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by

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Faithful teleportation of an unknown state $|\phi\rangle$ in $\mathbb{C}^d$ is considered, where a general entangled state ρ is served as the resource. The necessary conditions of mixed states to complete perfect teleportation are proved. Based on these results, the necessary and sufficient conditions of faithful teleportation channel ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ and $\mathbb{C}^d \otimes \mathbb{C}^n$ are derived. It is shown that for ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$, ρ must be a maximally entangled state, while for ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$, ρ must be a mixed or pure maximally entangled state. Furthermore, the sender's measurements must be all projectors of maximally entangled states. On the other hand, for m, $n > d$, we present some classes of states for faithful teleportation.

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I. INTRODUCTION

Quantum teleportation plays an important role in quantum information processing and can serve as an elementary operation in quantum computers and in a number of quantum communication protocols. It, employing classical communication and shared resource of entanglement, allows to transmit an unknown quantum state from a sender to a receiver who are spatially separated. Bennett *et. al.* [1] first demonstrated teleportation of an arbitrary single qubit, through an entangled channel of Einstein-Podolsky-Rosen (EPR) pair. For a general case, it is proved that only maximally entangled pure states in $\mathbb{C}^d \otimes \mathbb{C}^d$ could faithfully teleport an arbitrary pure state in $\mathbb{C}^d$[2, 3].

Moreover, multipartite states have also been introduced to fulfill the faithful and deterministic teleportation of an unknown state. For instance, it is shown that three-qubit GHZ state and a class of W states can be used for perfect teleportation of one qubit state [4, 5]. Another five-qubit state was proved to be capable of perfect teleportation of arbitrary two-qubit state [6]. They are all maximally entangled pure states in $\mathbb{C}^2 \otimes \mathbb{C}^2$ between Alice and Bob. In addition, the tensor products of two Bell states in [7] and the genuine four-qubit entangled states in [8] are also used for perfect teleportation of two-qubit states, in which the teleportation channels can be regarded as maximally entangled states in $\mathbb{C}^4 \otimes \mathbb{C}^4$ shared by the sender and receiver. Thus, multipartite states are basically analogous to higher dimensional bipartite states in quantum teleportation.

In a realistic situation, however, due to the decoherence, the pure maximally entangled states may be transformed to mixed entangled states, which could make the teleportation unperfect. Moreover, for the case of the teleportation channel ρ with higher dimension other than $\mathbb{C}^d \otimes \mathbb{C}^d$, there is no a general result on faithful teleportation.

In this paper, we study high dimensional bipartite pure or mixed states as the resource for perfect and deterministic teleportation. We investigate faithful teleportation of an arbitrary pure state $|\phi\rangle = \sum_{i=1}^d \alpha_i |i\rangle$ by using entangled state ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ ($m, n \geq d$). First we prove necessary conditions of mixed states to complete perfect teleportation. Then the necessary and sufficient conditions of faithful teleportation channel ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$ and $\mathbb{C}^d \otimes \mathbb{C}^n$ are obtained. We show that for ρ in $\mathbb{C}^m \otimes \mathbb{C}^d$, either pure or mixed, it must be maximally entangled. Furthermore, the sender's measurements must be all projectors of maximally entangled states. For ρ in $\mathbb{C}^d \otimes \mathbb{C}^n$, it must be pure maximally entangled state. Finally, for m, $n > d$, we present some classes of states for faithful teleportation.

II. TELEPORTATION OF AN ARBITRARY PURE STATE

Suppose Alice wants to teleport an unknown pure state $|\phi\rangle$ to Bob. They have at disposal only a classical communication channel and an entangled state ρ. To perform teleportation Alice needs to measure her two particles: one in state $|\phi\rangle$ and one part of the entangled state ρ. She informs Bob her measurement results via the classical communication channel. Then Bob chooses a corresponding unitary transformation on the other part of the entangled state ρ, which can transform the state of this part to be exactly the unknown state $|\phi\rangle$.

First we begin with the general case that Alice and Bob previously share a pair of particles in a mixed state ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ ($m, n \geq d$), and show a necessary condition of ρ to be able to used in perfect teleportation.

Theorem 1 *If mixed state ρ is the ideal resource for faithful teleportation, and $\{p_i, |\psi_i\rangle\}$ is any pure state decomposition of ρ, then every pure state $|\psi_i\rangle$ is the ideal resource for faithful teleportation.*

Proof. Let $\rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|$, $p_i \geq 0$, $\sum_{i=1}^k p_i = 1$, be any pure state decomposition of ρ. The initial state is $|\phi\rangle\langle\phi| \otimes \rho$. Assume Alice makes a complete measurement $\{|\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|\}_{j=1}^r$ with $\sum_{j=1}^r |\tilde{\psi}_j\rangle\langle\tilde{\psi}_j| = I_r$, where $\{|\tilde{\psi}_j\rangle\}_{j=1}^r$ is an orthonormal basis in $\mathbb{C}^r$, I_r is the $r \times r$ identity matrix. Bob chooses unitary operations $U^{(j)\dagger}$ on his part with respect to Alice's measurement result j . If ρ is the ideal resource for faithful teleportation, then one has

$$|\phi\rangle\langle\phi|\otimes\rho = \sum_{i=1}^{k} p_i|\phi\rangle|\psi_i\rangle\langle\psi_i|\langle\phi| \quad (1)$$
$$= \sum_{j,j'=1}^{r} q_j|\tilde{\psi}_j\rangle U^{(j)}|\phi\rangle\langle\phi|U^{(j')\dagger}\langle\tilde{\psi}_{j'}|$$

with $\sum_{j=1}^{r} q_j = 1$, $q_j \geq 0$, $j = 1,\cdots,r$. From Eq. (1) we obtain

$$\sum_{i=1}^{k} p_i\langle\tilde{\psi}_j|\phi\rangle|\psi_i\rangle\langle\psi_i|\langle\phi|\tilde{\psi}_j\rangle = q_j U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \quad (2)$$
$$\langle\tilde{\psi}_j|\phi\rangle|\psi_i\rangle\langle\psi_i|\langle\phi|\tilde{\psi}_j\rangle = f_{ij}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \quad (3)$$

with $\sum_{i=1}^{k}\sum_{j=1}^{r} f_{ij} = 1$, $f_{ij} \geq 0$, $i = 1,\cdots,k$, $j = 1,\cdots,r$. Hence we have

$$\sum_{j=1}^{r}\langle\tilde{\psi}_j|\phi\rangle|\psi_i\rangle\langle\psi_i|\langle\phi|\tilde{\psi}_j\rangle = \sum_{j=1}^{r} f_{ij}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \quad (4)$$

for $i = 1,\cdots,k$. Due to the completeness of $\{|\tilde{\psi}_j\rangle\langle\tilde{\psi}_j|\}$, f_{ij} satisfies $\sum_{j=1}^{r} f_{ij} = 1$ for each i. Therefore we arrive at that every pure state $|\psi_i\rangle$ must be an ideal resource for faithful teleportation. □

If we replace any pure state decomposition with the spectral decomposition in the above theorem, then we will get another necessary condition of mixed states for perfect teleportation.

Corollary 2 *If mixed state ρ is the ideal resource for faithful teleportation, then its eigenstates must be all ideal channels for faithful teleportation.*

Utilizing the necessary conditions of mixed states, we consider now which kind of states in $\mathbb{C}^m\otimes\mathbb{C}^n$ could be used as perfect channel for teleportation of an unknown state $|\phi\rangle$ in $\mathbb{C}^d$ explicitly. Here we divide the whole discussion into four cases.

Case i). *$n = d$, pure states.*

At first we consider pure states in case $n = d$. Since any pure state $|\psi\rangle$ in $\mathbb{C}^m\otimes\mathbb{C}^d$ can be transformed into some pure state in $\mathbb{C}^d\otimes\mathbb{C}^d$ by local unitary transformations of Alice and Bob, for this case, we only need consider a pure state $|\psi\rangle$ in $\mathbb{C}^d\otimes\mathbb{C}^d$ directly. It has been shown that only maximally entangled pure states can be used for perfect teleportation in $\mathbb{C}^d\otimes\mathbb{C}^d$ [2, 3], which is illustrated from optimal teleportation fidelity. Here we give a direct proof that the maximally entangled states are the only ones that fulfill faithful teleportation mathematically.

Theorem 3 *Pure state $|\psi\rangle$ in $\mathbb{C}^d\otimes\mathbb{C}^d$ is an ideal resource for faithful teleportation of $|\phi\rangle = \sum_{i=1}^{d}\alpha_i|i\rangle$, if and only if $|\psi\rangle$ is maximally entangled state. Furthermore, Alice's measurements must be all projectors of maximally entangled states.*

Proof. Let $|\psi\rangle = \sum_{i,j=1}^{d} a_{ij}|ij\rangle$ be the entangled pure state shared by Alice and Bob. To teleport the unknown state $|\phi\rangle$, Alice carries out complete measurement $\{|\psi_{st}\rangle\langle|\psi_{st}|\}_{s,t=1}^{d}$, where $|\psi_{st}\rangle = \sum_{p,q=1}^{d} U_{pq,st}|pq\rangle$ with $\langle\psi_{s',t'}|\psi_{s,t}\rangle = \sum_{p,q=1}^{d} U_{pq,st}U^*_{pq,s't'} = \delta_{s,s'}\delta_{t,t'}$. Then one has

$$|\phi\rangle|\psi\rangle = \sum_{i,j,k=1}^{d}\alpha_i a_{jk}|ijk\rangle$$
$$= \sum_{s,t=1}^{d}\sum_{i,j,k=1}^{d}\alpha_i a_{jk}|\psi_{st}\rangle\langle\psi_{st}|ijk\rangle$$
$$= \sum_{s,t=1}^{d}|\psi_{st}\rangle(\sum_{i,j,k=1}^{d}U^*_{ij,st}\alpha_i a_{jk}|k\rangle)$$
$$= \sum_{s,t=1}^{d}|\psi_{st}\rangle A^T V^\dagger_{st}|\phi\rangle \quad (5)$$

where $V_{st} = (U_{ij,st})$ and $A = (a_{ij})$ are the coefficient matrices of $|\psi_{st}\rangle$ and $|\psi\rangle$ respectively. If $|\psi\rangle$ is an ideal resource for faithful teleportation, then $A^T V^\dagger_{st}$ should be unitary up to a constant factor:

$$A^T V^\dagger_{st} = c_{st}W_{st}, \quad (6)$$

and

$$\sum_{s,t=1}^{d}|c_{st}|^2 = 1. \quad (7)$$

where W_{st} is unitary and $0 \leq c_{st} \leq 1$, $s,t = 1,\cdots,d$. Let $A^T = U_1\Lambda_1V_1$ and $V^\dagger_{st} = U_{2,st}\Lambda_{2,st}V_{2,st}$ be the singular decompositions of A^T and $V^\dagger_{st}$ respectively, with unitary $U_1, V_1, U_{2,st}, V_{2,st}$, and $\Lambda_1 = diag(\lambda_1,\cdots,\lambda_d)$, $\Lambda_{2,st} = diag(\mu_{1,st},\cdots,\mu_{d,st})$. Due to normality $trAA^\dagger = trV_{st}V^\dagger_{st} = 1$, one gets

$$\sum_{i=1}^{d}|\lambda_i|^2 = 1, \quad (8)$$
$$\sum_{i=1}^{d}|\mu_{i,st}|^2 = 1. \quad (9)$$

Then from Eq. (6), we have

$$c^2_{st}I_d = A^T V^\dagger_{st}V_{st}A^{T\dagger}$$
$$= U_1\Lambda_1V_1U_{2,st}\Lambda_{2,st}\Lambda^\dagger_{2,st}U^\dagger_{2,st}V^\dagger_1\Lambda^\dagger_1U^\dagger_1 \quad (10)$$

and

$$c^2_{st}(\Lambda_1\Lambda^\dagger_1)^{-1} = V_1U_{2,st}\Lambda_{2,st}\Lambda^\dagger_{2,st}U^\dagger_{2,st}V^\dagger_1, \quad (11)$$

which give rise to

$$|\lambda_i|^{-2} = \frac{1}{c^2_{st}}|\mu_{i,st}|^2 \quad (12)$$

and

$$|\lambda_i|^2 = \frac{c^2_{st}}{|\mu_{i,st}|^2} \quad (13)$$

by reordering $\{\lambda_i\}$ and $\{\mu_{i,st}\}$. From Eq. (8) and Eq. (12) we know

$$\frac{1}{c_{st}^2}\sum_{i=1}^{d}|\mu_{i,st}|^2=\sum_{i=1}^{d}|\lambda_i|^{-2}\leq d^2.$$

Combining this equation with Eq. (9), one has $c_{st}^2 \geq 1/d^2$ for $s,t=1,\cdots,d$, which results in

$$c_{st}^2=\frac{1}{d^2} \tag{14}$$

by taking into account Eq. (7). Inserting Eq. (14) into Eq. (13) and using Eq. (8) we get

$$\sum_{i=1}^{d}\frac{1}{|\mu_{i,st}|^2}=d^2. \tag{15}$$

Hence in terms of Eq. (9), Eq. (13) and Eq. (15) we obtain

$$|\lambda_i|^2=|\mu_{i,st}|^2=\frac{1}{d} \tag{16}$$

for $i,s,t=1,\cdots,d$, which are just the square of Schmidt coefficients of $|\psi\rangle$ and $|\psi_{st}\rangle$ respectively. Therefore $A^T=\frac{1}{\sqrt{d}}\tilde{U}$ and $V_{st}=\frac{1}{\sqrt{d}}\tilde{V}_{st}$ for some unitary matrices $\tilde{U}$ and $\tilde{V}_{st}$. As a result, we arrive at that the shared entangled state $|\psi\rangle$ and the projective measurements $\{|\psi_{st}\rangle\langle\psi_{st}|\}$ must be all maximally entangled states. At last,

$$|\phi\rangle|\psi\rangle=\frac{1}{d}\sum_{s,t=1}^{d}|\psi_{st}\rangle\tilde{U}_{st}|\phi\rangle, \tag{17}$$

where $\tilde{U}_{st}=dA^TV_{st}^\dagger$ is determined by the shared state $|\psi\rangle$ and the measurement operators $|\psi_{st}\rangle\langle\psi_{st}|$. Experimentally, Alice measures her particles by d^2 orthonormal projectors, and tells Bob her measurement results. Each result appears in her measurements with probability $\frac{1}{d^2}$. According to Alice's measurement result st, Bob fulfills faithful teleportation by applying unitary operation $\tilde{U}_{st}^\dagger$ on his part of the entangled resource.

On the other hand, if $|\psi\rangle$ is maximally entangled, then the pure maximally entangled states have the form $U_1\otimes U_2|\psi\rangle$ with $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. Here we use $|\psi\rangle$ as a channel to teleport $|\phi\rangle=\sum_{i=1}^{d}\alpha_i|i\rangle$. It is:

$$|\phi\rangle\otimes|\psi\rangle=\frac{1}{d}\sum_{s,t=1}^{d}|\tilde{\psi}_{st}\rangle\otimes U_{st}|\phi\rangle$$

with $|\tilde{\psi}_{st}\rangle=\frac{1}{\sqrt{d}}U_{st}\otimes I|\psi\rangle$, $s,t=1,2,\cdots,d$. Here $\{U_{st}\}$ is the basis of the unitary operators, i.e., $tr(U_{st}U_{s't'}^\dagger)=d\delta_{ss'}\delta_{tt'}$ and $tr(U_{st}U_{st}^\dagger)=I_d$. For instance, we could choose $U_{st}=h^tg^s$ with $d\times d$ matrices h and g such that $h|j\rangle=|(j+1)\ \mathrm{mod}\ d\rangle$, $g|j\rangle=\omega^j|j\rangle$, $\omega=\exp\{-2i\pi/d\}$, $s,t=1,2,\cdots,d$, as the basis of the unitary operators to perform the perfect teleportation. □

Case ii). *$n=d$, mixed states.*

After the consideration of the pure states as the ideal channel, now we will prove which kind of mixed state ρ in $\mathbb{C}^m\otimes\mathbb{C}^d$ can be used for perfect teleportation.

Theorem 4 *Mixed state ρ with rank k in $\mathbb{C}^m\otimes\mathbb{C}^d$ can be used for perfect teleportation of $|\phi\rangle=\sum_{i=1}^{d}\alpha_i|i\rangle$, if and only if $m=kd$ and ρ is the mixed maximally entangled state: $\rho=\sum_{x=1}^{k}p_x|\psi_x\rangle\langle\psi_x|$ with $|\psi_x\rangle$ is maximally entangled in $H_x\otimes\mathbb{C}^d$, $x=1,\cdots,k$ and $\{H_x\}$ are orthogonal to each other.*

Proof. It has been proved that the mixed maximally entangled state [9] could be used for perfect teleportation. Now we prove the counterpart. Suppose ρ in $\mathbb{C}^m\otimes\mathbb{C}^d$ is the ideal resource for teleportation, we have known that its k orthogonal eigenstates $\{|\psi_i\rangle\}$ are all maximally entangled by corollary 2. In fact, they can be constituted in the following way. We assume, without loss of generality, $|\psi_1\rangle$ is maximally entangled in $H_1\otimes\mathbb{C}^d$ with $H_1=\mathbb{C}^d=Span\{|1\rangle,\cdots,|d\rangle\}$. From Eq. (4) one has

$$\sum_{j=1}^{d^2}\langle\tilde{\psi}_j|\phi\rangle|\psi_1\rangle\langle\psi_1|\langle\phi|\tilde{\psi}_j\rangle=\sum_{j=1}^{d^2}f_{1j}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \tag{18}$$

with $f_{1j}=1/d^2$ for $j=1,\cdots,d^2$, and $f_{1j}=0$ for $j>d^2$. Here $\{|\tilde{\psi}_j\rangle\}_{j=1}^{d^2}$ are maximally entangled states and also constitute orthogonal basis in $\mathbb{C}^d\otimes H_1$. Similarly state $|\psi_2\rangle$ satisfies

$$\sum_{j=d^2+1}^{2d^2}\langle\tilde{\psi}_j|\phi\rangle|\psi_2\rangle\langle\psi_2|\langle\phi|\tilde{\psi}_j\rangle=\sum_{j=d^2+1}^{2d^2}f_{2j}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger} \tag{19}$$

with $f_{2j}=1/d^2$ for $j=d^2+1,\cdots,2d^2$, and $f_{2j}=0$ for others. From the orthogonality of $\{|\tilde{\psi}_j\rangle\}_{j=1}^{d^2}$ and $\{|\tilde{\psi}_j\rangle\}_{j=d^2+1}^{2d^2}$, we know $\{|\tilde{\psi}_j\rangle\}_{j=d^2+1}^{2d^2}$ are maximally entangled and also constitute orthogonal basis in $\mathbb{C}^d\otimes H_2$, where H_2 is orthogonal to H_1, $dimH_2=d$. Hence $|\psi_2\rangle$ is maximally entangled in $H_2\otimes\mathbb{C}^d$. For other eigenstates $|\psi_x\rangle$, $2\leq x\leq k$, they can be treated in a similar way,

$$\sum_{j=(x-1)d^2+1}^{xd^2}\langle\tilde{\psi}_j|\phi\rangle|\psi_x\rangle\langle\psi_x|\langle\phi|\tilde{\psi}_j\rangle=\sum_{j=(x-1)d^2+1}^{xd^2}f_{xj}U^{(j)}|\phi\rangle\langle\phi|U^{(j)\dagger}. \tag{20}$$

$\{|\tilde{\psi}_j\rangle\}_{j=(x-1)d^2+1}^{xd^2}$, which are maximally entangled, constitute the orthogonal basis in $\mathbb{C}^d\otimes H_x$, where $\{H_x\}$ are orthogonal to each other, $dimH_x=d$ for $x=1,\cdots,k$. Hence $|\psi_x\rangle$ is maximally entangled in $H_x\otimes\mathbb{C}^d$. From the above analysis, we get $m=kd$ with k the rank of ρ. The probabilities of each measurement result depend on the eigenvalues p_i, $i=1,\cdots,k$. Therefore a mixed state ρ in $\mathbb{C}^m\otimes\mathbb{C}^d$ that can be used for faithful teleportation of $|\phi\rangle$ in $\mathbb{C}^d$ must be a mixed maximally entangled state. □

As an example, we consider the perfect teleportation of $|\phi\rangle=\sum_{i=1}^{d}\alpha_i|i\rangle$ by using a mixed maximally entangled state $\rho=\frac{1}{2d}((\sum_{i=1}^{d}|ii\rangle)(\sum_{i=1}^{d}\langle ii|)+(\sum_{i=1}^{d}|d+i,i\rangle)(\sum_{i=1}^{d}\langle d+i,i|))$ in $\mathbb{C}^{2d}\otimes\mathbb{C}^d$. By straightforward calculations one has

$$\begin{aligned}|\phi\rangle\langle\phi|\otimes\rho=\ &\frac{1}{2d}(\sum_{s,t,s',t'}|\tilde{\psi}_{st}^1\rangle\langle\tilde{\psi}_{s't'}^1|\otimes U_{st}|\phi\rangle\langle\phi|U_{s't'}^\dagger\\&+|\tilde{\psi}_{st}^2\rangle\langle\tilde{\psi}_{s't'}^2|\otimes U_{st}|\phi\rangle\langle\phi|U_{s't'}^\dagger),\end{aligned}$$

where $|\tilde{\psi}^1_{st}\rangle = \frac{1}{\sqrt{d}}U_{st}\otimes I(\sum_{i=1}^d |i,i\rangle)$, $|\tilde{\psi}^2_{st}\rangle = \frac{1}{\sqrt{d}}U_{st}\otimes I(\sum_{i=1}^d |i,d+i\rangle)$, $\{U_{st}\}$ is the basis of unitary operators in $\mathbb{C}^d$.

Case iii). *$m = d$, pure or mixed states.*

In this case we show which kind of states in $\mathbb{C}^d\otimes\mathbb{C}^n$ are ideal resources for teleportation.

Theorem 5 *The state ρ in $\mathbb{C}^d\otimes\mathbb{C}^n$ could be used for perfect teleportation of $|\phi\rangle = \sum_{i=1}^d \alpha_i|i\rangle$, if and only if it is a maximally entangled pure state in $\mathbb{C}^d\otimes\mathbb{C}^d$.*

Proof. Suppose ρ with rank k is able to teleport $|\phi\rangle$ perfectly, then its eigenstates are all the ideal resources for teleportation by corollary 2. If so, the first subsystems of ρ's eigenstates must be orthogonal to each other by the proof of theorem 4, which means the dimension of the first subsystem of ρ is kd. Since we are considering the case with the first subsystem of ρ is d dimensional here, hence $k = 1$ and ρ is maximally entangled pure state. □

Case iv). *$m, n > d$, pure or mixed states.*

Although we have shown necessary conditions for faithful teleportation, there is no general result on which kind of pure states in $\mathbb{C}^m\otimes\mathbb{C}^n$ could fulfill perfect teleportation of an unknown state in $\mathbb{C}^d$ with $m, n > d$.

Here we introduce a class of states:

$$|\psi\rangle = c_1|\psi_1\rangle + \cdots + c_l|\psi_l\rangle, \tag{21}$$

where $|\psi_p\rangle \in H^A_p\otimes H^B_p$ is maximally entangled, $\{H^A_p$ are orthogonal to each other, $dimH^A_p = dimH^B_p = n_p \geq d$ for $p = 1,\cdots,l$, $\sum_{p=1}^l n_p \leq \min\{m,n\}$ and $\sum_{i=1}^l |c_i|^2 = 1$. Without loss of generality, we assume Alice's complete measurements are $\{|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|\}$, $s = 1,\cdots,d$, $t = 1,\cdots,n_p$, $p = 1,\cdots,l$. Here for $p = 1,\cdots,l$, $\{|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|\}$ are projectors onto $\mathbb{C}^d\otimes H^A_p$. Besides, $\{|\tilde{\psi}_{st,p}\rangle\}$ are maximally entangled states and they constitute orthogonal basis in $\mathbb{C}^d\otimes H^A_p$ for $p = 1,\cdots,l$. Therefore,

$$|\phi\rangle|\psi\rangle = \frac{1}{d}\sum_{p=1}^{l}\sum_{s=1}^{d}\sum_{t=1}^{n_p} c_p|\tilde{\psi}_{st,p}\rangle\tilde{U}_{st,p}|\phi\rangle, \tag{22}$$

where the unitary matrix $\tilde{U}_{st,p}$ depends on the shared resource state $|\psi\rangle$ and measurement operator $|\tilde{\psi}_{st,p}\rangle\langle\tilde{\psi}_{st,p}|$. The probability of getting a result in each Alice's measurement is $|c_p|^2/d^2$.

For example, we consider the teleportation of a qubit state $|\phi\rangle = \alpha_1|0\rangle + \alpha_2|1\rangle$ with nonmaximally entangled state $|\psi\rangle = \sqrt{a}|\eta\rangle + \sqrt{1-a}|\xi\rangle$ with $|\eta\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|\xi\rangle = \frac{1}{\sqrt{3}}(|22\rangle + |33\rangle + |44\rangle)$. We have

$$\begin{aligned} & |\phi\rangle\otimes|\psi\rangle \\ = \ & \frac{\sqrt{a}}{2}(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\otimes U_1|\phi\rangle + \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\otimes U_2|\phi\rangle \\ & +\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)\otimes U_3|\phi\rangle + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\otimes U_4|\phi\rangle) \\ & +\sqrt{\frac{1-a}{6}}(\frac{1}{\sqrt{2}}(|02\rangle+|13\rangle)\otimes V_1|\phi\rangle+\frac{1}{\sqrt{2}}(|02\rangle-|13\rangle)\otimes V_2|\phi\rangle \\ & +\frac{1}{\sqrt{2}}(|03\rangle + |14\rangle)\otimes V_3|\phi\rangle + \frac{1}{\sqrt{2}}(|03\rangle - |14\rangle)\otimes V_4|\phi\rangle \\ & +\frac{1}{\sqrt{2}}(|04\rangle + |12\rangle)\otimes V_5|\phi\rangle + \frac{1}{\sqrt{2}}(|04\rangle - |12\rangle)\otimes V_6|\phi\rangle), \end{aligned}$$

where $U_{1,2} = |0\rangle\langle 0| \pm |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$, $U_{3,4} = |1\rangle\langle 0| \pm |0\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4|$, $V_{1,2} = |0\rangle\langle 2| + |2\rangle\langle 0| \pm |1\rangle\langle 3| \pm |3\rangle\langle 1| + |4\rangle\langle 4|$, $V_{3,4} = |0\rangle\langle 3| + |3\rangle\langle 0| \pm |1\rangle\langle 4| \pm |4\rangle\langle 1| + |2\rangle\langle 2|$, $V_{5,6} = |0\rangle\langle 4| + |4\rangle\langle 0| \pm |1\rangle\langle 2| \pm |2\rangle\langle 1| + |3\rangle\langle 3|$. It is obvious that with respect to the Alice's measurement results, faithful teleportion can be realized by applying the corresponding unitary transformations U_i, $i = 1,...,4$, V_j, $j = 1,...,6$, on Bob's part.

For mixed states in $\mathbb{C}^m\otimes\mathbb{C}^n$, we consider ρ with all eigenstates belonging to this class. From the analysis at the beginning, we know that if such mixed state could be used for perfect teleportation, then any two of its eigenstates $|\xi\rangle = c_1|\xi_1\rangle + \cdots + c_p|\xi_p\rangle$ and $|\eta\rangle = c'_1|\eta_1\rangle + \cdots + c'_q|\eta_q\rangle$ satisfy $|\xi_i\rangle = |\eta_j\rangle$, or the first subsystems of $|\xi_i\rangle$ and $|\eta_j\rangle$ are orthogonal for $i = 1,\cdots,p$, $j = 1,\cdots,q$, which means that any superpositions of such eigenstates still belong to the class. For instance, $\rho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|$ with $|\psi_1\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{6}}(|22\rangle + |33\rangle + |44\rangle)$, $|\psi_2\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{2}(|52\rangle + |63\rangle)$, $p_1 + p_2 = 1$, can be used for perfect teleportation of one qubit state. Hence mixed states satisfying the above condition could be used for faithful teleportation.

Remark. For the case of multipartite states used in perfect teleportation protocols, such as the faithful teleportation of number d qubit state in [10], if we treat them as teleportation of pure states in $\mathbb{C}^{2^d}$, then it is easy to check the shared resources belong to this class of state (21).

III. TELEPORTATION AND ENTANGLEMENT

We next investigate the relation between the degree of entanglement of teleportation channel ρ and teleportation. Here we take the well-known entanglement measure, entanglement of formation [11].

For a pure bipartite state $|\psi\rangle_{AB}$, entanglement of formation is defined as the partial entropy of either of the two subsystems: $E(|\psi\rangle_{AB}) = -tr(\rho_{A(B)}\log_2\rho_{A(B)})$, where $\rho_{A(B)} = tr_{B(A)}(|\psi\rangle_{AB}\langle\psi|)$. For mixed state ρ with pure state decompositions $\{p_i, |\tilde{\phi}_i\rangle\}$ such that $\rho = \sum_i p_i|\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|$ with $\sum_i p_i = 1$, the entanglement of formation is defined as the average entanglement of the pure states in the decomposition, minimized over all pure state decompositions of ρ: $E(\rho) = \inf\sum_i p_iE(|\tilde{\phi}_i\rangle)$.

It can be verified that either the maximally entangled pure state $|\psi\rangle$ in $\mathbb{C}^d \otimes \mathbb{C}^d$, or the mixed maximally entangled state ρ in $\mathbb{C}^{kd} \otimes \mathbb{C}^d$, the entanglement of formation is $E = \log d$. Furthermore, entangled states with entanglement of formation $E = \log d$ must be pure or mixed maximally entangled states [9]. Hence we derive that the states in $\mathbb{C}^m \otimes \mathbb{C}^d$ $(m \geq d)$ can be used for perfect teleportation if and only if their entanglement of formation are $\log d$. However, states having $E = \log d$ but not in $\mathbb{C}^m \otimes \mathbb{C}^d$ $(m \geq d)$ may be not capable of carrying out faithful teleportation of a d dimensional pure state. As an example we consider $\rho_0 = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2|$, with $|\psi_1\rangle = \frac{1}{\sqrt{d}}(|1,1\rangle + \cdots + |d,d\rangle)$, $|\psi_2\rangle = \frac{1}{\sqrt{d}}(|1,d+1\rangle + \cdots + |d,2d\rangle)$. It can be verified that $E(\rho_0) = \log_2 d$ and it is a maximally entangled mixed state in $\mathbb{C}^d \otimes \mathbb{C}^{2d}$. Nevertheless this state can not be used to teleport $|\phi\rangle$ faithfully. For teleportation channel ρ in $\mathbb{C}^m \otimes \mathbb{C}^n$ with $m, n > d$, the entanglement of formation of ρ presented in this paper for perfect teleportation is larger than or equal to $\log d$. Therefore the above studies give the implication that the entanglement of formation for all ideal entangled resources are not less than $\log d$, which might be another necessary condition for quantum states to be used in perfect teleportation. But the converse is not always true, which can be illustrated by the state ρ_0 as a counterexample.

IV. CONLUSIONS

In summary, we have investigated which states in $\mathbb{C}^m \otimes \mathbb{C}^n$ $(m, n \geq d)$ can be used for faithful teleportation of $|\phi\rangle$ in $\mathbb{C}^d$ and proved the necessary conditions for such states. Furthermore, we have shown that for $n = d$, ρ can be used for faithful teleportation if and only if it is maximally entangled and $m = kd$, with k the rank of ρ. For $m = d$, ρ can be used for faithful teleportation of $|\phi\rangle$ if and only if it is a maximally entangled pure state. For $m, n > d$, we get a class of pure and mixed states that could be used for faithful teleportation respectively. From the point of view of experimental implementation of quantum teleportation [12], our results may help to understand the character of faithful teleportation and to facilitate experimental preparation of entangled resources.

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