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Fredholm Integro- Differential Equations of  
Fractional Order**

by

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# A CAS Wavelet Method for Solving Nonlinear Fredholm Integro- Differential Equations of Fractional Order

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## Abstract

*In this paper we present a computational method for solving a class of nonlinear Fredholm integro- differential equations of fractional order which is based on CAS (Cosine And Sine) wavelets. The CAS wavelet operational matrix of fractional integration is derived and used to transform the equation to a system of algebraic equations. some examples are included to demonstrate the validity and applicability of the technique.*

**Keywords:** fractional calculus, CAS wavelets, Fredholm integro- differential equations.

**2000 Mathematics Subject Classification:** 65R20, 65T60, 26A33.

# 1 Introduction

The conception of fractional calculus was first introduced in the middle of the 19th century. The use of fractional differential and integral operators in mathematical models has become increasingly widespread in recent years. However, it is in the past hundred years that the most fractional problems in engineering and scientific applications have been found. For example, the non-linear oscillation of earthquake can be modeled with fractional derivatives [14], and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [15], therefore differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [20, 18]. Also there are several techniques for solving such equations like Adomian decomposition method [23, 19], collocation method [22] and differential transform method [10].

Wavelets theory is a new and emerging area in mathematical research, it is very successfully used in signal analysis for waveform representation and segmentations, timefrequency analysis and fast algorithms for easy implementation [9, 1]. However, the interest to the wavelet treatment of various integral equations has recently increased due to promising applications of this method in computational chemistry [4, 5, 6, 11, 12, 13]. The aim of this work is to present a numerical method (CAS wavelet method) for approximating the solution of a nonlinear fractional integro-differential equation of the second kind:

$$D_*^\alpha f(x) - \lambda \int_0^1 k(x, t)[f(t)]^q dt = g(x), \quad q > 1, \quad (1.1)$$

with these supplementary conditions:

$$f^{(i)}(0) = \delta_i, \quad i = 0, 1, \dots, r-1, \quad \exists \quad r-1 < \alpha \leq r, \quad r \in \mathbb{N}, \quad (1.2)$$

where,  $g \in L^2([0, 1])$ ,  $k \in L^2([0, 1]^2)$  are known functions,  $f(x)$  is the unknown function,  $D_*^\alpha$  is the Caputo fractional differentiation operator and  $q$  is a positive integer.

There are several definitions of a fractional derivative of order  $\alpha > 0$ . The two most commonly used definitions are the Riemann- Liouville and Caputo. Each definition uses Riemann- Liouville fractional integration and derivatives of whole order. The Riemann- Liouville fractional integration of order  $\alpha$  is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad x > 0, \quad J^0 f(x) = f(x),$$

and the Caputo fractional derivatives of order  $\alpha$  is defined as  $D_*^\alpha f(x) = J^{m-\alpha} D^m f(x)$ , where  $D^m$  is the usual integer differential operator of order  $m$  and  $J^{m-\alpha}$  is the Riemann-

Liouville integral operator of order  $m - \alpha$  and  $m - 1 < \alpha \leq m$ . The relation between the Riemann- Liouville operator and Caputo operator is given by the following lemma [20]:

**Lemma 1.1.** *If  $m - 1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ , then  $D_*^\alpha J^\alpha f(x) = f(x)$ , and:*

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann- Liouville fractional derivative is computed in the reverse order. Therefore, the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem, but the Riemann- Liouville fractional derivative allows initial conditions in terms of fractional integrals and their derivatives.

So we first define CAS wavelets and Block Pulse Functions and approximating a function via them, then we introduce the operational matrix of fractional integration. After that the method is described and the error is analyzed. Finally numerical results are shown in figures and tables.

## 2 Function Approximation

In this section first we give some necessary definitions and mathematical preliminaries of CAS wavelets and Block Pulse Functions (BPFs) which are used further in this paper. Then function approximation via these two conceptions is introduced.

The CAS wavelets employed in this paper are defined as:

$$\psi_{n,m}(x) = \begin{cases} 2^{k/2} CAS_m(2^k x - n), & \text{if } \frac{n}{2^k} \leq x < \frac{n+1}{2^k}; \\ 0, & \text{otherwise,} \end{cases}$$

where:

$$CAS_m(x) = \cos(2m\pi x) + \sin(2m\pi x),$$

and  $n = 0, 1, \dots, 2^k - 1$ ,  $k \in \mathbb{N} \cup \{0\}$ ,  $m \in \mathbb{Z}$ .

It is clear that CAS wavelets have compact support i. e:

$$Supp(\psi_{n,m}(x)) = \overline{\{x : \psi_{n,m}(x) \neq 0\}} = \left[ \frac{n}{2^k}, \frac{n+1}{2^k} \right].$$

Let us introduce the following useful notation, corresponding to CAS wavelets here:

$$\tilde{\psi}_{n,m}(x) = \begin{cases} 2^{k/2} CAS_m(n - 2^k x), & \text{if } \frac{n}{2^k} \leq x < \frac{n+1}{2^k}; \\ 0, & \text{otherwise.} \end{cases}$$

An  $m$ -set of Block Pulse Functions (BPFs) over the interval  $[0, T)$  is defined as:

$$b_i(x) = \begin{cases} 1, & \text{if } \frac{iT}{m} \leq x < \frac{(i+1)T}{m}; \\ 0, & \text{otherwise,} \end{cases} \quad i = 0, 1, 2, \dots, m-1,$$

with a positive integer value for  $m$ . In this paper, it is assumed that  $T = 1$ , so BPFs are defined over  $[0, 1)$ . Now we explain some useful properties of BPFs:

- *Disjointness* :

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j; \\ 0, & i \neq j. \end{cases} \quad (2.1)$$

- *Orthogonality* :

$$\int_0^1 b_i(x)b_j(x)dx = \begin{cases} 1/m, & i = j; \\ 0, & i \neq j. \end{cases} \quad (2.2)$$

- *Completeness* :

For every  $f \in L^2([0, 1))$ , the sequence  $\{b_i\}$  is complete if  $\int b_i f = 0$  results in  $f = 0$  almost every where. Because of completeness of  $\{b_i(x)\}$ , Parseval's identity holds, i. e. we have  $\int_0^1 f^2(x)dx = \sum_{i=0}^{\infty} f_i^2 \|b_i(x)\|^2$ , for every real bounded function  $f(x) \in L^2([0, 1))$  and:

$$f_i = m \int_0^1 b_i(x)f(x)dx. \quad (2.3)$$

- BPFs have compact support i. e  $Supp(b_i(x)) = [\frac{i}{m}, \frac{i+1}{m}]$ .

The set of CAS wavelets forms an orthonormal basis for  $L^2([0, 1))$ . This implies that any function  $f(x)$  defined over  $[0, 1)$  can be expanded as:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \sum_{m \in \mathbb{Z}} c_{n,m} \psi_{n,m}(x) \\ &\cong \sum_{n=0}^{2^k-1} \sum_{m=-M}^M c_{n,m} \psi_{n,m}(x) \\ &= \mathbf{c}^T \mathbf{\Psi}(x), \end{aligned}$$

where  $c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle = \int_0^1 f(x)\psi_{n,m}(x)dx$ , and  $\langle f, g \rangle$  is the inner product of the function  $f$  and  $g$ ,  $\mathbf{c}$  and  $\mathbf{\Psi}$  are  $2^k(2M+1) \times 1$  vectors given by:

$$\mathbf{c} = [c_{0,-M}, c_{0,-M+1}, \dots, c_{0,M}, c_{1,-M}, \dots, c_{1,M}, \dots, c_{2^k-1,-M}, \dots, c_{2^k-1,M}]^T,$$

$$\mathbf{\Psi}(x) = [\psi_{0,-M}, \psi_{0,-M+1}, \dots, \psi_{0,M}, \psi_{1,-M}, \dots, \psi_{1,M}, \dots, \psi_{2^k-1,-M}, \dots, \psi_{2^k-1,M}].$$

*Notation.* From now we define  $m' = 2^k(2M + 1)$ , such that  $k, M \in \mathbb{N} \cup \{0\}$ .

Also from the orthogonality property of BPFs, it is possible to expand functions into their Block- Pulse series [21], this means that For every  $f(x) \in L^2([0, 1])$  we can write:

$$f(x) \cong \sum_{i=0}^{m-1} f_i b_i(x) = \mathbf{f}^T \mathbf{B}_m(x), \quad (2.4)$$

where:

$$\mathbf{f} = [f_0, f_1, \dots, f_{m-1}]^T,$$

$$\mathbf{B}_m(x) = [b_0(x), b_1(x), \dots, b_{m-1}(x)],$$

such that  $f_i$  for  $i = 0, 1, \dots, m - 1$  are obtained by Eq. (2.3).

### 3 Operational Matrix of Fractional Integration

Eq. (2.4) implies that CAS wavelets can be also expanded into an  $m'$ -term BPFs as:

$$\psi_{nm}(x) \cong \sum_{i=0}^{m'-1} f_i b_i(x).$$

By using the properties of CAS wavelets and Eq. (2.3) we have:

$$\begin{aligned} f_i &= m' \int_{i/m'}^{(i+1)/m'} \psi_{nm}(x) dx \\ &= \frac{m' 2^{k/2}}{2^{k/2+1} m \pi} \{ \sin(2m\pi(2^k \frac{i+1}{m'} - n)) - \sin(2m\pi(2^k \frac{i}{m'} - n)) \\ &\quad - \cos(2m\pi(2^k \frac{i+1}{m'} - n)) + \cos(2m\pi(2^k \frac{i}{m'} - n)) \} \\ &= \frac{m'}{2^{k/2+1} m \pi} \{ \tilde{\psi}_{nm}(\frac{i}{m'}) - \tilde{\psi}_{nm}(\frac{i+1}{m'}) \}, \end{aligned}$$

for  $i = n(2M + 1), \dots, (n + 1)(2M + 1) - 1$  and otherwise  $f_i = 0$ . Therefor we get:

$$\begin{aligned} \psi_{nm}(x) &\cong \frac{m'}{2^{k/2+1} m \pi} \underbrace{[0, 0, \dots, 0]}_{n(2M+1)}, \tilde{\psi}_{nm}(\frac{i}{m'}) - \tilde{\psi}_{nm}(\frac{i+1}{m'}), \dots, \tilde{\psi}_{nm}(\frac{i+2M}{m'}) - \tilde{\psi}_{nm}(\frac{i+2M+1}{m'}), \\ &0, \dots, 0] \mathbf{B}_{m'}(x) \end{aligned}$$

where  $i = n(2M + 1), n = 0, 1, \dots, 2^k - 1$  and  $m = -M, \dots, M$ . Therefore:

$$\mathbf{\Psi}(x) = \mathbf{\Phi}_{m' \times m'} \mathbf{B}_{m'}(x), \quad (3.1)$$

where  $\Phi_{m' \times m'} = \text{Diag}(\Phi_0, \Phi_1, \dots, \Phi_{2^k-1})$ , and  $\Phi_n$  for  $n = 0, 1, \dots, 2^k - 1$  is a  $(2M+1) \times (2M+1)$  matrix which is introduced as:

$$\Phi_n = \Lambda \left( \left[ \begin{array}{cccc} \tilde{\Psi}_n(\frac{i}{m'}) & \tilde{\Psi}_n(\frac{i+1}{m'}) & \dots & \tilde{\Psi}_n(\frac{i+2M}{m'}) \end{array} \right] - \left[ \begin{array}{cccc} \tilde{\Psi}_n(\frac{i+1}{m'}) & \tilde{\Psi}_n(\frac{i+2}{m'}) & \dots & \tilde{\Psi}_n(\frac{i+2M+1}{m'}) \end{array} \right] \right),$$

where:

$$\Lambda = \frac{m'}{2^{k/2+1}\pi} \begin{bmatrix} \frac{1}{-M} & \frac{1}{-M+1} & \dots & \frac{1}{M} \\ \frac{1}{-M} & \frac{1}{-M+1} & \dots & \frac{1}{M} \\ \vdots & \vdots & & \vdots \\ \frac{1}{-M} & \frac{1}{-M+1} & \dots & \frac{1}{M} \end{bmatrix},$$

$$\tilde{\Psi}_n(x) = \left[ \tilde{\psi}_{n,-M}(x) \quad \tilde{\psi}_{n,-M+1}(x) \quad \dots \quad \tilde{\psi}_{n,M}(x) \right]^T.$$

Kilicman and Al Zhour (see [16]) have given the Block Pulse operational matrix of fractional integration  $F^\alpha$  as follows:

$$(J^\alpha \mathbf{B}_{m'}) (x) \cong \mathbf{F}^\alpha \mathbf{B}_{m'}(x), \quad (3.2)$$

where:

$$\mathbf{F}^\alpha = \frac{1}{m'^\alpha} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m'-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m'-2} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m'-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and  $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ . Let:

$$(J^\alpha \Psi_{m'}) (x) \cong \mathbf{P}_{m' \times m'}^\alpha \Psi_{m'}(x), \quad (3.3)$$

where matrix  $\mathbf{P}_{m' \times m'}^\alpha$  is called the CAS wavelet operational matrix of fractional integration. Using Eqs. (3.1) and (3.2), we have:

$$(J^\alpha \Psi_{m'}) (x) \cong (J^\alpha \Phi_{m' \times m'} \mathbf{B}_{m'}) (x) = \Phi_{m' \times m'} (J^\alpha \mathbf{B}_{m'}) (x) \cong \Phi_{m' \times m'} \mathbf{F}^\alpha \mathbf{B}_{m'}(x). \quad (3.4)$$

By Eqs. (3.3) and (3.4), we get:

$$\mathbf{P}_{m' \times m'}^\alpha \Psi_{m'}(x) \cong \Phi_{m' \times m'} \mathbf{F}^\alpha \mathbf{B}_{m'}(x),$$

therefore the CAS wavelet operational matrix of fractional integration  $\mathbf{P}_{m' \times m'}^\alpha$  is given by:

$$\mathbf{P}_{m' \times m'}^\alpha = \Phi_{m' \times m'} \mathbf{F}^\alpha \Phi_{m' \times m'}^{-1}.$$

## 4 Application of the Method

Consider Eq. (1.1), by previous section, the two variable function  $k(x, t) \in L^2([0, 1])^2$  can be approximated as:

$$k(x, t) \cong \sum_{n=0}^{2^k-1} \sum_{l_1=-M}^M \sum_{m=0}^{2^k-1} \sum_{l_2=-M}^M k_{i,j} \psi_{n,l_1}(x) \psi_{m,l_2}(t),$$

for  $i = n(2M + 1) + l_1 + M + 1$ ,  $j = m(2M + 1) + l_2 + M + 1$ , or in the matrix form:

$$k(x, t) \cong \Psi^T(x) \mathbf{K} \Psi(t), \quad (4.1)$$

where  $\mathbf{K} = [k_{i,j}]$  and  $k_{i,j} = \langle \psi_{n,l_1}(x), \langle k(x, t), \psi_{m,l_2}(t) \rangle \rangle$ . Also the right hand side of Eq. (1.1) can be written as:

$$g(x) \cong \mathbf{g}^T \Psi(x). \quad (4.2)$$

Now, let:

$$D_*^\alpha f(x) \cong \mathbf{c}^T \Psi(x). \quad (4.3)$$

For simplicity, we can assume that  $\delta_i = 0$  (in the supplementary conditions (1.2)). Hence by using lemma 1.1 and Eqs. (4.3) and (3.3) we have:

$$f(x) = \mathbf{c}^T \mathbf{P}_{m' \times m'}^\alpha \Psi(x). \quad (4.4)$$

According to Eq. (3.1), from above equation we get:

$$f(x) = \mathbf{c}^T \mathbf{P}_{m' \times m'}^\alpha \Phi_{m' \times m'} \mathbf{B}_{m'}(x).$$

Define:

$$\mathbf{a} = [a_0, a_1, \dots, a_{m'-1}] = \mathbf{c}^T \mathbf{P}_{m' \times m'}^\alpha \Phi_{m' \times m'},$$

so,  $f(x) \cong \mathbf{a} \mathbf{B}_{m'}(x)$ . From the disjoint property of the BPFs, we have:

$$\begin{aligned} [f(x)]^2 &\cong [\mathbf{a} \mathbf{B}_{m'}(x)]^2 \\ &= [a_0 b_0(x) + a_1 b_1(x) + \dots + a_{m'-1} b_{m'-1}(x)]^2 \\ &= a_0^2 b_0(x) + a_1^2 b_1(x) + \dots + a_{m'-1}^2 b_{m'-1}(x) \\ &= [a_0^2, a_1^2, \dots, a_{m'-1}^2] \mathbf{B}_{m'}(x) \\ &= \tilde{\mathbf{a}}_2 \mathbf{B}_{m'}(x), \end{aligned}$$

and it is easy to show by induction that:

$$[f(x)]^q \cong [a_0^q, a_1^q, \dots, a_{m'-1}^q] \mathbf{B}_{m'}(x) = \tilde{\mathbf{a}}_q \mathbf{B}_{m'}(x), \quad (4.5)$$

where:

$$\tilde{\mathbf{a}}_q = [a_0^q, a_1^q, \dots, a_{m'-1}^q],$$

for all positive integers  $q$ . Using Eqs. (3.1), (4.1) and (4.5) we will have:

$$\begin{aligned} \int_0^1 k(x, t)[f(t)]^q dt &= \int_0^1 \Psi^T(x) \mathbf{K} \Psi(t) \mathbf{B}_{m'}^T(t) \tilde{\mathbf{a}}_q^T dt \\ &= \int_0^1 \Psi^T(x) \mathbf{K} \Phi_{m' \times m'} \mathbf{B}_{m'}(t) \mathbf{B}_{m'}^T(t) \tilde{\mathbf{a}}_q^T dt \\ &= \Psi^T(x) \mathbf{K} \Phi_{m' \times m'} \int_0^1 \mathbf{B}_{m'}(t) \mathbf{B}_{m'}^T(t) \tilde{\mathbf{a}}_q^T dt. \end{aligned} \quad (4.6)$$

By using properties Eqs. (2.1) and (2.2), we simplify the integral part of (4.6) as:

$$\begin{aligned} \int_0^1 \mathbf{B}_{m'}(t) \mathbf{B}_{m'}^T(t) \tilde{\mathbf{a}}_q^T dt &= \int_0^1 \left\{ \begin{bmatrix} b_0(t) & & & O \\ & b_1(t) & & \\ & & \ddots & \\ O & & & b_{m'-1}(t) \end{bmatrix} \begin{bmatrix} a_0^q \\ a_1^q \\ \vdots \\ a_{m'-1}^q \end{bmatrix} \right\} dt \\ &= \int_0^1 [a_0^q b_0(t), a_1^q b_1(t), \dots, a_{m'-1}^q b_{m'-1}(t)]^T dt \\ &= \frac{1}{m'} [a_0^q, a_1^q, \dots, a_{m'-1}^q]^T \\ &= \frac{1}{m'} \tilde{\mathbf{a}}_q. \end{aligned}$$

Thus in (4.6) we have:

$$\int_0^1 k(x, t)[f(t)]^q dt \cong \frac{1}{m'} \Psi^T(x) \mathbf{K} \Phi_{m' \times m'} \tilde{\mathbf{a}}_q. \quad (4.7)$$

By substituting the approximations (4.2), (4.3) and (4.7) into (1.1) we obtain:

$$\Psi(x)^T \mathbf{c} - \lambda \frac{1}{m'} \Psi^T(x) \mathbf{K} \Phi_{m' \times m'} \tilde{\mathbf{a}}_q \cong \Psi(x)^T \mathbf{g}. \quad (4.8)$$

Now, by multiplying two sides of (4.8) in  $\Psi(x)$  then integration in the interval  $[0, 1]$ , according to orthonormality of CAS wavelets we get:

$$\mathbf{c} - \lambda \frac{1}{m'} \mathbf{K} \Phi_{m' \times m'} \tilde{\mathbf{a}}_q = \mathbf{g}$$

which is a nonlinear system of algebraic equations. By solving this system we can obtain the approximate solution of Eq. (1.1) according to Eq. (4.4).

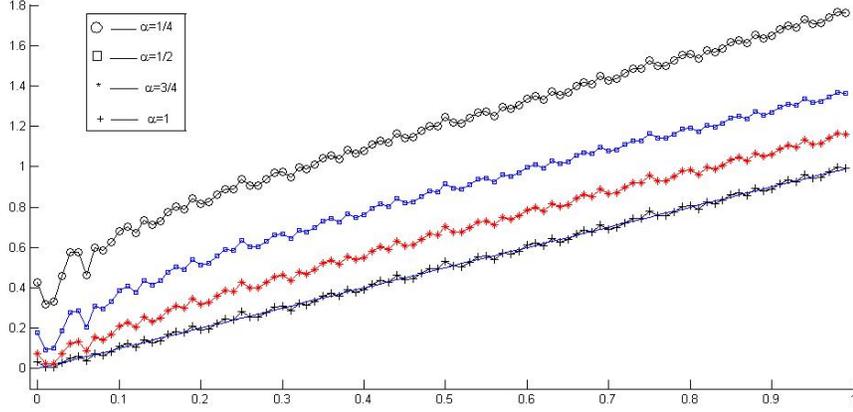


Figure 1.

## 5 Error Analysis

We can easily check the accuracy of the method. Since the truncated CAS wavelet series is an approximate solution of Eq. (1.1), when the approximate functions (4.2), (4.3) and (4.7) are substituted in Eq.(1.1), the resulting equation, (4.8), must be satisfied approximately, that is for  $x \in [0, 1)$

$$R_{m'}(x) = | \Psi(x)^T \mathbf{c} - \lambda \frac{1}{m'} \Psi^T(x) \mathbf{K} \Phi_{m' \times m'} \tilde{\mathbf{a}}_q - \Psi(x)^T \mathbf{g} | \cong 0.$$

If we set  $x = x_i$ , then our aim is to have  $R_{m'}(x_i) \leq 10^{-r_i}$ , where  $r_i$  is any positive integer. If we prescribe,  $Max\{r_i\} = 10^r$ , then we increase  $m'$  as long as the following inequality holds at each point  $x_r$ :

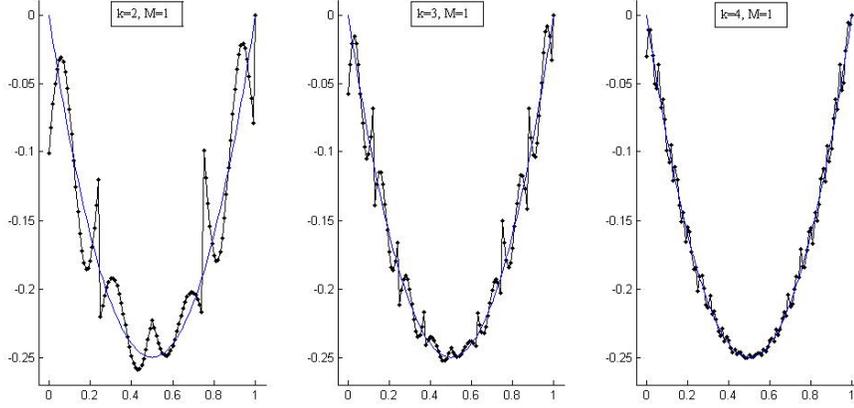
$$R_{m'}(x_i) \leq 10^{-r},$$

in other words, by increasing  $m'$  the error function  $R_{m'}(x_i)$  approaches zero. If  $R_{m'}(x) \rightarrow 0$  when  $m'$  is sufficiently large enough, then the error decreases.

## 6 Numerical Examples

To show the efficiency of the proposed method, we consider the following examples. Note that:

$$\|e_{m'}(x)\|_2 = \left( \int_0^1 e_{m'}^2(x) dx \right)^{1/2} \cong \left( \frac{1}{N} \sum_{i=0}^N e_{m'}^2(x_i) \right)^{1/2},$$



Figures 2.

where  $e_{m'}(x_i) = f(x_i) - f_{m'}(x_i)$ ,  $i = 0, 1, \dots, N$ .  $f(x)$  is the exact solution and  $f_{m'}(x)$  is the approximate solution which is obtained by Eq. (4.4).

And also consider that all of the computations have been done by MATLAB 7.8.

**Example 6.1.** Consider the following fractional nonlinear integro- differential equation:

$$D_*^\alpha f(x) - \int_0^1 xt[f(t)]^2 dt = 1 - \frac{x}{4}, \quad 0 \leq x < 1, \quad 0 < \alpha \leq 1,$$

with this supplementary condition  $f(0) = 0$ . Figure 1 shows the numerical results for  $k = 4$ ,  $M = 1$  and various  $0 < \alpha \leq 1$ . The comparisons show that as  $\alpha \rightarrow 1$ , the approximate solutions tend to  $f(x) = x$ , which is the exact solution of the equation in the case of  $\alpha = 1$ . The error in the case  $\alpha = 1$ , for different values of  $k$  and  $M$ , is shown in Table 1.

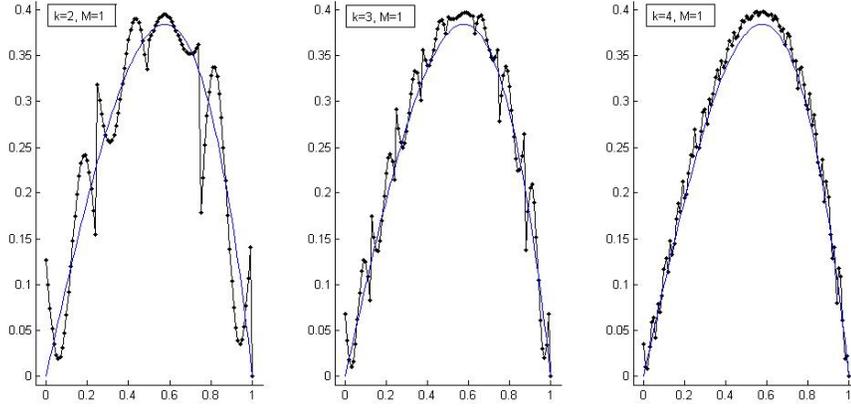
**Example 6.2.** Consider equation:

$$D_*^{\frac{1}{2}} f(x) - \int_0^1 xt[f(t)]^4 dt = g(x), \quad 0 \leq x < 1,$$

such that  $f(0) = 0$  and  $g(x) = \frac{1}{\Gamma(1/2)}(\frac{8}{3}\sqrt{x^3} - 2\sqrt{x}) - \frac{x}{1260}$ . From figure 2 and table 1, we can see the numerical solutions are in a very good agreement with the exact solution,  $f(x) = x^2 - x$ .

**Example 6.3.** Consider the following equation, of order  $\alpha = \frac{5}{6}$ :

$$D_*^{\frac{5}{6}} f(x) - \int_0^1 xe^t[f(t)]^2 dt = g(x), \quad 0 \leq x < 1,$$



Figures 3.

where  $g(x) = \frac{3}{\Gamma(1/6)}(2\sqrt[6]{x} - \frac{432}{91}\sqrt[6]{x^{13}}) + x(248e - 674)$ , with these supplementary conditions  $f(0) = f'(0) = 0$ . Figure 3 shows the behavior of the numerical solutions for various  $k$  and  $M$ , which are in agreement with the exact solution,  $f(x) = x - x^3$ . The error for different values of  $k$  and  $M$ , is shown in Table 1.

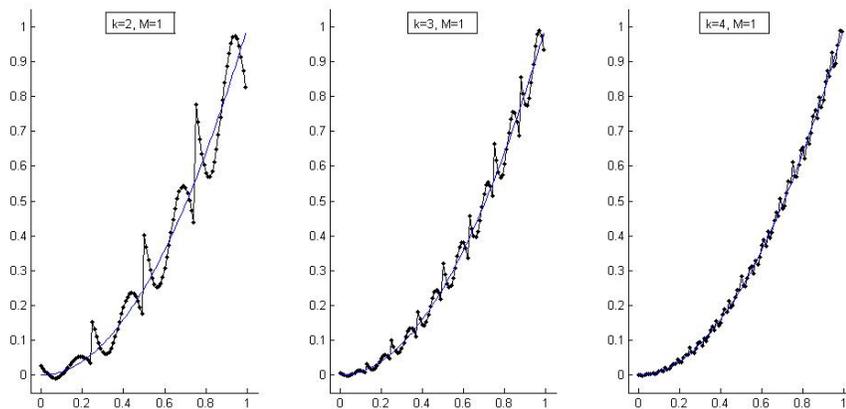
**Example 6.4.** Consider the following nonlinear Fredholm integro- differential equation, of order  $\alpha = \frac{5}{3}$ :

$$D_*^{\frac{5}{3}}f(x) - \int_0^1 (x+t)^2[f(t)]^3 dt = g(x), \quad 0 \leq x < 1,$$

where  $g(x) = \frac{6}{\Gamma(1/3)}\sqrt[3]{x} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}$ , with these supplementary conditions  $f(0) = f'(0) = 0$ . Figure 4 shows the numerical solutions for various  $k$  and  $M$ , with the exact solution,  $f(x) = x^2$ . The error for different values of  $k$  and  $M$ , is shown in Table 1.

Examples	$\ e_{12}\ _2$ ( $k = 2, M = 1$ )	$\ e_{24}\ _2$ ( $k = 3, M = 1$ )	$\ e_{48}\ _2$ ( $k = 4, M = 1$ )
Example 6.1	2.7133e-003	6.8179e-004	1.6745e-005
Example 6.2	7.711e-004	2.0755e-005	5.3445e-006
Example 6.3	2.0862e-003	6.3440e-004	2.5659e-004
Example 6.4	3.5560e-003	9.0145e-004	2.2537e-005

Table 1.



Figures 4.

## 7 Conclusion

In this work we derive CAS wavelet operational matrix of fractional integration, and use it to solve a class of nonlinear Fredholm integro- differential equation of fractional (arbitrary) order. Several examples are given to demonstrate the powerfulness of the proposed method. The solution is convergent, even though the size of increment may be large. Also this method can be used to obtain the numerical solutions of ordinary nonlinear integro- differential equations. The method is in the case of the fractional Fredholm integral equations, which is interest of current applications in computational chemistry [2, 3, 7, 8].

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