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*Kishore Kumar Naraparaju, and G. Naga Raju*

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# Nonconforming least-squares method for elliptic partial differential equations with smooth interfaces

N. Kishore Kumar\* and G. Naga Raju†

\*Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany.

†Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India.

## Abstract

In this paper a numerical method based on least-squares approximation is proposed for elliptic interface problems in two dimensions, where the interface is smooth. The underlying method is spectral element method. In the least-squares formulation a functional is minimized as defined in (4.1). The jump in the solution and its normal derivative across the interface are enforced (in an appropriate Sobolev norm) in the functional. The solution is obtained by solving the normal equations using preconditioned conjugate gradient method. Essentially the method is nonconforming, so a block diagonal matrix is constructed as a preconditioner based on the stability estimate where each diagonal block is decoupled. A conforming solution is obtained by making a set of corrections to the nonconforming solution as in [24] and an error estimate in  $H^1$ -norm is given which shows the exponential convergence of the proposed method.

**MSC:** 65M70, 65N35, 65Y05, 74B05

**Keywords:** Interface, Least-squares, Nonconforming, Spectral element, Preconditioner, Exponential accuracy

## 1 Introduction

Many problems in engineering are characterized by elliptic partial differential equations with discontinuous coefficients, steady state heat diffusion, electro static, multiphase and porous flow problems are the few examples. An interface problem is a special case of an elliptic partial differential equation with discontinuous coefficients. Such interface problems arise in different situations, for example, in heat conduction or in elasticity problems where the domain of definition is composed of different materials.

In the solution of the elliptic boundary value problems singularities may occur, when the boundary is not smooth or the boundary is smooth yet one or more of the given data of the problem are not smooth. The latter type of singularity typically arises in interface problems. The solution of the interface problem for the elliptic partial differential equation has interface singularity at the points which are either the intersection of interfaces or the intersection of interfaces with the boundary of the domain. For these interface problems the solution will also have singular behavior at the points where the interfaces cross each other.

When the interface is smooth enough the solution of the interface problem is also very smooth in the individual regions but the global regularity is low. Due to lack of the global regularity many standard finite difference algorithms do not apply for interface problems. Immersed-interface method in the framework of finite difference method has some disadvantages [21]. An immersed-interface finite element methods for elliptic interface problems have been introduced in [10, 20, 21]. The basic idea of an immersed-interface method is to incorporate the jump conditions in constructing basis functions. The jump conditions are enforced through

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\*Email: narapara@mis.mpg.de, †me.raju@gmail.com

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the construction of special finite element basis function that satisfies the homogeneous interface conditions. Optimal convergent rates ( $O(h)$  in  $H^1$  norm) have been obtained with linear finite elements.

In the early 70's the elliptic interface problem was addressed by Babuska [1] in the framework of finite element method. The interface problem was formulated as an equivalent minimization problem with all boundary and jump conditions incorporated in the cost functions. The elliptic interface problem has also been studied by Kellogg in [14, 15]. He considered the interface problem for Poisson equation in two independent variables.

In [3], two different finite element methods have been considered. First one is fitted mesh method in which the interface is approximated by the sides of isoparametric elements in the discretization. The interface should be approximated accurately to get optimal rates of convergence in these methods. Second one is unfitted mesh method, which is based on a mesh which is independent of the interface. In both cases optimal convergence rates (in  $h$ ) have been obtained. Only linear finite elements have been studied in most of the conforming finite element methods. In [22] a conforming higher order finite element method has been analyzed for the elliptic interface problems.

Most existing methods are basically conforming finite element methods and require the triangulations in different sub-regions to be matching on the interface. This may pose serious restrictions when the physical solutions of the interface problems are of different scales in different subregions. The nonconforming methods like mortar finite element method and discontinuous Galerkin methods are good alternatives to relax such restrictions. In [13] Huang and Zuo proposed mortar finite element method for elliptic interface problems. The optimal  $L^2$  and  $H^1$  error estimates were (when the interface is of arbitrary shape but smooth) achieved even though the regularity of the true solution is low. In [18] the interface problems have also been studied based on the techniques of mortar finite element method and dual Lagrange multipliers.

The interface problem has also been studied in the framework of least-squares finite element method. The obvious advantages of this class of methods is that the discrete problems are symmetric and positive definite. In [7] the interface problems are recast into a first order formulation and suitable least-squares formulation is applied and the method is nonconforming. A first order system least-squares method (FOSLS) has been proposed in [4, 5] for the interface problems. A discontinuous Galerkin method for interface problems has been proposed in [12]. They enforce the discontinuities in the solution and in its normal derivatives along the interfaces weakly in the DG formulation provided that the triangulation of the domain is fitted to the interface. Optimal convergent rates (in  $h$ ) in  $L^2$  norm have been obtained. In [23] unfitted DG method is studied to the interface problems. An error estimate (in  $H^1$  and  $L^2$  norm) have been obtained in both  $h$  and  $p$ .

In [8, 16] a nonconforming  $hp$ /spectral element method has been proposed. They proposed an exponentially accurate method for elliptic problems with mixed Neumann and Dirichlet boundary conditions on nonsmooth domains. A geometric mesh is used in the neighbourhood of the corners and the auxiliary map of the form  $z = \ln \zeta$  is introduced to remove the singularities at the corners. In the remaining part of the domain usual Cartesian coordinate system is used.

In this paper a least-squares spectral element method is proposed for elliptic interface problems with the smooth interface, in the same lines as the method proposed in [8, 16, 17]. The given domain is discretized into finite number of subdomains so that the division matches along the interface. The interface is resolved exactly using blending elements[11]. Higher order spectral elements are used to approximate the solution. The spectral elements are sum of tensor products of polynomials of degree  $W$  in each variable and the spectral element functions are nonconforming. The extension of the method to elliptic interface problems with the nonsmooth solutions is an ongoing work.

In the least-squares formulation of the method, a solution is sought which minimizes the sum of the squares of a squared norms of the residuals in the partial differential equation and the sum of the residuals in the boundary conditions in fractional Sobolev norms and the sum of the jumps in the value and its normal derivatives of the function across the interface in appropriate fractional Sobolev norms and enforce the continuity along the inter element boundaries by adding a term which measures the sum of the squares of the jump in the function and its derivatives in fractional Sobolev norms (4.1). The solution is obtained by solving the normal equations

using preconditioned conjugate gradient method. The integrals involved in the residual computations are evaluated efficiently. An error estimate in  $H^1$  norm (if the solution is continuous across the interface) is derived. If the given data is analytic, the proposed method is shown to be exponentially accurate.

The rest of the paper is organized as follows: In Section 2 the required function spaces and the elliptic interface problem are defined. The discretization of the domain is given in Section 3 along with the stability estimate. In Section 4 the numerical scheme is described which is based on the stability estimate and the error estimate is given. Finally in Section 5 numerical results are presented for various examples.

## 2 Elliptic Interface problems

Let  $\Omega$  and  $\Omega_1$  ( $\overline{\Omega}_1 \subset \Omega$ ) be bounded domains with boundaries  $\partial\Omega = \Gamma$  and  $\Gamma_0$  respectively. Assume that the boundary  $\Gamma_0$  is smooth. Further, let  $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ . Let  $u_1 = u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$ .

Denoting  $H^k(\Omega_i)$ , the usual Sobolev space of integer order  $k$  with the norm  $\|\cdot\|_{k,\Omega_i}$  as given below,

$$\|u_i(x, y)\|_{k,\Omega_i}^2 = \int_{\Omega_i} \sum_{\alpha_1 + \alpha_2 \leq k} |\partial_x^{\alpha_1} \partial_y^{\alpha_2} u_i(x, y)|^2 dx dy \text{ for } i = 1, 2.$$

Let

$$\|u(x, y)\|_{k,\Omega_1 \cup \Omega_2}^2 = \|u_1(x, y)\|_{k,\Omega_1}^2 + \|u_2(x, y)\|_{k,\Omega_2}^2.$$

Further, let

$$\|u_i\|_{s,J}^2 = \int_J u_i^2(x) dx + \int_J \int_J \frac{|u_i(x) - u_i(x')|^2}{|x - x'|^{1+2s}} dx dx' \text{ for } i = 1, 2,$$

denote the fractional Sobolev norm of order  $s$ , where  $0 < s < 1$ . Here  $J$  denotes an interval contained in  $\mathbb{R}$ .

Now consider the following interface problem

$$\begin{aligned} \mathcal{L}u = -\nabla \cdot (\beta \nabla u) &= f \text{ in } \Omega_1 \cup \Omega_2, \\ u &= g \text{ on } \Gamma, \end{aligned} \quad (2.1)$$

which satisfies the interface conditions

$$\begin{aligned} [u] &= q_0, \\ \left[ \beta \frac{\partial u}{\partial n} \right] &= q_1 \text{ on } \Gamma_0, \end{aligned} \quad (2.2)$$

where  $n = (n_1, n_2)$  is a unit outward normal vector to the interface  $\Gamma_0$  and  $[v]$  denotes the jump of a quantity  $v$  across the interface  $\Gamma_0$ , i.e.,  $[v](x) = v_1(x) - v_2(x)$ ,  $x \in \Gamma_0$ .

The coefficient  $\beta$  in (2.1) is piecewise constant, i.e.,

$$\beta = \begin{cases} \beta^- & \text{in } \Omega_1, \\ \beta^+ & \text{in } \Omega_2. \end{cases} \quad (2.3)$$

The interface problem (2.1-2.3) is well posed as the solution  $u$  satisfies the following regularity estimate [6].

**THEOREM 2.1 :-** *If  $f \in L_2(\Omega)$ ,  $q_0 \in H^{3/2}(\Gamma_0)$ ,  $q_1 \in H^{1/2}(\Gamma_0)$  and  $g \in H^{3/2}(\Gamma)$ , then the solution  $u \in H^2(\Omega_1 \cup \Omega_2)$  and*

$$\|u\|_{2,\Omega_1 \cup \Omega_2} \leq C \left( \|f\|_{0,\Omega} + \|g\|_{3/2,\Gamma} + \|q_0\|_{3/2,\Gamma_0} + \|q_1\|_{1/2,\Gamma_0} \right). \quad (2.4)$$

**Remark 1 :-** *If  $q_0 = 0$  on the interface  $\Gamma_0$ , the solution of the interface problem belongs to  $H^1(\Omega)$ .*

### 3 Discretization and the stability estimate

Considered the circular domain  $\Omega_1$  such that  $\overline{\Omega}_1 \subset \Omega$ , where  $\Omega$  is square whose boundary is  $\Gamma = \cup_{i=1}^4 \Gamma^i$  as shown in Figure 1, for brevity. Let  $\Omega_2 = \Omega \setminus \overline{\Omega}_1$  and the interface is  $\Gamma_0$  which is smooth as shown in Figure 1. The results presented are applicable to arbitrary smooth interfaces also.

Now the domain  $\Omega_1$  and  $\Omega_2$  are partitioned into finite number of quadrilateral subdomains (elements)  $\Omega_1^1, \Omega_1^2, \dots, \Omega_1^p$  and  $\Omega_2^1, \Omega_2^2, \dots, \Omega_2^q$  such that the subdomain divisions match on the interface. The division of the domain into quadrilaterals is more general as a triangle can be further subdivided into three quadrilaterals by joining midpoints of the sides of the triangle to the center of the triangle.

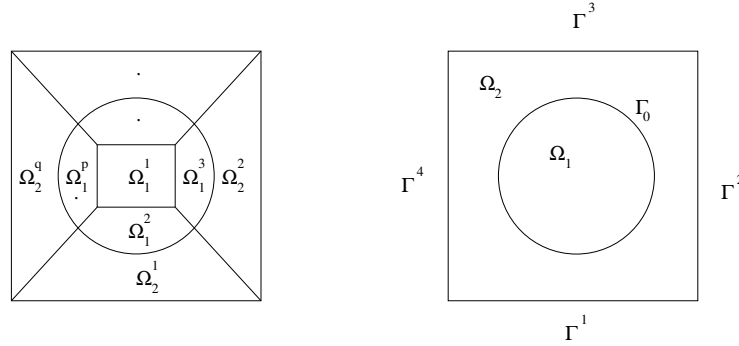


Figure 1: The discretization and the domain  $\Omega$ .

Define an analytic map  $M_i^l$  from the master square  $S = (-1, 1)^2$  to  $\Omega_i^l$  by (see [2, 11])

$$\begin{aligned} x &= X_i^l(\xi, \eta), \\ y &= Y_i^l(\xi, \eta), i = 1, 2. \end{aligned}$$

Here and in the rest of this section  $l = 1, \dots, p$  for  $i = 1$  and  $l = 1, \dots, q$  for  $i = 2$ .

Define the spectral element functions  $\{\tilde{u}_i^l\}_i$  as the tensor product of polynomials<sup>1</sup> of degree  $W$  in each variable  $\xi$  and  $\eta$  as

$$\tilde{u}_i^l(\xi, \eta) = \sum_{r=0}^W \sum_{s=0}^W t_{r,s}^i \xi^r \eta^s$$

Then  $\{u_i^l\}_i$  are given by

$$u_i^l(x, y) = \tilde{u}_i^l \left( (M_i^l)^{-1} \right).$$

#### Stability Estimate

Let  $J_i^l(\xi, \eta)$  be the Jacobian of the mapping  $M_i^l(\xi, \eta)$  from  $S = (-1, 1)^2$  to  $\Omega_i^l$  for  $i = 1, 2$ . Now

$$\int_{\Omega_i^l} |\mathcal{L}u_i^l|^2 dx dy = \int_S |\mathcal{L}u_i^l|^2 J_i^l d\xi d\eta.$$

Define  $\mathcal{L}_i^l \tilde{u}_i^l = \mathcal{L}u_i^l \sqrt{J_i^l}$ . Then

$$\int_{\Omega_i^l} |\mathcal{L}u_i^l|^2 dx dy = \int_S |\mathcal{L}_i^l \tilde{u}_i^l|^2 d\xi d\eta$$

<sup>1</sup>Lagrangian interpolating polynomials defined at Gauss-Legendre-Lobatto quadrature points in  $[-1, 1]^2$ .

where

$$\mathcal{L}_i^l \tilde{u} = \mathcal{A}_i^l \tilde{u}_{\xi\xi} + 2\mathcal{B}_i^l \tilde{u}_{\xi\eta} + \mathcal{C}_i^l \tilde{u}_{\eta\eta} + \mathcal{D}_i^l \tilde{u}_{\xi} + \mathcal{E}_i^l \tilde{u}_{\eta} + \mathcal{F}_i^l \tilde{u} \text{ for } i = 1, 2.$$

Here some notations are given which are needed to define the functional for the stability estimate.

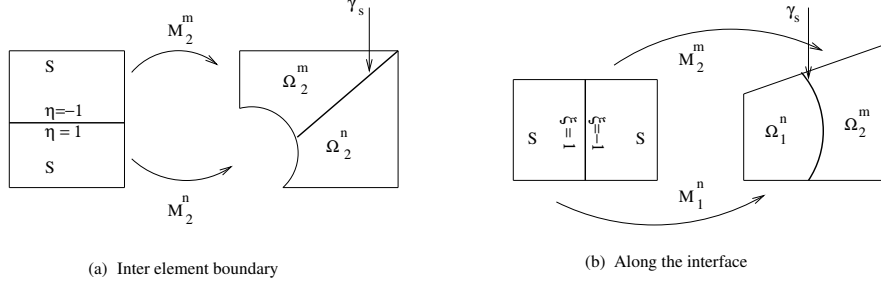


Figure 2: The elements with common edges

Let  $\gamma_s$  be a side common to the two adjacent elements  $\Omega_i^m$  and  $\Omega_i^n$ ,  $i = 1, 2$  (as shown in Fig. 2(a) for  $i = 2$ ). Assume that  $\gamma_s$  is the image of  $\eta = -1$  under the mapping  $M_i^m$  which maps  $S$  to  $\Omega_i^m$  and also the image of  $\eta = 1$  under the mapping  $M_i^n$  which maps  $S$  to  $\Omega_i^n$ . By chain rule

$$(u_i^m)_x = (\tilde{u}_i^m)_\xi \xi_x + (\tilde{u}_i^m)_\eta \eta_x, \text{ and} \\ (u_i^m)_y = (\tilde{u}_i^m)_\xi \xi_y + (\tilde{u}_i^m)_\eta \eta_y.$$

Then the jumps along the inter-element boundaries are defined as

$$\begin{aligned} \|[u_i]\|_{0,\gamma_s}^2 &= \|\tilde{u}_i^m(\xi, -1) - \tilde{u}_i^n(\xi, 1)\|_{0,I}^2, \\ \|[(u_i)_x]\|_{1/2,\gamma_s}^2 &= \|(u_i^m)_x(\xi, -1) - (u_i^n)_x(\xi, 1)\|_{1/2,I}^2, \text{ and} \\ \|[(u_i)_y]\|_{1/2,\gamma_s}^2 &= \|(u_i^m)_y(\xi, -1) - (u_i^n)_y(\xi, 1)\|_{1/2,I}^2. \end{aligned}$$

Here and in what follows,  $I$  is an interval  $(-1, 1)$ .

As the division of the domain into subdomains match along the interface, we define the jump across the interface by taking it (a part of interface) as the common edge. Consider the elements  $\Omega_1^n$  and  $\Omega_2^m$  (as shown in Fig. 2(b)) which have the common edge  $\gamma_s \subseteq \Gamma_0$ . Let  $\gamma_s$  be the image of  $\xi = 1$  under the mapping  $M_1^n$  which maps  $S$  to  $\Omega_1^n$  and also the image of  $\xi = -1$  under the mapping  $M_2^m$  which maps  $S$  to  $\Omega_2^m$ . Define

$$\|[u]\|_{\frac{3}{2},\gamma_s}^2 = \|u_1 - u_2\|_{\frac{3}{2},\gamma_s}^2 = \|\tilde{u}_1^m(-1, \eta) - \tilde{u}_2^n(1, \eta)\|_{0,I}^2 + \left\| \frac{\partial \tilde{u}_1^m}{\partial T}(-1, \eta) - \frac{\partial \tilde{u}_2^n}{\partial T}(1, \eta) \right\|_{1/2,I}^2,$$

$\frac{\partial u_1}{\partial T}$  and  $\frac{\partial u_2}{\partial T}$  are the tangential derivatives of  $u_1$  and  $u_2$  respectively. Let  $n = (n_1, n_2)$  be the unit outward normal to the interface  $\Gamma_0$ . From the given data  $\beta = \beta^-$  in  $\Omega_1$  and  $\beta = \beta^+$  in  $\Omega_2$ , then

$$\|[\beta \frac{\partial u}{\partial n}]\|_{\frac{1}{2},\gamma_s}^2 = \left\| \beta^- \frac{\partial u_1}{\partial n} - \beta^+ \frac{\partial u_2}{\partial n} \right\|_{\frac{1}{2},\gamma_s}^2 = \left\| \beta^- \frac{\partial \tilde{u}_1^m}{\partial n}(-1, \eta) - \beta^+ \frac{\partial \tilde{u}_2^n}{\partial n}(1, \eta) \right\|_{1/2,I}^2.$$

Now along the boundary  $\Gamma = \cup_{j=1}^4 \Gamma^j$ , let  $\gamma_s \subseteq \Gamma^j$  (for some  $j$ ) be the image of  $\xi = 1$  under the mapping  $M_2^m$  which maps  $S$  to  $\Omega_2^m$ . Then

$$\|u_2\|_{0,\gamma_s}^2 + \left\| \frac{\partial u_2}{\partial T} \right\|_{1/2,\gamma_s}^2 = \|\tilde{u}_2^m(1, \eta)\|_{0,I}^2 + \left\| \frac{\partial \tilde{u}_2^m}{\partial T}(1, \eta) \right\|_{1/2,I}^2.$$

Let  $\Pi^W = (\{\tilde{u}_1^k(\xi, \eta)\}_k, \{\tilde{u}_2^l(\xi, \eta)\}_l)$  be the space of spectral element functions. Now define the functional  $\mathcal{V}^W(\{\tilde{u}_1^k(\xi, \eta)\}_k, \{\tilde{u}_2^l(\xi, \eta)\}_l)$  as

$$\begin{aligned}
\mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) &= \sum_{k=1}^p \left\| (\mathcal{L}_1^k) \tilde{u}_1^k(\xi, \eta) \right\|_{0,S}^2 + \sum_{l=1}^q \left\| (\mathcal{L}_2^l) \tilde{u}_2^l(\xi, \eta) \right\|_{0,S}^2 \\
&+ \sum_{i=1}^2 \sum_{\gamma_s \subseteq \Omega_i} \left( \| [u_i] \|_{0,\gamma_s}^2 + \| [(u_i)_x] \|_{1/2,\gamma_s}^2 + \| [(u_i)_y] \|_{1/2,\gamma_s}^2 \right) \\
&+ \sum_{\gamma_s \subseteq \Gamma_0} \left( \| [u] \|_{3/2,\gamma_s}^2 + \left\| \left[ \beta \frac{\partial u}{\partial n} \right] \right\|_{1/2,\gamma_s}^2 \right) + \sum_{\gamma_s \subseteq \Gamma} \| u_2 \|_{0,\gamma_s}^2 + \left\| \frac{\partial u_2}{\partial T} \right\|_{1/2,\gamma_s}^2.
\end{aligned}$$

**THEOREM 3.1 :-** For  $W$  large enough there exists a constant  $C > 0$  such that

$$\sum_{k=1}^p \left\| \tilde{u}_1^k(\xi, \eta) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| \tilde{u}_2^l(\xi, \eta) \right\|_{2,S}^2 \leq C(\ln W)^2 \mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right).$$

**Proof:** By the Lemma 3.2 there exists  $\{ \{ \tilde{v}_1^k(\xi, \eta) \}_k, \{ \tilde{v}_2^l(\xi, \eta) \}_l \}$  (where  $\tilde{v}_1^k = 0$  on  $\Gamma_0$  for all  $k = 1, 2, \dots, p$ ,  $\tilde{v}_2^l = 0$  on  $\Gamma_0$  for all  $l = 1, 2, \dots, q$ ) such that  $w_i = u_i + v_i \in H^2(\Omega_i)$  for  $i = 1, 2$ . Moreover  $w_1 = u_1$  and  $w_2 = u_2$  on the interface  $\Gamma_0$ . Hence by the regularity result stated in Section 2

$$\| w_1 \|_{2,\Omega_1}^2 + \| w_2 \|_{2,\Omega_2}^2 \leq C \left( \| \mathcal{L}_1 w_1 \|_{0,\Omega_1}^2 + \| \mathcal{L}_2 w_2 \|_{0,\Omega_2}^2 + \| w_2 \|_{3/2,\Gamma}^2 + \| [w] \|_{3/2,\Gamma_0}^2 + \left\| \left[ \beta \frac{\partial w}{\partial n} \right] \right\|_{1/2,\Gamma_0}^2 \right).$$

The rest of the proof follows from the Lemma 3.1 and Lemma 3.2.

**Lemma 3.1 :-** Let  $\{ \{ \tilde{u}_1^k(\xi, \eta) \}_k, \{ \tilde{u}_2^l(\xi, \eta) \}_l \} \in \Pi^W$ . Then there exists  $\{ \{ \tilde{v}_1^k(\xi, \eta) \}_k, \{ \tilde{v}_2^l(\xi, \eta) \}_l \}$  (where  $\tilde{v}_1^k = 0$  on  $\Gamma_0$  for all  $k = 1, 2, \dots, p$ ,  $\tilde{v}_2^l = 0$  on  $\Gamma_0$  for all  $l = 1, 2, \dots, q$ ) such that  $v_1^k, v_2^l \in H^2(S)$ , for  $k = 1, 2, \dots, p, l = 1, 2, \dots, q$  and  $u_1 + v_1 \in H^2(\Omega_1), u_2 + v_2 \in H^2(\Omega_2)$ . Moreover the estimate

$$\sum_{k=1}^p \left\| \tilde{v}_1^k(\xi, \eta) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| \tilde{v}_2^l(\xi, \eta) \right\|_{2,S}^2 \leq C(\ln W)^2 \left( \sum_{i=1}^2 \sum_{\gamma_s \subseteq \Omega_i} \left( \| [u_i] \|_{0,\gamma_s}^2 + \| [(u_i)_x] \|_{1/2,\gamma_s}^2 + \| [(u_i)_y] \|_{1/2,\gamma_s}^2 \right) \right).$$

**Lemma 3.2 :-** Let  $w_i = u_i + v_i \in H^2(\Omega_i)$  for  $i = 1, 2$ . Here  $\{ \{ \tilde{u}_1^k(\xi, \eta) \}_k, \{ \tilde{u}_2^l(\xi, \eta) \}_l \} \in \Pi^W$  and  $\{ \{ \tilde{v}_1^k(\xi, \eta) \}_k, \{ \tilde{v}_2^l(\xi, \eta) \}_l \}$  is as defined in Lemma 3.1. Then the estimate

$$\| w_2 \|_{3/2,\Gamma}^2 \leq C(\ln W)^2 \left( \sum_{\gamma_s \subseteq \Gamma} \| u_2 \|_{0,\gamma_s}^2 + \left\| \frac{\partial u_2}{\partial T} \right\|_{1/2,\gamma_s}^2 + \sum_{\gamma_s \subseteq \Omega_i, i=1}^2 \left( \| [u_i] \|_{0,\gamma_s}^2 + \| [(u_i)_x] \|_{1/2,\gamma_s}^2 + \| [(u_i)_y] \|_{1/2,\gamma_s}^2 \right) \right).$$

Proof of the Lemma 3.1 and Lemma 3.2 easily follows from the Lemma 7.1 and Lemma 7.2 of [8].

## 4 Numerical Technique

Define  $f_1 = f|_{\Omega_1}$  and  $f_2 = f|_{\Omega_2}$ . Let  $J_1^l(\xi, \eta)$  be the Jacobian of the mapping  $M_1^l(\xi, \eta)$  from  $S = (-1, 1)^2$  to  $\Omega_1^l$  for  $l = 1, 2, \dots, p$ . Let  $f_1^l(\xi, \eta) = f_1(M_1^l(\xi, \eta))$  and define

$$F_1^l(\xi, \eta) = f_1^l(\xi, \eta) \sqrt{J_1^l(\xi, \eta)},$$

for  $l = 1, 2, \dots, p$ . Similarly one can define  $F_2^k(\xi, \eta)$  for  $k = 1, 2, \dots, q$ .



As defined earlier  $u_1 = u|_{\Omega_1}$  and  $u_2 = u|_{\Omega_2}$ , so the boundary condition  $u = g$  on  $\Gamma$  in the discrete form will be  $u_2 = g$  on  $\Gamma^j \cap \partial\Omega_2^m$ . Let  $\Gamma^j \cap \partial\Omega_2^m = c_2^m$  be the image of the mapping  $M_2^m$  of  $S$  onto  $\Omega_2^m$  corresponding to the side  $\xi = 1$  and

$$o_2^m(\eta) = g(M_2^m(1, \eta)),$$

where  $-1 \leq \eta \leq 1$ .

On an interface  $\Gamma_0$  we have  $[u] = q_0$  and  $[\beta \frac{\partial u}{\partial n}] = q_1$ . Let  $\gamma_s \subseteq \Gamma_0$  be the image of  $\xi = 1$  under the mapping  $M_1^n$  which maps  $S$  to  $\Omega_1^n$  and also the image of  $\xi = -1$  under the mapping  $M_2^m$  which maps  $S$  to  $\Omega_2^m$ . Let

$$l_0^{m,n}(\eta) = q_0(-1, \eta) = q_0(1, \eta) \text{ for } -1 \leq \eta \leq 1.$$

Similarly one can define

$$l_1^{m,n}(\eta) = q_1(-1, \eta) = q_1(1, \eta) \text{ for } -1 \leq \eta \leq 1.$$

Define the functional

$$\begin{aligned} \mathbf{r}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) &= \sum_{k=1}^p \left\| (\mathcal{L}_1^k) \tilde{u}_1^k(\xi, \eta) - F_1^k(\xi, \eta) \right\|_{0,S}^2 + \sum_{l=1}^q \left\| (\mathcal{L}_2^l) \tilde{u}_2^l(\xi, \eta) - F_2^l(\xi, \eta) \right\|_{0,S}^2 \\ &+ \sum_{i=1}^2 \sum_{\gamma_s \subseteq \Omega_i} \left( \left\| [u_i] \right\|_{0,\gamma_s}^2 + \left\| [(u_i)_x] \right\|_{1/2,\gamma_s}^2 + \left\| [(u_i)_y] \right\|_{1/2,\gamma_s}^2 \right) \\ &+ \sum_{\gamma_s \subseteq \Gamma_0} \left( \left\| [u] - l_0^{m,n} \right\|_{3/2,\gamma_s}^2 + \left\| \left[ \beta \frac{\partial u}{\partial n} \right] - l_1^{m,n} \right\|_{1/2,\gamma_s}^2 \right) \\ &+ \sum_{\gamma_s \subseteq \Gamma} \left( \left\| u_2 - o_2^m(\eta) \right\|_{0,\gamma_s}^2 + \left\| \left( \frac{\partial u_2}{\partial T} \right) - \left( \frac{\partial o_2^m}{\partial T} \right) \right\|_{1/2,\gamma_s}^2 \right). \end{aligned} \quad (4.1)$$

The approximate solution is chosen as the unique  $\{ \{ \tilde{z}_1^k(\xi, \eta) \}_k, \{ \tilde{z}_2^l(\xi, \eta) \}_l \} \in \Pi^W$ , which minimizes the functional  $\mathbf{r}^W(\{ \tilde{u}_1^k(\xi, \eta) \}_k, \{ \tilde{u}_2^l(\xi, \eta) \}_l)$  over all  $\{ \{ \tilde{u}_1^k(\xi, \eta) \}_k, \{ \tilde{u}_2^l(\xi, \eta) \}_l \}$ .

The minimization problem leads to a system of equations of the form

$$AZ = h. \quad (4.2)$$

Here  $A$  is symmetric, positive definite matrix and the vector  $Z$  is composed of the values of spectral element functions at Gauss-Legendre-Lobatto points. The system (4.2) is solved using preconditioned conjugate gradient method (PCGM). Since the method is nonconforming, there is no set of common boundary values to be obtained. The residuals in the normal equations can be computed efficiently and inexpensively [25]. In each iteration of the PCGM the jumps in the value of the function and its derivatives along the inter element boundaries as well as along the interface in the minimization, are carried out by interchanging the function and its derivative values between each element. An efficient preconditioner has been used which is proposed in [9] for the matrix  $A$  so that the condition number of the preconditioned system is as small as possible. The condition number of the preconditioned system is  $O((\ln W)^2)$ . The preconditioner is a block diagonal matrix, where each diagonal block is constructed using the separation of variable technique (A.1). The solution is obtained to an exponential accuracy using  $O(W \ln W)$  iterations of the PCGM.

After obtaining a nonconforming solution by solving the normal equations using PCGM a set of corrections are made to the solution similar to Lemma 4.57 of [24], so that the corrected solution is conforming. Then the following error estimate holds:

**THEOREM 4.1 :-** For  $W$  large enough

$$\|u - z\|_{1,\Omega} \leq C e^{-bW},$$

where  $C$  and  $b$  are constants and  $z$  is the corrected solution.

The proof is given in (A.2).

**Note:** The computational procedure of the integrals arising in the symmetric formulation are similar to the one given in the appendix of [25].

## 5 Numerical Results

The relative error  $\|e\|_{ER}$  is defined as  $\|e\|_{ER} = \frac{\|e\|_E}{\|u\|_E}$ , where  $\|\cdot\|_E$  denotes energy norm ( $H^1$ -norm). In all of the examples degree of the approximating polynomial is denoted by  $^2W$ , 'iters' means the total number of iterations required to compute the solution using PCGM and 'error' means  $u_{ex} - u_{num}$ . For Ex's 5.1-5.3 the discretization of the domain is as defined in Section 3.

**Example 5.1 :** Consider the following interface problem on a domain which is a square  $\Omega = [-1, 1]^2$  with a circle centered at the origin of radius  $s$  as an interface.

$$\begin{aligned} -\nabla \cdot (\nabla \beta u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

where

$$\beta = \begin{cases} 1 & \text{if } r < s, \\ p & \text{if } r \geq s. \end{cases}$$

Let the solution  $u$  satisfies the interface conditions

$$[u] = 0 \text{ and } [\beta \frac{\partial u}{\partial n}] = 0.$$

The data is chosen such that the given interface problem has the exact solution

$$u(x, y) = \begin{cases} r^\alpha & \text{if } r < s, \\ \frac{r^\alpha}{p} + \left(1 - \frac{1}{p}\right) s^\alpha & \text{if } r \geq s. \end{cases}$$

where  $r = \sqrt{x^2 + y^2}$  and  $\alpha = 2$ . Here we choose the radius of the circle  $s = 1/2$ . Note that the exact solution satisfies above defined interface conditions.

The conforming solution is obtained for various values of  $p$  and the relative error  $\|e\|_{ER}$  in percent, the iterations are tabulated in Table 1 and Table 2.

$p = 5$			$p = 10$			$p = 50$		
$W$	$\ e\ _{ER} \%$	iters	$\ e\ _{ER} \%$	iters		$\ e\ _{ER} \%$	iters	
2	1.03482E+01	6	3.18875E+01	7		4.27347E+01	12	
3	1.937462E-00	16	1.529332E-00	27		1.395556E-00	37	
4	3.623503E-01	22	9.286868E-02	35		1.622266E-01	73	
5	2.172657E-02	37	3.962240E-02	55		3.547243E-02	129	
6	2.058583E-03	48	2.521481E-03	67		2.223661E-03	162	
7	1.631193E-04	59	1.645520E-04	88		2.831054E-04	265	
8	2.470227E-05	73	2.731148E-05	114		1.262566E-05	304	
9	2.542338E-06	86	2.724790E-06	139		2.403579E-06	384	

Table 1: The relative error in percent and iterations for different  $W$

<sup>2</sup>For the simplicity of the programming  $W$  is assumed to be uniform.

$p = 100$			$p = 500$		
$W$	$\ e\ _{ER} \%$	iter	$\ e\ _{ER} \%$	iter	
2	2.576707E+01	16	4.268661E+01	16	
3	1.472354E-00	46	1.5222003E-00	78	
4	2.6243884E-01	103	1.3855314E-01	215	
5	2.1261933E-02	154	1.8194082E-02	308	
6	2.4964483E-03	208	1.2277585E-03	439	
7	2.6893833E-04	314	2.8411281E-04	601	
8	1.6674588E-05	412	2.6438164E-05	770	
9	2.6689012E-06	530	3.7427644E-06	985	

Table 2: The relative error in percent and iterations for  $p = 100, 500$

In Fig. 3 the exact and the numerical solution are drawn at the same number of nonuniform grid points for  $p = 500$  and  $W = 7$ , the grid points are shown in dots. In Fig. 4 the error in the solution is plotted for  $p = 500$  and  $W = 7$ .



Figure 3: Exact and numerical solution at same number of grid points

The log of relative error against  $W$  is drawn in Fig. 5 for  $p = 100, 500$ . The relation is almost linear. This shows the exponential accuracy of the method.

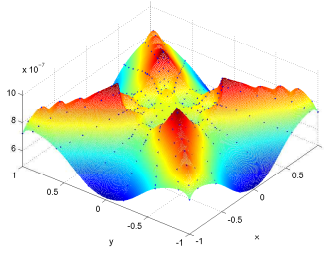


Figure 4: Error in the solution

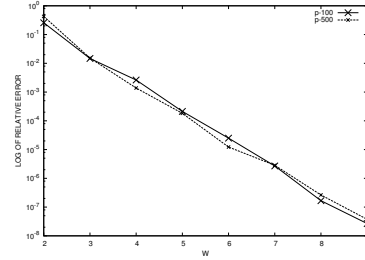


Figure 5: Log of relative error vs.  $W$

**Example 5.2 :** Consider the Laplace's equation  $\nabla^2 u = 0$  on a square domain  $\Omega = [-1, 1]^2$  with the interface as a circle  $x^2 + y^2 = 1/4$ . The jump conditions along the interface are

$$[u] = 0 \text{ and } [u_n] = 2.$$

This example can be considered as a problem where there is a singular source term along the interface, and the exact solution is given by

$$u(x, y) = \begin{cases} 1, & x^2 + y^2 < 1/4, \\ 1 + \log(2\sqrt{x^2 + y^2}), & x^2 + y^2 \geq 1/4. \end{cases}$$

The exterior boundary conditions are derived from the exact solution.

This problem has been addressed in [19, 27]. They have used IIM, EJIIM and shown the second order accuracy (in  $h$ ) in max norm. Table 3 contains the values of relative error in percentage and iterations against  $W$ , the degree of the approximating polynomial. The exact and the numerical solution at the same number of nonuniform grid points (shown as dots) are in Figure 6 for  $W = 12$ .

$W$	$\ e\ _{ER} \%$	iters
3	8.82794929E-00	13
4	1.14312663E-00	22
5	6.15649980E-01	27
6	1.58996531E-01	30
7	8.52831452E-02	33
8	2.53598439E-02	41
9	1.29475910E-02	43
10	4.08932952E-03	48
11	2.03647852E-03	50
12	6.65588003E-04	54

Table 3: Relative error in percentage and the number of iteration against  $W$

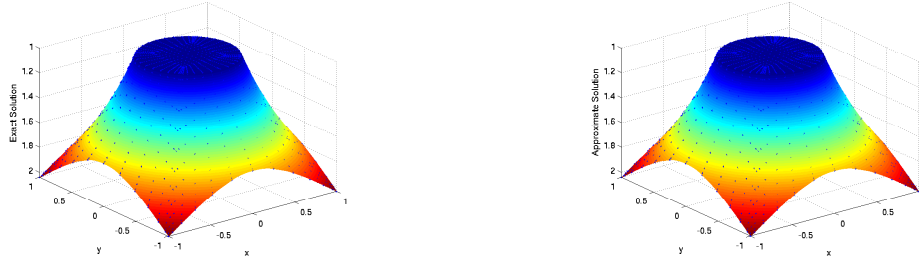


Figure 6: Exact solution and the numerical solution

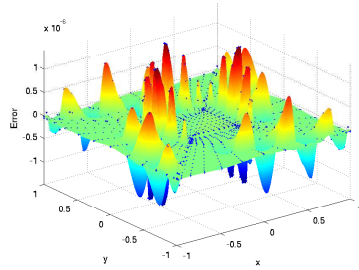


Figure 7: Error plot for the results in Ex. 5.2

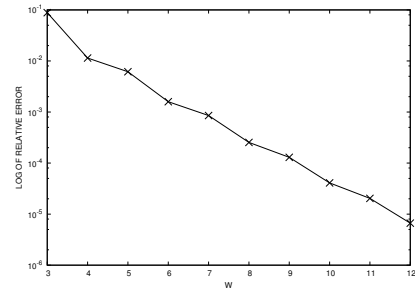


Figure 8: Log of relative error vs.  $W$

The graph shown in Figure 7 is the pointwise error for the degree of the approximating polynomial  $W = 12$ . In Figure 8 the plot of log of relative error against  $W$  is drawn, which is a straight line as  $W$  increases, shows the exponential convergence of the method.

**Example 5.3 :** Consider the Laplace's equation  $\nabla^2 u = 0$  on a square domain  $\Omega = [-1, 1]^2$  with the interface as a circle  $x^2 + y^2 = 1/4$ .

The exact solution is

$$u(x, y) = \begin{cases} \frac{2x}{\rho + 1 + s^2(\rho - 1)}, & r < s, \\ \frac{x(\rho + 1) - s^2(\rho - 1)x/r^2}{\rho + 1 + s^2(\rho - 1)}, & r \geq s, \end{cases}$$

and the diffusion coefficient is

$$\beta = \begin{cases} \beta^-, & r < s, \\ \beta^+, & \text{otherwise.} \end{cases}$$

Here  $\rho = \beta^-/\beta^+$  is the contrast ratio,  $r = \sqrt{x^2 + y^2}$  and  $s$  is the radius of the interface. Hence the jump conditions are

$$[u] = 0 \text{ and } [\beta u_n] = 0.$$

This example can be considered as a composite material problem with piecewise constant coefficients. The exterior boundary conditions are derived from the analytical solution.

This problem has been addressed in [27], shown that EJIIM is second order accurate (in  $h$ ) in max norm. The problem is solved for high contrast ratio's, likely  $\rho = 1/5000$  and  $\rho = 5000$ . Chosen  $s = 1/2$  in both the cases. In Table 4 and Table 5 the results containing the relative error and iterations against  $W$  for  $\rho = 1/5000$  and  $\rho = 5000$  are given.

$W$	$\ e\ _{ER} \%$	iters
3	6.8613422E+01	56
4	1.7514331E+01	101
5	3.0852021E+00	213
6	4.5581811E-01	281
7	1.6298455E-01	374
8	6.8485556E-02	424
9	2.9545968E-02	524
10	1.2798077E-02	594
11	5.4965594E-03	670
12	2.3688017E-03	751

Table 4: Results for  $\rho = 1/5000$

$W$	$\ e\ _{ER} \%$	iters	user time
3	1.9965679E+01	72	1.100068
4	6.61212224E-00	139	3.344209
5	1.90708701E-00	170	10.52066
6	6.28637729E-01	261	22.65742
7	2.39188629E-01	351	44.05475
8	9.96239378E-02	440	77.71285
9	4.27909532E-02	497	132.2283
10	1.8498735E-02	606	194.4122
11	7.9577535E-03	708	323.2722
12	3.43214350E-03	820	443.9918

Table 5: Results for  $\rho = 5000$

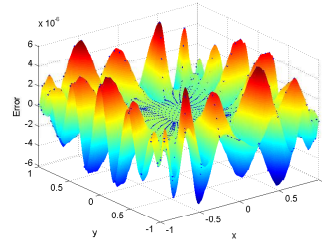
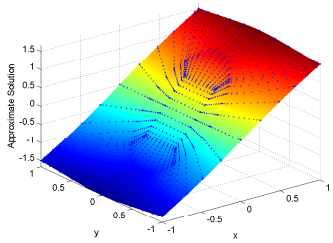


Figure 9: Numerical solution and the error plot for  $\rho = 1/5000$

The graphs of approximate solution and the error for  $W = 12$  are given in Fig. 9 for  $\rho = 1/5000$  and in Fig. 10 for  $\rho = 5000$ .



Figure 10: Numerical solution and the error plot for  $\rho = 5000$

In Table 5, in addition the user time(on screen time) in seconds is given as it shows the time taken for the computations are few minutes. The computations are performed on an intel dual-core processor machine with 2GB RAM. The same is true for the rest of the examples too. Fig. 11 shows the log of the relative error against  $W$  for the both cases that is  $\rho = 1/5000$  and  $\rho = 5000$ .

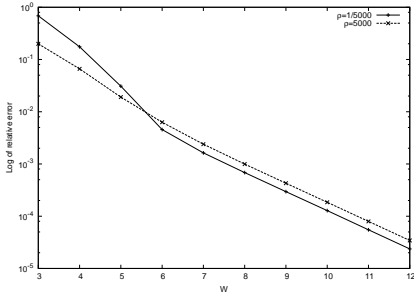


Figure 11: Log of the relative error against  $W$

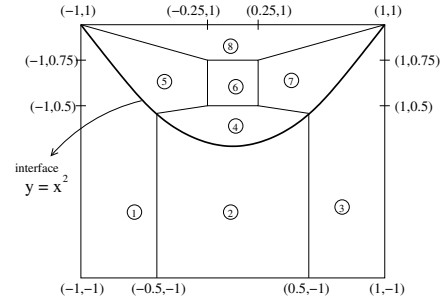


Figure 12: The domain and its discretization

**Example 5.4 :** Consider the following interface problem on the domain  $\Omega = [-1, 1]^2$  with homogeneous interface conditions

$$\begin{aligned} -\nabla \cdot (\nabla \beta u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

where

$$\beta = \begin{cases} \beta^- & \text{if } y < x^2, \\ \beta^+ & \text{if } y \geq x^2. \end{cases}$$

This example is an interface problem with curved interface. The interface is a parabola  $y = x^2$ , as shown in Fig. 12.

Chosen the data from the analytical solution

$$u = \begin{cases} \frac{(y - x^2)^2 - 5(y - x^2)}{\beta^-} & \text{if } y < x^2, \\ \frac{(y - x^2)^2 - 5(y - x^2)}{\beta^+} & \text{if } y \geq x^2. \end{cases}$$

The discretization of the domain into eight quadrilateral elements are shown in Fig. 12. The discretization matches along the interface. Two cases have been considered, in the first case the problem is solved with  $\beta^- = 10$  and  $\beta^+ = 1$ . In Table 6 the relative error and the iteration count against different values of approximating polynomial order  $W$  are given. In the second case  $\beta^- = 1$  and  $\beta^+ = 10$  is taken and the results are provided in Table 7.

$W$	$\ e\ _{ER} \%$	iters
2	2.836438E+01	11
3	8.873654E+00	34
4	1.619525E-01	59
5	1.020317E-02	86
6	1.743856E-03	107
7	1.470459E-04	141
8	1.208095E-05	171
9	1.580360E-06	199

Table 6: Results for Case I

$W$	$\ e\ _{ER} \%$	iters
2	1.6920600E+01	21
3	4.7569957E+00	49
4	1.9445176E-01	93
5	1.6722458E-02	112
6	1.5500771E-03	150
7	1.1095496E-04	191
8	1.4425672E-05	219
9	1.0296398E-06	258

Table 7: Results for Case II

In Fig. 13 the analytical and numerical solutions are shown at the same number of nonuniform grid points for Case I and  $W = 9$ . In Fig. 14 the error in the solution is plotted for Case I and  $W = 9$ . Fig. 15 shows the log of the relative error against  $W$  for both the cases.

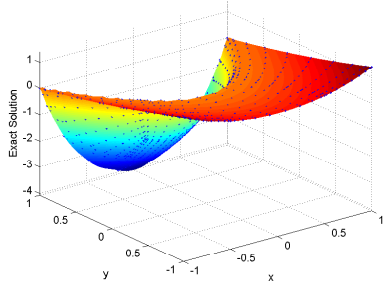


Figure 13: Exact and Numerical solution for the results in Case I

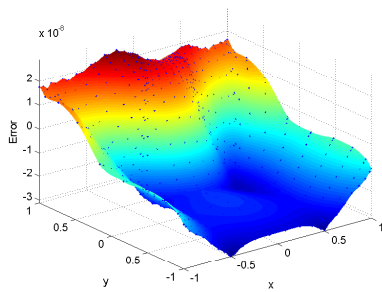
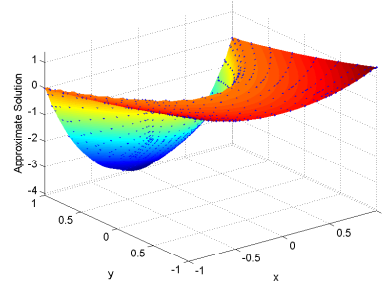


Figure 14: Error graph

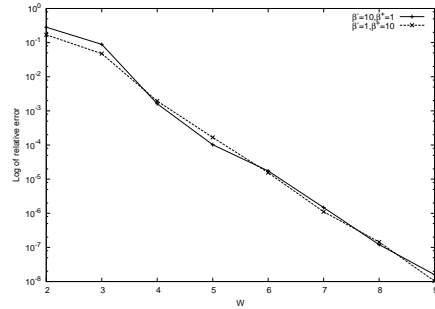


Figure 15: Log of relative error vs.  $W$  for Cases I & II

**Example 5.5 :** Consider the following problem on the domain  $\Omega = [0, 1]^2$  with the interface as a straight line, as shown in Fig. 16.

$$\begin{aligned} -\nabla \cdot (\nabla \beta u) &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

where

$$\beta = \begin{cases} 1 & \text{in } \Omega_1, \\ p & \text{in } \Omega_2. \end{cases}$$

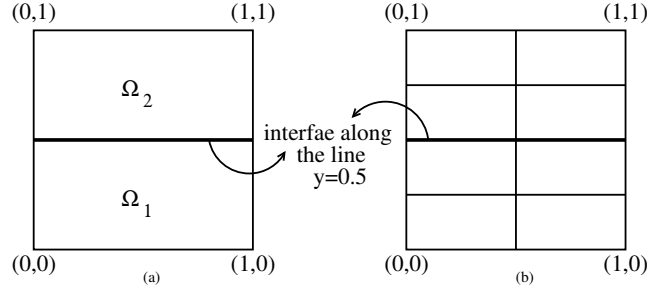


Figure 16: (a) The domain  $\Omega$  with an interface at  $y = 0.5$  (b) Discretization of the domain

The solution  $u$  satisfies the following homogeneous interface conditions at  $y = 0.5$

$$\begin{aligned} u(x, 0.5-) &= u(x, 0.5+) \\ \frac{\partial u}{\partial y}(x, 0.5-) &= p \frac{\partial u}{\partial y}(x, 0.5+) \end{aligned}$$

Choose the data such that the interface problem has the exact solution

$$u = \begin{cases} e^x (y^2 + (p-1)y) & \text{in } \Omega_1, \\ e^x \left( y^2 + \left( \frac{p-1}{2} \right) \right) & \text{in } \Omega_2. \end{cases}$$

Here the function  $f$  is given by

$$f = \begin{cases} e^x (y^2 + (p-1)y + 2) & \text{in } \Omega_1, \\ p e^x \left( y^2 + \left( \frac{p+3}{2} \right) \right) & \text{in } \Omega_2. \end{cases}$$

Divided the given domain as shown in Fig. 16(b) such that the discretization matches along the interface. The numerical solution is obtained for different values of  $p$ . The percentage in the relative error in  $H^1$ -norm and the number of iterations are tabulated in Table 8 and Table 9 against  $W$ . The error decays exponentially for all values of  $p$ . It has been noted that the number of iterations increases as the value of  $p$  increases, this is because of the approximation of high jumps in the minimization.

$p = 2$			$p = 10$			$p = 50$		
$W$	$\ e\ _{ER} \%$	iters	$\ e\ _{ER} \%$	iters	$\ e\ _{ER} \%$	iters		
2	2.382418E+00	12	1.627861E+00	18	2.107910E+00	23		
3	2.942283E-01	20	2.1180927E-01	35	1.579566E-01	50		
4	2.9571251E-02	30	2.4031170E-02	52	2.319688E-02	72		
5	2.3427500E-03	41	2.5334644E-03	70	1.223032E-03	103		
6	2.9779650E-04	49	1.9665916E-04	89	1.957943E-04	129		
7	3.0149553E-05	59	2.3130516E-05	112	2.374531E-05	160		
8	2.4204794E-06	73	2.373925E-06	132	2.488801E-06	193		
9	6.7775744E-08	85	2.1233162E-07	156	1.782141E-07	226		

Table 8: The relative error in percent, iterations against  $W$  for  $p = 2, 10, 50$ .



$p = 100$			$p = 500$		
$W$	$\ e\ _{ER} \%$	iters	$\ e\ _{ER} \%$	iters	
2	2.202174E+00	24	2.281236E+00	29	
3	2.1526836E-01	52	1.407257E-01	65	
4	2.522699E-02	77	1.542423E-02	99	
5	2.0123253E-03	108	1.467398E-03	135	
6	2.6720418E-04	138	1.238593E-04	173	
7	2.251335E-05	177	9.795864E-06	221	
8	9.2210359E-07	222	1.739342E-06	266	
9	1.6312305E-07	257	1.761961E-07	314	

Table 9: The relative error in percent, iterations against  $W$  for  $p = 100, 500$ .

The results shows the efficiency of the method. In Fig. 17 the log of relative error against the degree of the polynomial  $W$  is drawn for  $p = 100$  and  $p = 500$ . The relation is almost linear. This shows the exponential accuracy of the proposed method.

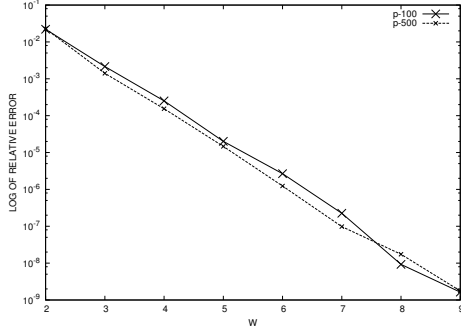


Figure 17: Plot for the result in Ex. 5.5

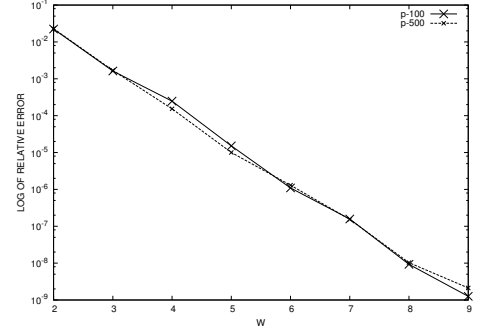


Figure 18: Plot for the result in Ex. 5.6

**Example 5.6 :** Consider the interface problem as in Ex. 5.5, such that the solution  $u$  satisfies the nonhomogeneous interface conditions at  $y = 0.5$

$$u(x, 0.5-) = u(x, 0.5+)$$

$$\frac{\partial u}{\partial y}(x, 0.5-) - p \frac{\partial u}{\partial y}(x, 0.5+) = -e^x.$$

Chosen the exact solution as

$$u = \begin{cases} e^x (y^2 + 2(p-1)y + 0.5) & \text{in } \Omega_1, \\ e^x (y^2 + y + (p-1)) & \text{in } \Omega_2, \end{cases}$$

which satisfies the interface conditions. Here the function  $f$  is given by

$$\begin{cases} e^x (y^2 + 2(p-1)y + 2.5) & \text{in } \Omega_1, \\ p e^x (y^2 + y + (p+1)) & \text{in } \Omega_2. \end{cases}$$

The discretization of the domain is as in Ex. 5.5. The relative error  $\|e\|_{ER}$  in percent and the number of iterations for various values of  $p$  are shown in Table 10.

Fig. 18 shows the graph of log of relative error against the degree of polynomial approximation  $W$  for  $p = 100$  and  $p = 500$ . The relation is almost linear, showing the exponential convergence.

$p = 10$			$p = 100$			$p = 500$		
$W$	$\ e\ _{ER} \%$	iters	$\ e\ _{ER} \%$	iters		$\ e\ _{ER} \%$	iters	
2	1.724767E+00	18	2.235594E+00	25		2.288274E+00	29	
3	1.916471E-01	36	1.629614E-01	54		1.687104E-01	64	
4	1.438651E-02	57	2.477020E-02	77		1.534507E-02	99	
5	1.066419E-03	74	1.518970E-03	109		1.006158E-03	137	
6	1.685558E-04	92	1.093721E-04	145		1.298757E-04	180	
7	8.027572E-06	119	1.573942E-05	182		1.537620E-05	218	
8	1.196847E-06	139	9.268662E-07	223		1.029490E-06	269	
9	1.622942E-07	158	1.256157E-07	258		2.123501E-07	312	

Table 10: The relative error in %, iterations against  $W$

**Conclusions** The method is non-conforming and exponentially accurate. The interface is resolved exactly using blending elements. The proposed preconditioner is decoupled block diagonal with optimal condition number and easily invertible on each element. A small data has to be interchanged in between the elements for each iteration of the PCGM and the residuals in the normal equations can be obtained efficiently and inexpensively. The proposed method is efficient even the jump in the coefficient is large. The numerical results shows that large differences in the coefficients leads to increase in the number of iterations of the PCGM but the time required for these computations are of few minutes. The method is applicable to arbitrary smooth interfaces too and the method can be extended to the singular case which is ongoing work.

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## A Appendix

### A.1 Preconditioner

In (4.2)  $A$  is symmetric, positive definite matrix and

$$\mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) = U^T A U.$$

Define the quadratic form

$$\mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) = \sum_{k=1}^p \left\| \tilde{u}_1^k \right\|_{2,S}^2 + \sum_{l=1}^q \left\| \tilde{u}_2^l \right\|_{2,S}^2.$$

Then for  $W$  large enough the following estimate holds

$$\mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) \leq c (\ln W)^2 \mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right). \quad (\text{A.1})$$

Here  $c$  is a constant.

Using the trace theorem for Sobolev spaces, the following inequality

$$\mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) \leq k \mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right), \quad (\text{A.2})$$

holds. Here  $k$  is a constant.

Hence using (A.1) and (A.2) it follows that there exists a constant  $K$  such that

$$\frac{1}{K} \mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) \leq \mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right) \leq K (\ln W)^2 \mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right).$$

Thus the two forms  $\mathcal{V}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right)$  and  $\mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right)$  are spectrally equivalent. We can use the quadratic form  $\mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right)$  which consists of a decoupled set of quadratic forms on each element as a preconditioner for  $A$ .

Here it is enough to consider the quadratic form  $\mathcal{B}(\tilde{v}) = \|\tilde{v}\|_{2,S}^2$ . It can be shown that  $\mathcal{C}(\tilde{v})$  is spectrally equivalent to  $\mathcal{B}(\tilde{v})$  which can be diagonalized using separation of variables similarity as shown in [9].

So the quadratic form  $\mathcal{U}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right)$  has a decoupled diagonal representation where each diagonal corresponds to the  $H^2$ -norm of the spectral element function representation on a particular element which is mapped onto the square  $S$ .

### A.2 Error estimate

Let  $u \in H^s(\Omega_1 \cup \Omega_2)$ , with the norm  $\|u\|_{s, \Omega_1 \cup \Omega_2}$  as defined earlier and  $f \in H^t(\Omega)$ . Without loss of generality assume that  $g, q_0$  and  $q_1$  to be zero. The result presented below is also true for the interface problems with non-homogeneous data.

**THEOREM A.1 :-** Let  $U_i^l(\xi, \eta) = u(M_i^l(\xi, \eta))$  for  $(\xi, \eta) \in S$ ,  $l = 1, \dots, p$  if  $i = 1$  and  $l = 1, 2, \dots, q$  if  $i = 2$ , and  $\left\{ \left\{ \tilde{z}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{z}_2^l(\xi, \eta) \right\}_l \right\} \in \Pi^W$  which minimizes  $\mathfrak{r}^W \left( \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right)$  over all  $\left\{ \left\{ \tilde{u}_1^k(\xi, \eta) \right\}_k, \left\{ \tilde{u}_2^l(\xi, \eta) \right\}_l \right\}$ .

Then for  $W$  large enough the estimate

$$\sum_{k=1}^p \left\| (\tilde{z}_1^k(\xi, \eta) - U_1^k(\xi, \eta)) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| (\tilde{z}_2^l(\xi, \eta) - U_2^l(\xi, \eta)) \right\|_{2,S}^2 \leq C_s e^{2s} W^{-2s+8} \ln W \text{ (I)} + C_t e^{2t} W^{-2t+8} \text{ (II)}$$

$$\text{holds, where } \mathbf{I} = \sum_{k=1}^p \left\| U_1^k(\xi, \eta) \right\|_{s,S}^2 + \sum_{l=1}^q \left\| U_2^l(\xi, \eta) \right\|_{s,S}^2 \quad \text{and} \quad \mathbf{II} = \sum_{k=1}^p \left\| F_1^k(\xi, \eta) \right\|_{t,S}^2 + \sum_{l=1}^q \left\| F_2^l(\xi, \eta) \right\|_{t,S}^2.$$

The proof is similar to the analysis given in [2, 26]. Here, briefly few main steps of the proof are described.

Using the results on approximation theory given in [2], there exists a polynomial  $\Phi_i^l(\xi, \eta)$  of degree  $W$  in each variable separately such that

$$\left\| U_i^l(\xi, \eta) - \Phi_i^l(\xi, \eta) \right\|_{2,S}^2 \leq C_s W^{-2s+8} \left\| U_i^l \right\|_{s,S}^2,$$

$W > s, l = 1, 2, \dots, p$  if  $i = 1$  and  $l = 1, 2, \dots, q$  if  $i = 2$ . Where  $C_s = C_1 e^{2s}$ .

Consider the set of functions  $\{\{\Phi_1^k(\xi, \eta)\}_k, \{\Phi_2^l(\xi, \eta)\}_l\}$ . Then the estimate

$$\mathfrak{r}^W(\{\Phi_1^k(\xi, \eta)\}_k, \{\Phi_2^l(\xi, \eta)\}_l) \leq C_s W^{-2s+8} \ln W \text{ (I)} + C_t W^{-2t+8} \text{ (II)}$$

holds.

Since  $\{\{\tilde{z}_1^k(\xi, \eta)\}_k, \{\tilde{z}_2^l(\xi, \eta)\}_l\}$  minimizes  $\mathfrak{r}^W(\{\tilde{u}_1^k(\xi, \eta)\}_k, \{\tilde{u}_2^l(\xi, \eta)\}_l)$ , the estimate

$$\mathfrak{r}^W(\{\tilde{z}_1^k(\xi, \eta)\}_k, \{\tilde{z}_2^l(\xi, \eta)\}_l) \leq C_s W^{-2s+8} \ln W \text{ (I)} + C_t W^{-2t+8} \text{ (II)}$$

holds. Using the stability estimate (Theo. 3.1),

$$\sum_{k=1}^p \left\| (\tilde{z}_1^k(\xi, \eta) - \Phi_1^k(\xi, \eta)) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| (\tilde{z}_2^l(\xi, \eta) - \Phi_2^l(\xi, \eta)) \right\|_{2,S}^2 \leq C_s W^{-2s+8} \ln W \text{ (I)} + C_t W^{-2t+8} \text{ (II)}. \quad (\text{A.3})$$

Then it easy to show that

$$\sum_{k=1}^p \left\| U_1^k(\xi, \eta) - \Phi_1^k(\xi, \eta) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| U_2^l(\xi, \eta) - \Phi_2^l(\xi, \eta) \right\|_{2,S}^2 \leq C_s W^{-2s+8} \ln W \text{ (I)} + C_t W^{-2t+8} \text{ (II)}. \quad (\text{A.4})$$

Using the above estimates (A.3) and (A.4),

$$\sum_{k=1}^p \left\| U_1^k(\xi, \eta) - \tilde{z}_1^k(\xi, \eta) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| U_2^l(\xi, \eta) - \tilde{z}_2^l(\xi, \eta) \right\|_{2,S}^2 \leq C_s W^{-2s+8} \ln W \text{ (I)} + C_t W^{-2t+8} \text{ (II)}. \blacksquare$$

With the proper choices of  $s$  and  $t$ , sterling's formula and by Remark 2, one can prove that there exists a constant  $b > 0$  such that

$$\sum_{k=1}^p \left\| U_1^k(\xi, \eta) - \tilde{z}_1^k(\xi, \eta) \right\|_{2,S}^2 + \sum_{l=1}^q \left\| U_2^l(\xi, \eta) - \tilde{z}_2^l(\xi, \eta) \right\|_{2,S}^2 \leq C e^{-bW}.$$

Hence the proof of Theorem 4.1.

**Remark 2** :- If  $f$  and  $u$  are analytic, then there exists constants  $C$  and  $d$  such that

$$\left\| U_i^l \right\|_{s,S}^2 \leq (C d^s s!)^2 \quad \text{and} \quad \left\| F_i^l \right\|_{t,S}^2 \leq (C d^t t!)^2$$

for  $l = 1, 2, \dots, p$  if  $i = 1$  and  $l = 1, 2, \dots, q$  if  $i = 2$ .