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Abstract

We show that the Eisenbud-Goto conjecture holds for seminormal simplicial affine semigroup rings. Moreover we prove an upper bound for the Castelnuovo-Mumford regularity in terms of the dimension, which is similar as in the normal case. Finally we compute explicitly the regularity of full Veronese rings.

1 Introduction

Let S be a homogeneous simplicial affine semigroup, i. e. S is the submonoid of $(\mathbb{N}^d, +)$ generated by a set $A := \{e_1, \ldots, e_d, a_1, \ldots, a_c\} \subset \mathbb{N}^d$, where

$$e_1 := (\alpha, 0, \ldots, 0), e_2 := (0, \alpha, 0, \ldots, 0), \ldots, e_d := (0, \ldots, 0, \alpha),$$

$$a_i = (a_{i[1]}, \ldots, a_{i[d]}), \text{ with } a_{i[1]} + \ldots + a_{i[d]} = \alpha, \ i = 1, \ldots, c.$$

Moreover we assume that the integers $a_{i[j]}$, $i = 1, \ldots, c$, $j = 1, \ldots, d$ are relatively prime and we assume that $d \geq 2, c \geq 1$ and $\alpha \geq 2$. Let K be an arbitrary field, by $K[S]$ we denote the affine semigroup ring of S. As usual we can identify the affine semigroup ring $K[S]$ with the subring of the polynomial ring $K[t_1, \ldots, t_d]$ generated by monomials $t^a := t_1^{a_{[1]}} \cdots t_d^{a_{[d]}}$, where $a = (a_{[1]}, \ldots, a_{[d]}) \in S$. In the following we study the $\mathbb{Z}$-grading on $K[S]$ which is induced by $\deg t^a = (\sum_{i=1}^d a_{[i]})/\alpha$. We note that $\dim K[S] = d$. By $R := K[x_1, \ldots, x_{d+c}]$ we denote the standard-graded polynomial ring over K, i. e. $\deg x_i = 1$. Thus we have a $\mathbb{Z}$-graded surjective K-algebra homomorphism:

$$\pi : K[x_1, \ldots, x_{d+c}] \to K[S],$$

given by $x_i \mapsto t_i^\alpha$, $i = 1, \ldots, d$ and $x_{d+j} \mapsto t^{a_j}$, $j = 1, \ldots, c$. Hence $K[S] \cong R/\ker\pi$, where $\ker\pi$ is a homogeneous prime ideal of R. Let m_R denote the maximal homogeneous ideal of R and $a(M) := \max\{n \mid M_n \neq 0\}$ with $a(M) := -\infty$ if $M = 0$, for a graded R-module M. As usual the Castelnuovo-Mumford regularity $\mathrm{reg}K[S]$ of $K[S]$ is defined by

$$\mathrm{reg}K[S] := \max\left\{i + a(H^i_{m_R}(K[S])) \mid 0 \leq i \leq \dim K[S]\right\}.$$

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Since the Eisenbud-Goto conjecture [3] is widely open in general, it would be nice to answer the following:

Question (Eisenbud-Goto)**.** Does reg$K[S] \leq$ deg$K[S]$ − codim$K[S]$ hold?

Where codim$K[S] := \dim_K K[S]_1 - \dim K[S] = c$ and deg$K[S]$ denotes the multiplicity of $K[S]$. By a result of Treger [20] the question has a positive answer, if $K[S]$ is Cohen-Macaulay; the Buchsbaum case was proven by Stückrad and Vogel in [19]. For projective monomial curves, i. e. $d = 2$, the Eisenbud-Goto conjecture holds by a result of Gruson Lazarsfeld and Peskine [6]. The case $c = 2$ was proven by Peeva and Sturmfels in [18]. Moreover in [8] Herzog and Hibi showed that the Eisenbud-Goto conjecture holds for (homogeneous) simplicial affine semigroup rings with isolated singularity (see Remark 3.7). In [9, Theorem 3.2] Hoa and Stückrad presented a bound for the regularity of $K[S]$ which is a "good" bound, in addition to this they provided some positive answers for the Eisenbud-Goto conjecture. But in fact the Eisenbud-Goto conjecture remains widely open for simplicial affine semigroup rings.

Let S be normal (see Definition 3.1), hence $K[S]$ is Cohen-Macaulay by [12, Theorem 1], i. e. the Eisenbud-Goto conjecture holds. In fact the ring $K[S]$ is not necessary Cohen-Macaulay or Buchsbaum, if S is seminormal (see Definition 3.1 and Example 3.6). By [9, Proposition 2.2] the Castelnuovo-Mumford regularity of $K[S]$ can be computed in terms of the regularity of certain monomial ideals by studying the intersection of the Apéry sets of the extremal rays of S, we call this set B_S. In [14, Theorem 4.1.1] Li characterized the seminormal property of S in terms of B_S. By this we show in Theorem 4.15: If S is seminormal, then

$$\mathrm{reg}K[S] \leq \mathrm{deg}K[S] - \mathrm{codim}K[S].$$

In fact this bound could be not sharp, since deg$K[S]$ could be equal to α^{d-1}. A subclass of seminormal simplicial affine semigroups with deg$K[S] = \alpha^{d-1}$ are full Veronese rings. Let $S_{d,\alpha} := \langle A_{d,\alpha}\rangle$ be the monoid generated by $A_{d,\alpha} := \{(a_{[1]}, \ldots, a_{[d]}) \in \mathbb{N}^d \mid \sum_{i=1}^{d} a_{[i]} = \alpha\}$, we have

$$\mathrm{deg}K[S_{d,\alpha}] - \mathrm{codim}K[S_{d,\alpha}] = \alpha^{d-1} - \begin{pmatrix} \alpha + d - 1 \\ d - 1 \end{pmatrix} + d,$$

by Remark 5.1. In Theorem 5.3 we show that

$$\mathrm{reg}K[S_{d,\alpha}] = \left\lfloor d - \tfrac{d}{\alpha} \right\rfloor.$$

So in this case the Eisenbud-Goto conjecture is not sharp, see Example 5.4. In fact $S_{d,\alpha}$ is normal and therefore reg$K[S_{d,\alpha}] \leq d - 1$, by Remark 4.6. In Section 4 we extend this bound to the seminormal case, we show in Theorem 4.7: If S is seminormal, then:

$$\mathrm{reg}K[S] \leq d - 1.$$

In Section 2 we fix the basic notation and the computation of the regularity of $K[S]$ in terms of the regularity of certain monomial ideals. In the following we study the seminormal case in Section 3. In Section 4 we provide several bounds for the regularity of seminormal simplicial affine semigroup rings. Finally we compute the regularity of full Veronese rings in Section 5. For unspecified notation we refer to [2, 16].

2 Basics

Let $G := G(S)$ be the group generated by S in $\mathbb{Z}^d$. By $x_{[i]}$ we denote the i-th component of x and we define $\deg x := (\sum_{j=1}^{d} x_{[j]})/\alpha$, for $x \in G$. Let $n \in S$, the Apéry set of n is defined by $S(n) := \{x \in S \,|\, x - n \notin S\}$. We set $B_S := \cap_{j=1}^{d} S(e_j)$, i.e. for $x \in B_S$ we have $x - e_i \notin S$ for all $i = 1, \dots, d$. We note that if $x \notin B_S$, then $x + y \notin B_S$, for all $x, y \in S$. Let $x \sim y$ if and only if $x - y \in \alpha\mathbb{Z}^d$, hence $\sim$ is an equivalence relation on G. It is obvious that every element in G is equivalent to an element in $G \cap D$, where $D := \{x \in \mathbb{Q}^d \,|\, 0 \le x_{[i]} < \alpha, \forall i\}$ and for all $x, y \in G \cap D$ with $x \neq y$ we have $x \not\sim y$. Hence the number of equivalence classes $f := \#(G \cap D)$ in G is finite. One can show that there are exactly $f \in \mathbb{N}$ equivalence classes in G, $G \cap D$, S, and in B_S (see [17, Section 2]). By $\Gamma_1, \dots, \Gamma_f$ we denote the equivalence classes on B_S. For $j = 1, \dots, f$ we define

$$h_j := (\min\left\{m_{[1]} \,|\, m \in \Gamma_j\right\}, \min\left\{m_{[2]} \,|\, m \in \Gamma_j\right\}, \dots, \min\left\{m_{[d]} \,|\, m \in \Gamma_j\right\}).$$

Let $T := K[y_1, \dots, y_d]$ be the polynomial ring graded by $\deg y_i = 1$. We set $\tilde{\Gamma}_j := \{y^{(x-h_j)/\alpha} \,|\, x \in \Gamma_j\}$, where $y^{(a_{[1]},\dots,a_{[d]})} := y_1^{a_{[1]}} \cdots y_d^{a_{[d]}}$. By construction $I_j := \tilde{\Gamma}_j T$ are monomial ideals in T, since $h_j \sim x$ for all $x \in \Gamma_j$. We note that $\mathrm{ht} I_j \ge 2$ (height), since $\gcd \tilde{\Gamma}_j = 1$, for all $j = 1, \dots, f$. We define m_T as the homogeneous maximal ideal of T and m_S as the homogenous maximal ideal of $K[S]$.

Proposition 2.1 ([9, Proposition 2.2])**.** *There are isomorphisms of $\mathbb{Z}$-graded T-modules:*

1.) $K[S] \cong \bigoplus_{j=1}^{f} I_j(-\deg h_j)$.

2.) $H^i_{m_S}(K[S]) \cong \bigoplus_{j=1}^{f} H^i_{m_T}(I_j)(-\deg h_j)$.

We note that this idea can be extended for arbitrary simplicial affine semigroups, see [17, Proposition 4.1]. Applying the fact $H^i_{m_R}(K[S]) \cong H^i_{m_S}(K[S])$ we have:

$$\mathrm{reg} K[S] = \max\left\{\mathrm{reg} I_j + \deg h_j \,|\, j = 1, \dots, f\right\}, \tag{1}$$

where $\mathrm{reg} I_j$ is the regularity of I_j considered as a $\mathbb{Z}$-graded T-module.

Remark 2.2. We note that $\mathrm{reg} K[S]$ is independent of K for $d \le 5$, by [1, Corollary 1.4] and (1). By Proposition 2.1 it follows that $\deg K[S] = f$. Since $\Gamma_j \subset B_S$, we have $\Gamma_j \subset \langle a_1, \dots, a_c\rangle$ for all $j = 1, \dots, f$. Moreover it is clear that $\{0, a_1, \dots, a_c\} \subseteq B_S$. Consider an element $x \in \{0, a_1, \dots, a_c\}$ and an element $y \in B_S$ with $x \neq y$. Suppose that $x \sim y$. Since $0 \le x_{[i]} < \alpha$, for all $i = 1, \dots, d$, we have $y \ge x$, meaning $y_{[k]} \ge x_{[k]}$ for all $k = 1, \dots, d$, and therefore $y \notin B_S$. This shows that $x \not\sim y$. W.l.o.g we therefore may assume that $\Gamma_1 = \{0\}, \Gamma_2 = \{a_1\}, \dots, \Gamma_{c+1} = \{a_c\}$.

Definition 2.3. For an element $x \in S$ we say that a sequence $b_1, \dots, b_n$ has $*$-property $:\Leftrightarrow$ $b_1, \dots, b_n \in \{a_1, \dots, a_c\}$ and $x - b_1 \in S, x - b_1 - b_2 \in S, \dots, x - (\sum_{j=1}^{n} b_j) \in S$. Moreover we define $x(i) := x - (\sum_{j=1}^{i} b_j)$ w.r.t. a sequence $b_1, \dots, b_n$ with $*$-property and $x(0) := x$.

Remark 2.4. Suppose that $x \in S$ has a sequence $b_1, \dots, b_{\deg x}$ with $*$-property, then we get $\deg x(i) = \deg x - i$ for $i = 0, \dots, \deg x$ and therefore $x(\deg x) = 0$. Hence the length of a sequence with $*$-property is bounded by $\deg x$. Moreover for $0 \le i < j \le \deg x$ we have $x(i) \ge x(j)$. There are elements in S with no sequence with $*$-property, e.g. e_j.

Proposition 2.5. *Let $x \in B_S \setminus \{0\}$.*

1) *There exists a sequence $b_1, \dots, b_{\deg x}$ with $*$-property.*
2) *Let $b_1, \dots, b_n$ be a sequence with $*$-property. Then there exists a sequence with $*$-property $b_1, \dots, b_n, b_{n+1}, \dots, b_{\deg x}$.*

Proof. 1) Suppose on the contrary that there is no sequence with $*$-property of length $\deg x$. Then $x \notin \langle a_1, \dots, a_c \rangle$, which contradicts to $x \in B_S$.

2) Suppose that $x(n) \notin B_S$, then $x \notin B_S$ which is a contradiction. Therefore we have $x(n) \in B_S$. By claim 1) we are done.

□

Proposition 2.6. *Let $x \in S$ and $b_1, \dots, b_n$ be a sequence with $*$-property. Let $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ be a bijection.*

1) *$b_{\sigma(1)}, \dots, b_{\sigma(n)}$ is a sequence with $*$-property.*
2) *$b_1, \dots, b_m$ is a sequence with $*$-property for all $1 \le m \le n$.*

Proof. 1) We need to show that $x(i) \in S$, for all $i = 1, \dots, n$ w.r.t $b_{\sigma(1)}, \dots, b_{\sigma(n)}$, since clearly $b_{\sigma(1)}, \dots, b_{\sigma(n)} \in \{a_1, \dots, a_c\}$. Let $i = n$, we have $x(n) = x - (\sum_{j=1}^{n} b_{\sigma(j)}) = x - (\sum_{j=1}^{n} b_j) \in S$ by assumption. Fix one $i < n$, then

$$x(i) = x - (\textstyle\sum_{j=1}^{i} b_{\sigma(j)}) = \underbrace{x - (\textstyle\sum_{j=1}^{n} b_{\sigma(j)})}_{\in S} + \underbrace{\textstyle\sum_{j=i+1}^{n} b_{\sigma(j)}}_{\in S} \in S.$$

2) This is obvious.

□

Lemma 2.7. *Let $x \in B_S \setminus \{0\}$ and $b_1, \dots, b_{\deg x}$ be a sequence with $*$-property.*

1) *$x(i) \in B_S$, for all $i = 0, \dots, \deg x$.*
2) *We have $x(i) \not\sim x(j)$, for all $0 \le i < j \le \deg x$.*

Proof. 1) Follows from the fact that if $x(i) \notin B_S$, then $x(i) + y \notin B_S$ for all $y \in S$.

2) Suppose on the contrary that $x(i) \sim x(j)$. We have $\deg x(i) > \deg x(j)$ and $x(i) \ge x(j)$, hence $x(i) \notin B_S$ which contradicts to claim 1).

□

Corollary 2.8 ([9, Theorem 1.1])**.** *We have $\deg x \le \deg K[S] - \operatorname{codim} K[S]$, for all $x \in B_S$.*

Proof. W.l.o.g. we may assume that $\deg x \ge 2$. By Lemma 2.7 and Remark 2.2 there is a set $L = \{0, a_1, \dots, a_c, x(0), \dots, x(\deg x - 2)\} \subseteq B_S$, such that for all $x, y \in L$ with $x \ne y$ we have $x \not\sim y$. Hence $f = \deg K[S] \ge \#L = \deg x + \operatorname{codim} K[S]$.

□

Remark 2.9. We note that this proof is a new short proof of [9, Theorem 1.1]. We define the reduction number $\mathrm{r}(K[S])$ of $K[S]$ by $\mathrm{r}(K[S]) := \max\{\deg x \mid x \in B_S\}$, see [9, Section 1 and first Remark in Section 2]. By Corollary 2.8 or [9, Theorem 1.1] we get

$$\mathrm{r}(K[S]) \leq \deg K[S] - \operatorname{codim} K[S], \tag{2}$$

i. e. the Eisenbud-Goto conjecture holds for the reduction number of $K[S]$. So whenever we have $\operatorname{reg} K[S] = \mathrm{r}(K[S])$ the Eisenbud-Goto conjecture holds. It should be mentioned that this property does not hold in general. Even for a monomial curve in $\mathbb{P}^3$ the equality does not hold. For $S = \langle (40,0),(0,40),(35,5),(11,29)\rangle$ we have $\operatorname{reg} K[S] = 13 > 11 = \mathrm{r}(K[S])$. Moreover it is obvious that $\mathrm{r}(K[S]) \leq \operatorname{reg} K[S]$, by (1).

Example 2.10. Let $S = \langle (4,0),(0,4),(3,1),(1,3)\rangle$. Using Macaulay2 [5] we have $B_S = \{(0,0),(3,1),(1,3),(6,2),(2,6)\}$ and therefore $\mathrm{r}(K[S]) = \max\{0,1,1,2,2\} = 2$. We get $\Gamma_1 = \{(0,0)\}, \Gamma_2 = \{(3,1)\}, \Gamma_3 = \{(1,3)\}, \Gamma_4 = \{(6,2),(2,6)\}$ and $h_1 = (0,0), h_2 = (3,1), h_3 = (1,3), h_4 = (2,2)$. By this we have $I_1 = I_2 = I_3 = T$ and $I_4 = (y_1, y_2)T$, hence

$$\operatorname{reg} K[S] = \max\{\operatorname{reg} T + 0, \operatorname{reg} T + 1, \operatorname{reg} T + 1, \operatorname{reg}(y_1,y_2)T + 1\} = \max\{0,1,1,2\} = 2.$$

Lemma 2.11. *Let $x \in B_S, t \in \mathbb{N}^+, k \in \{1,\dots,d\}$ and $x_{[k]} = t\alpha$. There is a sequence with $*$-property b such that $(t-1)\alpha < (x-b)_{[k]} < t\alpha$.*

Proof. By Proposition 2.5 there is a sequence $b_1,\dots,b_{\deg x}$ with $*$-property. We have $x(\deg x) = 0$ by Remark 2.4, hence there is a $p \in \{1,\dots,\deg x\}$ such that $b_{p[k]} > 0$. Since $b_p \in \{a_1,\dots,a_c\}$ we know that $b_{p[k]} < \alpha$. The assertion follows by Proposition 2.6. □

Lemma 2.12. *Let $J \subseteq \{1,\dots,d\}$ with $\#J \geq 1$. Let $x \in B_S$ such that $x_{[k]} = \alpha$, for all $k \in J$. There exists a sequence $b_1,\dots,b_{\deg x}$ with $*$-property such that: for all $i = 1,\dots,\#J$ there is at least one $k \in J$ such that $0 < x(i)_{[k]} < \alpha$.*

Proof. By Lemma 2.11 the case $\#J = 1$ is clear, assume that $\#J > 1$. Fix an arbitrary sequence with $*$-property $b_1,\dots,b_{\#J-1}$. By Remark 2.4 there is a $k \in J$ such that $x(i)_{[k]} > 0$, for all $i = 1,\dots,\#J-1$. By this, induction and Lemma 2.11 there is a sequence with $*$-property $b_1,\dots,b_{\#J-1}$ such that: for all $i = 1,\dots,\#J-1$ there is a $k \in J$ such that $0 < x(i)_{[k]} < \alpha$. By Lemma 2.11 we may assume that already $x(\#J-1)_{[k]} < \alpha$ for all $k \in J$. By Proposition 2.5 2) there is a sequence with $*$-property $b_1,\dots,b_{\#J-1},b_{\#J},\dots,b_{\deg x}$. Suppose on the contrary that $x(\#J)_{[k]} = 0$, for all $k \in J$. Since $\deg x(\#J) = \deg x - \#J$ and $x \geq x(\#J)$ we have $x(\#J) = x - (\sum_{k\in J} e_k)$ and therefore $x \notin B_S$, since $x(\#J) \in S$. □

3 The seminormal case

Let us consider an affine semigroup $U \subseteq \mathbb{N}^d$, i. e. U is a finitely generated submonoid of $(\mathbb{N}^d, +)$. By $G(U)$ we denote the group generated by U. There are two closely related definitions in this context:

Definition 3.1. 1. We call U seminormal, if $x \in G(U)$ and $2x, 3x \in U$ imply $x \in U$.

2. We call U normal, if $x \in G(U)$ and $tx \in U$ for some $t \in \mathbb{N}^+$ imply $x \in U$.

Remark 3.2. A Noetherian domain $\bar{R}$ is called seminormal if for an element x in the quotient field $Q(\bar{R})$ of $\bar{R}$ such that $x^2, x^3 \in \bar{R}$ we have $x \in \bar{R}$. By a result of Hochster and Roberts the ring $K[U]$ is seminormal if and only if U is seminormal, see [13, Proposition 5.32]. A similar result holds in the normal case, by [12].

To get new bounds for the regularity of $K[S]$, we need another characterization. We define the set $\text{Box} := \{x \in S \,|\, x = \sum_{i=1}^{d} \lambda_i e_i, \text{ for some } \lambda_i \in \mathbb{Q} \cap [0,1]\}$. So we have $\text{Box} = \{x \in S \,|\, x_{[i]} \leq \alpha, \forall i = 1, \ldots, d\}$.

Theorem 3.3 ([14, Theorem 4.1.1])**.** *The semigroup S is seminormal if and only if B_S is contained in* Box.

From now on we assume that S is seminormal. Let $I_j \neq T$ be an ideal which arises by the construction of Proposition 2.1. For $x \in \Gamma_j$ we have $0 \leq x_{[i]} \leq \alpha$ and therefore $((x - h_j)/\alpha)_{[i]} \in \{0, 1\}$. Hence I_j is a squarefree monomial ideal in T.

Lemma 3.4. *Let $i, t \in \mathbb{N}$ with $1 \leq i \leq d$ and $1 \leq t \leq f$.*

1) *Let $x, y \in \Gamma_t$ with $x \neq y$. If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.*
2) *Let $x, y \in \Gamma_t$ with $x \neq y$. If $0 < x_{[i]} < \alpha$, then $x_{[i]} = y_{[i]}$.*
3) *Let $x, y \in \Gamma_t$ with $x \neq y$. If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} \in \{0, \alpha\}$ and $y_{[i]} = \alpha - x_{[i]}$.*
4) *Let $x, y \in \Gamma_t$ with $x \neq y$, then $0 < x_{[i]} = y_{[i]} < \alpha$ and $0 < x_{[j]} = y_{[j]} < \alpha$ for some $i, j \in \{1, \ldots, d\}$ with $i \neq j$.*
5) *If $h_{t[i]} > 0$, then $h_{t[i]} = x_{[i]}$, for all $x \in \Gamma_t$.*

Proof. 1) We have $x_{[i]} - y_{[i]} \in \alpha\mathbb{Z}$ and $x_{[i]} - y_{[i]} \in [-\alpha, \alpha]$, since $0 \leq x_{[i]}, y_{[i]} \leq \alpha$. Hence $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.

2) We have $x_{[i]} - y_{[i]} \notin \{-\alpha, \alpha\}$ and therefore $x_{[i]} = y_{[i]}$ by claim 1).

3) By claim 1) we have $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$ and $x_{[i]} \in \{0, \alpha\}$, by claim 2). Hence $y_{[i]} = \alpha - x_{[i]}$.

4) By claim 2) it is sufficient to show that $0 < x_{[i]}, x_{[j]} < \alpha$ for some $i \neq j$. Suppose on the contrary that this is not true. If $x_{[i]} \in \{0, \alpha\}$ for all $i = 1, \ldots, d$ we have $x \sim 0$. Hence $0 \in \Gamma_t$ and therefore $\#\Gamma_t = 1$ which is a contradiction. Suppose that $0 < x_{[i]} < \alpha$ for exact one $i \in \{1, \ldots, d\}$, i. e. $x_{[j]} \in \{0, \alpha\}$ for all $j \in \{1, \ldots, d\} \setminus \{i\}$. By this we have $\sum_{j=1}^{d} x_{[j]} \notin \alpha\mathbb{N}$ which is a contradiction, since $x \in S$.

5) Let $x \in \Gamma_t$. We have $0 < h_{t[i]} \leq x_{[i]} \leq \alpha$ and therefore $h_{t[i]} = x_{[i]}$, since $h_{t[i]} - x_{[i]} \in \alpha\mathbb{Z}$, by construction. □

Corollary 3.5 ([15, Theorem 2.2])**.** *If $d \leq 3$, then S is Cohen-Macaulay.*

Proof. By [4, Proposition 8] we need to show that $\#\Gamma_t = 1$, for all $t = 1, \ldots, f$. By Lemma 3.4 4) the case $d = 2$ is trivial. Suppose on the contrary that $x, y \in \Gamma_t$ with $x \neq y$. By Lemma 3.4 4) we may assume that $0 < x_{[i]} = y_{[i]} < \alpha$ for $i = 1, 2$. By Lemma 3.4 3) we may assume that $x_{[3]} = \alpha$ and $y_{[3]} = 0$, since $x_{[3]} \neq y_{[3]}$. Then $x - e_3 = y \in S$ which contradicts to $x \in B_S$.

□

Example 3.6. Let us consider the semigroup

$$S = \langle e_1, \dots, e_6, (1,1,0,0,0,0), (1,0,1,0,0,0), (0,0,1,1,0,0), (0,1,0,1,0,0), (0,0,0,0,1,1) \rangle,$$

in $\mathbb{N}^6$ with $\alpha = 2$. We have $B_S \subseteq \mathrm{Box}$ thus S is seminormal by Theorem 3.3. One can show that $(0,1,1,0,0,0)+e_1, (0,1,1,0,0,0)+e_4 \in S$, but $(0,1,1,0,0,0)+(0,0,0,0,1,1) = (0,1,1,0,1,1) \notin S$. Hence $K[S]$ is not Buchsbaum by [21, Lemma 3]. By a similar example, one can show that Corollary 3.5 does not hold for $d = 4$. For a general discussion of the relation between the seminormal property and the Cohen-Macaulay property of affine semigroup rings we refer to [14].

Remark 3.7. Herzog and Hibi showed in [8] that the Eisenbud-Goto conjecture holds for simplicial affine semigroups with isolated singularity. This is equivalent to the statement that A (see Introduction) contains all points of type $(0, \dots, \alpha - 1, \dots, 1, \dots, 0)$, where $\alpha - 1, 1$ stay in the i-th and j-th positions, respectively, and the other coordinates are zero. By Example 3.6 we are studying a distinct class of simplicial affine semigroup rings.

4 Bounding the regularity

In this section we assume that S is seminormal. Keep in mind that I_j is a squarefree monomial ideal in T, for all $j = 1, \dots, f$.

Remark 4.1. By Theorem 3.3 S is seminormal, if and only if $B_S \subseteq \mathrm{Box}$. Clearly $\mathrm{r}(K[S]) \le d$. On the other hand there is only one element in Box with degree d, namely $(\alpha, \dots, \alpha)$, but $(\alpha, \dots, \alpha) \notin B_S$. Hence $\mathrm{r}(K[S]) \le d - 1$. In Theorem 4.7 we obtain a similar bound for the regularity of $K[S]$.

Definition 4.2. For a monomial $m = y_1^{b_1} \cdots y_d^{b_d}$ we define $\deg m = \sum_{j=1}^{d} b_j$. Let I be a monomial ideal in T with a minimal set of monomial generators $\{m_1, \dots, m_s\}$. Let F be the least common multiple of $\{m_1, \dots, m_s\}$, then we define $\mathrm{var}(I) := \deg F$.

Remark 4.3. Consider the squarefree monomial ideal $I = (y_1y_2, y_2y_3y_4, y_7)T$ in $T = K[y_1, \dots, y_7]$. Clearly $\mathrm{var}(I) = 5$. So in the squarefree case $\mathrm{var}(I)$ is equal to the number of variables, which occur in the generators of I. We note that $\tilde{\Gamma}_j$ is always a minimal set of monomial generators of I_j. Moreover every monomial ideal in T has a unique minimal set of monomial generators by [16, Lemma 1.2]. Since $\mathrm{ht} I_j \ge 2$ we have $\mathrm{var}(I_j) \ne 1$. Moreover for all $j = 1, \dots, f$ we get $I_j \ne T$, if and only if $\mathrm{var}(I_j) \ne 0$.

Lemma 4.4. $\mathrm{var}(I_j) \le d - 1 - \deg h_j$, *for all* $j = 1, \dots, f$.

Proof. Assume that $I_j = T$, then $\mathrm{var}(T) = 0$ and $\deg h_j \le d - 1$ by Remark 4.1. So we may assume that $\#\Gamma_j \ge 2$. By Lemma 3.4 4) we have $0 < x_{[k]}, x_{[l]} < \alpha$, for all $x \in \Gamma_j$ and some $k \ne l$. In particular $0 < h_{j[k]}, h_{j[l]} < \alpha$. Suppose on the contrary that $\mathrm{var}(I_j) \ge d - \deg h_j$. Then by Lemma 3.4 5) $h_{j[t]} = 0$ for all $t \in J$ for some $J \subseteq \{1, \dots, d\}$ with $\#J = d - \deg h_j$. Since $0 < h_{j[k]}, h_{j[l]} < \alpha$ for some $l \ne k$ and $0 \le h_{j[i]} \le \alpha$, for all $i = 1, \dots, d$ by Theorem 3.3, we get $\deg h_j < \deg h_j$ which is a contradiction. ∎

Theorem 4.5 ([11, Theorem 3.1])**.** *Let I be a proper monomial ideal in T. Then*

$$\mathrm{reg} I \leq \mathrm{var}(I) - \mathrm{ht} I + 1.$$

Remark 4.6. One can show that $\mathrm{reg} K[S] \leq d - 1$, if S is normal (by the proof of [10, Corollary 4.7] and [10, Corollary 3.8]). The next Theorem obtains a similar bound in the seminormal case.

Theorem 4.7.

$$\mathrm{reg} K[S] \leq d - 1$$

Proof. By Remark 4.1 and (1) we may assume that $\#\Gamma_j \geq 2$. We need to show that $\mathrm{reg} I_j + \deg h_j \leq d - 1$, for a fixed $j \in \{1, \dots, f\}$ such that $\#\Gamma_j \geq 2$. By Lemma 4.4 and Theorem 4.5 we get

$$\mathrm{reg} I_j \leq \mathrm{var}(I_j) - \mathrm{ht} I_j + 1 \leq d - 1 - \deg h_j - 2 + 1 = d - 2 - \deg h_j,$$

since $\mathrm{ht} I_j \geq 2$. Hence $\mathrm{reg} I_j + \deg h_j \leq d - 2$ and we are done. $\square$

Remark 4.8. We note that the bound established in Theorem 4.7 is sharp. Assume $\alpha \geq d$ in Theorem 5.3, by this we get $\mathrm{reg} K[S_{d,\alpha}] = d-1$ and of course $S_{d,\alpha}$ is seminormal.

Proposition 4.9. *If $d \leq 5$, then* $\mathrm{reg} K[S] = \mathrm{r}(K[S])$.

Proof. By Corollary 3.5 and [4, Proposition 8] the case $d \leq 3$ is clear. We show that $\mathrm{reg} I_j$ is equal to the maximal degree of a generator of I_j. By this we get:

$$\mathrm{reg} I_j + \deg h_j = \max \{\deg x \mid x \in \Gamma_j\},$$

for all $j = 1, \dots, f$ and we are done by (1). The case $\#\Gamma_j = 1$ is obvious. We therefore may assume that $\#\Gamma_j \geq 2$ and we fix such a $j \in \{1, \dots, f\}$. By Lemma 3.4 we get $\deg h_j \geq 1$. Let $d = 5$, by Lemma 4.4 we have to consider the cases $\mathrm{var}(I_j) \in \{2, 3\}$. Let $\mathrm{var}(I_j) = 2$. The ideal I_j is of the form $I_j = (y_k, y_l)T$ for some $k \neq l$ and $k, l \in \{1, \dots, 5\}$, since $\mathrm{ht} I_j \geq 2$. It follows that $\mathrm{reg} I_j = 1$. By a similar argument we get the assertion for the case $d = 4$ and $\mathrm{var}(I_j) = 2$. Let $d = 5$ and $\mathrm{var}(I_j) = 3$, i. e. $\deg h_j = 1$. Since $\mathrm{ht} I_j \geq 2$ and by Theorem 3.3 the only ideals possible are:

$$I_{j_1} = (y_k, y_l, y_m), I_{j_2} = (y_k y_l, y_m), I_{j_3} = (y_k y_l, y_k y_m, y_l y_m)$$

and $k, l, m \in \{1, \dots, 5\}$ are pairwise not equal. By Theorem 4.5 we get $\mathrm{reg} I_{j_1} = 1$ and $\mathrm{reg} I_{j_2} = \mathrm{reg} I_{j_3} = 2$. $\square$

By Corollary 2.8 the Eisenbud-Goto conjecture holds, if $d \leq 5$. The next Theorem shows that the Eisenbud-Goto conjecture holds in any dimension.

Remark 4.10. Proposition 4.9 could fail for $d \geq 6$. Let us consider the squarefree monomial ideal $I = (y_1 y_2, y_3 y_4)T$ with $\mathrm{var}(I) = 4$. So $\mathrm{reg} I = 3$ is bigger than the maximal degree of a generator of I, which is 2.

Definition 4.11. Let I be a proper monomial ideal in T with a minimal set of monomial generators $\{m_1, \ldots, m_s\}$. Let F be the least common multiple of $\{m_1, \ldots, m_s\}$, say $F = y_1^{b_1} \cdots y_d^{b_d}$. We define the set $\operatorname{supp}(I) \subseteq \{1, \ldots, d\}$ w.r.t. I by: $i \in \operatorname{supp}(I) :\Leftrightarrow b_i \neq 0$.

Remark 4.12. So $\operatorname{supp}(I)$ is the set of indices of the variables, which occur in one of the minimal generators of I. For the ideal $I = (y_1y_2, y_2y_3, y_5y_6)T$ in $T = K[y_1, \ldots, y_7]$, we have $F = y_1y_2y_3y_5y_6$, i.e. $\operatorname{supp}(I) = \{1, 2, 3, 5, 6\}$. Consider the ideal y_1y_2T, we get $\operatorname{supp}(y_1y_2T) = \{1, 2\}$.

Lemma 4.13. *Let* $\#\Gamma_j \geq 2$*,* $n \in \Gamma_j$ *and* $m \in \tilde{\Gamma}_j$ *such that* $m = y^{(n-h_j)/\alpha}$*. Then*

1) $n_{[q]} = 0$*, for all* $q \in \operatorname{supp}(I_j) \setminus \operatorname{supp}(mT)$*.*
2) $n_{[q]} = \alpha$*, for all* $q \in \operatorname{supp}(mT)$*.*

Proof. 1) Suppose on the contrary that there is a $q \in (\operatorname{supp}(I_j) \setminus \operatorname{supp}(mT)) \neq \emptyset$ such that $n_{[q]} > 0$. Since $q \in \operatorname{supp}(I_j)$ we have $h_{j[q]} = 0$ by Lemma 3.4 5) and therefore $n_{[q]} = \alpha$, since $h_{j[q]} - n_{[q]} \in \alpha\mathbb{Z}$ and $n_{[q]} \leq \alpha$. This implies $q \in \operatorname{supp}(mT)$, which is a contradiction.

2) Since $q \in \operatorname{supp}(mT)$, we have $n_{[q]} \geq \alpha$. By Theorem 3.3 $n_{[q]} \leq \alpha$. □

Remark 4.14. The above Lemma fails for an arbitrary S, like in Example 2.10. Consider $\Gamma_4 = \{(6, 2), (2, 6)\}$, i.e. $h_4 = (2, 2)$ and $\tilde{\Gamma}_4 = \{y_1, y_2\}$. For every $n \in \Gamma_4$ we have $n_{[i]} \neq 0, i = 1, 2$. But $\operatorname{supp}(I_4) = \{1, 2\}$ and $\#\operatorname{supp}(y_1T) = \#\operatorname{supp}(y_2T) = 1$.

Theorem 4.15.

$$\operatorname{reg}K[S] \leq \deg K[S] - \operatorname{codim}K[S]$$

Proof. By (1) we need to show that $\deg K[S] - c \geq \operatorname{reg}I_j + \deg h_j$, for all $j = 1, \ldots, f$. If $\#\Gamma_j = 1$ the assertion follows by Corollary 2.8. Let us fix a $j \in \{1, \ldots, f\}$ such that $\#\Gamma_j \geq 2$. We have $\Gamma_j = \{n_1, \ldots, n_{\#\Gamma_j}\}$ and $\tilde{\Gamma}_j = \{m_1, \ldots, m_{\#\Gamma_j}\}$. We may assume that $m_i = y^{(n_i - h_j)/\alpha}$. We set $J_k := (m_1, \ldots, m_k)T$ and $g(k) := \operatorname{var}(J_k) - \operatorname{ht}J_k + 1 + \deg h_j$, for $1 \leq k \leq \#\Gamma_j$. We show by induction on k with $1 \leq k \leq \#\Gamma_j$ that there is a set L_k:

(*i*) $L_k \subseteq B_S$.

(*ii*) $\#L_k \geq g(k) - 1$.

(*iii*) $x \not\sim y$, for all $x, y \in L_k$ with $x \neq y$.

(*iv*) $\deg x \geq 2$, for all $x \in L_k$.

(*v*) $x_{[q]} = 0$, for all $x \in L_k$ and for all $q \in \operatorname{supp}(I_j) \setminus \operatorname{supp}(J_k)$.

Let $k = 1$. We know that $\operatorname{ht}J_1 = 1$ and $\operatorname{var}(J_1) + \deg h_j = \deg n_1$, i.e. $g(1) = \deg n_1$. By Proposition 2.5 1) n_1 has a sequence $b_1, \ldots, b_{\deg n_1}$ with $*$-property, since $n_1 \in B_S$. Set

$$L_1 := \{n_1(0), \ldots, n_1(\deg n_1 - 2)\},$$

clearly $\#L_1 \geq \deg n_1 - 1 = g(1) - 1$, i.e. property (*ii*) is satisfied and by construction we get property (*iv*). By Lemma 2.7 1) $L_1 \subseteq B_S$ which shows (*i*) and by Lemma 2.7 2) property (*iii*) holds. By Lemma 4.13 1) property (*v*) holds for $n_1(0)$, hence for every element in L_1.

Using induction on $k \leq \#\Gamma_j - 1$ the properties (*i*)-(*v*) hold for $L_k = \{c_1, \ldots, c_p\}$. There could be two different cases:

<u>Case 1:</u> $\operatorname{supp}(J_k) \cap \operatorname{supp}(m_{k+1}T) \neq \emptyset$. (e.g. $k = 2$, $J_2 = (y_1y_2, y_2y_3y_4)T$, $m_3 = y_4y_5y_6$.)

(iii) We set $J := (\mathrm{supp}(m_{k+1}T) \setminus \mathrm{supp}(J_k))$. Since $\mathrm{supp}(J_k) \cap \mathrm{supp}(m_{k+1}T) \neq \emptyset$ we have $\deg n_{k+1} \geq \#J+2$, in particular $n_{k+1[q]} = \alpha$, for all $q \in J$, see Lemma 4.13 2). By Lemma 2.12 there is a sequence $b_1, \ldots, b_{\deg n_{k+1}}$ with $*$-property such that for all $q = 1, \ldots, \#J$ there is one $p \in J$ with $0 < n_{k+1}(q)_{[p]} < \alpha$. Let us fix a c_i. By property (v) $c_{i[p]} = 0$, hence $c_i \not\sim n_{k+1}(q)$, for all $q = 1, \ldots, \#J$. Set

$$L_{k+1} := \{c_1, \ldots, c_p, n_{k+1}(1), \ldots, n_{k+1}(\#J)\}.$$

In case that $J = \emptyset$, we set $L_{k+1} = L_k$. By Lemma 2.7 2) we get (iii).

(i) By Lemma 2.7 1) $n_{k+1}(1), \ldots, n_{k+1}(\#J) \in B_S$, since $n_{k+1} \in B_S$.

(iv) Since $\deg n_{k+1} \geq \#J + 2$.

(v) By induction $c_{i[q]} = 0$, for all $q \in (\mathrm{supp}(I_j) \setminus \mathrm{supp}(J_k)) \supseteq (\mathrm{supp}(I_j) \setminus \mathrm{supp}(J_{k+1}))$. By Lemma 4.13 1) we have $n_{k+1[q]} = 0$, for all $q \in (\mathrm{supp}(I_j) \setminus \mathrm{supp}(m_{k+1}T)) \supseteq (\mathrm{supp}(I_j) \setminus \mathrm{supp}(J_{k+1}))$, hence property (v) holds.

(ii) Since $\mathrm{ht}J_{k+1} \geq \mathrm{ht}J_k$ and $\mathrm{var}(J_{k+1}) = \mathrm{var}(J_k) + \#J$ we have

$$g(k+1) - 1 \leq \#J + \mathrm{var}(J_k) - \mathrm{ht}J_k + 1 + \deg h_j - 1 = \#J + g(k) - 1 \leq \#J + p.$$

<u>Case 2:</u> $\mathrm{supp}(J_k) \cap \mathrm{supp}(m_{k+1}T) = \emptyset$. (e.g. $k = 2$, $J_2 = (y_1y_2, y_2y_3y_4)T$, $m_3 = y_5y_6y_7$.)

(iii) Similar argument beside of the fact that $\deg n_{k+1} \geq \#J + 1$. Replace L_{k+1} by

$$L_{k+1} := \{c_1, \ldots, c_p, n_{k+1}(1), \ldots, n_{k+1}(\#J - 1)\}.$$

In case that $\#J = 1$, we set $L_{k+1} = L_k$.

$(i), (iv), (v)$ Analogous, replace $\#J$ by $\#J - 1$.

(ii) Since $\mathrm{supp}(J_k) \cap \mathrm{supp}(m_{k+1}T) = \emptyset$, $m_{k+1} + J_k$ is a non-zero-divisor of T/J_k. Hence $\mathrm{ht}J_{k+1} = \mathrm{ht}J_k + 1$, by Krull's Principal Ideal Theorem (e.g. see [2, Theorem 10.1]). So

$$g(k+1) - 1 = \#J + \mathrm{var}(J_k) - \mathrm{ht}J_k - 1 + 1 + \deg h_j - 1 = \#J + g(k) - 2 \leq \#J + p - 1.$$

By this we get a set $L_{\#\Gamma_j} = \{c_1, \ldots, c_p\} \subseteq B_S$ with the above properties, in particular $p \geq g(\#\Gamma_j) - 1 = \mathrm{var}(I_j) - \mathrm{ht}I_j + 1 + \deg h_j - 1 \geq \mathrm{reg}I_j + \deg h_j - 1$, by Theorem 4.5. By Remark 2.2 we get a set $L = \{0, a_1, \ldots, a_c, c_1, \ldots, c_p\} \subseteq B_S$ with $x \not\sim y$, for all $x, y \in L$ with $x \neq y$. So for all $j = 1, \ldots, f$ we have

$$f = \deg K[S] \geq \#L = c + p + 1 \geq c + \mathrm{reg}I_j + \deg h_j.$$

□

Remark 4.16. We note that Corollary 2.8 holds for any S. So Theorem 4.15 holds with the following assumption on S:

- If $\#\Gamma_j \geq 2$, then Γ_j is contained in Box.

5 Regularity of full Veronese rings

For $X, Y \subseteq \mathbb{N}^d$ we define $X + Y := \{x + y \,|\, x \in X, y \in Y\}$, $mX := X + \ldots + X$ (m-times) and $0X := 0$. Moreover we set $A_{d,\alpha} := \{(a_{[1]}, \ldots, a_{[d]}) \in \mathbb{N}^d \,|\, \sum_{i=1}^d a_{[i]} = \alpha\}$ and $S_{d,\alpha} = \langle A_{d,\alpha}\rangle$. For example $A_{3,2} = \{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}$. It is trivial that:

$$nA_{d,\alpha} = \{(a_{[1]}, \ldots, a_{[d]}) \in \mathbb{N}^d \,|\, \textstyle\sum_{i=1}^d a_{[i]} = n\alpha\}. \tag{3}$$

Hence there is an isomorphism of K-vector spaces:

$$K[S_{d,1}]_{n\alpha} = K[t_1, \ldots, t_d]_{n\alpha} \cong K[S_{d,\alpha}]_n.$$

We have $h_{K[t_1,\ldots,t_d]}(n) = \begin{pmatrix} n+d-1 \\ d-1 \end{pmatrix}$, where $h_{K[t_1,\ldots,t_d]}$ denotes the Hilbert polynomial of $K[t_1, \ldots, t_d]$ and therefore

$$h_{K[S_{d,\alpha}]}(n) = h_{K[t_1,\ldots,t_d]}(n\alpha) = \begin{pmatrix} n\alpha+d-1 \\ d-1 \end{pmatrix}. \tag{4}$$

Remark 5.1. By (4) $\deg K[S_{d,\alpha}] = \alpha^{d-1}$ and $\#A_{d,\alpha} = h_{K[S_{d,\alpha}]}(1) = \begin{pmatrix} \alpha+d-1 \\ d-1 \end{pmatrix}$, hence $\operatorname{codim} K[S_{d,\alpha}] = \begin{pmatrix} \alpha+d-1 \\ d-1 \end{pmatrix} - d$.

Since the semigroup $S_{d,\alpha}$ is normal, the ring $K[S_{d,\alpha}]$ is Cohen-Macaulay by [12, Theorem 1] and therefore $\#\Gamma_j = 1$ for all $j = 1, \ldots, f$ (see [10] or [4, Proposition 8]). Hence

$$\operatorname{reg} K[S_{d,\alpha}] = \mathrm{r}(K[S_{d,\alpha}]), \tag{5}$$

by (1). Now we compute the reduction number of $K[S_{d,\alpha}]$, which can be computed by $\mathrm{r}(K[S_{d,\alpha}]) = \min\{r \in \mathbb{N} \,|\, rA_{d,\alpha} + \{e_1, \ldots, e_d\} = (r+1)A_{d,\alpha}\}$, see [9, Section 1].

Lemma 5.2. *Let $r \in \mathbb{N}$. The following assertions are equivalent:*

1) $rA_{d,\alpha} + \{e_1, \ldots, e_d\} = (r+1)A_{d,\alpha}$.
2) $(r+1)\alpha > d(\alpha - 1)$.

Proof. 1) $\Rightarrow$ 2) Let us assume that $0 \leq (r+1)\alpha \leq d(\alpha-1)$. It is trivial that there is an element $x \in \mathbb{N}^d$ with $x_{[i]} \leq \alpha - 1$ for all $i = 1, \ldots, d$ and $\sum_{i=1}^d x_{[i]} = (r+1)\alpha$. We have $x \in (r+1)A_{d,\alpha}$, by (3). Now suppose that $x \in rA_{d,\alpha} + \{e_1, \ldots, e_d\}$ then $x = x' + e_j$, for some j and therefore $x_{[j]} \geq \alpha$ a contradiction, hence $x \notin rA_{d,\alpha} + \{e_1, \ldots, e_d\}$.

2) $\Rightarrow$ 1) Let $x \in (r+1)A_{d,\alpha}$ and suppose that $x_{[j]} \leq \alpha - 1$ for all j then $(r+1)\alpha = \sum_{i=1}^d x_{[i]} \leq d(\alpha - 1)$. Hence $x_{[j]} \geq \alpha$ for some j and therefore $x - e_j \in rA_{d,\alpha}$ by (3). Hence $(r+1)A_{d,\alpha} \subseteq rA_{d,\alpha} + \{e_1, \ldots, e_d\}$ and we are done.

□

Theorem 5.3.

$$\operatorname{reg} K[S_{d,\alpha}] = \left\lfloor d - \tfrac{d}{\alpha} \right\rfloor$$

Proof. We show that $\mathrm{r}(K[S_{d,\alpha}]) = \lfloor d - \frac{d}{\alpha} \rfloor$ and we are done by (5). We have

$$\left(\lfloor d - \tfrac{d}{\alpha} \rfloor + 1\right) \alpha > (d - \tfrac{d}{\alpha} + 1 - 1)\alpha = d(\alpha - 1),$$

hence $\mathrm{r}(K[S_{d,\alpha}]) \leq \lfloor d - \frac{d}{\alpha} \rfloor$, by Lemma 5.2. Without loss of generality assume that $\lfloor d - \frac{d}{\alpha} \rfloor \geq 1$. We have

$$\left(\lfloor d - \tfrac{d}{\alpha} \rfloor - 1 + 1\right) \alpha \leq \left(d - \tfrac{d}{\alpha}\right) \alpha = d(\alpha - 1).$$

hence $\mathrm{r}(K[S_{d,\alpha}]) > \lfloor d - \frac{d}{\alpha} \rfloor - 1$, by Lemma 5.2. □

Example 5.4. By Theorem 5.3 we are able to compute the Castelnuovo-Mumford regularity of full Veronese rings. For $S_{20,2}$ we know that $\mathrm{reg}K[S_{20,2}] = \lfloor 20 - \frac{20}{2} \rfloor = 10$ and $\deg K[S_{20,2}] - \mathrm{codim}K[S_{20,2}] = 2^{19} - \begin{pmatrix} 2+19 \\ 19 \end{pmatrix} + 20 = 524098$, by Remark 5.1.

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References

[1] W. Bruns and J. Herzog. Semigroup rings and simplicial complexes. *Journal of Pure and Applied Algebra*, 122(3):185–208, 1997.

[2] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry (Graduate Texts in Mathematics).* Springer, 2004.

[3] D. Eisenbud and S. Goto. Linear free resolutions and minimal multiplicity. *Journal of Algebra*, 88(1):89–133, 1984.

[4] P. A. García-Sánchez and J. C. Rosales. On Buchsbaum simplicial affine semigroups. *Pacific Journal of Mathematics*, 202(2):329–339, 2002.

[5] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[6] L. Gruson, R. Lazarsfeld, and C. Peskine. On a theorem of Castelnuovo, and the equations defining space curves. *Inventiones Mathematicae*, 72(3):491–506, 1983.

[7] M. Hellus, L. T. Hoa, and J. Stückrad. Gröbner Bases of simplicial toric ideals. *Nagoya Mathematical Journal*, 196:67–85, 2009.

[8] J. Herzog and T. Hibi. Castelnuovo-Mumford regularity of simplicial semigroup rings with isolated singularity. *Proceedings of the American Mathematical Society*, 131(9):2641–2647, 2003.

[9] L. T. Hoa and J. Stückrad. Castelnuovo-Mumford regularity of simplicial toric rings. *Journal of Algebra*, 259(1):127–146, 2003.

[10] L. T. Hoa and N. V. Trung. Affine semigroups and Cohen-Macaulay rings generated by monomials. *Transactions of the American Mathematical Society*, 298(1):145–167, 1986.

[11] L. T. Hoa and N. V. Trung. On the Castelnuovo-Mumford regularity and the arithmetic degree of monomial ideals. *Mathematische Zeitschrift*, 229(3):519–537, 1998.

[12] M. Hochster. Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes. *The Annals of Mathematics*, 96(2):318–337, 1972.

[13] M. Hochster and J. L. Roberts. The Purity of the Frobenius and Local Cohomology. *Advances in Mathematics*, 21(2):117–172, 1976.

[14] P. Li. *Seminormality and the Cohen-Macaulay Property.* PhD thesis, Queen's University, Kingston, Canada, 2004.

[15] P. Li and L. G. Roberts. Cohen-Macaulay and seminormal affine semigroups in dimension three (submitted). http://www.mast.queensu.ca/∼pingli/aboutme.html, August 2010.

[16] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra (Graduate Texts in Mathematics)*. Springer, 2005.

[17] M. J. Nitsche. Natural Cohen-Macaulayfication of simplicial affine semigroup rings (submitted). arXiv:1006.5670, 2010.

[18] I. Peeva and B. Sturmfels. Syzygies of codimension 2 lattice ideals. *Mathematische Zeitschrift*, 229(1):163–194, 1998.

[19] J. Stückrad and W. Vogel. Castelnuovo bounds for certain subvarieties in $\mathbb{P}^n$. *Mathematische Annalen*, 276(2):341–352, 1987.

[20] R. Treger. On equations defining arithmetically Cohen-Macaulay schemes. I. *Mathematische Annalen*, 261(2):141–153, 1982.

[21] N. V. Trung. Classification of the double projections of Veronese varieties. *Kyoto Journal of Mathematics*, 22(4):567–581, 1982.