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Abstract

We show that the Eisenbud-Goto conjecture holds for seminormal simplicial affine semigroup rings. Moreover we prove an upper bound for the Castelnuovo-Mumford regularity in terms of the dimension, which is similar as in the normal case. Finally we compute explicitly the regularity of full Veronese rings.

1 Introduction

Let $S$ be a homogeneous simplicial affine semigroup, i.e. $S$ is the submonoid of $(\mathbb{N}^d, +)$ generated by a set $A := \{e_1, \ldots, e_d, a_1, \ldots, a_c\} \subset \mathbb{N}^d$, where

$e_1 := (\alpha, 0, \ldots, 0), \quad e_2 := (0, \alpha, 0, \ldots, 0), \ldots, \quad e_d := (0, \ldots, 0, \alpha)$,

$a_i = (a_{i[1]}, \ldots, a_{i[d]}), \quad \text{with} \quad a_{i[1]} + \ldots + a_{i[d]} = \alpha, \quad i = 1, \ldots, c$.

Moreover we assume that the integers $a_{i[j]}, \ i = 1, \ldots, c, \ j = 1, \ldots, d$ are relatively prime and we assume that $d \geq 2, c \geq 1$ and $\alpha \geq 2$. Let $K$ be an arbitrary field, by $K[S]$ we denote the affine semigroup ring of $S$. As usual we can identify the affine semigroup ring $K[S]$ with the subring of the polynomial ring $K[t_1, \ldots, t_d]$ generated by monomials $t^a := t_1^{a[1]} \cdots t_d^{a[d]}$, where $a = (a_{i[1]}, \ldots, a_{i[d]}) \in S$. In the following we study the $\mathbb{Z}$-grading on $K[S]$ which is induced by $\deg t^a = (\sum_{i=1}^d a_{i[j]})/\alpha$. We note that $\dim K[S] = d$.

By $R := K[x_1, \ldots, x_{d+c}]$ we denote the standard-graded polynomial ring over $K$, i.e. $\deg x_i = 1$. Thus we have a $\mathbb{Z}$-graded surjective $K$-algebra homomorphism:

$$\pi : K[x_1, \ldots, x_{d+c}] \twoheadrightarrow K[S],$$

given by $x_i \mapsto t_i^a, \ i = 1, \ldots, d$ and $x_{d+j} \mapsto t_j^a, \ j = 1, \ldots, c$. Hence $K[S] \cong R/\ker \pi$, where $\ker \pi$ is a homogeneous prime ideal of $R$. Let $m_R$ denote the maximal homogeneous ideal of $R$ and $a(M) := \max \{n \mid M_n \neq 0\}$ with $a(M) := -\infty$ if $M = 0$, for a graded $R$-module $M$. As usual the Castelnuovo-Mumford regularity $\reg K[S]$ of $K[S]$ is defined by

$$\reg K[S] := \max \left\{ i + a(H_{m_R}^i(K[S])) \mid 0 \leq i \leq \dim K[S] \right\}.$$
Since the Eisenbud-Goto conjecture is widely open in general, it would be nice to answer the following:

**Question** (Eisenbud-Goto). Does \( \text{reg} K[S] \leq \deg K[S] - \text{codim} K[S] \) hold?

Where \( \text{codim} K[S] := \dim K[S] - \dim K[S] = c \) and \( \deg K[S] \) denotes the multiplicity of \( K[S] \). By a result of Treger the question has a positive answer, if \( K[S] \) is Cohen-Macaulay; the Buchsbaum case was proven by St¨uckrad and Vogel in [19]. For projective monomial curves, i.e. \( d = 2 \), the Eisenbud-Goto conjecture holds by a result of Gruson Lazarsfeld and Peskine [6]. The case \( c = 2 \) was proven by Peeva and Sturmfels in [18]. Moreover in [8] Herzog and Hibi showed that the Eisenbud-Goto conjecture holds for (homogeneous) simplicial affine semigroup rings with isolated singularity (see Remark 3.7). In [9, Theorem 3.2] Hoa and St ¨uckrad presented a bound for the regularity of \( K[S] \) which is a “good” bound, in addition to this they provided some positive answers for the Eisenbud-Goto conjecture. But in fact the Eisenbud-Goto conjecture remains widely open for simplicial affine semigroup rings.

Let \( S \) be normal (see Definition 3.1), hence \( K[S] \) is Cohen-Macaulay by [12, Theorem 1], i.e. the Eisenbud-Goto conjecture holds. In fact the ring \( K[S] \) is not necessary Cohen-Macaulay or Buchsbaum, if \( S \) is seminormal (see Definition 3.1 and Example 3.6). By [9, Proposition 2.2] the Castelnuovo-Mumford regularity of \( K[S] \) can be computed in terms of the regularity of certain monomial ideals by studying the intersection of the Ap´ery sets of the extremal rays of \( S \), we call this set \( B[S] \). In [14, Theorem 4.1.1] Li characterized the seminormal property of \( S \) in terms of \( B[S] \). By this we show in Theorem 4.15: If \( S \) is seminormal, then

\[
\text{reg} K[S] \leq \deg K[S] - \text{codim} K[S].
\]

In fact this bound could be not sharp, since \( \deg K[S] \) could be equal to \( \alpha^{d-1} \). A subclass of seminormal simplicial affine semigroups with \( \deg K[S] = \alpha^{d-1} \) are full Veronese rings. Let \( S_{d,\alpha} := \langle A_{d,\alpha} \rangle \) be the monoid generated by \( A_{d,\alpha} := \{(a_{[1]}, \ldots, a_{[d]}) \in \mathbb{N}^d \mid \sum_{i=1}^d a_{[i]} = \alpha\} \), we have

\[
\deg K[S_{d,\alpha}] - \text{codim} K[S_{d,\alpha}] = \alpha^{d-1} - \left( \frac{\alpha + d - 1}{d - 1} \right) + d,
\]

by Remark 5.1 In Theorem 5.3 we show that

\[
\text{reg} K[S_{d,\alpha}] = \lfloor d - \frac{d}{\alpha} \rfloor.
\]

So in this case the Eisenbud-Goto conjecture is not sharp, see Example 5.4. In fact \( S_{d,\alpha} \) is normal and therefore \( \text{reg} K[S_{d,\alpha}] \leq d - 1 \), by Remark 4.6. In Section 4 we extend this bound to the seminormal case, we show in Theorem 4.7: If \( S \) is seminormal, then:

\[
\text{reg} K[S] \leq d - 1.
\]

In Section 2 we fix the basic notation and the computation of the regularity of \( K[S] \) in terms of the regularity of certain monomial ideals. In the following we study the seminormal case in Section 3. In Section 4 we provide several bounds for the regularity of seminormal simplicial affine semigroup rings. Finally we compute the regularity of full Veronese rings in Section 5. For unspecified notation we refer to [2, 10].
2 Basics

Let $G := G(S)$ be the group generated by $S$ in $\mathbb{Z}^d$. By $x_{[i]}$ we denote the $i$-th component of $x$ and we define $\deg x := \left(\sum_{j=1}^{d} x_{[j]}\right)/\alpha$, for $x \in G$. Let $n \in S$, the Apéry set of $n$ is defined by $S(n) := \{ x \in S \mid x - n \notin S \}$. We set $B_S := \bigcap_{j=1}^{d} S(ε_j)$, i.e. for $x \in B_S$ we have $x - ε_i \notin S$ for all $i = 1, \ldots, d$. We note that if $x \notin B_S$, then $x + y \notin B_S$, for all $x, y \in S$. Let $x \sim y$ if and only if $x - y \in α\mathbb{Z}^d$, hence $\sim$ is an equivalence relation on $G$. It is obvious that every element in $G$ is equivalent to an element in $G \cap D$, where $D := \{ x \in \mathbb{Q}^d \mid 0 \leq x_{[i]} < α, \forall i \}$ and for all $x, y \in G \cap D$ with $x \neq y$ we have $x \not\sim y$. Hence the number of equivalence classes $f := \#(G \cap D)$ is finite. One can show that there are exactly $f \in \mathbb{N}$ equivalence classes in $G$, $G \cap D$, $S$, and in $B_S$ (see [17 Section2]). By $Γ_1, \ldots, Γ_f$ we denote the equivalence classes on $B_S$. For $j = 1, \ldots, f$ we define

$$h_j := (\min \{m_{[1]} \mid m \in Γ_j\}, \min \{m_{[2]} \mid m \in Γ_j\}, \ldots, \min \{m_{[d]} \mid m \in Γ_j\}).$$

Let $T := K[y_1, \ldots, y_d]$ be the polynomial ring graded by $\deg y_i = 1$. We set $Γ_j := \{y^{(x-h_j)/α} \mid x \in Γ_j\}$, where $y^{(u_1,\ldots,u_d)} := y_1^{u_1} \cdots y_d^{u_d}$. By construction $I_j := Γ_j \cap T$ are monomial ideals in $T$, since $h_j \sim x$ for all $x \in Γ_j$. We note that $ht I_j \geq 2$ (height), since $gcd I_j = 1$, for all $j = 1, \ldots, f$. We define $m_T$ as the homogeneous maximal ideal of $T$ and $m_S$ as the homogenous maximal ideal of $K[S]$.

**Proposition 2.1** ([9 Proposition 2.2]). There are isomorphisms of $\mathbb{Z}$-graded $T$-modules:

1) $K[S] \cong \bigoplus_{j=1}^{f} I_j(-\deg h_j)$.

2) $H^i_{m_T}(K[S]) \cong \bigoplus_{j=1}^{f} H^i_{m_T}(I_j)(-\deg h_j)$.

We note that this idea can be extended for arbitrary simplicial affine semigroups, see [17 Proposition 4.1]. Applying the fact $H^i_{m_T}(K[S]) \cong H^i_{m_S}(K[S])$ we have:

$$\text{reg} K[S] = \max \{\text{reg} I_j + \deg h_j \mid j = 1, \ldots, f\},$$

where $\text{reg} I_j$ is the regularity of $I_j$ considered as a $\mathbb{Z}$-graded $T$-module.

**Remark 2.2.** We note that $\text{reg} K[S]$ is independent of $K$ for $d \leq 5$, by [11 Corollary 1.4] and [1]. By Proposition 2.1 it follows that $\deg K[S] = f$. Since $Γ_j \subset B_S$, we have $Γ_j \subset \{a_1, \ldots, a_d\}$ for all $j = 1, \ldots, f$. Moreover it is clear that $\{0, a_1, \ldots, a_d\} \subset B_S$. Consider an element $x \in \{0, a_1, \ldots, a_d\}$ and an element $y \in B_S$ with $x \neq y$. Suppose that $x \sim y$. Since $0 \leq x_{[i]} < α$, for all $i = 1, \ldots, d$, we have $y \geq x$, meaning $y_{[i]} \geq x_{[i]}$ for all $k = 1, \ldots, d$, and therefore $y \notin B_S$. This shows that $x \not\sim y$. W.l.o.g we therefore may assume that $Γ_1 = \{0\}, Γ_2 = \{a_1\}, \ldots, Γ_{c+1} = \{a_c\}$.

**Definition 2.3.** For an element $x \in S$ we say that a sequence $b_1, \ldots, b_n$ has $*$-property $\iff b_1, \ldots, b_n \in \{a_1, \ldots, a_c\}$ and $x - b_1 \in S, x - b_1 - b_2 \in S, \ldots, x - (\sum_{j=1}^{n} b_j) \in S$. Moreover we define $x(i) := x - (\sum_{j=1}^{i} b_j)$ w.r.t. a sequence $b_1, \ldots, b_n$ with $*$-property and $x(0) := x$.

**Remark 2.4.** Suppose that $x \in S$ has a sequence $b_1, \ldots, b_{\deg x}$ with $*$-property, then we get $\deg x(i) = \deg x - i$ for $i = 0, \ldots, \deg x$ and therefore $x(\deg x) = 0$. Hence the length of a sequence with $*$-property is bounded by $\deg x$. Moreover for $0 \leq i < j \leq \deg x$ we have $x(i) \geq x(j)$. There are elements in $S$ with no sequence with $*$-property, e.g. $e_j$. 

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Proposition 2.5. Let $x \in B_S \setminus \{0\}$.

1) There exists a sequence $b_1, \ldots, b_{\deg x}$ with $\ast$-property.

2) Let $b_1, \ldots, b_n$ be a sequence with $\ast$-property. Then there exists a sequence with $\ast$-property $b_1, \ldots, b_n, b_{n+1}, \ldots, b_{\deg x}$.

Proof. 1) Suppose on the contrary that there is no sequence with $\ast$-property of length $\deg x$. Then $x \notin \langle a_1, \ldots, a_c \rangle$, which contradicts to $x \in B_S$.

2) Suppose that $x(n) \notin B_S$, then $x \notin B_S$ which is a contradiction. Therefore we have $x(n) \in B_S$. By claim 1) we are done.

Proposition 2.6. Let $x \in S$ and $b_1, \ldots, b_n$ be a sequence with $\ast$-property. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijection.

1) $b_{\sigma(1)}, \ldots, b_{\sigma(n)}$ is a sequence with $\ast$-property.

2) $b_1, \ldots, b_m$ is a sequence with $\ast$-property for all $1 \leq m \leq n$.

Proof. 1) We need to show that $x(i) \in S$, for all $i = 1, \ldots, n$ w.r.t $b_{\sigma(1)}, \ldots, b_{\sigma(n)}$, since clearly $b_{\sigma(1)}, \ldots, b_{\sigma(n)} \in \{a_1, \ldots, a_c\}$. Let $i = n$, we have $x(n) = x - (\sum_{j=1}^n b_{\sigma(j)}) = x - (\sum_{j=1}^n b_j) \in S$ by assumption. Fix one $i < n$, then

$$x(i) = x - (\sum_{j=1}^i b_{\sigma(j)}) = x - (\sum_{j=1}^n b_{\sigma(j)}) + \sum_{j=i+1}^n b_{\sigma(j)} \in S.$$ 

2) This is obvious.

Lemma 2.7. Let $x \in B_S \setminus \{0\}$ and $b_1, \ldots, b_{\deg x}$ be a sequence with $\ast$-property.

1) $x(i) \in B_S$, for all $i = 0, \ldots, \deg x$.

2) We have $x(i) \neq x(j)$, for all $0 \leq i < j \leq \deg x$.

Proof. 1) Follows from the fact that if $x(i) \notin B_S$, then $x(i) + y \notin B_S$ for all $y \in S$.

2) Suppose on the contrary that $x(i) \sim x(j)$. We have $\deg x(i) > \deg x(j)$ and $x(i) \geq x(j)$, hence $x(i) \notin B_S$ which contradicts to claim 1).

Corollary 2.8 ([9, Theorem 1.1]). We have $\deg x \leq \deg K[S] - \codim K[S]$, for all $x \in B_S$.

Proof. W.l.o.g. we may assume that $\deg x \geq 2$. By Lemma 2.7 and Remark 2.2 there is a set $L = \{0, a_1, \ldots, a_c, x(0), \ldots, x(\deg x - 2)\} \subseteq B_S$, such that for all $x, y \in L$ with $x \neq y$ we have $x \neq y$. Hence $f = \deg K[S] \geq \#L = \deg x + \codim K[S]$.

\[ \]

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Remark 2.9. We note that this proof is a new short proof of [9, Theorem 1.1]. We define the reduction number \(r(K[S])\) of \(K[S]\) by \(r(K[S]) := \max \{\deg x \mid x \in B_S\}\), see [9, Section 1 and first Remark in Section 2]. By Corollary 2.8 or [9, Theorem 1.1] we get
\[
r(K[S]) \leq \deg K[S] - \text{codim}K[S],
\]
i.e. the Eisenbud-Goto conjecture holds for the reduction number of \(K[S]\). So whenever we have \(\text{reg}K[S] = r(K[S])\) the Eisenbud-Goto conjecture holds. It should be mentioned that this property does not hold in general. Even for a monomial curve in \(\mathbb{P}^3\) the equality does not hold. For \(S = \langle (40, 0), (0, 40), (35, 5), (11, 29) \rangle\) we have \(\text{reg}K[S] = 13 > 11 = r(K[S]).\) Moreover it is obvious that \(r(K[S]) \leq \text{reg}K[S]\), by (1).

Example 2.10. Let \(S = \langle (4, 0), (0, 4), (3, 1), (1, 3) \rangle\). Using Macaulay2 [5] we have \(B_S = \{(0, 0), (3, 1), (1, 3), (6, 2), (2, 6)\}\) and therefore \(r(K[S]) = \max \{0, 1, 1, 2, 2\} = 2\). We get \(\Gamma_1 = \{(0, 0)\}, \Gamma_2 = \{(3, 1)\}, \Gamma_3 = \{(1, 3)\}, \Gamma_4 = \{(6, 2), (2, 6)\}\) and \(h_1 = (0, 0), h_2 = (3, 1), h_3 = (1, 3), h_4 = (2, 2)\). By this we have \(I_1 = I_2 = I_3 = T\) and \(I_4 = (y_1, y_2)T\), hence
\[
\text{reg}K[S] = \max \{\text{reg}T + 0, \text{reg}T + 1, \text{reg}T + 1, \text{reg}(y_1, y_2)T + 1\} = \max \{0, 1, 1, 2\} = 2.
\]

Lemma 2.11. Let \(x \in B_S, t \in \mathbb{N}^+, k \in \{1, \ldots, d\}\) and \(x_{[k]} = ta\). There is a sequence with \(*\)-property \(b\) such that \((t-1)\alpha < (x-b)[k] < ta\).

Proof. By Proposition 2.5 there is a sequence \(b_1, \ldots, b_{\deg x}\) with \(*\)-property. We have \(x(\deg x) = 0\) by Remark 2.4, hence there is a \(p \in \{1, \ldots, \deg x\}\) such that \(b_p[k] > 0\). Since \(b_p \in \{a_1, \ldots, a_c\}\) we know that \(b_p[k] < \alpha\). The assertion follows by Proposition 2.6.

Lemma 2.12. Let \(J \subseteq \{1, \ldots, d\}\) with \(#J \geq 1\). Let \(x \in B_S\) such that \(x_{[k]} = \alpha\), for all \(k \in J\). There exists a sequence \(b_1, \ldots, b_{\deg x}\) with \(*\)-property such that: for all \(i = 1, \ldots, \#J\) there is at least one \(k \in J\) such that \(0 < x(i)[k] < \alpha\).

Proof. By Lemma 2.11 the case \(#J = 1\) is clear, assume that \(#J > 1\). Fix an arbitrary sequence with \(*\)-property \(b_1, \ldots, b_{\#J-1}\). By Remark 2.4 there is a \(k \in J\) such that \(x(i)[k] > 0\), for all \(i = 1, \ldots, \#J-1\). By this, induction and Lemma 2.11 there is a sequence with \(*\)-property \(b_1, \ldots, b_{\#J-1}\) such that: for all \(i = 1, \ldots, \#J-1\) there is a \(k \in J\) such that \(0 < x(i)[k] < \alpha\). By Lemma 2.11 we may assume that already \(x(\#J-1)[k] < \alpha\) for all \(k \in J\). By Proposition 2.5 there is a sequence with \(*\)-property \(b_1, \ldots, b_{\#J-1}, b_{\#J}, \ldots, b_{\deg x}\). Suppose on the contrary that \(x(\#J)[k] = 0\), for all \(k \in J\). Since \(\deg x(\#J) = \deg x - \#J\) and \(x \geq x(\#J)\) we have \(x(\#J) = x(\sum_{k \in J} e_k)\) and therefore \(x \notin B_S\), since \(x(\#J) \in S\).

3 The seminormal case

Let us consider an affine semigroup \(U \subseteq \mathbb{N}^d\), i.e. \(U\) is a finitely generated submonoid of \((\mathbb{N}^d, +)\). By \(G(U)\) we denote the group generated by \(U\). There are two closely related definitions in this context:
Definition 3.1. 1. We call $U$ seminormal, if $x \in G(U)$ and $2x, 3x \in U$ imply $x \in U$.
2. We call $U$ normal, if $x \in G(U)$ and $tx \in U$ for some $t \in \mathbb{N}^+$ imply $x \in U$.

Remark 3.2. A Noetherian domain $\bar{R}$ is called seminormal if for an element $x$ in the quotient field $Q(\bar{R})$ of $\bar{R}$ such that $x^2, x^3 \in \bar{R}$ we have $x \in \bar{R}$. By a result of Hochster and Roberts the ring $K[U]$ is seminormal if and only if $U$ is seminormal, see [13, Proposition 5.32]. A similar result holds in the normal case, by [12].

To get new bounds for the regularity of $K[S]$, we need another characterization. We define the set Box := $\{ x \in S \mid x = \sum_{i=1}^{d} \lambda_i e_i \}$, for some $\lambda_i \in \mathbb{Q} \cap [0,1]$. So we have Box = $\{ x \in S \mid x[i] \leq \alpha, \forall i = 1, \ldots, d \}$.

Theorem 3.3 ([13, Theorem 4.1.1]). The semigroup $S$ is seminormal if and only if $B_S$ is contained in Box.

From now on we assume that $S$ is seminormal. Let $I_j \neq T$ be an ideal which arises by the construction of Proposition 2.1. For $x \in \Gamma_j$, we have $0 \leq x[i] \leq \alpha$ and therefore $((x - h_j)/\alpha)[i] \in [0,1]$. Hence $I_j$ is a squarefree monomial ideal in $T$.

Lemma 3.4. Let $i, t \in \mathbb{N}$ with $1 \leq i \leq d$ and $1 \leq t \leq f$.

1) Let $x, y \in \Gamma_t$ with $x \neq y$. If $x[i] \neq y[i]$, then $x[i] - y[i] \in \{-\alpha, \alpha\}$. If $0 < x[i] < \alpha$, then $x[i] = y[i]$.
2) Let $x, y \in \Gamma_i$ with $x \neq y$. If $0 < x[i] < \alpha$, then $x[i] = y[i]$.
3) Let $x, y \in \Gamma_i$ with $x \neq y$. If $x[i] \neq y[i]$, then $x[i] \in \{0, \alpha\}$ and $y[i] = \alpha - x[i]$.
4) Let $x, y \in \Gamma_i$ with $x \neq y$. If $0 < x[i] = y[i] < \alpha$ and $0 < x[i] = y[i] < \alpha$ for some $i, j \in \{1, \ldots, d\}$ with $i \neq j$.
5) If $h_{\ell}[i] > 0$, then $h_{\ell}[i] = x[i]$, for all $x \in \Gamma_t$.

Proof. 1) We have $x[i] - y[i] \in \alpha \mathbb{Z}$ and $x[i] - y[i] \in [-\alpha, \alpha]$, since $0 \leq x[i], y[i] \leq \alpha$. Hence $x[i] - y[i] \in \{-\alpha, \alpha\}$.
2) We have $x[i] - y[i] \notin \{-\alpha, \alpha\}$ and therefore $x[i] = y[i]$ by claim 1).
3) By claim 1) we have $x[i] - y[i] \in \{-\alpha, \alpha\}$ and $x[i] \in \{0, \alpha\}$, by claim 2). Hence $y[i] = \alpha - x[i]$.
4) By claim 2) it is sufficient to show that $0 < x[i], x[j] < \alpha$ for some $i \neq j$. Suppose on the contrary that this is not true. If $x[i] \in \{0, \alpha\}$ for all $i = 1, \ldots, d$ we have $x \sim 0$. Hence $0 \in \Gamma_i$ and therefore $\# \Gamma_i = 1$ which is a contradiction. Suppose that $0 < x[i] < \alpha$ for exact one $i \in \{1, \ldots, d\}$, i.e. $x[i] \in \{0, \alpha\}$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$. By this we have $\sum_{j=1}^{d} x[j] \notin \alpha \mathbb{N}$ which is a contradiction, since $x \in S$.
5) Let $x \in \Gamma_i$. We have $0 < h_{\ell}[i] \leq x[i] \leq \alpha$ and therefore $h_{\ell}[i] = x[i]$, since $h_{\ell}[i] - x[i] \in \alpha \mathbb{Z}$, by construction.

Corollary 3.5 ([13, Theorem 2.2]). If $d \leq 3$, then $S$ is Cohen-Macaulay.

Proof. By [4, Proposition 8] we need to show that $\# \Gamma_i = 1$, for all $t = 1, \ldots, f$. By Lemma 3.4(4) the case $d = 2$ is trivial. Suppose on the contrary that $x, y \in \Gamma_i$ with $x \neq y$.
By Lemma 3.4(4) we may assume that $0 < x[i] = y[i] < \alpha$ for $i = 1, 2$. By Lemma 3.4(3) we may assume that $x[3] = \alpha$ and $y[3] = 0$, since $x[3] \neq y[3]$. Then $x - e_3 = y \in S$ which contradicts to $x \in B_S$. 

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Example 3.6. Let us consider the semigroup
\[ S = \langle e_1, \ldots, e_6, (1, 1, 0, 0, 0, 0), (1, 0, 1, 0, 0, 0), (0, 0, 1, 1, 0, 0), (0, 1, 0, 1, 0, 0), (0, 0, 0, 0, 1, 1) \rangle, \]
in \( \mathbb{N}^6 \) with \( \alpha = 2 \). We have \( B_S \subseteq \text{Box} \) thus \( S \) is seminormal by Theorem 3.3. One can show that \( (0, 1, 1, 0, 0, 0) + e_1, (0, 1, 0, 0, 0) + e_2 \in S \), but \( (0, 1, 1, 0, 0, 0) + (0, 0, 0, 0, 1, 1) = (0, 1, 1, 0, 1, 1) \notin S \). Hence \( K[S] \) is not Buchsbaum by [21, Lemma 3]. By a similar example, one can show that Corollary 3.5 does not hold for \( d = 4 \). For a general discussion of the relation between the seminormal property and the Cohen-Macaulay property of affine semigroup rings we refer to [14].

Remark 3.7. Herzog and Hibi showed in [8] that the Eisenbud-Goto conjecture holds for simplicial affine semigroups with isolated singularity. This is equivalent to the statement that \( A \) (see Introduction) contains all points of type \((0, \ldots, \alpha - 1, \ldots, 1, \ldots, 0)\), where \( \alpha - 1, 1 \) stay in the \( i \)-th and \( j \)-th positions, respectively, and the other coordinates are zero. By Example 3.6 we are studying a distinct class of simplicial affine semigroup rings.

4 Bounding the regularity

In this section we assume that \( S \) is seminormal. Keep in mind that \( I_j \) is a squarefree monomial ideal in \( T \), for all \( j = 1, \ldots, f \).

Remark 4.1. By Theorem 3.3 \( S \) is seminormal, if and only if \( B_S \subseteq \text{Box} \). Clearly \( r(K[S]) \leq d \). On the other hand there is only one element in \( \text{Box} \) with degree \( d \), namely \( (\alpha, \ldots, \alpha) \), but \( (\alpha, \ldots, \alpha) \notin B_S \). Hence \( r(K[S]) \leq d - 1 \). In Theorem 4.7 we obtain a similar bound for the regularity of \( K[S] \).

Definition 4.2. For a monomial \( m = y_1^{b_1} \cdots y_d^{b_d} \) we define \( \deg m = \sum_{j=1}^{d} b_j \). Let \( I \) be a monomial ideal in \( T \) with a minimal set of monomial generators \( \{m_1, \ldots, m_s\} \). Let \( F \) be the least common multiple of \( \{m_1, \ldots, m_s\} \), then we define \( \var(I) := \deg F \).

Remark 4.3. Consider the squarefree monomial ideal \( I = (y_1y_2, y_2y_3y_4, y_7)T \) in \( T = K[y_1, \ldots, y_7] \). Clearly \( \var(I) = 5 \). So in the squarefree case \( \var(I) \) is equal to the number of variables, which occur in the generators of \( I \). We note that \( I \) is always a minimal set of monomial generators of \( I_j \). Moreover every monomial ideal in \( T \) has a unique minimal set of monomial generators by [16, Lemma 1.2]. Since \( \text{ht} I_j \geq 2 \) we have \( \var(I_j) \neq 1 \). Moreover for all \( j = 1, \ldots, f \) we get \( I_j \neq T \), if and only if \( \var(I_j) \neq 0 \).

Lemma 4.4. \( \var(I_j) \leq d - 1 - \deg h_j \), for all \( j = 1, \ldots, f \).

Proof. Assume that \( I_j = T \), then \( \var(T) = 0 \) and \( \deg h_j \leq d - 1 \) by Remark 4.1. So we may assume that \( \# \Gamma_j \geq 2 \). By Lemma 3.4 we have \( 0 < x_{[i]}, x_{[l]} < \alpha \), for all \( x \in \Gamma_j \) and some \( k \neq l \). In particular \( 0 < h_{j[k]}, h_{j[l]} < \alpha \). Suppose on the contrary that \( \var(I_j) \geq d - \deg h_j \). Then by Lemma 3.4 \( h_{j[t]} = 0 \) for all \( t \in J \) for some \( J \subseteq \{1, \ldots, d\} \) with \( \# J = d - \deg h_j \). Since \( 0 < h_{j[k]}, h_{j[l]} < \alpha \) for some \( l \neq k \) and \( 0 \leq h_{j[i]} \leq \alpha \), for all \( i = 1, \ldots, d \) by Theorem 3.3 we get \( \deg h_j < \deg h_j \) which is a contradiction. \( \square \)
Theorem 4.5 (II, Theorem 3.1). Let $I$ be a proper monomial ideal in $T$. Then
\[ \text{reg} I \leq \text{var}(I) - \text{ht} I + 1. \]

Remark 4.6. One can show that $\text{reg} K[S] \leq d - 1$, if $S$ is normal (by the proof of [10, Corollary 4.7] and [10, Corollary 3.8]). The next Theorem obtains a similar bound in the seminormal case.

Theorem 4.7.
\[ \text{reg} K[S] \leq d - 1 \]

Proof. By Remark 4.1 and (1) we may assume that $\#\Gamma_j \geq 2$. We need to show that $\text{reg} I_j + \deg h_j \leq d - 1$, for a fixed $j \in \{1, \ldots, f\}$ such that $\#\Gamma_j \geq 2$. By Lemma 4.4 and Theorem 4.5 we get
\[ \text{reg} I_j \leq \text{var}(I_j) - \text{ht} I_j + 1 \leq d - 1 - \deg h_j - 2 + 1 = d - 2 - \deg h_j, \]
since $\text{ht} I_j \geq 2$. Hence $\text{reg} I_j + \deg h_j \leq d - 2$ and we are done.

Remark 4.8. We note that the bound established in Theorem 4.7 is sharp. Assume \( \alpha \geq d \) in Theorem 5.3, by this we get $\text{reg} K[S_{d,\alpha}] = d - 1$ and of course $S_{d,\alpha}$ is seminormal.

Proposition 4.9. If $d \leq 5$, then $\text{reg} K[S] = r(K[S])$.

Proof. By Corollary 3.3 and [1] Proposition 8 the case $d \leq 3$ is clear. We show that $\text{reg} I_j$ is equal to the maximal degree of a generator of $I_j$. By this we get:
\[ \text{reg} I_j + \deg h_j = \max \{\deg x \mid x \in \Gamma_j\}, \]
for all $j = 1, \ldots, f$ and we are done by [1]. The case $\#\Gamma_j = 1$ is obvious. We therefore may assume that $\#\Gamma_j \geq 2$ and we fix such a $j \in \{1, \ldots, f\}$. By Lemma 3.3 we get $\deg h_j \geq 1$. Let $d = 5$, by Lemma 4.4 we have to consider the cases $\text{var}(I_j) \in \{2, 3\}$. Let $\text{var}(I_j) = 2$. The ideal $I_j$ is of the form $I_j = (y_k, y_l)T$ for some $k \neq l$ and $k, l \in \{1, \ldots, 5\}$, since $\text{ht} I_j \geq 2$. It follows that $\text{reg} I_j = 1$. By a similar argument we get the assertion for the case $d = 4$ and $\text{var}(I_j) = 2$. Let $d = 5$ and $\text{var}(I_j) = 3$, i.e. $\deg h_j = 1$. Since $\text{ht} I_j \geq 2$ and by Theorem 3.3 the only ideals possible are:
\[ I_{j_1} = (yk, y_l, y_m), I_{j_2} = (yk y_l, y_m), I_{j_3} = (yk y_l, yk y_m, y_l y_m) \]
and $k, l, m \in \{1, \ldots, 5\}$ are pairwise not equal. By Theorem 4.3 we get $\text{reg} I_{j_1} = 1$ and $\text{reg} I_{j_2} = \text{reg} I_{j_3} = 2$.

By Corollary 2.3 the Eisenbud-Goto conjecture holds, if $d \leq 5$. The next Theorem shows that the Eisenbud-Goto conjecture holds in any dimension.

Remark 4.10. Proposition 4.9 could fail for $d \geq 6$. Let us consider the squarefree monomial ideal $I = (y_1 y_2, y_3 y_4)T$ with $\text{var}(I) = 4$. So $\text{reg} I = 3$ is bigger than the maximal degree of a generator of $I$, which is 2.
Definition 4.11. Let $I$ be a proper monomial ideal in $T$ with a minimal set of monomial generators $\{m_1, \ldots, m_s\}$. Let $F$ be the least common multiple of $\{m_1, \ldots, m_s\}$, say $F = y_1^{c_1} \cdots y_d^{c_d}$. We define the set $\supp(I) \subseteq \{1, \ldots, d\}$ w.r.t. $I$: $i \in \supp(I) \iff b_i \neq 0$.

Remark 4.12. So $\supp(I)$ is the set of indices of the variables, which occur in one of the minimal generators of $I$. For the ideal $I = (y_1 y_2 y_3 y_4, y_5 y_6)T$ in $T = K[y_1, \ldots, y_7]$, we have $F = y_1 y_2 y_3 y_4 y_5 y_6$, i.e. $\supp(I) = \{1, 2, 3, 5, 6\}$. Consider the ideal $y_1 y_2 T$, we get $\supp(y_1 y_2 T) = \{1, 2\}$.

Lemma 4.13. Let $\#\Gamma_j \geq 2$, $n \in \Gamma_j$ and $m \in \tilde{\Gamma}_j$ such that $m = y^{(n-h_i)/\alpha}$. Then

1) $n_{[q]} = 0$, for all $q \in \supp(I_j) \setminus \supp(mT)$.
2) $n_{[q]} = \alpha$, for all $q \in \supp(mT)$.

Proof. 1) Suppose on the contrary that there is a $q \in (\supp(I_j) \setminus \supp(mT)) \neq \emptyset$ such that $n_{[q]} > 0$. Since $q \in \supp(I_j)$ we have $h_{j[q]} = 0$ by Lemma 3.15 and therefore $n_{[q]} = \alpha$, since $h_{j[q]} - n_{[q]} \in \alpha \mathbb{Z}$ and $n_{[q]} \leq \alpha$. This implies $q \in \supp(mT)$, which is a contradiction.
2) Since $q \in \supp(mT)$, we have $n_{[q]} \geq 0$. By Theorem 3.3 $n_{[q]} \leq \alpha$. \qed

Remark 4.14. The above Lemma fails for an arbitrary $S$, like in Example 2.10 Consider $\Gamma_4 = \{(6, 2), (2, 6)\}$, i.e. $h_4 = (2, 2)$ and $\tilde{\Gamma}_4 = \{y_1, y_2\}$. For every $n \in \Gamma_4$ we have $n_{[i]} \neq 0$, $i = 1, 2$. But $\supp(I_4) = \{1, 2\}$ and $\#\supp(y_1 T) = \#\supp(y_2 T) = 1$.

Theorem 4.15.

$$\text{reg}K[S] \leq \text{deg}K[S] - \text{codim}K[S]$$

Proof. By (i) we need to show that $\text{deg}K[S] - c \geq \text{reg}I_j + \deg h_j$, for all $j = 1, \ldots, f$. If $\#\Gamma_j = 1$ the assertion follows by Corollary 2.8. Let us fix a $j \in \{1, \ldots, f\}$ such that $\#\Gamma_j \geq 2$. We have $\Gamma_j = \{n_1, \ldots, n_{\#\Gamma_j}\}$ and $\tilde{\Gamma}_j = \{m_1, \ldots, m_{\#\tilde{\Gamma}_j}\}$. We may assume that $m_i = y^{(n_i-h_i)/\alpha}$. We set $J_k := (m_1, \ldots, m_k)T$ and $g(k) := \var(J_k) - \text{ht}J_k + 1 + \deg h_j$, for $1 \leq k \leq \#\Gamma_j$. We show by induction on $k$ with $1 \leq k \leq \#\Gamma_j$ that there is a set $L_k$:

(i) $L_k \subseteq B_S$.
(ii) $\#L_k \geq g(k) - 1$.
(iii) $x \neq y$, for all $x, y \in L_k$ with $x \neq y$.
(iv) $\deg x \geq 2$, for all $x \in L_k$.
(v) $x_{[i]} = 0$, for all $x \in L_k$ and for all $q \in \supp(I_j) \setminus \supp(J_k)$.

Let $k = 1$. We know that $\text{ht}J_1 = 1$ and $\var(J_1) + \deg h_j = \deg n_1$, i.e. $g(1) = \deg n_1$. By Proposition 2.5 (1) $n_1$ has a sequence $b_1, \ldots, b_{\deg n_1}$ with $*$-property, since $n_1 \in B_S$. Set $L_1 := \{n_1(0), \ldots, n_1(\deg n_1 - 2)\}$, clearly $\#L_1 \geq \deg n_1 - 1 = g(1) - 1$, i.e. property (ii) is satisfied and by construction we get property (iv). By Lemma 2.7 (1) $L_1 \subseteq B_S$ which shows (i) and by Lemma 2.7 (2) property (iii) holds. By Lemma 4.13 (1) property (v) holds for $n_1(0)$, hence for every element in $L_1$.

Using induction on $k \leq \#\Gamma_j - 1$ the properties (i)-(v) hold for $L_k = \{c_1, \ldots, c_p\}$. There could be two different cases:

Case 1: $\supp(J_k) \cap \supp(m_{k+1} T) \neq \emptyset$. (e.g. $k = 2$, $J_2 = (y_1 y_2, y_2 y_3 y_4)T$, $m_3 = y_4 y_5 y_6$).
(iii) We set \( J := (\text{supp}(m_{k+1}T) \setminus \text{supp}(J_k)) \). Since \( \text{supp}(J_k) \cap \text{supp}(m_{k+1}T) \neq \emptyset \) we have \( \deg n_{k+1} \geq \# J + 2 \), in particular \( n_{k+1}[q] = \alpha \), for all \( q \in J \), see Lemma 2.12. By Lemma 2.12 there is a sequence \( b_1, \ldots, b_{\deg n_{k+1}} \) with \( \ast \)-property such that for all \( q = 1, \ldots, \# J \) there is one \( p \in J \) with \( 0 < n_{k+1}(q)[p] < \alpha \). Let us fix a \( c_i \). By property (v) \( c_{i[p]} = 0 \), hence \( c_i \neq n_{k+1}(q) \), for all \( q = 1, \ldots, \# J \). Set

\[
L_{k+1} := \{c_1, \ldots, c_p, n_{k+1}(1), \ldots, n_{k+1}(\# J)\}.
\]

In case that \( J = \emptyset \), we set \( L_{k+1} = L_k \). By Lemma 2.7 we get (iii).

(i) By Lemma 2.7 \( n_{k+1}(1), \ldots, n_{k+1}(\# J) \in B_S \), since \( n_{k+1} \in B_S \).

(iv) Since \( \deg n_{k+1} \geq \# J + 2 \).

(v) By induction \( c_{i[q]} = 0 \), for all \( q \in (\text{supp}(I_j) \setminus \text{supp}(J_k)) \). By Lemma 1.13 we have \( n_{k+1}[q] = 0 \), for all \( q \in (\text{supp}(I_j) \setminus \text{supp}(m_{k+1}T)) \cap (\text{supp}(I_j) \setminus \text{supp}(J_{k+1})) \), hence property (v) holds.

(ii) Since \( \text{ht} J_{k+1} \geq \text{ht} J_k \) and \( \text{var}(J_{k+1}) = \text{var}(J_k) + \# J \) we have

\[
g(k + 1) - 1 \leq \# J + \text{var}(J_k) - \text{ht} J_k + 1 + \deg h_j - 1 = \# J + g(k) - 1 \leq \# J + p.
\]

Case 2: \( \text{supp}(J_k) \cap \text{supp}(m_{k+1}T) = \emptyset \). (e.g. \( k = 2, J_2 = (y_1 y_2, y_2 y_3 y_4)T, m_3 = y_5 y_6 y_7 \).)

(iii) Similar argument beside of the fact that \( \deg n_{k+1} \geq \# J + 1 \). Replace \( L_{k+1} \) by

\[
L_{k+1} := \{c_1, \ldots, c_p, n_{k+1}(1), \ldots, n_{k+1}(\# J - 1)\}.
\]

In case that \( \# J = 1 \), we set \( L_{k+1} = L_k \).

(i), (iv), (v) Analogous, replace \# \( J \) by \# \( J - 1 \).

(ii) Since \( \text{supp}(J_k) \cap \text{supp}(m_{k+1}T) = \emptyset \), \( m_{k+1} + J_k \) is a non-zero-divisor of \( T/J_k \). Hence \( \text{ht} J_{k+1} = \text{ht} J_k + 1 \), by Krull’s Principal Ideal Theorem (e.g. see [2, Theorem 10.1]). So

\[
g(k + 1) - 1 = \# J + \text{var}(J_k) - \text{ht} J_k - 1 + 1 + \deg h_j - 1 = \# J + g(k) - 2 \leq \# J + p - 1.
\]

By this we get a set \( L_{\# J} = \{c_1, \ldots, c_p\} \subseteq B_S \) with the above properties, in particular \( p \geq g(\# J) - 1 = \text{var}(I_j) - \text{ht} I_j + 1 + \deg h_j - 1 \geq \text{reg} I_j + \deg h_j - 1 \), by Theorem 4.5. By Remark 2.2 we get a set \( L = \{0, a_1, \ldots, a_c, c_1, \ldots, c_p\} \subseteq B_S \) with \( x \neq y \), for all \( x, y \in L \) with \( x \neq y \). So for all \( j = 1, \ldots, J \) we have

\[
f = \deg K[S] \geq \# L = c + p + 1 \geq c + \text{reg} I_j + \deg h_j.
\]

\[\square\]

Remark 4.16. We note that Corollary 2.8 holds for any \( S \). So Theorem 4.15 holds with the following assumption on \( S \):

- If \( \# \Gamma_j \geq 2 \), then \( \Gamma_j \) is contained in Box.
5 Regularity of full Veronese rings

For $X, Y \subseteq \mathbb{N}^d$ we define $X + Y := \{x + y | x \in X, y \in Y\}$, $mX := X + \ldots + X$ (m-times) and $0X := 0$. Moreover we set $A_{d, \alpha} := \{(a_1[1], \ldots, a_d[1]) \in \mathbb{N}^d | \sum_{i=1}^d a_i[1] = \alpha\}$ and $S_{d, \alpha} = \langle A_{d, \alpha} \rangle$. For example $A_{3, 2} = \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$.

It is trivial that:

$$nA_{d, \alpha} = \{(a_1, \ldots, a_d) \in \mathbb{N}^d | \sum_{i=1}^d a_i = n\alpha\}. \tag{3}$$

Hence there is an isomorphism of $K$-vector spaces:

$$K[S_{d, 1}]_n = K[t_1, \ldots, t_d]_n \cong K[S_{d, \alpha}]_{n\alpha}.$$

We have $h_{K[t_1, \ldots, t_d]}(n) = \left(\frac{n + d - 1}{d - 1}\right)$, where $h_{K[t_1, \ldots, t_d]}$ denotes the Hilbert polynomial of $K[t_1, \ldots, t_d]$ and therefore

$$h_{K[S_{d, \alpha}]}(n) = h_{K[t_1, \ldots, t_d]}(n\alpha) = \left(\frac{n\alpha + d - 1}{d - 1}\right). \tag{4}$$

**Remark 5.1.** By $\{4\}$ deg$K[S_{d, \alpha}] = \alpha^{d-1}$ and $\#A_{d, \alpha} = h_{K[S_{d, \alpha}]}(1) = \left(\frac{\alpha + d - 1}{d - 1}\right)$, hence codim$K[S_{d, \alpha}] = \left(\frac{\alpha + d - 1}{d - 1}\right) - d$.

Since the semigroup $S_{d, \alpha}$ is normal, the ring $K[S_{d, \alpha}]$ is Cohen-Macaulay by [12, Theorem 1] and therefore $\#\Gamma_j = 1$ for all $j = 1, \ldots, f$ (see [10] or [3, Proposition 8]). Hence

$$\text{reg}K[S_{d, \alpha}] = r(K[S_{d, \alpha}]_1), \tag{5}$$

by $\{1\}$. Now we compute the reduction number of $K[S_{d, \alpha}]$, which can be computed by $r(K[S_{d, \alpha}]) = \min \{r \in \mathbb{N} | rA_{d, \alpha} + \{e_1, \ldots, e_d\} = (r + 1)A_{d, \alpha}\}$, see [9, Section 1].

**Lemma 5.2.** Let $r \in \mathbb{N}$. The following assertions are equivalent:

1) $rA_{d, \alpha} + \{e_1, \ldots, e_d\} = (r + 1)A_{d, \alpha}$.
2) $(r + 1)\alpha > d(\alpha - 1)$.

**Proof.** 1) $\Rightarrow$ 2) Let us assume that $0 \leq (r + 1)\alpha \leq d(\alpha - 1)$. It is trivial that there is an element $x \in \mathbb{N}^d$ with $x[i] \leq \alpha - 1$ for all $i = 1, \ldots, d$ and $\sum_{i=1}^d x[i] = (r + 1)\alpha$. We have $x \in (r + 1)A_{d, \alpha}$, by $\{3\}$. Now suppose that $x \in rA_{d, \alpha} + \{e_1, \ldots, e_d\}$ then $x = x' + e_j$, for some $j$ and therefore $x[j] \geq \alpha$ a contradiction, hence $x \notin rA_{d, \alpha} + \{e_1, \ldots, e_d\}$.

2) $\Rightarrow$ 1) Let $x \in (r + 1)A_{d, \alpha}$ and suppose that $x[i] \leq \alpha - 1$ for all $j$ then $(r + 1)\alpha = \sum_{i=1}^d x[i] \leq (\alpha - 1)$, hence $x[j] \geq \alpha$ for some $j$ and therefore $x - e_j \in rA_{d, \alpha}$ by $\{3\}$. Hence $(r + 1)A_{d, \alpha} \subseteq rA_{d, \alpha} + \{e_1, \ldots, e_d\}$ and we are done.

**Theorem 5.3.**

$$\text{reg}K[S_{d, \alpha}] = \left[d - \frac{d}{\alpha}\right]$$
Proof. We show that $r(K[S_{d,\alpha}]) = \lfloor d - \frac{d}{\alpha} \rfloor$ and we are done by (5). We have
$$
\left( \lfloor d - \frac{d}{\alpha} \rfloor + 1 \right) \alpha > (d - \frac{d}{\alpha} + 1 - 1)\alpha = d(\alpha - 1),
$$
hence $r(K[S_{d,\alpha}]) \leq \lfloor d - \frac{d}{\alpha} \rfloor$, by Lemma 5.2. Without loss of generality assume that $\lfloor d - \frac{d}{\alpha} \rfloor \geq 1$. We have
$$
\left( \lfloor d - \frac{d}{\alpha} \rfloor - 1 + 1 \right) \alpha \leq \left( d - \frac{d}{\alpha} \right) \alpha = d(\alpha - 1),
$$
hence $r(K[S_{d,\alpha}]) > \lfloor d - \frac{d}{\alpha} \rfloor - 1$, by Lemma 5.2.

Example 5.4. By Theorem 5.3 we are able to compute the Castelnuovo-Mumford regularity of full Veronese rings. For $S_{20,2}$ we know that $\text{reg} K[S_{20,2}] = \lfloor 20 - \frac{20}{2} \rfloor = 10$ and $\deg K[S_{20,2}] - \text{codim} K[S_{20,2}] = 2^{19} - \left( \begin{array}{c} 2 + 19 \\ 19 \end{array} \right) + 20 = 524098$, by Remark 5.1.

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References


