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Abstract

This note deals with quasi-states on the two-dimensional torus. Quasi-states are certain quasi-linear functionals (introduced by Aarnes) on the space of continuous functions. Grubb constructed a quasi-state on the torus, which is invariant under the group of area-preserving diffeomorphisms, and which moreover vanishes on functions having support in an open disk. Knudsen asserted the uniqueness of such a quasi-state; for the sake of completeness, we provide a proof. We calculate the value of Grubb's quasi-state on Morse functions with distinct critical values via their Reeb graphs. The resulting formula coincides with the one obtained by Py in his work on quasi-morphisms on the group of area-preserving diffeomorphisms of the torus. Included is a short introduction to the link between quasi-states and quasi-morphisms in symplectic geometry.

1 Introduction and formulation of the result

1.1 Quasi-states and quasi-morphisms in symplectic geometry

Following Aarnes [Aa], we give the following definition:

Definition 1.1. If Z is a compact (Hausdorff) space, let $C(Z)$ denote the Banach algebra of all real-valued continuous functions on Z. Denote by $C(F)$ the closed subalgebra of $C(Z)$ generated by F, that is $C(F) = \{\phi \circ F \,|\, \phi \in C(\operatorname{im} F)\}$. A functional $\eta\colon C(Z) \to \mathbb{R}$ is called a quasi-state if it satisfies

(i) $\eta(1) = 1$;

(ii) $\eta(F) \geq 0$ for $F \geq 0$;

(iii) for each $F \in C(Z)$ the restriction $\eta|_{C(F)}$ is linear.

Remark 1.2. Note for further use the nontrivial fact that a quasi-state is monotone, that is $\eta(F) \leq \eta(G)$ for $F \leq G$, and consequently it is Lipschitz with respect to the uniform norm, see [Aa]. In particular, it is continuous in the C^0-topology on $C(Z)$.

Before stating the results, we would like to briefly go over a certain way quasi-states arise on symplectic manifolds. For additional details, the reader is referred to [EP2] and references therein. To this end, the following definition is relevant.

Definition 1.3. If $\mathcal{G}$ is a group, a function $\mu\colon \mathcal{G} \to \mathbb{R}$ is a homogeneous quasi-morphism on $\mathcal{G}$ if

$$\sup_{a,b\in\mathcal{G}} |\mu(ab) - \mu(a) - \mu(b)| < \infty$$

and moreover $\mu(a^n) = n\mu(a)$ for all $n \in \mathbb{Z}, a \in \mathcal{G}$.

Remark 1.4. It easily follows that a homogeneous quasi-morphism is invariant under conjugation and is a homomorphism when restricted to any abelian subgroup of $\mathcal{G}$. Henceforth all quasi-morphisms are homogeneous and are referred to simply as quasi-morphisms.

Now assume that Z is a closed $2n$-dimensional manifold endowed with a symplectic form ω, where we assume $\int_Z \omega^n = 1$, and let $\mathcal{G}$ be the universal cover of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(Z, \omega)$ of Z. There is a natural map $\exp\colon C^\infty(Z) \to \mathcal{G}$, $\exp H$ being the element of $\mathcal{G}$ generated by the Hamiltonian flow of H for time 1. Pulling back a homogeneous quasi-morphism μ on $\mathcal{G}$, we obtain a functional $\mu \circ \exp\colon C^\infty(Z) \to \mathbb{R}$, which however attains the value zero on constant functions. To amend this, we define $\zeta(H) = \int_Z H\omega^n + \mu(\exp H)$. It immediately follows that ζ is additive on Poisson commutative subalgebras of $C^\infty(Z)$, Poisson commuting functions generating commuting elements of $\mathcal{G}$, and that $\zeta(1) = 1$. In order for ζ to be a candidate for a quasi-state, however, it needs to be monotone, see remark 1.2. Assuming for the moment that it is true, it can be shown that it is Lipschitz with respect to the C^0 norm on $C^\infty(Z)$ and hence admits a unique Lipschitz-continuous extension to $\zeta\colon C(Z) \to \mathbb{R}$, for which axiom (iii) of the definition of a quasi-state is readily established, since functions belonging to the same singly generated subalgebra (that is, a subset of the form $C(F)$ for $F \in C(Z)$) can be approximated in the C^0 norm by smooth Poisson commuting functions.

Monotonicity does not follow from the definition of a quasi-morphism on $\mathcal{G}$ and so needs to be proven separately. Entov, Polterovich, and the author do so in [EPZ] for the so-called Calabi quasi-morphisms, introduced in [EP1], using a special property of those quasi-morphisms, the so-called stability, also established in [EP1], which implies the monotonicity of ζ.

Note that any subgroup of the group $\mathrm{Symp}(Z, \omega)$ of symplectomorphisms of Z acts on $C^\infty(Z)$ by pulling back and on $\mathcal{G}$ by conjugation,[1)] exp intertwining the two actions. If μ is invariant under the action of such a group, it follows that ζ is. For example, μ is always invariant under the action of the group $\mathrm{Ham}(Z, \omega)$. In special cases, such as the one considered in section 1.2, μ may be invariant under the whole group of symplectomorphisms, and then ζ inherits this invariance.

1.2 Py's quasi-morphisms and quasi-states

In [Py2], Py constructs remarkable quasi-morphisms on $\mathrm{Ham}(Z, \omega)$, where Z is a closed surface of genus ≥ 2. Whether his quasi-morphisms have the stability property is a difficult open question. The easier question whether the functional on $C^\infty(Z)$ induced from these quasi-morphisms is (the restriction of) a quasi-state can be asked. Rosenberg [Ro] proves that this is the case, directly showing that this functional is monotone. See also [Za] for an alternative proof.

Here we consider the case of the torus $\mathbb{T}^2$ endowed with an area form ω of total area $\int_{\mathbb{T}^2} \omega = 1$. In [Py1], Py constructs a quasi-morphism μ: $\mathrm{Symp}_0(\mathbb{T}^2, \omega) \to \mathbb{R}$, where $\mathrm{Symp}_0(\mathbb{T}^2, \omega)$ is the identity component of the group of symplectomorphisms. We show that the functional θ: $C^\infty(\mathbb{T}^2) \to \mathbb{R}$ defined by $\theta(H) = \int_{\mathbb{T}^2} H\omega - \frac{1}{2}\mu(\exp H)$ is indeed the restriction to $C^\infty(\mathbb{T}^2)$ of a quasi-state.

We note first that μ is invariant under the action of the whole group $\mathrm{Symp}(\mathbb{T}^2, \omega)$ and hence θ is. Further, for $H \in C^\infty(\mathbb{T}^2)$ with support in a disk Py computes $\mu(\exp H) = 2\int_{\mathbb{T}^2} H\omega$, hence $\theta(H) = 0$. There is a unique quasi-state on the torus which is invariant under area-preserving diffeomorphisms and vanishes on functions with support in a disk. Its uniqueness was asserted by Knudsen in [Kn]; for the sake of completeness, we provide a proof below. This is Grubb's quasi-state [Gr]. It follows that if Py's functional can indeed be extended to a quasi-state, it must be Grubb's quasi-state.

Our purpose now is to calculate the value of Grubb's quasi-state ζ on a Morse function $H \in C^\infty(\mathbb{T}^2)$ with distinct critical values in terms of its Reeb graph.

Definition 1.5. The Reeb graph Γ_H of H is obtained from $\mathbb{T}^2$ by collapsing connected components of level sets of H into points.

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1) If a group K acts by continuous automorphisms on a connected topological group G, the action naturally extends to the universal cover of G.

We let $\pi\colon \mathbb{T}^2 \to \Gamma_H$ denote the quotient map. It can be easily seen that the graph Γ_H has a unique circular subgraph Γ with vertices $s_1, \ldots, s_k$, and pairwise disjoint trees $T_1, \ldots, T_k \subset \Gamma_H$, such that each T_j intersects Γ precisely at s_j, and $\Gamma_H = \Gamma \cup \bigcup_j T_j$. For $j = 1, \ldots, k$ let $\alpha_j := H(s_j)$. Then

Theorem 1.6.

$$\zeta(H) = \int_{\mathbb{T}^2} H\omega + \sum_{j=1}^{k} \int_{\pi^{-1}(T_j)} (\alpha_j - H)\omega\,.$$

The proof is given below. The same formula (up to a factor of 2) was obtained by Py in [Py1] for the value $\theta(H)$, provided that H satisfies the additional requirement of having zero mean, $\int_{\mathbb{T}^2} H\omega = 0$. Denote by $C_0^\infty(\mathbb{T}^2)$ the set of smooth functions on $\mathbb{T}^2$ having zero mean. Then we obtain

Corollary 1.7. *Py's functional is the restriction to $C_0^\infty(\mathbb{T}^2)$ of a quasi-state, namely, Grubb's quasi-state.*

The corollary follows from the discussion, except one technical point, which is accounted for below.

Remark 1.8. This note arose as an attempt to understand Py's functional on the torus and to extend the results of Rosenberg [Ro] on surfaces of higher genus to the case of the torus.

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2 Proofs

In order to carry out the calculation of theorem 1.6, we need another definition from [Aa].

Definition 2.1. For a compact space Z let $\mathcal{A}$ denote the collection of subsets of Z which are open or closed. A function $\tau\colon \mathcal{A} \to [0, 1]$ is called a topological measure if it satisfies

(i) $\tau(Z) = 1$;

(ii) $\tau(Z - K) + \tau(K) = 1$ for compact $K \subset Z$;

(iii) $\tau(K \cup K') = \tau(K) + \tau(K')$ for disjoint compact $K, K' \subset Z$;

(iv) $\tau(K) \leq \tau(K')$ for compact $K, K' \in \mathcal{A}$ such that $K \subset K'$;

(v) for open $U \subset Z$, $\tau(U) = \sup\{\tau(K) \mid K \subset U,\ K \text{ compact}\}$.

Quasi-states are in bijection with topological measures – this is Aarnes's representation theorem [Aa]. In one direction, if τ is a topological measure, then the value of the corresponding quasi-state η_τ on a function $F \in C(Z)$ is

$$\eta_\tau(F) = \max_Z F - \int_{\min_Z F}^{\max_Z F} b_F(t)\, dt\,,$$

where $b_F(t) = \tau(\{F \leq t\})$.

Grubb actually constructed a topological measure and the aforementioned quasi-state is, of course, provided by Aarnes's representation theorem. In what follows Grubb's topological measure is denoted by τ. It can be can be described as follows. A topological measure is completely determined by its values on compact submanifolds (with boundary) of full dimension, in our case, on compact subsurfaces, see [Za]. If $W \subset \mathbb{T}^2$ is a compact subsurface, let $\widehat{W} \subset \mathbb{T}^2$ denote the unique subsurface with no contractible boundary components such that (i) $\partial\widehat{W} \subset \partial W$, (ii) $\overline{\operatorname{Int} W \cap \operatorname{Int} \widehat{W}} \supset \partial\widehat{W}$, and (iii) $\overline{W \triangle \widehat{W}}$ is contained in a (finite) disjoint union of closed disks. Then $\tau(W) = |\widehat{W}|$, where $|\cdot|$ is the Lebesgue measure on $\mathbb{T}^2$ corresponding to ω. That τ thus defined extends to a topological measure is proved in [Gr].[2)] The uniqueness of $\widehat{W}$, as well as certain properties of the correspondence $W \mapsto \widehat{W}$ are established in [Za]; the notation $\operatorname{ed} W$ is used there instead of $\widehat{W}$. An illustration of this correspondence is given in figure 1.

Maintain the notations of subsection 1.2. Since the function in question, H, has only finitely many critical values, it suffices to compute $b_H(t) = \tau(\{H \leq t\})$ for a regular value t of H. Let $K_j := \pi^{-1}(s_j)$, $D_j := \operatorname{Int} \pi^{-1}(T_j)$ for $j = 1, \ldots, k$ and $S := \mathbb{T}^2 - \bigcup_j D_j = \pi^{-1}(\Gamma)$. Note that the K_j are figures-eight and the D_j are open disks. For $t \in \mathbb{R}$ let $(\mathbb{T}^2)^t := \{H \leq t\}$.

[2)] Unfortunately, it is not so easy to extract this characterization of Grubb's topological measure from his paper. It might help the reader if we mention that a subsurface of $\mathbb{T}^2$ is a "solid two-sided set" in the terminology of [Gr] if and only if it is nonempty, connected, and has no contractible boundary components. With this remark, it is relatively easy to understand the classes of subsets introduced *ibid.* on page 2068 in the case of $\mathbb{T}^2$, and the definition of a topological measure appearing on page 2070.

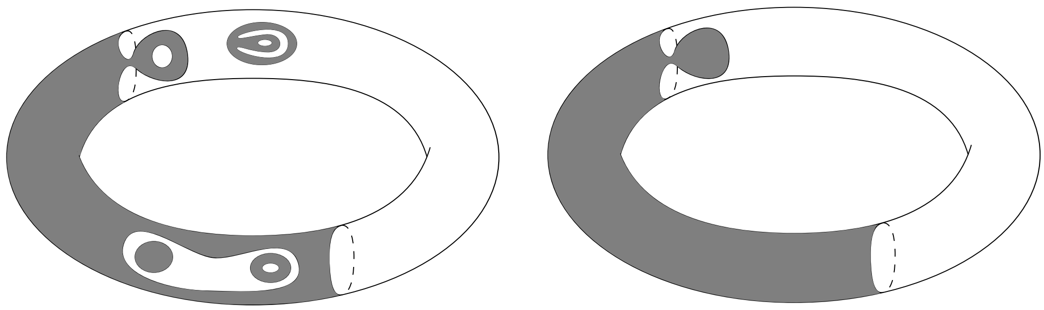

Figure 1: The correspondence $W \mapsto \widehat{W}$: W (filled) is a sample surface on the left; $\widehat{W}$ is on the right.

Lemma 2.2. *For a regular value t of H we have*

$$b_H(t) = |(\mathbb{T}^2)^t \cap S| + \sum_{j:\alpha_j > t} |D_j|\,. \tag{1}$$

Assuming the lemma for the moment, we proceed to the

Proof (of theorem 1.6)**.** Without loss of generality, $\alpha_1 < \cdots < \alpha_k$. Denote also $M = \max_{\mathbb{T}^2} H$ and $m = \min_{\mathbb{T}^2} H$. Then we have

$$\begin{aligned}
\zeta(H) &= M - \int_m^M b_H(t)\,dt = M - \int_m^M \Big[|(\mathbb{T}^2)^t \cap S| + \sum_{j:t>\alpha_j} |D_j|\Big]\,dt \\
&= M - (M-\alpha_k)|S| - \int_{\alpha_1}^{\alpha_k} |(\mathbb{T}^2)^t \cap S|\,dt \\
&\qquad - \sum_{j=1}^{k-1} (\alpha_{j+1} - \alpha_j)\Big|\bigcup_{i=1}^{j} D_i\Big| - (M - \alpha_k)\sum_{j=1}^{k} |D_j| \\
&= M - M|S| + \alpha_k|S| - M(1-|S|) - \int_{\alpha_1}^{\alpha_k} |(\mathbb{T}^2)^t \cap S|\,dt + \sum_{j=1}^{k} \alpha_j |D_j| \\
&= \alpha_k|S| - \int_{\alpha_1}^{\alpha_k} |(\mathbb{T}^2)^t \cap S|\,dt + \sum_{j=1}^{k} \alpha_j|D_j| = \int_S H\omega + \sum_{j=1}^{k} \alpha_j |D_j| \\
&= \int_{\mathbb{T}^2} H\omega + \sum_{j=1}^{k} \Big(\alpha_j|D_j| - \int_{D_j} H\omega\Big) = \int_{\mathbb{T}^2} H\omega + \sum_{j=1}^{k} \int_{D_j} (\alpha_j - H)\omega\,,
\end{aligned}$$

as required. Here we used the fact that if (Y,σ) is a finite measure space, G is a bounded measurable function on Y with maximum β and minimum α, then

$$\int_Y G\,d\sigma = \beta\sigma(Y) - \int_\alpha^\beta \sigma(\{G \le t\})\,dt\,;$$

here we have $Y = S$, $\sigma = |\cdot|$, $G = H|_S$, $\alpha = \alpha_1$, $\beta = \alpha_k$. □

Proof (of the lemma)**.** In order to establish formula (1), we need to compute $\tau(\{H \le t\})$.

Therefore we need to find $\widehat{(\mathbb{T}^2)^t}$ for a regular value t of H. If $\gamma \subset \mathbb{T}^2$ is a simple closed contractible curve, then it bounds a unique closed disk, which we denote $D(\gamma)$.

Definition 2.3. Let $W \subset \mathbb{T}^2$ be a subsurface. A contractible boundary component $\gamma \subset \partial W$ is called exterior (with respect to W) if $\overline{\operatorname{Int} D(\gamma) \cap \operatorname{Int} W} \supset \gamma$ and interior otherwise. A contractible boundary component $\delta \subset \partial W$ is called maximal if for any other contractible boundary component δ' we have either $D(\delta') \subset D(\delta)$ or $D(\delta') \cap D(\delta) = \varnothing$.

The following lemma is proved in [Za]:

Lemma 2.4. *Let $W \subset \mathbb{T}^2$ be a subsurface, and assume that $\gamma_1, \dots, \gamma_r$ are the maximal exterior boundary components of W, while $\delta_1, \dots, \delta_s$ are the maximal interior boundary components of W. Then*

$$\widehat{W} = \Big(W \cup \bigcup_i D(\delta_i)\Big) - \bigcup_j D(\gamma_j)\,.$$

□

Thus in order to compute $\widehat{(\mathbb{T}^2)^t}$, we need to know the maximal boundary components of $\partial(\mathbb{T}^2)^t = H^{-1}(t)$, and also which components are exterior and interior with respect to $(\mathbb{T}^2)^t$. Note that if $\gamma \subset H^{-1}(t)$ is a contractible curve, then it must be contained in one of the disks D_j. Before determining the exterior and interior maximal components of $(\mathbb{T}^2)^t$, we need an auxiliary result, which also appears in [Za]:

Lemma 2.5. *Let $W \subset \mathbb{T}^2$ be a subsurface contained in an open disk. Then any boundary component of W is contractible and any maximal boundary component of W is exterior with respect to W.* □

Lemma 2.6. *Let $\gamma \subset H^{-1}(t) = \partial(\mathbb{T}^2)^t$ be a maximal boundary component. Then it is exterior with respect to $(\mathbb{T}^2)^t$ if and only if $t < \alpha_j$ where j is the unique index such that $\gamma \subset D_j$.*

Proof. Let $W = D_j \cap (\mathbb{T}^2)^t$. If $\alpha_j > t$, then $(\mathbb{T}^2)^t$, and hence also W, is disjoint from K_j, which contains the topological boundary of D_j. Hence W is a compact subsurface of $\mathbb{T}^2$ contained in an open disk. Since γ is a maximal boundary component of $(\mathbb{T}^2)^t$, it is a maximal boundary component of W and hence is exterior with respect to W, by the previous lemma, therefore it is exterior with respect to $(\mathbb{T}^2)^t$.

To obtain the other direction, repeat the argument for $\{H \geq t\}$ and note that a maximal component $\delta \subset H^{-1}(t) = \partial(\mathbb{T}^2)^t = \partial\{H \geq t\}$ is exterior with respect to $(\mathbb{T}^2)^t = \{H \leq t\}$ if and only if it is interior with respect to $\{H \geq t\}$. □

It follows from lemmas 2.4, 2.6 that if $\alpha_j > t$, then the disk D_j is disjoint from $\widehat{(\mathbb{T}^2)^t}$ while if $\alpha_j < t$, then the disk D_j is entirely contained in $\widehat{(\mathbb{T}^2)^t}$. Consequently

$$\widehat{(\mathbb{T}^2)^t} = ((\mathbb{T}^2)^t \cap S) \cup \bigcup_{j:t>\alpha_j} D_j \,,$$

and so

$$b_H(t) = \tau((\mathbb{T}^2)^t) = |\widehat{(\mathbb{T}^2)^t}| = |(\mathbb{T}^2)^t \cap S| + \sum_{j:t>\alpha_j} |D_j| \,,$$

as asserted. □

Proof (of corollary 1.7)**.** So far we have shown that Py's functional θ and Grubb's quasi-state ζ coincide on the subset of $C_0^\infty(\mathbb{T}^2)$ consisting of Morse functions with distinct critical values. Since ζ is a quasi-state, it is continuous in the C^0-topology. Using the techniques of section 8 of [Ro], it is possible to show that θ is continuous in the C^2-topology. The set of Morse functions with distinct critical values is C^2-dense in $C_0^\infty(\mathbb{T}^2)$, *a fortiori* C^0-dense, hence we are done. □

We now turn to the aforementioned uniqueness. Grubb's topological measure is easily seen to be invariant under any symplectomorphism. We now show

Proposition 2.7. *Let σ be a topological measure on $\mathbb{T}^2$ which vanishes on disks and is invariant under symplectic isotopies. Then σ equals Grubb's topological measure τ.*

Proof. Since σ vanishes on disks, $\sigma(W) = \sigma(\widehat{W})$ for any subsurface $W \subset \mathbb{T}^2$. Thus it suffices to show that if W is a subsurface with no contractible boundary components, then $\sigma(W) = |W|$. Using additivity, it is enough to show this in case W is a closed smoothly embedded annulus with non-contractible boundary circles.

Choose a system of coordinates (p, q) on $\mathbb{T}^2$ such that $\omega = dp \wedge dq$. Call an embedded non-contractible circle linear if it has a parametrization of the form $\mathbb{R}/\mathbb{Z} \ni t \mapsto (p_0 + kt, q_0 + lt)$, where k, l is a pair of mutually prime integers, the slope of the circle. A closed smoothly embedded annulus is linear if its boundary circles are (in this case they must have the same slope). By a standard, thought somewhat lengthy, argument, it can be shown that any closed smoothly embedded annulus can be symplectically isotoped to a linear one of the same area. We shall show that the value of σ on any such annulus equals its area.

Clearly translations are symplectic isotopies so that all the linear annuli of the same slope and the same area are symplectically isotopic and so share the same value of σ.

Fix a slope; in what follows all annuli will be linear and have the chosen slope. Consider an annulus W of area $\frac{1}{n}$ where $n \geq 2$ is a natural number. Since we can fit $n-1$ pairwise disjoint translates of W into $\mathbb{T}^2$, we have, by the additivity and monotonicity of σ, $(n-1)\sigma(W) \leq 1$ so that

$$\sigma(W) \leq \frac{1}{n-1}\,.$$

Now consider an annulus W of area $\frac{1}{2N}$ where N is a natural number and let $O = \operatorname{Int} W$. The whole torus $\mathbb{T}^2$ is the disjoint union of N translates of W and N translates of O, so we obtain $N(\sigma(W) + \sigma(O)) = 1$, whence

$$\sigma(W) \geq \sigma(O) = \frac{1}{N} - \sigma(W) \geq \frac{1}{N} - \frac{1}{2N-1}\,,$$

by the previous inequality.

Finally, let A be an annulus of arbitrary area $\alpha \in (0,1)$. For any natural N let $m_N = \lfloor 2N\alpha \rfloor$. Note that $\lim_{N\to\infty} \frac{m_N}{2N} = \alpha$. We can fit at least $m_N - 1$ translates of an annulus of area $\frac{1}{2N}$ into A. This means that

$$\sigma(A) \geq (m_N - 1)\left(\frac{1}{N} - \frac{1}{2N-1}\right)\,.$$

On the other hand, we can fit at least $2N - 1 - m_N$ translates of an annulus of area $\frac{1}{2N}$ into the complement $\mathbb{T}^2 - A$, so that

$$\sigma(A) = 1 - \sigma(\mathbb{T}^2 - A) \leq 1 - (2N - 1 - m_N)\left(\frac{1}{N} - \frac{1}{2N-1}\right)\,.$$

It follows easily from the last two inequalities that for large N

$$\left|\sigma(A) - \frac{m_N}{2N}\right| \leq \frac{\text{const}}{N}\,.$$

Therefore $\sigma(A) = \lim_{N\to\infty} \frac{m_N}{2N} = \alpha$, as desired. □

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