# Max-Planck-Institut
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**Commutability of homogenization and linearization at identity in finite elasticity and applications**

by


*Antoine Gloria, and Stefan Neukamm*



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# COMMUTABILITY OF HOMOGENIZATION AND LINEARIZATION AT IDENTITY IN FINITE ELASTICITY AND APPLICATIONS

ANTOINE GLORIA AND STEFAN NEUKAMM

ABSTRACT. In this note we prove under some general assumptions on elastic energy densities (namely, frame indifference, minimality at identity, non-degeneracy and existence of a quadratic expansion at identity) that homogenization and linearization commute at identity. This generalizes a recent result by S. Müller and the second author by dropping their assumption of periodicity. As a first application, we extend their $\Gamma$-convergence commutation diagram for linearization and homogenization to the stochastic setting under standard growth conditions. As a second application, we prove that the $\Gamma$-closure is local at identity for this class of energy densities.

**Keywords:** homogenization, nonlinear elasticity, linearization, $\Gamma$-closure.

**2010 Mathematics Subject Classification:** 35B27, 49J45, 74E30, 74Q05, 74Q20.

## CONTENTS



## 1. INTRODUCTION

This note is devoted to the study of the commutability of linearization and homogenization at identity in finite elasticity. We consider an open bounded Lipschitz domain $D \subset \mathbb{R}^d$, and a family of integral functionals

$$\mathcal{I}_\varepsilon : H^1(D) \to [0, +\infty], \qquad u \mapsto \int_D W_\varepsilon(x, \nabla u(x))\,\mathrm{d}x$$

where $W_\varepsilon : D \times \mathbb{M}^d \to [0, +\infty]$ is a Borel function. As it is common in finite elasticity, we assume that $W_\varepsilon$ is frame indifferent and minimal at identity. Moreover, we assume that $W_\varepsilon$ is non-degenerate and admits a quadratic expansion at identity with quadratic term $Q_\varepsilon$; as a consequence, in situations when the deformation is close to a rigid-body motion, say when $|\nabla u - \mathrm{Id}\,| \sim h \ll 1$, we can accurately describe the functional $\mathcal{I}_\varepsilon$ (after scaling by $h^{-2}$) by the quadratic functional

$$\mathcal{E}_\varepsilon : H^1(D) \to [0, +\infty], \quad g \mapsto \int_D Q_\varepsilon(x, \nabla g(x))\,\mathrm{d}x \quad \text{with } g(x) := h^{-1}(u(x) - x).$$

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Since $Q_\varepsilon(\cdot,F)$ genuinely only depends on the symmetric part of the strain gradient $F$, the energy $\mathcal{E}_\varepsilon$ corresponds to linear elasticity. On the other hand, if $\mathcal{I}_\varepsilon$ has some specific structure in space rescaled by $\varepsilon$ (think of periodicity for instance), we may expect a homogenization property to hold as $\varepsilon$ vanishes, which justifies to replace the nonlinear oscillating-in-space energy density $(x,F)\mapsto W_\varepsilon(x,F)$ by a nonlinear homogeneous-in-space energy density $F\mapsto W_{\rm hom}(F)$ (or more generally by an energy density $(x,F)\mapsto W^*(x,F)$ whose oscillations in $x$ are independent of $\varepsilon$).

In this paper, we address the commutability of both limits (in $h$ and $\varepsilon$), and prove that they indeed commute in the following sense: The quadratic expansion of the homogenized energy $W_{\rm hom}$ (resp. $W^*$) at identity coincides with the homogenization of the quadratic expansion $Q_\varepsilon$ of the heterogeneous energy density at identity. In Theorem 2.1 we study functionals with standard growth and prove that the commutability holds (both, on the level of densities and on the level of the functionals), provided $\mathcal{I}_\varepsilon$ can be homogenized in the sense that $\mathcal{I}_\varepsilon$ $\Gamma(L^2)$-converges to a functional of the form

$$u\mapsto \int_D W^*(x,\nabla u)\,\mathrm{d}x.$$

In Theorem 2.2 we study unbounded energies and show that the commutability still holds provided both $\mathcal{I}_\varepsilon$ and $\mathcal{E}_\varepsilon$ can be homogenized. This theorem covers in particular the case when $W_\varepsilon(x,F)=+\infty$ if $\det F\leq 0$ — as it is desirable in finite elasticity. Our results generalize a recent work by S. Müller and the second author in [MN] (see also [Neu10]) by relaxing the periodicity assumption on $W_\varepsilon$ (as well as the growth condition from above). In [MN] the central object in the analysis is a multi-cell homogenization formula that allows in the periodic setting to compute the homogenized density $W_{\rm hom}$ by solving a sequence of periodic minimization problems on cubic domains invading $\mathbb{R}^d$. In [Neu10] the commutability of homogenization and linearization (solely as a $\Gamma$-convergence statement on the level of the energies) has been extended in the periodic case to energy densities without growth condition from above by extensive use of two-scale convergence methods. In the general situation considered in the present contribution, both the multi-cell homogenization formula and two-scale convergence approaches do not apply. Instead, we study the asymptotic formula

$$W_D(F):=\lim_{\varepsilon\to 0}\left\{\inf_{v\in H^1_0(D)}\mathcal{I}_\varepsilon(\varphi_F+v)\right\}$$

which is well defined whenever $\mathcal{I}_\varepsilon$ $\Gamma$-converges and is equi-coercive. In Proposition 2.3 we establish a quadratic expansion at identity for $W_D$ — which is the key insight in our analysis.

As a first application of Theorem 2.1, we show that linearization and stochastic homogenization commute at identity for energy densities which satisfy standard growth conditions (see Theorem 3.2). In a nutshell, what holds in [MN] in the periodic setting is also proved to hold here in the stochastic setting. This shows that the arguments used by S. Müller and the second author in [MN] are quite stable with respect to the structure assumption which ensures homogenization — at the core of the proof the quantitative rigidity estimate of [FJM02].

As a second application of Theorem 2.1, we prove a "weak local property" of the $\Gamma$-closure of a class of integral functionals at identity. The problem of $\Gamma$-closure consists in characterizing all the energy densities which can be reached by $\Gamma$-convergence starting from a composite made of a finite number of constituents with prescribed volume fraction. In particular, the $\Gamma$-closure is said to be local in some class of integrands if and only if any such "homogenized" energy density is the pointwise limit of a sequence of homogenized energy densities obtained by *periodic* homogenization. In the linear case, this property has been proved independently by Tartar in [Tar85] and Lurie and Cherkaev in [LC84]. The corresponding locality property of the

$G$-closure for monotone operators is due to Raitums in [Rai01] (generalizing an unpublished work by Dal Maso and Kohn). Related results of locality of the $\Gamma$-closure in the class of convex integrands can be found in [BB09]. Yet, the local character of the $\Gamma$-closure is an open question in the class of quasiconvex nonconvex integrands satisfying standard growth conditions. We focus here on a smaller class. In particular, we consider energy densities which are frame indifferent, non-degenerate, minimal at identity, admit a quadratic Taylor expansion at identity, and satisfy standard growth conditions. Then, we show that for any $F \mapsto W^*(F)$ in the $\Gamma$-closure of this set, there exists a sequence of periodic energy densities whose homogenized energy densities have quadratic Taylor expansions arbitrary close to the Taylor expansion of $W^*$ at identity (see Theorem 4.1). This can be seen as a weak version of the local character of the $\Gamma$-closure in this set at identity. Although quite restricted, this is the first such result for quasiconvex nonconvex energy densities.

This article is organized as follows: In Section 2 we state and prove our main theorem, the commutability of linearization and homogenization at identity. In Section 3 we apply this result to stochastic homogenization. The last section is dedicated to the local character of the $\Gamma$-closure at identity.

We will make use of the following notation throughout the text:

- $\mathbb{R}^+ := [0, +\infty)$ is the set of non-negative real numbers;
- $d$ is the dimension;
- $\mathbb{M}^d$ denotes the space of $d \times d$ real matrices, and for all $F \in \mathbb{M}^d$, $\operatorname{sym} F = 1/2(F + F^T)$ is the associated symmetric matrix, and $\operatorname{skw} F = F - \operatorname{sym} F$ the associated skew-symmetric matrix;
- $SO(d)$ is the set of rotations of $\mathbb{R}^d$;
- $\mathbb{T}^d_{\mathrm{sym}}$ denotes the space of symmetric fourth order tensors on $\mathbb{R}^d$;
- $D$ denotes an open bounded subset of $\mathbb{R}^d$ with a Lipschitz boundary (except for Theorem 2.2 and Proposition 2.3 in Section 2 where $D$ is further assumed to be $C^1$);
- $U = (0, 1)^d$ is the unit cell;
- for all $F \in \mathbb{M}^d$, we define the function $\varphi_F : \mathbb{R}^d \to \mathbb{R}^d$ as $\varphi_F : x \mapsto Fx$;
- for all $p \in [1, +\infty]$, $L^p(D)$, $H^1(D)$, $W^{1,p}(D)$, $H^1_0(D)$, and $W^{1,p}_0(D)$ denote the standard Lebesgue, Hilbert and Sobolev spaces of maps from $D$ to $\mathbb{R}^d$, and the associated subspaces of functions vanishing on the boundary $\partial D$ (in the sense of traces);
- $\varepsilon$ and $h$ denote generic elements of vanishing families of positive numbers $(\varepsilon)$ and $(h)$, respectively.
- $\rho$ (and $\rho'$) denotes a modulus of approximation, i.e. $\rho$ is an increasing function from $\mathbb{R}^+$ to $[0, +\infty]$ such that $\lim_{h\to 0} \rho(h) = 0$.

## 2. General commutability results

Throughout this article, we make the following assumptions on the energy densities.

*Definition* 1. For all $\alpha > 0$ and every modulus of approximation $\rho$, we denote by $\mathcal{W}_{\alpha,\rho}$ the set of measurable energy densities $W : \mathbb{M}^d \to [0, +\infty]$ which satisfy the following three properties:

(W1) $W$ is frame indifferent, i.e.

$$W(RF) = W(F) \qquad \text{for all } F \in \mathbb{M}^d,\ R \in \mathrm{SO}(d);$$

(W2) $W$ is non degenerate, i.e.

$$W(F) \geq \tfrac{1}{\alpha}\operatorname{dist}^2(F, \mathrm{SO}(d)) \qquad \text{for all } F \in \mathbb{M}^d;$$

(W3) $W$ is minimal at Id and admits the following quadratic expansion at Id:

$$\sup_{0<|G|\leq\delta} \frac{|W(\mathrm{Id}+G) - Q(G)|}{|G|^2} \leq \rho(\delta) \qquad \text{for all } \delta > 0,$$

where $Q : \mathbb{M}^d \to [0, \infty)$ is a quadratic form satisfying

$$0 \leq Q(G) \leq \alpha\,|G|^2 \qquad \text{for all } G \in \mathbb{M}^d.$$

For some results (in particular Theorem 2.1) we consider continuous energy densities that additionally satisfy standard growth conditions:

*Definition* 2. For all $p \in [1, +\infty)$ and $\alpha > 0$, we denote by $\mathcal{W}^p_\alpha$ the set of continuous energy densities $W : \mathbb{M}^d \to \mathbb{R}$ which satisfy the following standard growth condition of order $p$:

(W4)
$$\forall F \in \mathbb{M}^d \;:\; \tfrac{1}{\alpha}|F|^p - \alpha \;\leq\; W(F) \;\leq\; \alpha(|F|^p + 1).$$

In addition, we set $\mathcal{W}^p_{\alpha,\rho} := \mathcal{W}_{\alpha,\rho} \cap \mathcal{W}^p_\alpha$ for every modulus of approximation $\rho$. Note that $\mathcal{W}^p_{\alpha,\rho} = \emptyset$ for $p < 2$, and $\mathcal{W}^p_{\alpha,\rho} \neq \emptyset$ for $p \geq 2$.

*Remark* 1. Let $W \in \mathcal{W}_{\alpha,\rho}$ and let $Q$ denote the quadratic form associated with $W$ through (W3). Because of (W1) – (W3) the quadratic form $Q$ generically satisfies conditions that are common in linear elasticity; namely, the growth and ellipticity condition

(Q1)
$$\forall G \in \mathbb{M}^d \;:\; \tfrac{1}{\alpha'}\,|\mathrm{sym}\,G|^2 \leq Q(G) \leq \alpha'\,|G|^2$$

for some positive constant $\alpha'$ that only depends on $\alpha$, and

(Q2)
$$\forall G \in \mathbb{M}^d \;:\; Q(\mathrm{skw}\,G) = 0.$$

*Definition* 3. We denote by $\mathcal{Q}_{\alpha'}$ the set of non-negative quadratic forms $Q : \mathbb{M}^d \to \mathbb{R}^+$ satisfying (Q1) and (Q2).

Throughout this article we consider measurable maps $W$ from $D$ to $\mathcal{W}_{\alpha,\rho}$ such that $W(\cdot,\cdot)$ is a Borel function on $D \times \mathbb{M}^d$ (or equivalent to a Borel function), so that $x \mapsto W_\varepsilon(x, \nabla u(x))$ is a measurable function for all $u \in W^{1,1}(D)$. We call such maps "admissible energy densities" from $D$ to $\mathcal{W}_{\alpha,\rho}$. Note that measurable maps from $D$ to $\mathcal{W}^p_{\alpha,\rho}$ are Carathéodory functions and therefore admissible.

Let us consider a family $(W_\varepsilon)$ of admissible energy densities from $D$ to $\mathcal{W}_{\alpha,\rho}$. For almost every $x \in D$, we denote by $Q_\varepsilon(x,\cdot)$ the quadratic form associated with $W_\varepsilon(x,\cdot)$ through (W3); thus, $Q_\varepsilon$ can be written as the pointwise limit

$$(x, G) \mapsto Q_\varepsilon(x, G) := \lim_{h\to 0} \frac{1}{h^2} W_\varepsilon(x, \mathrm{Id}+hG),$$

and therefore inherits the measurability properties of $W_\varepsilon$. We then define two families of integral functionals, namely $\mathcal{I}_\varepsilon : H^1(D) \to [0, +\infty]$ characterized by

(1)
$$\mathcal{I}_\varepsilon(u) := \int_D W_\varepsilon(x, \nabla u(x))\,\mathrm{d}x,$$

and $\mathcal{E}_\varepsilon : H^1(D) \to [0, +\infty)$ characterized by

$$\mathcal{E}_\varepsilon(u) := \int_D Q_\varepsilon(x, \nabla u(x))\,\mathrm{d}x. \tag{2}$$

The main theorem of this paper is the following result, which generalizes [MN, Theorems 1 & 2] to the non-periodic setting.

**Theorem 2.1.** *Let $2 \le p < +\infty$, and let $(W_\varepsilon)$ denote a family of measurable energy densities from $D$ to $\mathcal{W}^p_{\alpha,\rho}$. Assume that the associated family of energy functionals $\mathcal{I}_\varepsilon$ defined in (1) $\Gamma(L^p)$-converges to an integral functional $\mathcal{I}^*$ on $W^{1,p}(D)$, defined by*

$$\mathcal{I}^*(u) := \int_D W^*(x, \nabla u(x))\,\mathrm{d}x,$$

*where $W^*$ is a Carathéodory function on $D \times \mathbb{M}^d$ with $W^*(x,\cdot) \in \mathcal{W}^p_\alpha$ for almost every $x \in D$. Then*

- *(a) $W^*(x,\cdot) \in \mathcal{W}^p_{\alpha',\rho'}$ for almost every $x \in D$, where $\alpha' > 0$ and the modulus of approximation $\rho'$ only depend on $\alpha$ and $\rho$;*
- *(b) there exist $\alpha'' > 0$ and a measurable map $Q^* : D \to \mathcal{Q}_{\alpha''}$ such that the energy functionals $\mathcal{E}_\varepsilon$ defined in (2) $\Gamma(L^2)$-converge to $\mathcal{E}^* : H^1(D) \to [0, +\infty)$ defined by*

$$\mathcal{E}^*(u) := \int_D Q^*(x, \nabla u(x))\,\mathrm{d}x;$$

- *(c) the expansion*

$$W^*(x, \mathrm{Id} + G) = Q^*(x, G) + o(|G|^2)$$

*holds for almost every $x \in D$ and for all $G \in \mathbb{M}^d$;*
- *(d) the following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}_{h,\varepsilon} & \xrightarrow{(1)} & \mathcal{E}_\varepsilon \\ {\scriptstyle (2)}\downarrow & & \downarrow{\scriptstyle (3)} \\ \mathcal{G}_{\mathrm{hom},h} & \xrightarrow[(4)]{} & \mathcal{E}_{\mathrm{hom}} \end{array}$$

*where $\mathcal{G}_{h,\varepsilon}$ and $\mathcal{G}_{\mathrm{hom}.h}$ denote the functionals from $H^1_0(D)$ to $[0, +\infty]$ defined as*

$$\mathcal{G}_{h,\varepsilon}(g) := \frac{1}{h^2}\mathcal{I}_\varepsilon(\varphi_{\mathrm{Id}} + hg), \qquad \mathcal{G}_{\mathrm{hom},h}(g) := \frac{1}{h^2}\mathcal{I}_{\mathrm{hom}}(\varphi_{\mathrm{Id}} + hg);$$

*and (1),(4), and (2),(3) mean $\Gamma$-convergence in $H^1_0(D)$ with respect to the strong topology of $L^2(D)$ as $h \to 0$ and $\varepsilon \to 0$, respectively. Moreover, the families $(\mathcal{I}_\varepsilon)$ and $(\mathcal{E}_\varepsilon)$ are equi-coercive w. r. t. weak convergence in $H^1_0(D)$.*

*Remark* 2. Due to the compactness of integral functionals with standard $p$-growth conditions w. r. t. $\Gamma(L^p)$-convergence (see for instance [BD98, Theorem 12.5]), the assumptions on $\mathcal{I}_\varepsilon$ are always satisfied up to extraction of a subsequence.

*Remark* 3. If $(W_\varepsilon)$ satisfies a growth condition from below of order $p \ge 2$ (uniformly in $\varepsilon$) then $\mathcal{I}_\varepsilon \equiv +\infty$ on $H^1(D) \setminus W^{1,p}(D)$ and it is natural to study the restricted functionals $\mathcal{I}_\varepsilon|_{W^{1,p}(D)}$ w. r. t. the strong topology in $L^p(D)$. In particular, $\mathcal{I}_\varepsilon|_{W^{1,p}(D)}$ is sequentially weakly lower semicontinuous in $W^{1,p}(D)$ if and only if it is lower semicontinuous w. r. t. strong convergence in $L^p(D)$. Note that due to condition (W2), $(W_\varepsilon)$ generically satisfies a uniform growth condition from below of order $p = 2$.

*Remark* 4. As in [MN], in Theorem 2.1 part (d), we can replace the function space $H^1_0(D)$ by the space

$$\mathcal{A}_\gamma := \{\, g\in H^1(D)\,:\, g=0 \text{ on } \gamma\,\},$$

where $\gamma$ denotes a closed subset of $\partial D$ with positive $d-1$-dimensional Hausdorff measure, and regular enough so that $\mathcal{A}_\gamma\cap W^{1,\infty}(D)$ is dense in $\mathcal{A}_\gamma$ (see [MN, Proof of Proposition 1] and [DMNP02] for details).

In finite elasticity it is desirable to consider energy densities with the physical behavior

$$W(x,F)=+\infty \qquad \text{for all } F\in\mathbb{M}^d \text{ with } \det F\le 0.$$

In order to allow such energy densities we have to drop the $p$-growth condition from above. The following theorem shows that for unbounded energy densities a linearization statement holds as well — although in that case homogenization is an open problem.

**Theorem 2.2.** *Suppose that the domain $D$ is $C^1$. Let $(W_\varepsilon)$ denote a family of admissible energy densities from $D$ to $\mathcal{W}_{\alpha,\rho}$. Suppose that there exist $p\ge 2$, and homogeneous-in-space energy densities $W_{\rm hom}:\mathbb{M}^d\to[0,+\infty]$ and $Q_{\rm hom}:\mathbb{M}^d\to\mathbb{R}$, such that as $\varepsilon\to 0$*

*(i) the energy functionals $\mathcal{I}_\varepsilon|_{W^{1,p}(D)}$ defined in (1) $\Gamma(L^p)$-converge to*

$$\mathcal{I}_{\rm hom}\,:\,W^{1,p}(D)\to[0,+\infty],\qquad u\mapsto\int_D W_{\rm hom}(\nabla u(x))\,\mathrm{d}x.$$

*(ii) the quadratic energy functionals $\mathcal{E}_\varepsilon$ defined in (2) $\Gamma(L^2)$-converge to $\mathcal{E}_{\rm hom}:H^1(D)\to[0,+\infty)$ given by*

$$\mathcal{E}_{\rm hom}\,:\,H^1(D)\to\mathbb{R},\qquad u\mapsto\int_D Q_{\rm hom}(\nabla u(x))\,\mathrm{d}x.$$

*If the homogenized density $W_{\rm hom}$ satisfies for all $F\in\mathbb{M}^d$ the asymptotic formula*

$$W_{\rm hom}(F)\;=\;\lim_{\varepsilon\to 0}\tfrac{1}{|D|}\inf\{\mathcal{I}_\varepsilon(\varphi_F+v)\;:\;v\in W^{1,p}_0(D)\},\tag{3}$$

*then $W_{\rm hom}$ admits a quadratic expansion at $\mathrm{Id}$ given by $Q_{\rm hom}$, i.e. for all $G\in\mathbb{M}^d$, there holds*

$$W_{\rm hom}(\mathrm{Id}+G)=Q_{\rm hom}(G)+o(|G|^2).\tag{4}$$

*Remark* 5. The quadratic expansion (4) of $W_{\rm hom}$ does not depend on the exponent for which the $\Gamma(L^p)$-convergence holds.

Theorems 2.1 & 2.2 follow from a result which is somewhat unrelated to homogenization, and establishes a quadratic expansion at $\mathrm{Id}$ for the asymptotic formula

$$W_D(F):=\lim_{\varepsilon\to 0}\left\{\inf_{v\in W^{1,p}_0(D)}\mathcal{I}_\varepsilon(\varphi_F+v)\right\}\tag{5}$$

if it exists.

**Proposition 2.3.** *Let $2\le p<+\infty$, let the domain $D$ be $C^1$, and let $(W_\varepsilon)$ be a family of admissible energy densities from $D$ to $\mathcal{W}_{\alpha,\rho}$. Suppose that the limit (5) exists in $[0,+\infty]$ (where $\mathcal{I}_\varepsilon$ is as in (1) ) for all $F\in\mathbb{M}^d$, and that the functionals $\mathcal{E}_\varepsilon$ defined in (2) $\Gamma(L^2)$-converge to a functional $\mathcal{E}^*$ on $H^1(D)$. Then there exist a constant $\alpha'>0$ that only depends on $\alpha$ and a modulus of approximation $\rho'$ that additionally depends on $\rho$ and on the geometry of $D$, such that $\frac{1}{|D|}W_D\in\mathcal{W}_{\alpha',\rho'}$ and*

$$\frac{\left|W_D(\mathrm{Id}+G)-\inf_{v\in H^1_0(D)}\mathcal{E}^*(\varphi_G+v)\right|}{|G|^2}\;\le\;|D|\rho'(|G|)\tag{6}$$

*for all $G\in\mathbb{M}^d$.*

*Remark* 6. In the proof of Proposition 2.3 we make $\rho'$ explicit:

$$\rho'(h) = C\max\left\{ \rho(h+\sqrt{h})(1+\alpha+\rho(h)) + h^{\frac{\mu}{4(2+\mu)}}(1+\alpha+\rho(h))^{\frac{4+3\mu}{4(2+\mu)}},\right. \\ \left. h^{2\mu} + \rho(h+\sqrt{h})(1+h^{2\mu})\right\}, \tag{7}$$

where the constants $C, \mu > 0$ only depend on $\alpha$ and on the geometry of $D$, i.e. $C$ and $\mu$ are invariant under dilation, rotation, and translation of $D$. Note that for $h \ll 1$, (7) reduces to

$$\rho'(h) \sim C(\rho(\sqrt{h}) + h^{\frac{\mu}{4(2+\mu)}}).$$

*Remark* 7. The assumption on $\mathcal{E}_\varepsilon$ is no restriction. In particular, by the compactness of $G$-convergence (see for instance [JKO94, Section 12.2]), we can always extract a subsequence of $(\varepsilon)$ to which Proposition 2.3 applies.

In the proof of Proposition 2.3 we will make use of the following higher integrability and Lipschitz truncation result for minimizers of quadratic functionals:

**Proposition 2.4.** *Let $\alpha > 0$, $G \in \mathbb{M}^d$, and $Q:\ D \to \mathcal{Q}_\alpha$ be a measurable map. Set*

$$\mathcal{E}_G:\ H^1_0(D) \to [0,+\infty), \qquad \mathcal{E}_G(g) := \int_D Q(x, G+\nabla g)\,\mathrm{d}x.$$

*(a) The functional $\mathcal{E}_G$ admits a unique minimizer $g^* \in H^1_0(D)$, characterized by the Euler-Lagrange equation*

$$\int_D \langle \mathbb{L}(x)(G+\nabla g^*,\ \nabla\varphi\rangle\ \mathrm{d}x = 0 \qquad \textit{for all } \varphi \in H^1_0(D)$$

*where $\mathbb{L} \in L^\infty(D, \mathbb{T}^d_{\mathrm{sym}})$ is defined by*

$$\langle \mathbb{L}(x)A,\ B\rangle = \frac{Q(x,A+B) - Q(x,A) - Q(x,B)}{2}$$

*for all $A, B \in \mathbb{M}^d$ and almost every $x \in D$.*

*(b) (Meyers' estimate). If in addition the domain $D$ is $C^1$, then there exists a Meyers' exponent $\mu > 0$ and a positive constant $C$ such that*

$$\|\nabla g^*\|^{2+\mu}_{L^{2+\mu}(D)} \le C\,|D|\,|G|^{2+\mu}.$$

*The exponent $\mu$ and the constant $C$ only depend on $\alpha$ and on the geometry of the domain $D$.*

*(c) (Lipschitz truncation). Let $\lambda > 0$. If in addition the domain $D$ is $C^1$, then there exists a map $g \in W^{1,\infty}_0(D)$ such that*

$$|\nabla g(x)| \le \lambda \qquad \textit{for a.e. } x \in D,$$
$$\mathcal{E}_G(g) - \mathcal{E}_G(g^*) \le C\lambda^{-\mu}\,|D|\,|G|^{2+\mu},$$

*where $\mu$ is a Meyers' exponent, and the constant $C$ only depends on $\alpha$ and on the geometry of the domain $D$ (in particular, it is independent of $\lambda$, $G$, and $\mu$).*

The first statement of Proposition 2.4 is standard and relies on Korn's inequality. The second statement is a higher integrability result for gradients in linear elasticity, as proved in [SW94]. This is the only place where we use the regularity of the domain. The third statement is essentially a combination of Meyers' estimate and of a Lipschitz truncation argument from [FJM02]. The constants $C$ and $\mu$ only depend on the geometry of the domain in the sense that they can be chosen invariant under translation, rotation, and dilation of the domain. The proof of this statement is deferred to the appendix.

*Proof of Proposition 2.3.* We divide the proof in three steps. In the first step, we introduce a quadratic form associated with $W_D$. The last two steps are dedicated to the proof of (6) proper.

*Step* 1. Definition of the quadratic form $Q_D$.
By the assumptions on $W_\varepsilon$ the associated quadratic form $Q_\varepsilon$ is a measurable map from $D$ to $\mathcal{Q}_{\tilde\alpha'}$, where the $\tilde\alpha' > 0$ only depends on $\alpha$. Remark 1 and Korn's inequality on $D$ thus imply that the quadratic energies $\mathcal{E}_\varepsilon$ are equi-coercive functionals on $H^1_0(D)$, so that the associated elliptic operators are compact w. .r. t. $G$-convergence (see for instance [JKO94, Section 12.2]). In particular, this yields $\Gamma$-convergence of the energy functionals to an integral functional (see for instance [GMT93, Subsection 4.4]): There exist $\tilde\alpha''$ depending only on $\alpha$, and a measurable map $Q^*$ from $D$ to $\mathcal{Q}_{\tilde\alpha''}$ such that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges (up to extraction) to the functional $\mathcal{E}^* : H^1(D) \to \mathbb{R}$ characterized by

$$\mathcal{E}^* : u \mapsto \int_D Q^*(x, \nabla u(x))\,\mathrm{d}x. \tag{8}$$

This shows that $\mathcal{E}^*$ is a quadratic integral functional.

We are now in position to define the map $Q_D : \mathbb{M}^d \to [0, +\infty)$ as

$$Q_D(G) := \inf_{v\in H^1_0(D)} \mathcal{E}^*(\varphi_G + v).$$

Because of the representation (8), the map $Q_D$ is a quadratic form of class $\mathcal{Q}_{\tilde\alpha}$ for a positive constant $\tilde\alpha$ depending only on $\alpha$.

We claim that $\frac{1}{|D|}W_D$ is of class $\mathcal{W}_{\alpha',\rho'}$ where $\rho'$ is defined by (7). It is clear that $\frac{1}{|D|}W_D$ is frame indifferent. It also satisfies a property of type (W2) as an application of [MN, Lemma 2] (its proof actually does not use periodicity, but only the asymptotic formula (5)). The expansion property (W3) is equivalent to (6). As in [MN] we notice that it is sufficient to prove the following: For all families $(G_h) \in \mathbb{R}^d$ with $|G_h| = 1$, we have:

$$\text{(lower bound)} \qquad \frac{1}{h^2}W_D(\mathrm{Id} + hG_h) \geq Q_D(G_h) - \tfrac{|D|}{2}\rho'(h),$$

$$\text{(upper bound)} \qquad \frac{1}{h^2}W_D(\mathrm{Id} + hG_h) \leq Q_D(G_h) + \tfrac{|D|}{2}\rho'(h).$$

We prove both statements separately.

*Step* 2. Proof of the lower bound.
By definition of $W_D$ (see (5)), for all $h > 0$,

$$0 \leq W_D(\mathrm{Id} + hG_h) \leq \limsup_{\varepsilon\to 0} \int_D W_\varepsilon(x, \mathrm{Id} + hG_h).$$

From (W3), the fact that $Q_\varepsilon(x, G) \leq \alpha\,|G|^2$ for a.e. $x \in D$, and the assumption $|G_h| = 1$, we infer that

$$0 \leq \frac{1}{h^2}W_D(\mathrm{Id} + hG_h) \leq |D|(\alpha + \rho(h)). \tag{9}$$

By definition of $W_D$, there exists a sequence $(u_{h,\varepsilon}) \in W^{1,p}(D)$ with the properties

$$u_{h,\varepsilon} - \varphi_{\mathrm{Id} + hG_h} \in W^{1,p}_0(D) \subset H^1_0(D), \tag{10}$$

$$\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon(u_{h,\varepsilon}) = W_D(\mathrm{Id} + hG_h). \tag{11}$$

We then define the following sequence of scaled displacements

$$g_{h,\varepsilon} := \frac{u_{h,\varepsilon} - \varphi_{\mathrm{Id} + hG_h}}{h}.$$

By construction $g_{h,\varepsilon} \in H^1_0(D)$ and the uniform non-degeneracy assumption (W2) on $W_\varepsilon$ yields the estimate

$$\frac{1}{h^2}\int_D \operatorname{dist}^2(\mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x)), \mathrm{SO}(d))\,\mathrm{d}x \le \alpha \frac{1}{h^2}\mathcal{I}_\varepsilon(u_{h,\varepsilon}).$$

The quantitative geometric rigidity estimate (see [FJM02, Theorem 3.1]) implies the existence of a rotation $R_{h,\varepsilon} \in \mathrm{SO}(d)$ such that

$$\|F_{h,\varepsilon} + \nabla g_{h,\varepsilon}\|^2_{L^2(D)} \le C\frac{1}{h^2}\mathcal{I}_\varepsilon(u_{h,\varepsilon}) \qquad \text{with} \qquad F_{h,\varepsilon} := \frac{\mathrm{Id} - R_{h,\varepsilon}}{h} + G_h.$$

Except otherwise stated, $C$ denotes a positive constant that may vary from line to line, but can be chosen only depending on $\alpha$ and on the geometry of $D$. Because $g_{h,\varepsilon}$ vanishes on $\partial D$, an integration by parts shows that $\nabla g_{h,\varepsilon}$ and the (constant) matrix $F_{h,\varepsilon}$ are orthogonal w. r. t. the inner product in $L^2(D;\mathbb{M}^d)$:

$$\|F_{h,\varepsilon} + \nabla g_{h,\varepsilon}\|^2_{L^2(D)} \;=\; |D||F_{h,\varepsilon}|^2 + \|\nabla g_{h,\varepsilon}\|^2_{L^2(D)} \;\ge\; \|\nabla g_{h,\varepsilon}\|^2_{L^2(D)}.$$

Hence, the rigidity estimate, (9), and (11) yield

$$\limsup_{\varepsilon\to 0} \|\nabla g_{h,\varepsilon}\|^2_{L^2(D)} \le C|D|(\alpha + \rho(h)). \tag{12}$$

Next, in order to make use of the quadratic expansion in (W3), we focus on the set where $h(G_h + \nabla g_{h,\varepsilon})$ is bounded. To this end, we let $\chi_{h,\varepsilon}$ denote the indicator function of the set $X_{h,\varepsilon} := \{\, x \in D \,:\, |\nabla g_{h,\varepsilon}| \le h^{-1/2}\}$, and note that

$$\begin{aligned}
\frac{1}{h^2}\mathcal{I}_\varepsilon(u_{h,\varepsilon}) =& \frac{1}{h^2}\int_D W_\varepsilon(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x\\
\ge& \frac{1}{h^2}\int_D \chi_{h,\varepsilon}(x) W_\varepsilon(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x\\
=& \frac{1}{h^2}\int_D W_\varepsilon(x, \mathrm{Id} + h\chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x
\end{aligned}$$

by the non-negativity of $W_\varepsilon$ and the fact that $W_\varepsilon$ vanishes at $\mathrm{Id}$. We then write the r. h. s. in the form

$$\begin{aligned}
\frac{1}{h^2}&\int_D W_\varepsilon(x, \mathrm{Id} + h\chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x\\
&= \int_D \Big( Q_\varepsilon\big(\,x, \chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))\,\big) + \mathrm{r}_{h,\varepsilon}(x)\Big)\,\mathrm{d}x,
\end{aligned}$$

where, using assumption (W3), the remainder $\mathrm{r}_{h,\varepsilon}$ satisfies for all $x \in X_{h,\varepsilon}$

$$\begin{aligned}
|\mathrm{r}_{h,\varepsilon}(x)| \;&=\; |G_h + \nabla g_{h,\varepsilon}(x)|^2\\
&\qquad \times \frac{|W_\varepsilon(x, \mathrm{Id} + h(G_h + \nabla g_{h,\varepsilon}(x))) - Q_\varepsilon(x, h(G_h + \nabla g_{h,\varepsilon}(x)))|}{h^2|G_h + \nabla g_{h,\varepsilon}(x)|^2}\\
&\le\; \rho(h\,|G_h| + \sqrt{h})\,|G_h + \nabla g_{h,\varepsilon}(x)|^2\\
&=\; \rho(h + \sqrt{h})\,|G_h + \nabla g_{h,\varepsilon}(x)|^2,
\end{aligned}$$

and $r_{h,\varepsilon}(x) = 0$ for a. e. $x \in D \setminus X_{h,\varepsilon}$. Thus, we conclude that

$$\frac{1}{h^2}\mathcal{I}_\varepsilon(u_{h,\varepsilon}) \ge \int_D Q_\varepsilon\Big(x, \chi_{h,\varepsilon}(G_h + \nabla g_{h,\varepsilon})\Big)\,\mathrm{d}x - \rho(h + \sqrt{h})\,\|G_h + \nabla g_{h,\varepsilon}\|^2_{L^2(D)}\,. \tag{13}$$

Appealing to (12) and using $|G_h| = 1$, (13) turns into

$$\liminf_{\varepsilon\to 0} \frac{1}{h^2}\mathcal{I}_\varepsilon(u_{h,\varepsilon}) \geq \liminf_{\varepsilon\to 0}\int_D Q_\varepsilon(x,\chi_{h,\varepsilon}(G_h+\nabla g_{h,\varepsilon}))\,\mathrm{d}x - C\,|D|\,\rho(\sqrt{h}+h)(1+\alpha+\rho(h)). \tag{14}$$

Next, we wish to replace the integral term on the r. h. s. of (14) by the infimum of $\mathcal{E}_\varepsilon$ on the set $\varphi_{G_h} + H^1_0(D)$. By coercivity of $\mathcal{E}_\varepsilon$ on this set, this infimum problem is well-posed, and there exists $g^*_{h,\varepsilon} \in H^1_0(D)$ such that $v_{h,\varepsilon} := \varphi_{G_h} + g^*_{h,\varepsilon}$ satisfies

$$\mathcal{E}_\varepsilon(v_{h,\varepsilon}) = \int_D Q_\varepsilon(x,\nabla v_{h,\varepsilon}(x))\,\mathrm{d}x \;=\; \inf_{v\in H^1_0(D)}\int_D Q_\varepsilon(x,G_h+\nabla v(x))\,\mathrm{d}x. \tag{15}$$

Since $\mathcal{E}_\varepsilon$ is equi-coercive on $\varphi_{G_h} + H^1_0(D)$, and $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}^*$ on $H^1(D)$, the $\Gamma$-limit is coercive, and the sequence of minima converges to the minimum of $\mathcal{E}_{\mathrm{hom}}$ on $\varphi_{G_h} + H^1_0(D)$. This yields

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_{h,\varepsilon}) = \inf_{v\in\varphi_{G_h}+H^1_0(D)}\mathcal{E}^*(v) = Q_D(G_h). \tag{16}$$

We shall actually prove that there exists $\mu > 0$ depending only on $\alpha$ and on the geometry of $D$ such that

$$\liminf_{\varepsilon\to 0}\int_D \Big(\, Q_\varepsilon\Big(x,\chi_{h,\varepsilon}(G+\nabla g_{h,\varepsilon})\Big) - Q_\varepsilon(x,\nabla v_{h,\varepsilon}(x))\Big)\,\mathrm{d}x \geq -C|D|h^{\frac{\mu}{4(2+\mu)}}(1+\alpha+\rho(h))^{\frac{4+3\mu}{4(2+\mu)}}. \tag{17}$$

Combined with (11), (14) and (16), (17) yields the desired lower bound.

The proof of (17) is the heart of the matter. Let $\mathbb{L}_\varepsilon \in L^\infty(D,\mathbb{T}^d_{\mathrm{sym}})$ denote the unique symmetric 4th order tensor associated with $Q_\varepsilon$, i.e.

$$\langle \mathbb{L}_\varepsilon(x)A,\,B\rangle = \frac{Q_\varepsilon(x,A+B)-Q_\varepsilon(x,A)-Q_\varepsilon(x,B)}{2} \tag{18}$$

for all $A, B \in \mathbb{M}^d$ and a.e. $x \in D$. Note that $(\mathbb{L}_\varepsilon)$ is uniformly bounded in $L^\infty(D,\mathbb{T}^d_{\mathrm{sym}})$ because the operator norm of $Q_\varepsilon(x,\cdot)$ on $\mathbb{M}^d$ is bounded by $\alpha$ for all $\varepsilon > 0$ and a.e. $x \in D$. Since $Q_\varepsilon(x,\cdot)$ is a non-negative quadratic form, the inequality

$$Q_\varepsilon(x,A) - Q_\varepsilon(x,B) \geq 2\,\langle\mathbb{L}_\varepsilon(x)(A-B),\,B\rangle$$

holds for all $A, B \in \mathbb{M}^d$ and a.e. $x \in D$. We use this estimate with $A = \chi_{h,\varepsilon}(x)(G_h + \nabla g_{h,\varepsilon}(x))$ and $B = \nabla v_{h,\varepsilon}(x)$, which yields by integration over $D$:

$$\int_D \Big(Q_\varepsilon(x,\chi_{h,\varepsilon}(G_h+\nabla g_{h,\varepsilon})(x)) - Q_\varepsilon(x,\nabla v_{h,\varepsilon}(x))\Big)\,\mathrm{d}x \geq 2\int_D \big\langle \mathbb{L}_\varepsilon\big(\,\chi_{h,\varepsilon}(G_h+\nabla g_{h,\varepsilon}) - \nabla v_{h,\varepsilon}\,\big),\,\nabla v_{h,\varepsilon}\big\rangle\,\mathrm{d}x. \tag{19}$$

Along the lines of [MN, Proof of Theorem 1], we rewrite the r. h. s. as

$$\int_D \big\langle \mathbb{L}_\varepsilon\big(\,\chi_{h,\varepsilon}(G_h+\nabla g_{h,\varepsilon}) - \nabla v_{h,\varepsilon}\,\big),\,\nabla v_{h,\varepsilon}\big\rangle\,\mathrm{d}x \;=\; I^{(1)}_{h,\varepsilon} \;-\; I^{(2)}_{h,\varepsilon}\,,$$

with

$$\begin{aligned} I^{(1)}_{h,\varepsilon} &:= \int_D \langle \mathbb{L}_\varepsilon(G_h+\nabla g_{h,\varepsilon}-\nabla v_{h,\varepsilon}),\,\nabla v_{h,\varepsilon}\rangle\,\mathrm{d}x,\\ I^{(2)}_{h,\varepsilon} &:= \int_D \langle \mathbb{L}_\varepsilon(1-\chi_{h,\varepsilon})(G_h+\nabla g_{h,\varepsilon}),\,\nabla v_{h,\varepsilon}\rangle\,\mathrm{d}x. \end{aligned}$$

Because $I^{(1)}_{h,\varepsilon}$ is the weak form of the Euler-Lagrange equation of the minimization problem in (15) with admissible test-function $\varphi_{G_h} + g_{h,\varepsilon} - v_{h,\varepsilon} \in H^1_0(D)$, the first term $I^{(1)}_{h,\varepsilon}$ vanishes identically. We now deal with the second term, and claim that

$$\limsup_{\varepsilon\to 0} |I^{(2)}_{h,\varepsilon}| \le C|D|h^{\frac{\mu}{4(2+\mu)}}(1+\alpha+\rho(h))^{\frac{4+3\mu}{4(2+\mu)}}. \tag{20}$$

Combined with (19) and $I^{(1)}_{h,\varepsilon} \equiv 0$, this implies the desired estimate (17). To prove (20), we apply the higher integrability result of Proposition 2.4 part (b) to $\nabla v_{h,\varepsilon}$. In particular, there exists a Meyers' exponent $\mu > 0$ and a positive constant $C$ such that

$$\int_D |\nabla v_{h,\varepsilon}|^{2+\mu}\,\mathrm{d}x \le C|D|\,|G_h|^{2+\mu} = C|D|. \tag{21}$$

By Cauchy-Schwarz' and Hölder's inequalities, we may estimate $I^{(2)}_{h,\varepsilon}$ by

$$\begin{aligned} \left|I^{(2)}_{h,\varepsilon}\right| &\le C\,\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}\,\|(1-\chi_{h,\varepsilon})\nabla v_{h,\varepsilon}\|_{L^2(D)} \\ &\le C\,\|G_h + \nabla g_{h,\varepsilon}\|_{L^2(D)}\,\|1-\chi_{h,\varepsilon}\|_{L^q(D)}\,\|\nabla v_{h,\varepsilon}\|_{L^{2+\mu}(D)} \end{aligned} \tag{22}$$

with $q := \frac{2(2+\mu)}{\mu} \in (1,\infty)$. By definition of $\chi_{h,\varepsilon}$ there holds

$$|1-\chi_{h,\varepsilon}(x)| \le \sqrt{h}(1-\chi_{h,\varepsilon}(x))\,|\nabla g_{h,\varepsilon}(x)| \le \sqrt{h}\,|\nabla g_{h,\varepsilon}(x)|$$

for a.e. $x \in D$, so that

$$\int_D |1-\chi_{h,\varepsilon}|^q\,\mathrm{d}x = \int_D |1-\chi_{h,\varepsilon}|\,\mathrm{d}x \le \sqrt{h}\int_D |\nabla g_{h,\varepsilon}|\,\mathrm{d}x.$$

Hence, by Cauchy-Schwarz' inequality,

$$\|1-\chi_{h,\varepsilon}\|_{L^q(D)} \le Ch^{\frac{1}{2q}}\,|D|^{\frac{1}{2q}}\,\|g_{h,\varepsilon}\|^{\frac{1}{q}}_{H^1(D)} = C|D|^{\frac{\mu}{4(2+\mu)}}\,h^{\frac{\mu}{4(2+\mu)}}\,\|g_{h,\varepsilon}\|^{\frac{\mu}{2(2+\mu)}}_{H^1(D)},$$

which, combined with (12), (21) & (22), proves (20). This concludes the proof of the lower bound.

*Step* 3. Proof of the upper bound.
As usual, the proof of the upper bound relies on an explicit construction. As a first step we apply the Lipschitz truncation argument of Proposition 2.4 part (c): There exists a doubly indexed sequence $(g_{h,\varepsilon}) \subset H^1_0(D)$ and some $\mu > 0$ (only depending on $\alpha$ and the geometry of $D$) such that

$$\begin{cases} \|\nabla g_{h,\varepsilon}\|_{L^\infty(D)} \le h^{-1/2}, \\ \mathcal{E}_\varepsilon(\varphi_{G_h} + g_{h,\varepsilon}) - Q_D(G_h) \le Ch^{2\mu}|D|. \end{cases} \tag{23}$$

Here and below, $C$ denotes a positive constant that may vary from line to line, but only depends on $\alpha$ and on the geometry of $D$.

Since for all $\varepsilon > 0$ the quadratic form $\mathcal{E}_\varepsilon$ is Korn-elliptic with some constant $\alpha'$ depending only on $\alpha$, the second property in (23) and Poincaré's inequality imply that the sequence $(g_{h,\varepsilon})_\varepsilon$ is bounded in $H^1(D)$. Using in addition Step 1 in the form of $Q_D(G_h) \le \tilde{\alpha}|D|$, this yields the estimate

$$\|G_h + \nabla g_{h,\varepsilon}\|^2_{L^2(D)} \le C(1+h^{2\mu})\,|D|\,. \tag{24}$$

We set

$$u_{h,\varepsilon} := \varphi_{\mathrm{Id}+hG_h} + hg_{h,\varepsilon}.$$

By definition we have

$$W_D(\operatorname{Id}+hG_h)=\lim_{\varepsilon\to 0}\left\{\inf_{v\in H^1_0(D)}\mathcal{I}_\varepsilon(\varphi_{\operatorname{Id}+hG_h}+v)\right\}\leq\liminf_{\varepsilon\to 0}\mathcal{I}_\varepsilon(u_{h,\varepsilon})$$
$$=\liminf_{\varepsilon\to 0}\int_D W_\varepsilon(x,\operatorname{Id}+h(G_h+\nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x. \tag{25}$$

As in the proof of the lower bound, we expand the r. h. s. as

$$\int_D W_\varepsilon(x,\operatorname{Id}+h(G_h+\nabla g_{h,\varepsilon}(x)))\,\mathrm{d}x=h^2\int_D\Big(Q_\varepsilon(x,G_h+\nabla g_{h,\varepsilon}(x))+\mathrm{r}_{h,\varepsilon}(x)\Big)\,\mathrm{d}x, \tag{26}$$

where, using assumption (W3) and property (23), the remainder is estimated by

$$\int_D|\mathrm{r}_{h,\varepsilon}(x)|\,\mathrm{d}x\;\leq\;\rho(h+\sqrt{h})\,\|G_h+\nabla g_{h,\varepsilon}\|^2_{L^2(D)}\,.$$

The combination of (25), (26), (24), and the second property in (23) then yields

$$\begin{aligned}\frac{1}{h^2}W_D(\operatorname{Id}+hG_h)\quad&\leq\quad\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(\varphi_{G_h}+g_{h,\varepsilon})+\limsup_{\varepsilon\to 0}\rho(h+\sqrt{h})\,\|G_h+\nabla g_{h,\varepsilon}\|^2_{L^2(D)}\\&\leq\quad Q_D(G_h)+C\,|D|\,\Big(h^{2\mu}+\rho(h+\sqrt{h})(1+h^{2\mu})\Big)\,.\end{aligned}$$

This proves the upper bound, and concludes the proof of the proposition. □

Theorem 2.2 is an immediate consequence Proposition 2.3:

*Proof of Theorem 2.2.* By assumption the $\Gamma$-limits $\mathcal{I}_{\hom}$ and $\mathcal{E}_{\hom}$ are integral functionals with homogeneous integrands $W_{\hom}$ and $Q_{\hom}$, respectively. Thus, the expansion (6) in Proposition 2.3 simplifies to

$$\inf_{v\in W^{1,p}_0(D)}\int_D W_{\hom}(\operatorname{Id}+G+\nabla v(x))\,\mathrm{d}x=\inf_{v\in H^1_0(D)}\int_D Q_{\hom}(G+\nabla v(x))\,\mathrm{d}x+o(\big|G^2\big|).$$

The functional $\mathcal{I}_{\hom}$ is (as a $\Gamma(L^p)$-limit) lower semicontinuous w. r. t. strong convergence in $L^p(D)$. Hence, $W_{\hom}$ is $W^{1,p}$-quasiconvex, and

$$\inf_{v\in W^{1,p}_0(D)}\int_D W_{\hom}(\operatorname{Id}+G+\nabla v(x))\,\mathrm{d}x=|D|\,W_{\hom}(\operatorname{Id}+G).$$

By convexity of $Q_{\hom}$, we also have

$$\inf_{v\in H^1_0(D)}\int_D Q_{\hom}(G+\nabla v(x))\,\mathrm{d}x=|D|\,Q_{\hom}(G).$$

This proves (4). □

The proof of Theorem 2.1 relies on the quantitative version of Proposition 2.3 (see Remark 6) and on a localization argument allowed by the $p$-growth condition.

*Proof of Theorem 2.1.* We split the proof into four steps.

*Step* 1. Localization of the energy $\mathcal{I}_\varepsilon$.
Let $\mathcal{B}$ denote the collection of all open balls contained in $D$, and define for all $B\in\mathcal{B}$ and all $u\in W^{1,p}(B)$ the localized functionals

$$\mathcal{I}_\varepsilon(u;B):=\int_B W_\varepsilon(x,\nabla u(x))\,\mathrm{d}x\qquad\text{and}\qquad\mathcal{I}^*(u;B):=\int_B W^*(x,\nabla u(x))\,\mathrm{d}x.$$

Since $W_\varepsilon$ satisfies the standard $p$-growth conditions, $\Gamma$-convergence is local (see [BD98, Theorem 12.5]), and for all $B\in\mathcal{B}$ the functionals $\mathcal{I}_\varepsilon(\cdot;B)$ $\Gamma(L^p)$-converge to $\mathcal{I}^*(\cdot;B)$.

*Step* 2. Localization of $\mathcal{E}_\varepsilon$.
We consider a subsequence (not relabeled) such that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to a functional $\mathcal{E}^*$ on $H^1(D)$. As in Step 1 of the proof of Proposition 2.3, $\mathcal{E}^*$ is of the form

$$\mathcal{E}^*(g) = \int_D Q^*(x, \nabla g(x))\,\mathrm{d}x$$

for some measurable map $Q^*$ from $D$ to $\mathcal{Q}_{\alpha''}$, where $\alpha''$ only depends on $\alpha$. Moreover, for all $B \in \mathcal{B}$ the localized functionals

$$\mathcal{E}_\varepsilon(g; B) := \int_B Q_\varepsilon(x, \nabla g(x))\,\mathrm{d}x$$

$\Gamma(L^2)$-converge on $H^1(B)$ to

$$\mathcal{E}^*(g; B) := \int_B Q^*(x, \nabla g(x))\,\mathrm{d}x.$$

*Step* 3. Characterization of $Q^*$.
For all $B \in \mathcal{B}$ and $G \in \mathbb{M}^d$ we define

$$\begin{aligned} Q_B(G) &:= \inf_{g \in H^1_0(B)} \int_B Q^*(x, G + \nabla g(x))\,\mathrm{d}x, \\ W_B(G) &:= \inf_{v \in W^{1,p}_0(B)} \int_B W^*(x, G + \nabla v(x))\,\mathrm{d}x. \end{aligned}$$

Since $\mathcal{I}^*(\cdot; B)$ is the $\Gamma(L^p)$-limit of the sequence $\mathcal{I}_\varepsilon(\cdot; B)$, the functional $\mathcal{I}^*(\cdot; B)$ is lower semicontinuous and its energy density $W^*$ satisfies a $p$-growth condition from below. Hence, infima converge, and we have:

$$W_B(G) = \lim_{\varepsilon \to 0} \left\{ \inf_{v \in W^{1,p}_0(B)} \mathcal{I}_\varepsilon(\varphi_G + v; B) \right\}, \tag{27}$$

which proves that (5) is well-defined. Since $B \in \mathcal{B}$ is of class $C^1$, we can apply Proposition 2.3 to each of the functionals $\mathcal{I}_\varepsilon(\cdot; B)$; and since each $B \in \mathcal{B}$ can be obtained by translation and dilation of the unit ball in $\mathbb{R}^d$, we deduce that there exist a constant $\alpha'$ and a modulus of approximation $\rho'$ (both only depending on $\alpha$, and on $\rho$), such that the following two properties are fulfilled: For all $B \in \mathcal{B}$ and $G \in \mathbb{M}^d$ there holds

$$\frac{1}{|B|} W_B \in \mathcal{W}_{\alpha', \rho'}, \tag{28}$$

$$\left| \tfrac{1}{|B|} W_B(\mathrm{Id} + G) - \tfrac{1}{|B|} Q_B(G) \right| \le \rho'(|G|)\,|G|^2. \tag{29}$$

In particular, (29) holds for all balls $B(x, r)$ with center $x \in D$ and sufficiently small radius $r > 0$. Because the l. h. s. of (29) is independent of $x$ and $r$, and since for almost every $x \in D$

$$\begin{aligned} &\lim_{r \to 0} \tfrac{1}{|B(x,r)|} W_{B(x,r)}(\mathrm{Id} + G) = W^*(x, \mathrm{Id} + G), \\ &\lim_{r \to 0} \tfrac{1}{|B(x,r)|} Q_{B(x,r)}(\mathrm{Id} + G) = Q^*(x, G), \end{aligned}$$

(see e.g. [DMM86a]), the estimate

$$|W^*(x, \mathrm{Id} + G) - Q^*(x, G)| \le \rho'(|G|)\,|G|^2 \tag{30}$$

holds for all $G \in \mathbb{M}^d$ and almost every $x \in D$. On the one hand, this implies that $W^*$ is of class $\mathcal{W}^p_{\alpha', \rho'}$. On the other hand, this proves that $Q^*$ can be characterized by

$$Q^*(x, G) := \lim_{h \to 0} \frac{1}{h^2} W^*(x, \mathrm{Id} + hG).$$

The limit on the r. h. s. does not depend on the extraction of Step 2, so that the entire sequence $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}^*$.

*Step* 4. Commutation diagram.
The proof of the diagram, which closely follows [MN, Section 6], is left to the reader. $\square$

## 3. Application to stochastic homogenization

Let us first recall a standard stochastic homogenization result (see the original contribution [DMM86b] in the convex case, and its generalization [MM94] to the quasiconvex case).

**Theorem 3.1.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\tau_z)_{z\in\mathbb{R}^d}$ be a strongly continuous measure-preserving ergodic translation group, and let $W : \mathbb{R}^d \times \mathbb{M}^d \times \Omega \to \mathbb{R}^+$ be a map such that*

*(i) $W$ is Lebesgue-measurable in its first variable,*
*(ii) $W$ is $\mathcal{F}$-measurable in its third variable,*
*(iii) $W(x, \cdot, \omega) \in \mathcal{W}^p_\alpha$ for $\mathbb{P}$-almost every $\omega \in \Omega$, almost every $x \in \mathbb{R}^d$ and some $p \in (1, \infty)$,*
*(iv) $W$ is stationary in the sense that for $\mathbb{P}$-almost every $\omega \in \Omega$, almost every $x \in \mathbb{R}^d$, every $F \in \mathbb{M}^d$ and every $z \in \mathbb{R}^d$*

$$W(x + z, F, \omega) \,=\, W(x, F, \tau_z\omega).$$

*Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the integral functional $\mathcal{I}_\varepsilon(\omega) : W^{1,p}(D) \to \mathbb{R}^+$ given for all $\varepsilon > 0$ by*

$$\mathcal{I}_\varepsilon(\omega)(u) \,=\, \int_D W(x/\varepsilon, \nabla u(x), \omega)\,\mathrm{d}x$$

*$\Gamma(L^p)$-converges, as $\varepsilon$ vanishes, to the integral functional $\mathcal{I}_{\mathrm{hom}} : W^{1,p}(D) \to \mathbb{R}^+$ given by*

$$\mathcal{I}_{\mathrm{hom}}(u) \,=\, \int_D W_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x,$$

*where the deterministic homogeneous-in-space energy density $W_{\mathrm{hom}}$ is quasiconvex, satisfies (W4) and the asymptotic formula*

$$W_{\mathrm{hom}}(F) \,=\, \lim_{R\to\infty} \frac{1}{R^d} \inf\left\{ \int_{(0,R)^d} W(x, F + \nabla\phi(x), \omega)\,\mathrm{d}x,\ \phi \in W^{1,p}_0((0,R)^d) \right\} \tag{31}$$

*for all $F \in \mathbb{M}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$.*

The combination of Theorems 2.2 & 3.1 yields

**Theorem 3.2.** *Let $W$ and $W_{\mathrm{hom}}$ be as in Theorem 3.1 and assume in addition that for some $p \geq 2$ and a modulus of approximation $\rho$*

$$W(x, \cdot, \omega) \in \mathcal{W}^p_{\alpha,\rho}$$

*for almost every $x \in \mathbb{R}^d$ and $\mathbb{P}$-almost every $\omega \in \Omega$. Let $Q$ denote the quadratic term of the Taylor expansion of $W$ at identity. Then*

*(a) the density $W_{\mathrm{hom}}$ is of class $\mathcal{W}^p_{\alpha',\rho'}$, with $\alpha'$ and $\rho'$ as in Theorem 2.1;*
*(b) the energy functionals*

$$\mathcal{E}_\varepsilon(\omega) :\ H^1(D) \to \mathbb{R}^+, \qquad u \mapsto \int_D Q_\varepsilon(x/\varepsilon, \nabla u(x), \omega)\,\mathrm{d}x$$

*$\Gamma(L^2)$-converge for $\mathbb{P}$-almost every $\omega \in \Omega$ to*

$$\mathcal{E}_{\mathrm{hom}} :\ H^1(D) \to \mathbb{R}, \qquad u \mapsto \int_D Q_{\mathrm{hom}}(\nabla u(x))\,\mathrm{d}x$$

*where $Q_{\rm hom}$ is the deterministic homogeneous-in-space quadratic energy density that is determined by the expansion*

$$\forall G \in \mathbb{M}^d \,:\, W_{\rm hom}(\mathrm{Id}+G) \;=\; Q_{\rm hom}(G)+o(|G|^2);$$

*(c) For $\mathbb{P}$-almost every $\omega \in \Omega$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}_{h,\varepsilon}(\omega) & \xrightarrow{(1)} & \mathcal{E}_\varepsilon(\omega) \\ {\scriptstyle (2)}\big\downarrow & & \big\downarrow{\scriptstyle (3)} \\ \mathcal{G}_{{\rm hom},h} & \xrightarrow[(4)]{} & \mathcal{E}_{\rm hom} \end{array}$$

*where $\mathcal{G}_{h,\varepsilon}(\omega)$ and $\mathcal{G}_{{\rm hom},h}$ denote the functionals from $H^1_0(D)$ to $[0,+\infty]$ defined as*

$$\mathcal{G}_{h,\varepsilon}(\omega)(g) := \frac{1}{h^2}\int_D W\Big(x/\varepsilon, \mathrm{Id}+h\nabla g(x), \omega\Big)\,\mathrm{d}x$$
$$\mathcal{G}_{{\rm hom},h}(g) := \frac{1}{h^2}\int_D W_{\rm hom}\Big(\mathrm{Id}+h\nabla g(x)\Big)\,\mathrm{d}x$$

*and (1),(4), and (2),(3) mean $\Gamma$-convergence in $H^1_0(D)$ with respect to the strong topology of $L^2(D)$ as $h \to 0$ and $\varepsilon \to 0$, respectively. Moreover, the families $(\mathcal{I}_\varepsilon(\omega))$ and $(\mathcal{E}_\varepsilon(\omega))$ are equi-coercive w. r. t. weak convergence in $H^1_0(D)$ (for $\mathbb{P}$-almost every $\omega\in\Omega$).*

*Proof.* Let $\mathcal{I}_\varepsilon(\omega)$ and $\mathcal{I}_{\rm hom}(\omega)$ be defined as in Theorem 3.1. Then $\mathcal{I}_\varepsilon(\omega)$ $\Gamma(L^p)$-converges to $\mathcal{I}_{\rm hom}(\omega)$ for $\mathbb{P}$-almost every $\omega\in\Omega$. Now, the statement is a direct consequence of Theorem 2.1 which applies for $\mathbb{P}$-almost every $\omega\in\Omega$. □

**Corollary 3.3.** *Within the notation and assumptions of Theorem 3.2, we also have*

$$Q_{\rm hom}(G) \;=\; \lim_{R\to\infty}\frac{1}{R^d}\inf\left\{\int_{(0,R)^d} Q(x, G+\nabla\phi(x),\omega)\,\mathrm{d}x,\ \phi\in H^1_0((0,R)^d)\right\} \tag{32}$$

*for all $G\in\mathbb{M}^d$, and for $\mathbb{P}$-almost every $\omega\in\Omega$, where*

$$Q(y,G,\omega) := \liminf_{h\to 0}\frac{W(y,\mathrm{Id}+hG,\omega)}{h^2}.$$

*Proof.* Once we know that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to $\mathcal{E}_{\rm hom}$, the uniform coercivity of $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\rm hom}$ implies the convergence of the infima, which yields the desired formula (32). The $\Gamma$-convergence result is either a consequence of Theorem 3.2 part (b), or of the $G$-convergence of the associated elliptic operator proved for instance in [JKO94, Section 12.3] (by definition, $Q(y,G,\omega)$ is stationary for the ergodic translation group). □

*Remark* 8. As can be easily seen, Theorem 3.2 holds as well in the almost-periodic case (see for instance [Bra85] or [BD98, Section 17.2]) and in variants of the stochastic case (see for instance [Glo08]).

## 4. Locality of the $\Gamma$-closure at identity

This section is devoted to the locality of the $\Gamma$-closure in $\mathcal{W}^p_{\alpha,\rho}$ at identity. Given $k$ homogeneous energy densities $\{W_i\}_{i\in\{1,\dots,k\}}\in\mathcal{W}^p_{\alpha,\rho}$, we are interested in characterizing the set of maps $W^*(x,\cdot)$ that can be reached as energy densities of $\Gamma(L^p)$-limits $\mathcal{I}^*: W^{1,p}(D)\to\mathbb{R}$

$$u\mapsto \mathcal{I}^*(u) \;=\; \int_D W^*(x,\nabla u(x))\,\mathrm{d}x, \tag{33}$$

of energy functionals $\mathcal{I}_{\chi^n}: W^{1,p}(D) \to \mathbb{R}$ of the form

$$u \mapsto \mathcal{I}_{\chi^n}(u) = \int_D \sum_{i=1}^k W_i(\nabla u(x))\chi_i^n(x)\,\mathrm{d}x \tag{34}$$

as $n$ goes to infinity. Above, $\chi^n \in L^\infty(D,\{0,1\}^k)$ denotes a vector field with $\sum_{i=1}^k \chi_i^n \equiv 1$ and satisfies $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D,[0,1]^k)$. The components of $\chi^n$ can be seen as the characteristic functions of the $k$ phases. Note that also in the limit we have $\sum_{i=1}^k \theta_i \equiv 1$.

The $\Gamma$-closure of $\{W_i\}_{i\in\{1,\dots,k\}}$ is said to be local if the set of such integrands $W^*(x,\cdot)$ for almost every $x \in D$ coincides with the closure for the pointwise convergence of the set of energy densities $W_{\text{hom}}$ obtained by periodic homogenization of mixtures of $\{W_i\}_{i\in\{1,\dots,k\}}$ in the proportions $\{\theta_i(x)\}_{i\in\{1,\dots,k\}}$. To turn this into a rigorous statement, let us recall some definitions related to periodic homogenization.

*Definition* 4 (see [Mül87] and [Bra85]). Let $1 < p < \infty$, and $W$ be a $U = (0,1)^d$ periodic, measurable density from $\mathbb{R}^d$ to $\mathcal{W}^p_\alpha$. The homogenized energy density associated with $W$ is denoted by $W_{\text{hom}}: \mathbb{M}^d \to \mathbb{R}$ and characterized by

$$W_{\text{hom}}(F) := \lim_{R\to\infty} \frac{1}{R^d} \inf\left\{ \int_{(0,R)^d} W(y, F + \nabla u(y))\,\mathrm{d}y, u \in W_0^{1,p}((0,R)^d) \right\}. \tag{35}$$

We are now in position to define the set of periodic homogenized energy densities.

*Definition* 5. Let $1 < p < \infty$, $\{W_i\}_{i\in\{1,\dots,k\}} \in \mathcal{W}^p_\alpha$, and $\theta \in [0,1]^k$ be such that $\sum_{i=1}^k \theta_i = 1$. We define the set of periodic homogenized energy densities associated with $\{W_i, \theta_i\}_{i\in\{1,\dots,k\}}$ as

$$\mathcal{P}_\theta = \Bigg\{ (W_\chi)_{\text{hom}}: \mathbb{M}^d \to \mathbb{R} \,:\, \exists \chi \in L^\infty(\mathbb{R}^d, \{0,1\}^k) \text{ such that } \chi \text{ is } U\text{-periodic with } \int_U \chi_i \,\mathrm{d}y = \theta_i \text{ and } (W_\chi)_{\text{hom}} \text{ is associated with } W_\chi : (y,F) \mapsto \sum_{i=1}^k W_i(F)\chi_i(y) \text{ through (35)} \Bigg\},$$

and its closure for the pointwise convergence by

$$\mathcal{G}_\theta = \left\{ W^* : \mathbb{M}^d \to \mathbb{R} \,:\, \exists (W_{\chi^n})_{\text{hom}} \in \mathcal{P}_\theta\,,\ (W_{\chi^n})_{\text{hom}} \to W^* \text{ pointwise} \right\}.$$

The definition of locality of the $\Gamma$-closure now reads:

*Definition* 6. Let $1 < p < \infty$, $\{W_i\}_{i\in\{1,\dots,k\}} \in \mathcal{W}^p_\alpha$. We say that the $\Gamma$-closure of $\{W_i\}_{i\in\{1,\dots,k\}}$ is local if and only if for every sequence $\chi^n \in L^\infty(D,\{0,1\}^k)$ with $\sum_{i=1}^k \chi_i^n \equiv 1$ and such that

- $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D,[0,1]^k)$,
- the functional $\mathcal{I}_{\chi^n}: W^{1,p}(D) \to \mathbb{R}$ defined in (34) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^*: W^{1,p}(D) \to \mathbb{R}$ defined in (33),

one has

$$W^*(x,\cdot) \in \mathcal{G}_{\theta(x)}$$

for almost every $x \in D$.

If the $k$ energy densities $\{W_i\}_{i\in\{1,\ldots,k\}}$ are convex functions, then the associated $\Gamma$-closure is local (see for instance [BB09, Theorem 5.1]). In the case of quasiconvex non-convex functions, the locality (or non-locality) of the $\Gamma$-closure is an open problem. In the specific case when $W_i \in \mathcal{W}^p_{\alpha,\rho}$ for all $i \in \{1,\ldots,k\}$, Theorem 2.2 allows us to prove that the $\Gamma$-closure is "local at identity". This notion is made precise by the following two definitions.

*Definition* 7. Let $2 \le p < \infty$, $\{W_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{W}^p_{\alpha,\rho}$, and $\theta \in [0,1]^k$ such that $\sum_{i=1}^k \theta_i = 1$. We define the set of periodic homogenized energy densities associated with $\{W_i,\theta_i\}_{i\in\{1,\ldots,k\}}$ at identity as

$$\mathcal{P}^{\mathrm{Id}}_\theta = \Bigg\{ W^* : \mathbb{M}^d \to \mathbb{R} \,:\, \exists (W_\chi)_{\mathrm{hom}} \in \mathcal{P}_\theta \qquad \text{such that } \big|W^*(\mathrm{Id}+G) - (W_\chi)_{\mathrm{hom}}(\mathrm{Id}+G)\big| = o(|G|^2) \Bigg\},$$

and its closure:

$$\mathcal{G}^{\mathrm{Id}}_\theta = \Bigg\{ W^* : \mathbb{M}^d \to \mathbb{R} \,:\, \text{there exists a sequence } (W_{\chi^n})_{\mathrm{hom}} \in \mathcal{P}_\theta \qquad \text{such that } \big|W^*(\mathrm{Id}+G) - \lim_{n\to 0}(W_{\chi^n})_{\mathrm{hom}}(\mathrm{Id}+G)\big| = o(|G|^2) \Bigg\}.$$

*Definition* 8. Let $2 \le p < \infty$, $\{W_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{W}^p_{\alpha,\rho}$. We say that the $\Gamma$-closure of $\{W_i\}_{i\in\{1,\ldots,k\}}$ is local at identity if and only if for every sequence $\chi^n \in L^\infty(D,\{0,1\}^k)$ with $\sum_{i=1}^k \chi_i^n \equiv 1$ and such that

- $\chi^n \rightharpoonup^* \theta$ weakly-* in $L^\infty(D,[0,1]^k)$,
- the functional $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ defined in (34) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$ defined in (33),

one has

$$W^*(x,\cdot) \in \mathcal{G}^{\mathrm{Id}}_{\theta(x)}$$

for almost every $x \in D$.

The above definition is a weakened version of the locality of the $\Gamma$-closure of Definition 6 obtained by restricting the property of approximation by periodic homogenized energy densities to a neighborhood of identity via a Taylor expansion. We have:

**Theorem 4.1.** *Let $2 \le p < \infty$ and $\{W_i\}_{i\in\{1,\ldots,k\}} \in \mathcal{W}^p_{\alpha,\rho}$, then the $\Gamma$-closure of $\{W_i\}_{i\in\{1,\ldots,k\}}$ is local at identity.*

*Proof.* By [BB09, Theorem 3.5], it is enough to prove the locality property in the so-called homogeneous case, that is with a repartition function $\chi^n \in L^\infty(D,\{0,1\}^k)$ such that

- $\chi^n$ weakly-* converges to a constant function $\theta$ in $L^\infty(D,[0,1]^k)$,
- the functional $\mathcal{I}_{\chi^n} : W^{1,p}(D) \to \mathbb{R}$ defined in (34) $\Gamma(L^p)$-converges to the functional $\mathcal{I}^* : W^{1,p}(D) \to \mathbb{R}$ defined in (33), where $W^*$ does not depend on the space variable.

Let $\mathcal{E}_{\chi^n} : H^1(D) \to \mathbb{R}^+$ denote the quadratic energy functional associated with $\mathcal{I}_{\chi^n}$, that is

$$\mathcal{E}_{\chi^n}(u) := \int_D \sum_{i=1}^k Q_i(\nabla u(x))\chi_i^n(x)\,\mathrm{d}x$$

where $Q_i \in \mathcal{Q}_{\alpha'}$ denotes the quadratic form associated with $W_i$ through (W3). We then apply Theorem 2.1 and deduce that $\mathcal{E}_\varepsilon$ $\Gamma(L^2)$-converges to

$$\mathcal{E}^* : H^1(D) \to [0,+\infty), \qquad \mathcal{E}^*(u) := \int_D Q^*(\nabla u(x))\,\mathrm{d}x,$$

where $Q^* \in \mathcal{Q}_{\hat\alpha}$ for some $\hat\alpha > 0$, and is characterized by the expansion

$$W^*(\mathrm{Id}+G) = Q^*(G) + o(|G|^2). \tag{36}$$

Next, we appeal to the locality of the $\Gamma$-closure for convex linear problems. In particular, there exists a $U$-periodic sequence $\tilde\chi^n \in L^\infty(\mathbb{R}^d, \{0,1\}^k)$ satisfying $\int_Q \tilde\chi_i^n(y)\,\mathrm{d}y = \theta_i$ for all $n \in \mathbb{N}$ and all $i \in \{1,\cdots,k\}$, and such that the homogenized quadratic functions $\widetilde{Q}^n_{\mathrm{hom}}$ associated with the periodic quadratic energy densities $\widetilde{Q}^n : \mathbb{R}^d \times \mathbb{M}^d \to \mathbb{R}$

$$\widetilde{Q}^n : (y,G) \mapsto \sum_{i=1}^k Q_i(G)\tilde\chi_i^n(y)$$

approximate $Q^*$ in the sense that for all $G \in \mathbb{M}^d$,

$$\lim_{n\to\infty} Q^n_{\mathrm{hom}}(G) \;=\; Q^*(G). \tag{37}$$

We are now in position to prove the claim. To this aim, we define a sequence of periodic integrands $\widetilde{W}^n : \mathbb{R}^d \times \mathbb{M}^d \to \mathbb{R}$ as

$$\widetilde{W}^n : (y,G) \mapsto \sum_{i=1}^k W_i(G)\tilde\chi_i^n(y).$$

With this sequence of periodic integrands we associate a sequence of homogenized integrands $W^n_{\mathrm{hom}}$ through (35) with $\widetilde{W}^n$ in place of $W$. Combined with standard periodic homogenization results (see for instance [Mül87], [Bra85] or [BD98, Section 14.2]), Theorem 2.1 then shows that

$$\frac{|W^n_{\mathrm{hom}}(\mathrm{Id}+G) - Q^n_{\mathrm{hom}}(G)|}{|G|^2} \;\leq\; \rho'(|G|),$$

and the theorem follows from (36), (37), and the uniformity of the validity of the Taylor expansion since for all $n$, $W^n_{\mathrm{hom}}$ and $W^*$ are of class $\mathcal{W}^p_{\alpha',\rho'}$ for the same function $\rho'$. □

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## Appendix A. Proof of Proposition 2.4

The proof is divided in three steps. In the first two steps we prove the statement for a fixed domain $D$ by combining Meyers' estimate and the Lipschitz truncation argument of [FJM02, Proposition A.3]. We then prove in the third step that the constants can be chosen only depending on $\alpha$ and on the geometry of $D$.

*Step* 1. Control of energy differences.
Let $g^* \in H^1_0(D)$ denote the unique minimizer of the functional $\mathcal{E}_G$ on $H^1_0(D)$. We claim that for all $g \in H^1_0(D)$ there holds

$$\int_D Q(x, G+\nabla g)\,\mathrm{d}x - \int_D Q(x, G+\nabla g^*)\,\mathrm{d}x \leq \alpha\,\|\nabla g - \nabla g^*\|^2_{L^2(D)}\,. \tag{38}$$

To prove this we first expand the formula for $Q(x, \nabla g - \nabla g^*)$ (use (18) with $A = G + \nabla g^*$ and $B = \nabla g - \nabla g^*$):

$$Q(x, G + \nabla g) - Q(x, G + \nabla g^*) = Q(x, \nabla g - \nabla g^*) + \langle \mathbb{L}(x)(G + \nabla g^*), \nabla g - \nabla g^* \rangle.$$

We integrate this identity over $D$ and note that the second term on the r. h. s. vanishes as the first variation of the minimization problem characterizing $g^*$. Thus, (38) follows from the fact that $Q(x, \cdot) \in \mathcal{Q}_\alpha$ for a.e. $x \in D$.

*Step* 2. Proof of the claim for a fixed domain.
We assert that there exist $C > 0$ and $\mu > 0$ such that for all $\lambda > 0$ there exists a map $g \in W_0^{1,\infty}(D)$ that satisfies

$$\|g\|_{W^{1,\infty}(D)} \leq \lambda, \tag{39}$$

$$\|\nabla g - \nabla g^*\|_{L^2(D)}^2 \leq C\lambda^{-\mu} \|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}. \tag{40}$$

We construct the map as follows. By Meyers' estimate (see Proposition 2.4 part (b)), there exist $C > 0$ and $\mu > 0$ depending only on $\alpha$ and $D$ such that for all $G \in \mathbb{M}^d$,

$$\|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu} \leq C\,|D|\,|G|^{2+\mu}.$$

Hence, [FJM02, Proposition A.2] yields a map $g \in W_0^{1,\infty}(D)$ that satisfies (39), and the estimate

$$|D_\lambda| \leq \frac{C}{\lambda^{2+\mu}} \|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}, \qquad D_\lambda := \{\, x \in D \,:\, g(x) \neq g^*(x) \,\} \tag{41}$$

for some $C$ independent of $\lambda$ and $g^*$. Let us prove that $g$ also satisfies (40). From Hölder's inequality with exponents $(\frac{2+\mu}{2}, \frac{2+\mu}{\mu})$, we have

$$\|\nabla g - \nabla g^*\|_{L^2(D)}^2 = \int_{D_\lambda} |\nabla g - \nabla g^*|^2 \,\mathrm{d}x \leq |D_\lambda|^{\frac{\mu}{2+\mu}} \left( \int_D |\nabla g - \nabla g^*|^{2+\mu} \,\mathrm{d}x \right)^{\frac{2}{2+\mu}}.$$

On the one hand, the combination of (39) and (41) yields

$$\begin{aligned}
\int_D |\nabla g - \nabla g^*|^{2+\mu} \,\mathrm{d}x &= \int_{D_\lambda} |\nabla g - \nabla g^*|^{2+\mu} \,\mathrm{d}x \\
&\leq C \left( \int_{D_\lambda} |\nabla g|^{2+\mu} \,\mathrm{d}x + \int_D |\nabla g^*|^{2+\mu} \,\mathrm{d}x \right) \\
&\leq C \left( |D_\lambda|\,\lambda^{2+\mu} + \int_D |\nabla g^*|^{2+\mu} \,\mathrm{d}x \right) \\
&\leq C \|\nabla g^*\|_{L^{2+\mu}(D)}^{2+\mu}.
\end{aligned}$$

On the other hand, (41) also implies

$$|D_\lambda|^{\frac{\mu}{2+\mu}} \leq C\lambda^{-(2+\mu)\frac{\mu}{2+\mu}} \|\nabla g^*\|_{L^{2+\mu}(D)}^{(2+\mu)\frac{\mu}{2+\mu}} = C\lambda^{-\mu} \|\nabla g^*\|_{L^{2+\mu}(D)}^{\mu}.$$

Estimate (39) follows from these last three inequalities.

*Step* 3. Dependence of the constants on $D$.
Step 2 provides a function $g$ satisfying the desired properties with some constants $\mu$ and $C$ which only depend on $\alpha$ and the domain $D$. Let us quickly show that both constants can be chosen invariant under dilations, translations, and rotations of $D$. Assume that $D$ is a translated, rotated, and dilated version of some reference domain $D_0$, i.e. $D := \tau + rRD_0$ with $\tau \in \mathbb{R}^d$, $R \in SO(d)$, and $r > 0$. We shall prove that $C$ and $\mu$ only depend on $\alpha$ and $D_0$. To this end we set

$$g_0^*(x) := \frac{1}{r} g^*(\tau + rRx) \quad \text{and} \quad Q_0(x, G) := Q(\tau + rRx, G).$$

Then $g_0^*$ is the unique minimizer of

$$H^1_0(D_0) \ni g_0 \mapsto \int_{D_0} Q_0(x, G + \nabla g_0)\,\mathrm{d}x.$$

For all $\lambda > 0$, Step 2 yields a map $g_0 \in W^{1,\infty}_0(D_0)$ with

$$\begin{aligned} &|\nabla g_0(x)| \leq \lambda \qquad \text{for a.e. } x \in D_0, \\ &\int_{D_0} Q_0(x, G + \nabla g_0(x))\,\mathrm{d}x - \int_{D_0} Q_0(x, G + \nabla g_0^*(x))\,\mathrm{d}x \quad \leq C_0 \lambda^{-\mu_0} |D_0|\,|G|^{2+\mu_0}, \end{aligned}$$

where $\mu_0$ and $C_0$ only depend on $\alpha$ and $D_0$. A simple change of variables shows that the map

$$g(x) := r g_0\left(R^{-1}\frac{x-\tau}{r}\right)$$

satisfies (39) & (40) with $\mu = \mu_0$ and $C = C_0$.

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STEFAN NEUKAMM, MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22-26, D-04103 LEIPZIG, GERMANY
*E-mail address*: neukamm@mis.mpg.de

ANTOINE GLORIA, PROJECT-TEAM SIMPAF, INRIA LILLE - NORD EUROPE, VILLENEUVE D'ASCQ, FRANCE
*E-mail address*: antoine.gloria@inria.fr