Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

Nonlocality of two-qubit and three-qubit Schmidt-Correlated states

by

Ming-Jing Zhao, Zong-Guo Li, Bo Li, Shao-Ming Fei, Zhi-Xi Wang, and Xianqing Li-Jost

Preprint no.: 47 2010

Nonlocality of two-qubit and three-qubit Schmidt-Correlated states

Ming-Jing Zhao[1], Zong-Guo Li[2], Bo Li[1], Shao-Ming Fei[1,3], Zhi-Xi Wang[1], and Xian-Qing Li-Jost[3]
[1] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[2] *College of Science, Tianjin University of Technology, Tianjin, 300191, China*
[3] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

We investigate the nonlocality of Schmidt-correlated (SC) states, and present analytical expressions of the maximum violation value of Bell inequalities. It is shown that the violation of Clauser-Horne-Shimony-Holt (CHSH) inequality is necessary and sufficient for the nonlocality of two-qubit SC states, whereas the violation of the Svetlichny inequality is only a sufficient condition for the genuine nonlocality of three-qubit SC states. Furthermore, the relations among the maximum violation values, concurrence and relative entropy entanglement are discussed.

I. INTRODUCTION

Einstein, Podolsky, and Rosen (EPR) [1] believed that the results of measurements on a local subsystem of a composite physical system which can be predicted with certainty would be determined by the local variables of the subsystems. However, the violation of Bell inequality [2] rules out all putative local hidden-variable (LHV) theories, and indicates that quantum nonlocality of entangled states is one of the most profound characters inherent in quantum mechanics. Moreover, Clauser, Horne, Shimony and Holt derived the well-known CHSH inequality, which provides a way of experimental testing of the LHV model [3].

Actually the nonlocality is intimately related to quantum entanglement. It is shown that the CHSH inequality is satisfied for every separable pure two-qubit state, but violated for all entangled pure two-qubit states, with the amount of violation increasing with the entanglement [4, 5]. Nevertheless, this conclusion is not true for mixed entangled states, as Werner presented a mixed entangled state satisfying the CHSH inequality [6]. Hence CHSH inequality is just a necessary, but not sufficient condition for separability of two-qubit states. Starting with the Bell and CHSH inequalities, many Bell type inequalities are also proposed with respect to different quantum systems [7]. For three-qubit system, Svetlichny introduced an inequality whose violation is a sufficient condition for genuine tripartite nonlocality [8]. Ghose *et al.* further derived the analytical expressions of violation of Svetlichny inequality for states in Greenberger, Horne and Zeilinger (GHZ) class [9]. However it is still intractable to determine whether a given state, especially mixed state, violates a certain Bell inequality or not, as one has to find the mean value of the related Bell operators for suitable observables [10].

As an important class of mixed states from a quantum dynamical perspective, Schmidt-correlated (SC) states have been paid much attention to [11–14]. Just as Khasin *et al.* [15] proposed, the bipartite SC states naturally appear in a system dynamics with additive integrals of motion. In fact, SC states $\rho = \sum_{m,n=0}^{N-1} a_{mn}|m\cdots m\rangle\langle n\cdots n|$, $\sum_{m=0}^{N-1} a_{mm} = 1$, are defined as the mixtures of pure states, sharing the same Schmidt basis [11, 14]. The SC states exhibit some elegant properties. For example, for any local quantum measurement on SC states, the result does not depend on which party the measurement is performed. Moreover, their separability is determined by the positivity of partial transposition [14]. In this paper we investigate the violation of the CHSH inequality and Svetlichny inequality for SC states. By presenting an analytical expression of the maximum expectation value F_{max} of CHSH inequality for two-qubit systems, we show that whether an SC state violates CHSH inequality is equivalent to whether it is entangled. For three-qubit systems, we give an analytical expression of the maximum expectation value S_{max} of the Svetlichny inequality, and prove that there exist genuine entangled SC states which obey Svetlichny inequality. Furthermore, the relations between F_{max} and concurrence [16], S_{max} and relative entropy entanglement [17] for SC states are derived. At last we illustrate F_{max} and S_{max} are not monotonic under local operations and classical communications (LOCC) by explicit examples.

This paper is organized as follows: in section II, we introduce the CHSH inequality and investigate the maximum expectation value F_{max} for two-qubit SC states. Then the relation between F_{max} and concurrence is provided. In Sec. III, the maximum expectation value S_{max} of the Svetlichny inequality and its relation to the relative entropy entanglement are studied for three-qubit SC states. Finally, we conclude with a summary of our results in Sec. IV.

II. TWO-QUBIT SC STATES

The well-known CHSH inequality is shown to be both necessary and sufficient for the separability of a two-qubit pure state. The corresponding Bell operator for the CHSH inequality is given by

$$F = AB + AB' + A'B - A'B', \tag{1}$$

where the observables $A = \vec{a}\cdot\vec{\sigma}$ and $A' = \vec{a}'\cdot\vec{\sigma}$ are associated with the first qubit, $B = \vec{b}\cdot\vec{\sigma}$ and $B' = \vec{b}'\cdot\vec{\sigma}$ are associated with the second qubit, while $\vec{a}$, $\vec{a}'$, $\vec{b}$ and $\vec{b}'$ are unit vectors, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ with σ_x, σ_y, σ_z the

Pauli matrices. $|\langle\psi|F|\psi\rangle| \leq 2$ holds if and only if the pure state $|\psi\rangle$ is separable.

For any mixed two-qubit state ρ, the expectation value $F(\rho) = Tr(\rho F)$ satisfies

$$|F(\rho)| \leq 2 \tag{2}$$

if ρ admits local hidden variable model. Violation of the inequality (2) implies that the state ρ is entangled. Let $F_{max}(\rho) = \max_{A,A',B,B'} F(\rho)$ be the maximal value of $F(\rho)$ under all possible observables A, A', B and B'. One can then decide whether a state ρ is entangled in terms of the maximum expectation value.

To find the maximum expectation value F_{max} for a given state ρ, we define $\vec{a} = (\sin\theta_a \cos\phi_a, \sin\theta_a \sin\phi_a, \cos\theta_a)$, and similarly for the unit vectors $\vec{a}'$, $\vec{b}$ and $\vec{b}'$. In addition, we define unit vectors $\vec{d}, \vec{d}'$ such that $\vec{b} + \vec{b}' = 2\vec{d}\cos\phi$ and $\vec{b} - \vec{b}' = 2\vec{d}'\sin\phi$. Thus

$$\vec{d} \cdot \vec{d}' = \cos\theta_d \cos\theta_{d'} + \sin\theta_d \sin\theta_{d'} \cos(\phi_d - \phi_{d'}) = 0. \tag{3}$$

Set $D = \vec{d} \cdot \vec{\sigma}$ and $D' = \vec{d}' \cdot \vec{\sigma}$, the expectation value $F(\rho)$ can be written as

$$\begin{aligned} F(\rho) &= \langle AB\rangle + \langle AB'\rangle + \langle A'B\rangle - \langle A'B'\rangle \\ &= \langle A(B+B')\rangle + \langle A'(B-B')\rangle \\ &= 2(\langle AD\rangle \cos\phi + \langle A'D'\rangle \sin\phi) \\ &\leq 2(\langle AD\rangle^2 + \langle A'D'\rangle^2)^{1/2}, \end{aligned} \tag{4}$$

where we have used the fact that

$$x\cos\theta + y\sin\theta \leq (x^2 + y^2)^{1/2}, \tag{5}$$

with the equality holding when $\tan\theta = y/x$.

For a two-qubit SC state ρ_1:

$$\rho_1 = a_1|00\rangle\langle 00| + a_2|00\rangle\langle 11| + a_2^*|11\rangle\langle 00| + a_4|11\rangle\langle 11|,$$

with $a_1, a_4 \geq 0$, $a_1 + a_4 = 1$ and $a_1 a_4 \geq |a_2|^2$. The first term in Eq. (4) with respect to this mixed state ρ_1 turns out to be

$$\begin{aligned} \langle AD\rangle &= \cos\theta_a \cos\theta_d + 2(\mathrm{Re}(a_2)\cos(\phi_a + \phi_d) \\ &\quad -\mathrm{Im}(a_2)\sin(\phi_a + \phi_d))\sin\theta_a \sin\theta_d \\ &\leq \{\cos^2\theta_d + 4[\mathrm{Re}(a_2)\cos(\phi_a + \phi_d) \\ &\quad -\mathrm{Im}(a_2)\sin(\phi_a + \phi_d)]^2 \sin^2\theta_d\}^{1/2} \\ &\leq \left[\cos^2\theta_d + 4|a_2|^2 \sin^2\theta_d\right]^{1/2} \\ &= \left[(1 - 4|a_2|^2)\cos^2\theta_d + 4|a_2|^2\right]^{1/2}, \end{aligned} \tag{6}$$

where the inequality (5) has been taken into account. From Eq. (4) and Eq. (6) we have

$$\begin{aligned} F(\rho_1) &\leq 2[(1 - 4|a_2|^2)(\cos^2\theta_d + \cos^2\theta_{d'}) + 8|a_2|^2]^{1/2} \\ &\leq 2[1 + 4|a_2|^2]^{1/2}. \end{aligned} \tag{7}$$

Here we have employed the fact that the maximum of $\cos^2\theta_d + \cos^2\theta_{d'}$ is 1 according to Eq. (3). The equality in Eq. (7) holds when $\vec{a} = \vec{z}$, $\vec{a}' = \vec{x}$, $\vec{b} = \sin\phi\cos\phi_d\,\vec{x} + \sin\phi\sin\phi_d\,\vec{y} + \cos\phi\,\vec{z}$ and $\vec{b}' = -\sin\phi\cos\phi_d\,\vec{x} - \sin\phi\sin\phi_d\,\vec{y} + \cos\phi\,\vec{z}$ with $\tan\phi = 2|a_2|$ and $\tan\phi_d = -\frac{\mathrm{Re}(a_2)}{\mathrm{Im}(a_2)}$. Therefore, we obtain

$$F_{max}(\rho_1) = 2\{1 + 4|a_2|^2\}^{1/2}. \tag{8}$$

Furthermore, the maximum expectation value $F_{max}(\rho_1)$ has a direct relation with its concurrence [16], which is an entanglement measure. The concurrence for a bipartite pure state $|\psi\rangle$ is defined by $C(|\psi\rangle) = \sqrt{2(1 - Tr\rho_A^2)}$, where the reduced density matrix ρ_A is given by $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$. The concurrence is then extended to mixed states ρ by the convex roof, $C(\rho) \equiv \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$, for all possible ensemble realizations $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$. For the state ρ_1 one has $C(\rho_1) = 2|a_2|$. Hence we get

$$F_{max}(\rho_1) = 2[1 + C^2(\rho_1)]^{1/2}, \tag{9}$$

which shows that $F_{max}(\rho_1)$ increases monotonically with $C(\rho_1)$.

The violation of the CHSH inequality has also relations to the dense coding, which uses previously shared entangled states to send possibly more information than classical information encoding. The capacity of dense coding for a given shared bipartite state ρ^{AB} is given by $\chi = \log_2 d_A + S(\rho_A) - S(\rho)$, with $S(\rho) = -tr(\rho\log_2\rho)$ [18]. ρ is useful for dense coding if its capacity is larger than $\log_2 d_A$. It is straightforwardly verified that for two-qubit SC state ρ_1,

$$\begin{aligned} \chi = \;& 1 - a_1\log_1 a_1 - a_4\log_1 a_4 \\ &+ \Big(\frac{1+\sqrt{1-4a_1a_4+4|a_2|^2}}{2}\log_2\frac{1+\sqrt{1-4a_1a_4+4|a_2|^2}}{2} \\ &+ \frac{1-\sqrt{1-4a_1a_4+4|a_2|^2}}{2}\log_2\frac{1-\sqrt{1-4a_1a_4+4|a_2|^2}}{2}\Big), \end{aligned}$$

which also increases monotonically with the maximum expectation value $F_{max}(\rho_1)$ for given a_1 and a_4. Hence one has the following equivalent statements for the SC state ρ_1: (i) it is entangled, (ii) it's concurrence is greater than zero; (iii) it violates CHSH inequality; (iv) it is useful for dense coding.

Now we generalize two-qubit SC state ρ_1 to mixed state ρ_2

$$\begin{aligned} \rho_2 \; &= b_1|00\rangle\langle 00| + b_2|01\rangle\langle 01| + b_3|10\rangle\langle 10| \\ &+ b_4|11\rangle\langle 11| + c_1|00\rangle\langle 11| + c_1^*|11\rangle\langle 00| \end{aligned} \tag{10}$$

with $b_i \geq 0$, $i = 1,2,3,4$, $\sum_{i=1}^4 b_i = 1$, $b_1 b_4 \geq |c_1|^2$. Nevertheless by similar calculation we can get its maximum expectation value

$$F_{max}(\rho_2) = 2\{(b_1 + b_4 - b_2 - b_3)^2 + 4|c_1|^2\}^{1/2}, \tag{11}$$

which can be obtained by $\vec{a} = \vec{z}$, $\vec{a}' = \vec{x}$, $\vec{b} = \sin\phi\cos\phi_d\vec{x} + \sin\phi\sin\phi_d\vec{y} + \cos\phi\vec{z}$ and $\vec{b}' =$

$-\sin\phi\cos\phi_d\vec{x} - \sin\phi\sin\phi_d\vec{y} + \cos\phi\vec{z}$ with $\tan\phi = \frac{2|c_1|}{b_1+b_4-b_2-b_3}$ and $\tan\phi_d = -\frac{\mathrm{Re}(c_1)}{\mathrm{Im}(c_1)}$.

Although the amount of maximum violation of CHSH inequalities increases with the entanglement for the SC states, the maximum expectation value F_{max} is not a legitimate entanglement measure for two-qubit states, because it does not decrease monotonically under LOCC. For example, considering a transverse noise channel [19] operating on Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the output state takes the following form, $\rho_3 = \sum_{i,j=1,2} K_i \otimes K_j|\psi\rangle\langle\psi|K_i^\dagger \otimes K_j^\dagger$, where the Kraus operators K_1 and K_2 denote the transverse noise channel,

$$K_1 = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}, \tag{12}$$

with time-dependent parameters $\gamma = \exp(-\Gamma t/2)$, $\omega = \sqrt{1-\gamma^2}$. By a simplification, the final state, $\rho_3 = \frac{1}{2}[\gamma^4|00\rangle\langle 00| + \gamma^2(|00\rangle\langle 11| + |11\rangle\langle 00|) + (1+\omega^4)|11\rangle\langle 11| + \gamma^2\omega^2(|01\rangle\langle 01| + |10\rangle\langle 10|)]$, is just of the form in Eq. (10). Therefore the maximum expectation value of ρ_3 is given by

$$F_{max}(\rho_3) = 2\{(2\gamma^4 - 2\gamma^2 + 1)^2 + \gamma^4\}^{1/2}. \tag{13}$$

It is obvious that the maximum expectation value F_{max} is not a monotonic function of γ from Eq. (13). Hence it is not monotonic with time under LOCC, i.e., F_{max} is not a legitimate entanglement measure. On the other hand, we can obtain the concurrence of ρ_3, $C(\rho_3) = \gamma^4$, is monotonic with γ. For $t > 0.265805/\Gamma$, ρ_3 does not violate the CHSH inequality (see FIG. 1). Thus, CHSH inequality can not detect entanglement of such states, though in fact some of these states are distillable [22], as shown in the experimental demonstration of the "hidden nonlocality" in [23].

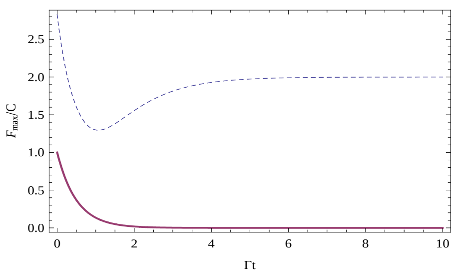

FIG. 1: Dashed line: $F_{max}(\rho_3)$ versus Γt. Solid line: concurrence $C(\rho_3)$ versus Γt.

III. THREE-QUBIT SC STATES

For three-qubit SC states, we take into account the Svetlichny inequality. The Svetlichny operator is defined by

$$\begin{aligned} S &= ABC + ABC' + AB'C - AB'C' \\ &\quad + A'BC - A'BC' - A'B'C - A'B'C', \end{aligned}$$

where observables $A = \vec{a}\cdot\vec{\sigma}$ and $A' = \vec{a}'\cdot\vec{\sigma}$ are associated with the qubit 1, $B = \vec{b}\cdot\vec{\sigma}$ and $B' = \vec{b}'\cdot\vec{\sigma}$ with qubit 2, and $C = \vec{c}\cdot\vec{\sigma}$ and $C' = \vec{c}'\cdot\vec{\sigma}$ with qubit 3. If a theory is consistent with a hybrid model of nonlocal-local realism, then the expectation value for any three-qubit state is bounded by Svetlichny inequality: $|S(\rho)| \le 4$, where $S(\rho) = Tr(S\rho)$ is the expectation value of S with respect to state ρ. In this section we are going to derive the analytical expression of maximum expectation value $S_{max}(\rho) = \max_{A,A',B,B',C,C'} S(\rho)$ for three-qubit SC states.

In order to find the maximum expectation value S_{max}, we implement the same transformation for $\vec{b}$ and $\vec{b}'$ as in the two-qubit case. The expectation value $S(\rho)$ can be written as:

$$\begin{aligned} S(\rho) &= \langle ABC\rangle + \langle ABC'\rangle + \langle AB'C\rangle - \langle AB'C'\rangle \\ &\quad + \langle A'BC\rangle - \langle A'BC'\rangle - \langle A'B'C\rangle - \langle A'B'C'\rangle \\ &= \langle A(B+B')C\rangle + \langle A(B-B')C'\rangle \\ &\quad + \langle A'(B-B')C\rangle - \langle A'(B+B')C'\rangle \\ &= 2(\cos\phi\langle ADC\rangle + \sin\phi\langle AD'C'\rangle \\ &\quad + \sin\phi\langle A'D'C\rangle - \cos\phi\langle A'DC'\rangle) \\ &\le 2[(\langle ADC\rangle^2 + \langle AD'C'\rangle^2)^{1/2} \\ &\quad + (\langle A'D'C\rangle^2 + \langle A'DC'\rangle^2)^{1/2}], \end{aligned} \tag{14}$$

where we have made use of Eq. (5) again.

For the three-qubit SC state:

$$\begin{aligned} \rho_4 &= a_1|000\rangle\langle 000| + a_2|000\rangle\langle 111| \\ &\quad + a_2^*|111\rangle\langle 000| + a_4|111\rangle\langle 111| \end{aligned}$$

with $a_1, a_4 \ge 0$, $a_1 + a_4 = 1$ and $a_1 a_4 \ge |a_2|^2$. The first term in Eq. (14) with respect to ρ_4 is given by

$$\begin{aligned} &\langle ADC\rangle \\ &= (a_1 - a_4)\cos\theta_a\cos\theta_d\cos\theta_c \\ &\quad + 2[\mathrm{Re}(a_2)\cos(\phi_a + \phi_d + \phi_c) \\ &\quad - \mathrm{Im}(a_2)\sin(\phi_a + \phi_d + \phi_c)]\sin\theta_a\sin\theta_d\sin\theta_c \\ &\le [(a_1 - a_4)^2\cos^2\theta_a\cos^2\theta_d + 4|a_2|^2\sin^2\theta_a\sin^2\theta_d]^{\frac{1}{2}}. \end{aligned} \tag{15}$$

From Eq. (14) and Eq. (15) we get

$$\begin{aligned} S(\rho_4) &\le 2\{[(a_1 - a_4)^2\cos^2\theta_a(\cos^2\theta_d + \cos^2\theta_{d'}) \\ &\quad + 4|a_2|^2\sin^2\theta_a(\sin^2\theta_d + \sin^2\theta_{d'})]^{1/2} \\ &\quad + [(a_1 - a_4)^2\cos^2\theta_{a'}(\cos^2\theta_d + \cos^2\theta_{d'}) \\ &\quad + 4|a_2|^2\sin^2\theta_{a'}(\sin^2\theta_d + \sin^2\theta_{d'})]^{1/2}\}. \end{aligned} \tag{16}$$

Due to the constraint condition Eq. (3), one has $\cos^2\theta_d + \cos^2\theta_{d'} \le 1$ and $\sin^2\theta_d + \sin^2\theta_{d'} \le 2$. Therefore we arrive at

$$S_{max}(\rho_4) = \max\{4|1 - 2a_1|, 8\sqrt{2}|a_2|\} \tag{17}$$

from the fact that

$$x\cos^2\theta + y\sin^2\theta \le \begin{cases} x, & x \ge y; \\ y, & x \le y, \end{cases} \tag{18}$$

where the equality holds when $\theta = 0$ for the first case, and when $\theta = \pi/2$ for the second case. Accordingly, $S_{max}(\rho_4) = 4|1-2a_1|$ holds when $\vec{a}$, $\vec{a}'$, $\vec{b}$, $\vec{b}'$ are all aligned along $\vec{z}$, $\vec{c} = sign(1-2a_1)\vec{z}$ and $\vec{c}' = -\vec{c}$, whereas $S_{max}(\rho_4) = 8\sqrt{2}|a_2|$ holds when all the measurement vectors lie in the $x-y$ plane with $\tan(\phi_a + \phi_d + \phi_c) = \tan(\phi_a + \phi_{d'} + \phi_{c'}) = \tan(\phi_{a'} + \phi_{d'} + \phi_c) = -\frac{\mathrm{Im}(a_2)}{\mathrm{Re}(a_2)}$, $\tan(\phi_{a'} + \phi_d + \phi_{c'}) = \pi$, $\phi_d - \phi_{d'} = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$. Eq. (17) implies that ρ_4 violates the Svetlichny inequality if and only if $|a_2| > \frac{1}{2\sqrt{2}}$. However ρ_4 is always genuine tripartite entangled for nonzero a_2. Hence the violation of the Svetlichny inequality is only a sufficient condition for the genuine nonlocality of three-qubit SC states.

Now we contrast the violation of Svetlichny inequality with entanglement. In terms of the reference [20], the generalized concurrence [21] of three-qubit SC state ρ_4 can be obtained, $C(\rho_4) = \sqrt{6}|a_2|$. Then, the Svetlichny inequality does not hold when $C(\rho_4) \geq \frac{\sqrt{3}}{2}$, and its violation satisfies the following equation

$$S_{max}(\rho_4) = \frac{8C(\rho_4)}{\sqrt{3}}. \tag{19}$$

Moreover, $S_{max}(\rho_4)$ has also direct relations to the relative entropy entanglement, $E(\rho) = \min_{\sigma\in D} S(\rho \parallel \sigma) = \min_{\sigma\in D} Tr[\rho\log\rho - \rho\log\sigma]$, where D is the set of all fully separable states. It has been proven that $\varrho = a_1|000\rangle\langle 000| + a_4|111\rangle\langle 111|$ is the optimal separable state for ρ_4 such that $E(\rho_4) = \min_{\sigma\in D} S(\rho_4 \parallel \sigma) = S(\rho_4 \parallel \varrho)$ [14]. Hence, when ρ_4 violates Svetlichny inequality, we have

$$\begin{aligned} E(\rho_4) &= f(a_1, a_4, a_2, a_2^*) \\ &\quad - f(a_1\log_2 a_1, a_4\log_2 a_4, a_2\log_2 a_4, a_2^*\log_2 a_1) \\ &= g(a_1, a_4, S_{max}^2) \\ &\quad - g(a_1\log_2 a_1, a_4\log_2 a_4, S_{max}^2\log_2 a_1\log_2 a_4), \end{aligned}$$

where $f(x_1, x_2, x_3, x_4) = f_+\log_2 f_+ + f_-\log_2 f_-$, $f_\pm = [(x_1 + x_2) \pm \sqrt{(x_1-x_2)^2 + 4x_3x_4}]/2$ and $g(x_1, x_2, x_3) = g_+\log_2 g_+ + g_-\log_2 g_-$, $g_\pm = [(x_1+x_2) \pm \sqrt{(x_1-x_2)^2 + \frac{x_3}{32}}]/2$.

Now we consider the generalization of the three-qubit SC state ρ_4 to mixed state ρ_5:

$$\begin{aligned} \rho_5 &= b_1|000\rangle\langle 000| + b_2|001\rangle\langle 001| + b_3|010\rangle\langle 010| \\ &\quad + b_4|100\rangle\langle 100| + b_5|011\rangle\langle 011| + b_6|101\rangle\langle 101| \\ &\quad + b_7|110\rangle\langle 110| + b_8|111\rangle\langle 111| + c_1|000\rangle\langle 111| \\ &\quad + c_1^*|111\rangle\langle 000|. \end{aligned} \tag{20}$$

For such state, the S_{max} becomes

$$\begin{aligned} & S_{max}(\rho_5) \\ &= \max\{4|b_1 - b_2 - b_3 - b_4 + b_5 + b_6 + b_7 - b_8|, 8\sqrt{2}|c_1|\}. \end{aligned}$$

Thus ρ_5 violates the Svetlichny inequality when $|c_1| > \frac{1}{2\sqrt{2}}$. Here $S_{max}(\rho_5) = 4|b_1 - b_2 - b_3 - b_4 + b_5 + b_6 + b_7 - b_8|$ holds when $\vec{a}$, $\vec{a}'$, $\vec{b}$, $\vec{b}'$ are all aligned along $\vec{z}$, $\vec{c} = sign(b_1 - b_2 - b_3 - b_4 + b_5 + b_6 + b_7 - b_8)\vec{z}$ and $\vec{c}' = -\vec{c}$. $S_{max}(\rho_5) = 8\sqrt{2}|c_1|$ holds when all the measurement directions lie in the $x-y$ plane with $\tan(\phi_a + \phi_d + \phi_c) = \tan(\phi_a + \phi_{d'} + \phi_{c'}) = \tan(\phi_{a'} + \phi_{d'} + \phi_c) = -\frac{\mathrm{Im}(c_1)}{\mathrm{Re}(c_1)}$, $\tan(\phi_{a'} + \phi_d + \phi_{c'}) = \pi$, $\phi_d - \phi_{d'} = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$.

In particular, let's consider a transverse noise channel operating on the GHZ state $|\phi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. Then the final state $\rho_6 = \sum_{i,j,l=1,2} K_i \otimes K_j \otimes K_l|\phi\rangle\langle\phi|K_i^\dagger \otimes K_j^\dagger \otimes K_l^\dagger = \frac{1}{2}[\gamma^6|000\rangle\langle 000| + \gamma^4\omega^2(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100|) + \gamma^2\omega^4(|011\rangle\langle 011| + |101\rangle\langle 101| + |110\rangle\langle 110|) + (1+\omega^6)|111\rangle\langle 111| + \gamma^3(|000\rangle\langle 111| + |111\rangle\langle 000|)]$, which is just of the form in Eq. (20). Therefore we have

$$\begin{aligned} & S_{max}(\rho_6) \\ &= \max\{2|\gamma^6 + 3\gamma^2\omega^4 - 3\gamma^4\omega^2 - 1 - \omega^6|, 4\sqrt{2}\gamma^3\} \\ &= \begin{cases} 2(1-\gamma^6-3\gamma^2\omega^4+3\gamma^4\omega^2+\omega^6), & 0\leq\gamma\leq\frac{1}{\sqrt{2}}; \\ 4\sqrt{2}\gamma^3, & \frac{1}{\sqrt{2}}\leq\gamma\leq 1, \end{cases} \end{aligned} \tag{21}$$

which shows that ρ_6 violates the Svetlichny inequality when $t < 0.693147/\Gamma$. Namely the Svetlichny inequality can not detect the hidden nonlocality any more for $t > 0.693147/\Gamma$. From Eq. (21) and FIG. 2, we can see that $S_{max}(\rho_6)$ is not a monotonic function of time; accordingly we assert that S_{max} is also not a suitable entanglement measure.

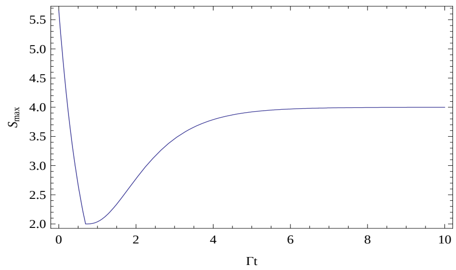

FIG. 2: $S_{max}(\rho_6)$ versus $\Gamma\, t$

IV. CONCLUSIONS

In summary, we have obtained an analytical formula of maximum expectation value F_{max} of CHSH inequality for two-qubit SC states, from which we have shown that this inequality is both necessary and sufficient for the nonlocality of two-qubit SC states, though this is not true for general two-qubit mixed states. In addition, the relations between F_{max}, entanglement and capacity of dense coding for SC states have been also derived. Moreover, unlike the entanglement measure, F_{max} is not monotonic with time under LOCC. For three-qubit systems, we have demonstrated that the violation of the Svetlichny inequality is only a sufficient condition for the genuine nonlocality of three-qubit SC states. Furthermore we have presented a relation between S_{max} and relative entropy entanglement, which gives a way to determine the relative entropy entanglement of SC states experimentally.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Physics (Long Island City, N. Y.) **1**, 195 (1964).
[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880(1969).
[4] N. Gisin, Phys. Lett. A **154**, 201 (1991).
[5] S. Popescu and D. Rohrlich, Phys. Lett. A **166**, 293 (1992).
[6] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[7] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); N. Gisin and A. Peres, Phys. Lett. A. **162**, 15 (1992); R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001); M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002); M. Żukowski and Č. Brukner, W. Laskowski, and M. Wieśniak, Phys. Rev. Lett. **88**, 210402 (2002); J. L. Chen, C. F. Wu, L. C. Kwek, and C.H. Oh, Phys. Rev. Lett. **93**, 140407 (2004); K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. A **74**, 050101(R)(2006); M. Li and S. M. Fei, Phys. Rev. Lett. **104**, 240502 (2010).
[8] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).
[9] S. Ghose, N. Sinclair, S. Debnath, P. Rungta, and R. Stock, Phys. Rev. Lett. **102**, 250404 (2009).
[10] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[11] E. M. Rains, Phys. Rev. A **60**, 179 (1999).
[12] E. M. Rains, Phys. Rev. A **63**, 019902 (2000).
[13] S. Virmani, M. F. Sacchi, M. B. Plenio, and D. Markham, Phys. Lett. A **288**, 62 (2001).
[14] M. J. Zhao, S. M. Fei, and Z. X. Wang, Phys. Lett. A **372**, 2552 (2008).
[15] M. Khasin and R. Kosloff, Phys. Rev. A **76**, 012304 (2007).
[16] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
[17] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. **78**, 2275 (1997); V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619 (1998).
[18] T. Hiroshima, J. Phys. A **34**, 6907 (2001); D. Bruß, G. M. D'Ariano, M. Lewenstein, C. Macchiavello, A. Sen(De), and U. Sen, Phys. Rev. Lett. **93**, 210501 (2004).
[19] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
[20] Z. G. Li and M. J. Zhao, in prepration.
[21] S. Albeverio and S. M. Fei, J. Opt. B **3**, 223 (2001).
[22] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).
[23] P. G. Kwiat, S. Barraza-Lopez, A. Stefanov, and N. Gisin, Nature, **409**, 1014 (2001).