{\pi}(\sqrt{f''(q_1)\cdot|f''(q_4)|}+\sqrt{f''(q_1)\cdot|f''(q_2)|})+O(h) = 6\sqrt{5}+O(h).$$

$$b_{0,-9} = a_{1,16}r + a_{2,25}r^2 a_{3,34}r^3 =$$

$$= \left\{-\frac{7}{150\sqrt{3}\pi}[6\sqrt{5}]^2+\frac{4}{15^2\sqrt{15}\pi}[6\sqrt{5}]^3\right\}\ln\, h+O(h^0) = \frac{18\sqrt{3}}{5\pi}\ln\, h+O(h^0).$$

Remember that $b_{1,0} = -a_{1,25} = \frac{2}{\sqrt{75}}+O(h)$.

Now, we have the first subdominant term in the spectrum. Namely,

$$E_r = e^{-\frac{9}{8\pi h}}\left(-\frac{a_{1,25}}{a_{2,34}}+E_1\right) =$$

$$= e^{-\frac{9}{8\pi h}}\left(-\frac{a_{1,25}}{a_{3,24}}+e^{-\frac{9}{8\pi h}}\left[-\frac{b_{0,-9}}{b_{1,0}}+\mathcal{E}^{-\frac{7}{8\pi}}\right]\right) =$$

$$= e^{-\frac{9}{8\pi h}}(6\sqrt{5}+o(h^0))+e^{-\frac{18}{8\pi h}}(-\frac{27}{\pi}\ln\, h+O(h^0))+\mathcal{E}^{-\frac{25}{8\pi}}. \tag{27}$$

Remark. Calculaton of the next term in h in the above asymptotic expansions multiplying $e^{-\frac{9}{8\pi h}}$ or $e^{-\frac{18}{8\pi h}}$ requires taking the integrals as in formulas (13), (14), which in our example can be done by hand. Performing this calculation, however, did not bring the author any new insight.

5.2 Asymptotic expansion of the eigenfunction corresponding to the nonzero low-lying eigenvalue.

We need to calculate two resurgent symbols A_+ and A_- that solve the equation

$$(F-Id)\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = 0.$$

According to the equation (16), and since, when E_r satisfies the quantization condition, F_Id is a rank one 2×2 matrix, the question reduces to calculating $[G_0]_{11}-(1+E_rk)^{-1}$ and $[G_0]_{12}$ in this case.

Recall that

$$\tau_1 = \tau_4 \sim e^{-\frac{9}{8\pi h}} E_r; \quad \tau_2 = \tau_3 \sim e^{-\frac{25}{8\pi h}} E_r.$$

Let us write down the exponential orders of the various summands in $[G_0]_{11} - (1 + E_r k)^{-1}$ and in $[G_0]_{12}$. Namely,

$$\begin{aligned} [G_0]_{11} - (1 + E_r k)^{-1} \ &= \ \overbrace{(\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1 + E_r k)^{-1})}^{\hat{\sim} e^{-\frac{9}{8\pi h}}} + \overbrace{\tau_4 \tau_2 \tau_3^{-1}}^{\hat{\sim} e^{-\frac{18}{8\pi h}}} + \overbrace{\tau_4 \tau_2 \tau_1^{-1}}^{\hat{\sim} e^{-\frac{34}{8\pi h}}} + \overbrace{\tau_4 \tau_2}^{\hat{\sim} e^{-\frac{52}{8\pi h}}} \\ &+ \underbrace{\tau_4 \tau_1^{-1} \tau_3^{-1} \mu_3 \mu_2}_{\hat{\sim} e^{\frac{16}{8\pi h}}} + \underbrace{\tau_4 \tau_1^{-1} \mu_2}_{\hat{\sim} e^{-\frac{9}{8\pi h}}} + \underbrace{\tau_3^{-1} \tau_4 \mu_3}_{\hat{\sim} e^{\frac{7}{8\pi h}}} + \underbrace{\tau_4}_{\hat{\sim} e^{-\frac{18}{8\pi h}}} . \end{aligned}$$

Thus, we have two main terms $\tau_4 \tau_1^{-1} \tau_3^{-1} \mu_3 \mu_2$ and $\tau_3^{-1} \tau_4 \mu_3$.
Furthermore,

$$\frac{[G_0]_{11} - (1 + E_r k)^{-1}}{\tau_3^{-1} \tau_4 \mu_3} \ = \ \tau_1^{-1} \mu_2 + 1 + \frac{\tau_4 \tau_1^{-1} \mu_2 + (\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1 + E_r k)^{-1})}{\tau_3^{-1} \tau_4 \mu_3} + \mathcal{E}^{-\frac{25}{8\pi}}$$

and the third summand $\sim e^{-\frac{16}{8\pi h}}$.

$$\begin{aligned} [G_0]_{12} \ &= \ \overbrace{\mu_1 \tau_1^{-1} \tau_2 \tau_3^{-1} \tau_4}^{\hat{\sim} e^{-\frac{9}{8\pi h}}} + \overbrace{\tau_2 \tau_3^{-1} \tau_4}^{\hat{\sim} e^{-\frac{18}{8\pi h}}} + \overbrace{\mu_1 \tau_1^{-1} \tau_2 \tau_4}^{\hat{\sim} e^{-\frac{43}{8\pi h}}} + \overbrace{\tau_2 \tau_4}^{\hat{\sim} e^{-\frac{52}{8\pi h}}} \\ &+ \underbrace{\mu_1 \mu_2 \mu_3 \tau_1^{-1} \tau_3^{-1} \tau_4}_{\hat{\sim} e^{\frac{7}{8\pi h}}} + \underbrace{\mu_3 \tau_3^{-1} \tau_4}_{e^{\frac{7}{8\pi h}}} + \underbrace{\mu_1 \mu_2 \tau_1^{-1} \tau_4}_{\hat{\sim} e^{-\frac{18}{8\pi h}}} + \underbrace{\tau_4}_{\hat{\sim} e^{-\frac{18}{8\pi h}}} . \end{aligned}$$

Main terms are $\mu_1 \mu_2 \mu_3 \tau_1^{-1} \tau_3^{-1} \tau_4$ and $\mu_3 \tau_3^{-1} \tau_4$, and we have

$$\frac{[G_0]_{12}}{\mu_3 \tau_3 \tau_4^{-1}} \ = \ 1 + \underbrace{\mu_1 \mu_2 \tau_1^{-1}}_{\hat{\sim} e^{0}} + \underbrace{\frac{\mu_1 \tau_1^{-1} \tau_2}{\mu_3}}_{\hat{\sim} e^{-\frac{16}{8\pi h}}} + \mathcal{E}^{-\frac{25}{8\pi h}}.$$

Put

$$\begin{pmatrix} Z_+^{(0)} \\ Z_-^{(0)} \end{pmatrix} \ = \ \begin{pmatrix} 1 + \mu_1 \mu_2 \tau_1^{-1} + \frac{\mu_1 \tau_1^{-1} \tau_2}{\mu_3} + \mathcal{E}^{-\frac{25}{8\pi h}} \\ -(\tau_1^{-1} \mu_2 + 1 + \frac{\tau_4 \tau_1^{-1} \mu_2 + (\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1 + E_r k)^{-1})}{\tau_3^{-1} \tau_4 \mu_3} + \mathcal{E}^{-\frac{25}{8\pi}}) \end{pmatrix}$$

Before writing down the explicit expressions for $Z_\pm^{(j)}$, let us derive the following consequence of the quantization condition. Using the explicit form of Tr G_0, assuming E_r satisfies (26), and keeping only the largest terms in (26), we obtain

$$(\mu_2 \mu_3 \tau_4 \tau_3^{-1} \tau_1^{-1} + \mu_3 \mu_4 \tau_3^{-1}) + \mu_1 \mu_2 \mu_3 \mu_4 \tau_1^{-1} \tau_3^{-1} = \mathcal{E}^{\frac{7}{8\pi}}$$

which, taking into account $\tau_1 = \tau_4$ and $\mu_2 = \mu_4$, simplifies to

$$2 + \mu_1 \mu_2 \tau_1^{-1} = \mathcal{E}^{-\frac{9}{8\pi}} \tag{28}$$

Now, applying formulas (17)-(20), obtain successively:

$$\begin{pmatrix} Z_+^{(1)} \\ Z_-^{(1)} \end{pmatrix} = \begin{pmatrix} 1-\mu_1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}+\mathcal{E}^{-\frac{25}{8\pi}} \\ -\tau_1^{-1}\mu_2+\mu_1\mu_2\tau_1^{-1}+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}-\frac{\tau_4\tau_1^{-1}\mu_2+(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1}-(1+E_rk)^{-1})}{\tau_3^{-1}\tau_4\mu_3}+\mathcal{E}^{-\frac{25}{8\pi}} \end{pmatrix},$$

which is convenient to rewrite as

$$\begin{pmatrix} Z_+^{(1)} \\ Z_-^{(1)} \end{pmatrix} = \left(1-\mu_1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}+\mathcal{E}^{-\frac{25}{8\pi}}\right)\begin{pmatrix} 1 \\ \tau_1^{-1}\mu_2\left[-1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}\right](1+\mathcal{E}^{-\frac{25}{8\pi}}) \end{pmatrix}.$$

$$\begin{pmatrix} Z_+^{(2)} \\ Z_-^{(2)} \end{pmatrix} = \begin{pmatrix} \tau_1^{-1}Z_+^{(1)}+Z_-^{(1)} \\ \mu_2\tau_1^{-1}Z_+^{(1)}+Z_-^{(1)} \end{pmatrix} =$$

$$= Z_+^{(1)}\begin{pmatrix} \tau_1^{-1}+\tau_1^{-1}\mu_2\left[-1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}\right](1+\mathcal{E}^{-\frac{25}{8\pi}}) \\ \mu_2\tau_1^{-1}+\tau_1^{-1}\mu_2\left[-1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}\right]+\mathcal{E}^{-\frac{16}{8\pi}} \end{pmatrix}$$

$$= Z_+^{(1)}\begin{pmatrix} \tau_1^{-1}+\tau_1^{-1}\mu_2\left[-1+\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}\right]+\mathcal{E}^{-\frac{16}{8\pi}} \\ \tau_1^{-1}\mu_2\frac{\mu_1\tau_1^{-1}\tau_2}{\mu_3}+\mathcal{E}^{-\frac{16}{8\pi}} \end{pmatrix}.$$

Remark 5.2 It is interesting to note that in the calculation of $Z_-^{(2)}$ the contributions from the leading exponential orders in $Z_+^{(1)}$ and $Z_-^{(1)}$ cancel and the nonzero value of $Z_-^{(2)}$ is due purely to subdominant exponentials in $Z^{(1)}$. Neglecting these subdominant terms would break down the rest of the calculation. This little algebraic detail is philosophically very important: it shows that constructing asymptotic expansions of an eigenfunction of the Witten Laplacian on all intervals (q_j, q_{j+1}) must be difficult without methods of resurgent analysis.

$$\begin{pmatrix} Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix} = \tau_2\tau_1^{-1}Z_+^{(1)}\begin{pmatrix} (1+\tau_1^{-1}\mu_2\mu_1)(1+\mathcal{E}^{-\frac{9}{8\pi}}) \\ \tau_1^{-1}\mu_2\frac{\mu_1}{\mu_3}(1+\mathcal{E}^{-\frac{9}{8\pi}}) \end{pmatrix}$$

$$= \tau_2\tau_1^{-1}Z_+^{(1)}\begin{pmatrix} -1+\mathcal{E}^{-\frac{9}{8\pi}} \\ \tau_1^{-1}\mu_2\frac{\mu_1}{\mu_3}+\mathcal{E}^0 \end{pmatrix},$$

where we have used (28) in the last step

Finally,

$$\begin{pmatrix} Z_+^{(4)} \\ Z_-^{(4)} \end{pmatrix} = \begin{pmatrix} \tau_3^{-1}Z_+^{(3)}+Z_-^{(3)} \\ \mu_4\tau_3^{-1}Z_+^{(3)}+Z_-^{(3)} \end{pmatrix} =$$

$$= \tau_2\tau_1^{-1}Z_+^{(1)}\begin{pmatrix} \tau_3^{-1}(-1+\mathcal{E}^{-\frac{9}{8\pi}})+\tau_1^{-1}\mu_2\frac{\mu_1}{\mu_3}+\mathcal{E}^0) \\ \mu_4\tau_3^{-1}(-1+\mathcal{E}^{-\frac{9}{8\pi}})+\tau_1^{-1}\mu_2\frac{\mu_1}{\mu_3}+\mathcal{E}^0 \end{pmatrix}.$$

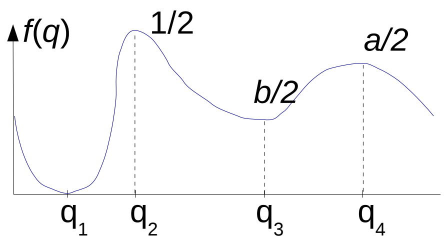

Figure 13: Graph of $f(q)$ in Example 2.

6 Global expressions for the eigenfunction – Example 2.

Let $f(q)$ be a trigonometric polynomial with two local minima and two local manima q_1, q_3 and two local maxima q_2, q_4 on the period $[0,1]$, where $0 < q_1 < q_2 < q_3 < q_4 < 1$. Up to shifting q by a constant we can assume that q_1 is the global minimum of f, and up to changing $f(q)$ into $f(2q_1 - q)$, that q_2 is its global maximum. Changing further $f(q)$ by an affine linear transformation $f \mapsto Af + B$, we can assume $f(q_1) = 0$, $f(q_2) = \frac{1}{2}$, $f(q_3) = \frac{b}{2}$, $f(q_4) = \frac{a}{2}$, where $0 \le b < a \le 1$, figure 13. All these transformations of f produce easily controllable changes in the eigenvalues and eigenfunctions of the Witten Laplacian.

We will actually assume that the inequalities are strict:

$$\text{Assume:} \quad 0 \ < \ a \ < \ b < \ 1, \tag{29}$$

and will gradually put more restrictions on a and b more specific as we progress through this section.

In our situation

$$\tau_1 \hat{\sim} E_r e^{-\frac{1}{h}}; \quad \tau_2 \hat{\sim} E_r e^{-\frac{1-b}{h}}; \quad \tau_3 \hat{\sim} E_r e^{-\frac{a-b}{h}}; \quad \tau_4 \hat{\sim} E_r e^{-\frac{a}{h}}.$$

In order to find the two low-lying eigenvalues of the Witten Laplacian (one of which equals, as we know already, to zero), we need to solve the same old quantization condition,

$$-\frac{1}{1+E_r k} \ + \ \text{Tr}\ G_0 \ - \ (1+E_r k)\det G_0 \ = \ 0. \tag{30}$$

where Tr G_0 has the same form as (25).

In the loose sense explained in the Example 1, we have now

$$\begin{array}{rrrrr} TrG_0 \ \hat{\sim} & 1 & +E_r e^{-\frac{1}{h}} & +E_r e^{\frac{b-a}{h}} & +E_r^2 e^{\frac{b-a-1}{h}} \\ & +E_r e^{\frac{1-b}{h}} & +E_r e^{-\frac{b}{h}} & +E_r e^{\frac{1-a}{h}} & +E_r e^{-\frac{a}{h}} \\ & +E_r e^{\frac{a}{h}} & +E_r e^{\frac{a-1}{h}} & +E_r e^{\frac{b}{h}} & +E_r e^{\frac{b-1}{h}} \\ & +E_r^2 e^{\frac{1+a-b}{h}} & +E_r e^{\frac{1}{h}} & +E_r e^{\frac{a-b}{h}} & +1. \end{array}$$

The Newton polygon corresponding to (30) will thus be as shown on fig.14, with the $E_r^2 e^{\frac{1+a-b}{h}}$-term coming from the $\mu_1\mu_2\mu_3\mu_4\tau_1^{-1}\tau_3^{-1}$ summand, and the $E_r e^{\frac{1}{h}}$ term – from the $\mu_1\mu_2\tau_1^{-1}$ summand. We conclude that the nonzero low-lying eigenvalue will have the exponential type $E_r \hat{\sim} e^{\frac{b-a}{h}}$.

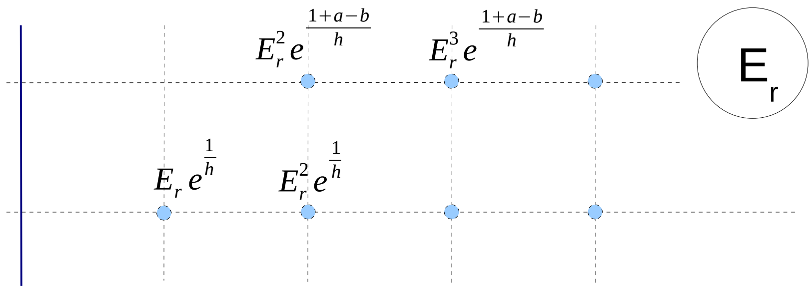

Figure 14: The Newton polygon of (30), situation of Example 2.

As $G_0 - (1+E_r k)^{-1} Id$ is a 2×2 matrix of rank 1, a nonzero vector in its kernel is proportional to $([G_0-(1+E_rk)^{-1}]_{12}, -[G_0-(1+E_rk)^{-1}]_{11})^T$, i.e. to $([G_0]_{12}, -[G_0]_{11}+(1+E_rk)^{-1})^T$.

For $E_r \tilde{\sim} e^{\frac{b-a}{h}}$, the exponential types of various summands in $[G_0-(1+E_rk)^{-1}]_{11}$, $[G_0]_{12}$ are as follows:

$$[G_0-(1+E_rk)^{-1}]_{11} = \underbrace{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1}-1)}_{\mathcal{E}^{b-a}} + \underbrace{\tau_4\tau_2\tau_3^{-1}}_{\tilde{\sim} e^{\frac{-1+b-a}{h}}} + \underbrace{\tau_4\tau_2\tau_1^{-1}}_{\tilde{\sim} e^{\frac{2b-2a}{h}}} + \underbrace{\tau_4\tau_2}_{\tilde{\sim} e^{\frac{-1+3b-3a}{h}}} +$$
$$+ \underbrace{\tau_4\tau_1^{-1}\tau_3^{-1}\mu_3\mu_2}_{\tilde{\sim} e^{\frac{1-a}{h}}} + \underbrace{\tau_4\tau_1^{-1}\mu_2}_{\tilde{\sim} e^{\frac{1+b-2a}{h}}} + \underbrace{\tau_3^{-1}\tau_4\mu_3}_{\tilde{\sim} e^{-\frac{a}{h}}} + \underbrace{\tau_4}_{\tilde{\sim} e^{\frac{b-2a}{h}}} .$$

(The first summand should typically be $\tilde{\sim} e^{\frac{b-a}{h}}$, but it is conceivable that its exponential type is actually smaller for a special choice of f.)

$$[G_0]_{12} = \underbrace{\mu_1\tau_1^{-1}\tau_2\tau_3^{-1}\tau_4}_{\tilde{\sim} e^{\frac{b-a}{h}}} + \underbrace{\tau_2\tau_3^{-1}\tau_4}_{\tilde{\sim} e^{\frac{-1+b-a}{h}}} + \underbrace{\mu_1\tau_1^{-1}\tau_2\tau_4}_{\tilde{\sim} e^{\frac{3b-3a}{h}}} + \underbrace{\tau_2\tau_4}_{\tilde{\sim} e^{\frac{-1+3b-3a}{h}}} +$$
$$+ \underbrace{\mu_1\mu_2\mu_3\tau_1^{-1}\tau_3^{-1}\tau_4}_{\tilde{\sim} e^{\frac{1+b-2a}{h}}} + \underbrace{\mu_3\tau_3^{-1}\tau_4}_{\tilde{\sim} e^{\frac{2b-3a}{h}}} + \underbrace{\mu_1\mu_2\tau_1^{-1}\tau_4}_{\tilde{\sim} e^{\frac{1+2b-3a}{h}}} + \underbrace{\tau_4}_{\tilde{\sim} e^{\frac{b-2a}{h}}} .$$

The two largest summands in the above formulas are thus $\mu_2\mu_3\tau_1^{-1}\tau_3^{-1}\tau_4 + \mu_2\tau_1^{-1}\tau_4$ and $\mu_1\mu_2\mu_3\tau_1^{-1}\tau_3^{-1}\tau_4 + \mu_1\mu_2\tau_1^{-1}\tau_4$, respectively; so it is reasonable to take

$$\begin{pmatrix} Z_+^{(0)} \\ Z_-^{(0)} \end{pmatrix} = \frac{1}{\mu_2\tau_1^{-1}\tau_3^{-1}\tau_4(\mu_3+\tau_3)} \begin{pmatrix} [G_0]_{12} \\ -[G_0]_{11}+(1+E_rk)^{-1} \end{pmatrix}.$$

There are too many summands in the entries of G_0 for us to be able to get an enlightening exposition, so we will artificially impose additional assumptions on (a,b). These assumptions will help us select the dominant exponential, the first subdominant, the second subdominant, etc, terms in every hyperasymptotic expression we are going to write down in a moment. There might be a combinatorial structure to various inequalities between (a,b) we are going to introduce, but we are not ready to comment on it at the present time.

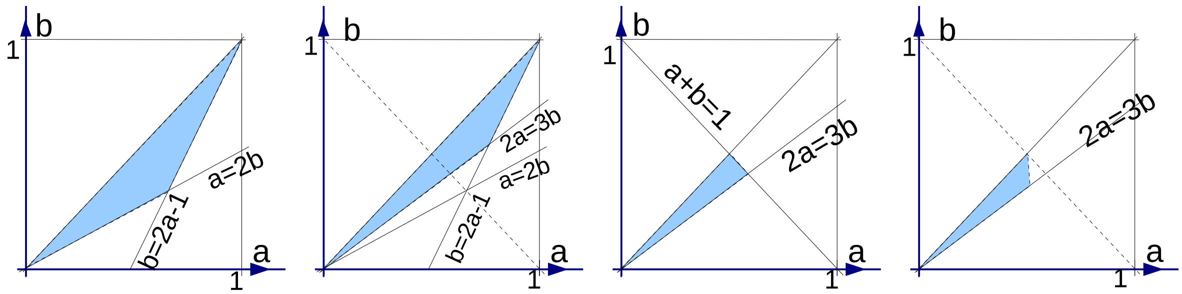

Figure 15: Values of (a, b) satisfying, from left to right, (31), (32), (33), (34).

Under

$$\text{additional assumptions: } a < 2b; \quad b > 2a - 1 \tag{31}$$

(first part of the figure 15), we can write

$$\frac{[G_0]_{11} - (1 + E_r k)^{-1}}{\mu_2 \tau_1^{-1} \tau_3^{-1} \tau_4 (\mu_3 + \tau_3)} = 1 + \frac{(\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1 + E_r k)^{-1})}{\mu_2 \tau_1^{-1} \tau_4 \mu_3 \tau_3^{-1}} + \mathcal{E}^{-1+2b-a},$$

$$\frac{[G_0]_{12}}{\mu_2 \tau_1^{-1} \tau_3^{-1} \tau_4 (\mu_3 + \tau_3)} = 1 + \underbrace{\boxed{\frac{\tau_1}{\mu_1 \mu_2}}}_{\hat{\sim} e^{\frac{-1+a-b}{h}}} + \frac{\tau_2}{\mu_2 \mu_3} - \frac{\tau_2 \tau_3}{\mu_2 \mu_3^2} + \mathcal{E}^{-1+2b-a}.$$

Restricting further to

$$\text{additional assumptions: } 2a < 3b; \quad b > 2a - 1, \tag{32}$$

see the second part of the figure 32, we can absorb the boxed term into the error $\mathcal{E}^{-1+2b-a}$.

We conclude that

$$\begin{pmatrix} Z_+^{(0)} \\ Z_-^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_1 \left(1 + \frac{\tau_2}{\mu_2 \mu_3} - \frac{\tau_2 \tau_3}{\mu_2 \mu_3^2} + \mathcal{E}^{-1+2b-a}\right) \\ -\left(1 + \frac{(\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1+E_r k)^{-1})}{\mu_2 \tau_1^{-1} \tau_4 \mu_3 \tau_3^{-1}} + \mathcal{E}^{-1+2b-a}\right) \end{pmatrix}.$$

Using (17),

$$\begin{pmatrix} Z_+^{(1)} \\ Z_-^{(1)} \end{pmatrix} = \begin{pmatrix} \mu_1 \left(\frac{\tau_2}{\mu_2 \mu_3} - \frac{(\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1+E_r k)^{-1})}{\mu_2 \tau_1^{-1} \tau_4 \mu_3 \tau_3^{-1}} - \frac{\tau_2 \tau_3}{\mu_2 \mu_3^2} + \mathcal{E}^{-1+2b-a}\right) \\ -1 + \mu_1 + \mathcal{E}^{-1+b} \end{pmatrix},$$

with the following exponential orders of the ingredients of $Z^{(1)}$:

$$\frac{\tau_2}{\mu_2 \mu_3} \hat{\sim} e^{\frac{-1+a}{h}}; \quad \frac{(\tau_4 \tau_2 \tau_1^{-1} \tau_3^{-1} - (1 + E_r k)^{-1})}{\mu_2 \tau_1^{-1} \tau_4 \mu_3 \tau_3^{-1}} \hat{\sim} e^{\frac{-1+b}{h}}; \quad \frac{\tau_2 \tau_3}{\mu_2 \mu_3^2} \hat{\sim} e^{\frac{-1+b}{h}}.$$

We have

$$\frac{Z_-^{(1)}}{\tau_1^{-1} Z_+^{(1)}} = \underbrace{-\frac{\tau_1 \mu_2 \mu_3}{\tau_2 \mu_1}}_{\hat{\sim} e^{-\frac{a}{h}}} + \mathcal{E}^{b-2a},$$

hence by (18)

$$\begin{pmatrix} Z_+^{(2)} \\ Z_-^{(2)} \end{pmatrix} = \frac{\mu_1}{\tau_1}\left(\frac{\tau_2}{\mu_2\mu_3} - \frac{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1} - (1+E_rk)^{-1})}{\mu_2\tau_1^{-1}\tau_4\mu_3\tau_3^{-1}} - \frac{\tau_2\tau_3}{\mu_2\mu_3^2} + \mathcal{E}^{-1+2b-a}\right)\begin{pmatrix} 1+\mathcal{E}^{-a} \\ \mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1} + \mathcal{E}^{b-2a} \end{pmatrix}$$

where the coefficient in front of this vector nothing but $\tau_1^{-1}Z_+^{(1)}$.

The formula (19) gives

$$\begin{pmatrix} Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix} = \frac{\mu_1}{\tau_1}\left(\frac{\tau_2}{\mu_2\mu_3} - \frac{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1} - (1+E_rk)^{-1})}{\mu_2\tau_1^{-1}\tau_4\mu_3\tau_3^{-1}} - \frac{\tau_2\tau_3}{\mu_2\mu_3^2} + \mathcal{E}^{-1+2b-a}\right)\begin{pmatrix} \tau_2 + \mathcal{E}^{-1+2b-2a} + \mu_3\left[\mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1}\right] + \mathcal{E}^{2b-3a} \\ \tau_2 + \mathcal{E}^{-1+2b-2a} + \mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1} + \mathcal{E}^{b-2a} \end{pmatrix}.$$

Under

$$\text{additional assumption:} \quad b+a<1, \quad 2a<3b \tag{33}$$

we have $\tau_2 \in \mathcal{E}^{b-2a}$ and therefore

$$\begin{pmatrix} Z_+^{(3)} \\ Z_-^{(3)} \end{pmatrix} = \frac{\mu_1}{\tau_1}\left(\frac{\tau_2}{\mu_2\mu_3} - \frac{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1} - (1+E_rk)^{-1})}{\mu_2\tau_1^{-1}\tau_4\mu_3\tau_3^{-1}} - \frac{\tau_2\tau_3}{\mu_2\mu_3^2} + \mathcal{E}^{-1+2b-a}\right)\begin{pmatrix} \tau_2 + \mu_3\left[\mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1}\right] + \mathcal{E}^{2b-3a} \\ \mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1} + \mathcal{E}^{b-2a} \end{pmatrix}.$$

Finally, use (20) to obtain:

$$\begin{pmatrix} Z_+^{(4)} \\ Z_-^{(4)} \end{pmatrix} = \frac{\mu_1}{\tau_1}\left(\frac{\tau_2}{\mu_2\mu_3} - \frac{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1} - (1+E_rk)^{-1})}{\mu_2\tau_1^{-1}\tau_4\mu_3\tau_3^{-1}} - \frac{\tau_2\tau_3}{\mu_2\mu_3^2} + \mathcal{E}^{-1+2b-a}\right)\times$$

$$\times\begin{pmatrix} \tau_3^{-1}(\mu_2\mu_3 + \mathcal{E}^{b-2a}) + \mu_2 + \mathcal{E}^{-a} \\ \mu_4\tau_3^{-1}(\tau_2 + \mu_3\left[\mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1}\right] + \mathcal{E}^{2b-3a}) + \mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1} + \mathcal{E}^{b-2a} \end{pmatrix}.$$

Under one more

$$\text{additional assumption: } a<\frac{1}{2}; \quad 2a<3b, \tag{34}$$

we have $\mu_4\tau_3^{-1}\tau_2 \hat{\sim} e^{\frac{b-1}{h}} \in \mathcal{E}^{b-2a}$, and the expression simplifies:

$$\begin{pmatrix} Z_+^{(4)} \\ Z_-^{(4)} \end{pmatrix} = \frac{\mu_1}{\tau_1}\left(\frac{\tau_2}{\mu_2\mu_3} - \frac{(\tau_4\tau_2\tau_1^{-1}\tau_3^{-1} - (1+E_rk)^{-1})}{\mu_2\tau_1^{-1}\tau_4\mu_3\tau_3^{-1}} - \frac{\tau_2\tau_3}{\mu_2\mu_3^2} + \mathcal{E}^{-1+2b-a}\right)\begin{pmatrix} \tau_3^{-1}\mu_2\mu_3 + \mu_2 + \mathcal{E}^{-b} \\ [\mu_4\tau_3^{-1}\mu_3 + 1]\cdot\left[\mu_2 - \frac{\tau_1\mu_2\mu_3}{\tau_2\mu_1}\right] + \mathcal{E}^{b-2a} \end{pmatrix}.$$

We will see now that the bracket $[\mu_4\tau_3^{-1}\mu_3+1]$ in the expression for $Z_-^{(4)}$ is not $\hat{\sim} e^{\frac{0}{h}}$ as would appear from the first glance, but is of a smaller exponential type. Indeed, the quantization condition (30) and the explicit form (25) of Tr G_0 imply

$$(\mu_3\tau_3^{-1}+1)\mu_2\tau_4\tau_1^{-1} + \mu_1\mu_2\tau_1^{-1}(1+\mu_3\mu_4\tau_3^{-1}) = \mathcal{E}^a,$$

or

$$(1+\mu_3\mu_4\tau_3^{-1}) = -(\mu_3\tau_3^{-1}+1)\frac{\tau_4}{\mu_1} + \mathcal{E}^{-1+2a-b} \hat{\sim} e^{-\frac{b}{h}}.$$

Remark 6.1 Here we observe again the cancelation of the leading exponential terms and stress again the importance of subdominant exponentials for the calculation of the asymptotics of eigenfunctions on all intervals (q_j, q_{j+1}).

Remark 6.2 [2] There exist trigonometric polynomials f satisfying assumptions of figure 15, i.e. having two local minima and two local maxima satisfying inequalities (31), or (32), or (33), or (34). Indeed, one should take any Morse C^∞ function f_0 with two local minima and two local maxima satisfying the inequalities, say,

$$f_0(q_1) < f_0(q_3) < f_0(q_4) < f_0(q_2);\quad 2[f_0(q_4) - f_0(q_1)] < 3[f_0(q_3) - f_0(q_1)];\quad 2[f_0(q_4) - f_0(q_1)] < f_0(q_2) - f_0(q_1), \tag{35}$$

that are, up to shift and rescaling, correspond to the conjunction of (34) and (29). Then the Fourier series of f_0 will converge to f_0 uniformly together with all derivatives, and an n-th partial sum f_n of that Fourier series for sufficiently large n will have critical points and critical values arbitrarily close to those of f. Since our conditions (35) are open, f_n will satisfy them for n large enough. With a little more work one can produce a trigonometric polynomial with exactly prescribed critical points and critical values. Alternatively, one can generate examples of trigonometric polynomials satisfying (34) and (29) using a computer algebra system.

A Useful formulae

For $A > 0$ and $E > 0$ and $E \to 0+$ we have the following asymptotics of various integrals:

$$\text{arccosh } (A/\sqrt{E}) = \text{Ln } 2A - \frac{1}{2}\text{Ln } E - \frac{E}{4A^2} + o(E) \tag{36}$$

$$\int_{\text{arccosh } (\frac{A}{\sqrt{E}})}^{0} \cosh^k t dt = \begin{cases} -\text{arccosh } (A/\sqrt{E}) & \text{if } k = 0 \\ -\frac{A^2}{2E} + \frac{1}{4} + \frac{1}{16}\frac{E}{A^2} - \frac{1}{2}\text{arccosh } (A/\sqrt{E}) + o(E) & \text{if } k = 2 \\ -\frac{A^4}{4E^2} - \frac{A^2}{4E} + \frac{7}{32} - \frac{3}{8}\text{arccosh } (A/\sqrt{E}) + o(E^0) & \text{if } k = 4 \\ -\frac{1}{k}A^k E^{-\frac{k}{2}} - \frac{1}{2(k-2)}A^{k-2}E^{1-\frac{k}{2}} - \frac{3}{8(k-4)}A^{k-4}E^{2-\frac{k}{2}} + o(E^{2-\frac{k}{2}}) & \text{if } k = 1, 3 \text{ or } \geq 5 \end{cases} \tag{37}$$

$$\int_{\text{arccosh } (\frac{A}{\sqrt{E}})}^{0} \sinh^2 t \cosh^k t dt = \begin{cases} -\frac{A^2}{2E} + \frac{1}{4} + \frac{E}{16A^2} + \frac{1}{2}\text{arccosh } (A/\sqrt{E}) + o(E) & \text{if } k = 0 \\ -\frac{A^4}{4E^2} + \frac{A^2}{4E} - \frac{1}{32} + \frac{1}{8}\text{arccosh } (A/\sqrt{E}) + o(E^0) & \text{if } k = 2 \\ E^{-1-\frac{k}{2}}\left(-\frac{A^{k+2}}{k+2} + \frac{A^k E}{2k} + \frac{A^{k-2}E^2}{8(k-2)} + o(E^2)\right) & \text{if } k = 1 \text{ or } \geq 3 \end{cases} \tag{38}$$

The following formulae are simple integration by parts used in section 2.5.

$$\int \frac{du}{(u^2-E)^{3/2}} = -\frac{1}{E}\frac{u}{\sqrt{u^2-E}}\ ;\qquad \int \frac{udu}{(u^2-E)^{3/2}} = -\frac{1}{\sqrt{u^2-E}} \tag{39}$$

$$\int \frac{u^k du}{(u^2-E)^{3/2}} = -\frac{u^{k-1}}{\sqrt{u^2-E}} + (k-1)\int \frac{u^{k-2}}{\sqrt{u^2-E}}du, \quad k \geq 2 \tag{40}$$

$$\int \frac{du}{(u^2-E)^{\frac{5}{2}}} = \frac{1}{E^2}\frac{u}{\sqrt{u^2-E}} - \frac{1}{E^2}\frac{u^3}{3(u^2-E)^{\frac{3}{2}}}\ ;\qquad \int \frac{udu}{(u^2-E)^{\frac{5}{2}}} = -\frac{1}{3(u^2-E)^{\frac{3}{2}}} \tag{41}$$

[2]The material contained in this remark was explained to the author by Prof. A.Gabrielov.

$$\int \frac{u^k du}{(u^2 - E)^{\frac{5}{2}}} \;=\; -\frac{u^{k-1}}{3(u^2 - E)^{\frac{3}{2}}} + \frac{k-1}{3} \int \frac{u^{k-2}}{(u^2 - E)^{\frac{3}{2}}} du, \; k \geq 2 \tag{42}$$

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