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by

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Abstract

We will give a pure combinatorial proof of the Eisenbud-Goto conjecture for arbitrary monomial curves. In addition to this, we show that the conjecture holds for certain simplicial affine semigroup rings.

1 Introduction

Let S be a homogeneous simplicial affine semigroup, i. e., (up to isomorphism) S is the submonoid of $(\mathbb{N}^d,+)$ generated by a set $A := \{e_1,\ldots,e_d,a_1,\ldots,a_c\} \subset \mathbb{N}^d$, where

$$e_1 := (\alpha,0,\ldots,0), e_2 := (0,\alpha,0,\ldots,0),\ldots,e_d := (0,\ldots,0,\alpha),$$

$$a_i = (a_{i[1]},\ldots,a_{i[d]}),\ \text{with } a_{i[1]}+\ldots+a_{i[d]} = \alpha,\ i = 1,\ldots,c.$$

Further we assume that the integers $a_{i[j]}$, $i = 1,\ldots,c$, $j = 1,\ldots,d$ are relatively prime and we assume that $d \geq 2, c \geq 1$ and $\alpha \geq 2$. Let K be an arbitrary field; by $K[S]$ we denote the affine semigroup ring of S and we identify the ring $K[S]$ with the subring of the polynomial ring $K[t_1,\ldots,t_d]$ generated by monomials $t^a := t_1^{a_{[1]}}\cdots t_d^{a_{[d]}}$, for $a = (a_{[1]},\ldots,a_{[d]}) \in S$. In the following we study the $\mathbb{Z}$-grading on $K[S]$ which is induced by $\deg t^a = (\sum_{i=1}^d a_{[i]})/\alpha$. We note that $\dim K[S] = d$. By $R := K[x_1,\ldots,x_{d+c}]$ we denote the standard-graded polynomial ring over K, i. e., $\deg x_i = 1$ for all $i = 1,\ldots,d+c$. Thus, we have a $\mathbb{Z}$-graded surjective K-algebra homomorphism:

$$\pi : K[x_1,\ldots,x_{d+c}] \to K[S],$$

given by $x_i \mapsto t_i^\alpha,\ i = 1,\ldots,d$ and $x_{d+j} \mapsto t^{a_j},\ j = 1,\ldots,c$. Hence $K[S] \cong R/\ker\pi$, where $\ker\pi$ is a homogeneous prime ideal of R. Let m_R denote the maximal homogeneous ideal of R. For a graded R-module M, we set $a(M) := \max\{n \mid M_n \neq 0\}$ with $a(M) := -\infty$ if $M = 0$. As usual the Castelnuovo-Mumford regularity $\mathrm{reg}K[S]$ of $K[S]$ is defined by

$$\mathrm{reg}K[S] := \max\left\{i + a(H^i_{m_R}(K[S])) \mid 0 \leq i \leq \dim K[S]\right\}.$$

Since the Eisenbud-Goto conjecture [2] is widely open in general, it would be nice to answer the following:

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Question (Eisenbud-Goto)**.** Does $\text{reg}K[S] \leq \text{deg}K[S] - \text{codim}K[S]$ hold?

Where $\text{codim}K[S] := \dim_K K[S]_1 - \dim K[S] = c$ and $\deg K[S]$ denotes the multiplicity of $K[S]$. By a result of Treger [18] the question has a positive answer if $K[S]$ is Cohen-Macaulay; the Buchsbaum case was proven by Stückrad and Vogel in [17]. For projective monomial curves, i. e., $d = 2$, the Eisenbud-Goto conjecture holds by a result of Gruson Lazarsfeld and Peskine [5]. The case $c = 2$ was proven by Peeva and Sturmfels in [16]. Moreover, in [7], Herzog and Hibi showed that the Eisenbud-Goto conjecture holds for (homogeneous) simplicial affine semigroup rings with isolated singularity. In addition to this the question has a positive answer if the ring $K[S]$ is seminormal, see [14]. We also refer to the paper of Lazarsfeld [10] for a proof of the Eisenbud-Goto conjecture for smooth surfaces in characteristic zero. In [8, Theorem 3.2] Hoa and Stückrad presented a very good bound for the regularity of $K[S]$; in addition to this they provided some positive answers for the Eisenbud-Goto conjecture. However, the Eisenbud-Goto conjecture is still widely open even for simplicial affine semigroup rings.

In case that $\dim K[S] = 2$ there are much better bounds than $\alpha - c$, in [9] L'vovsky showed that the regularity of $K[S]$ is bounded by $\#L + \#L' + 1$, where L and L' are the longest and the second longest gap of S. If we further assume that $(1, \alpha-1), (\alpha-1, 1) \in S$ we even get a better bound, namely $\text{reg}K[S] \leq \#L+1$ where L is the longest gap of S, by a result of Hellus, Hoa, and Stückrad [6]. For further details we refer to [6, Introduction]. However, the combinatorial bound in [6] needs the assumption that the corresponding monomial curve is smooth; it should be mentioned that even this bound is far from sharp for $c \geq 4$ (see [6, 13]). Moreover, in [10], Giaimo showed that the Eisenbud-Goto conjecture still holds for connected reduced curves.

In [8], Hoa and Stückrad introduced a decomposition of the ring $K[S]$ into a direct sum of certain monomial ideals. By using this they were able to show that the regularity of $K[S]$ is bounded by $d(\deg K[S] - c - 2) + 2$, provided that $\deg K[S] \geq c + 2$, see [8, Theorem 3.5]. Recently in [14] the author used this decomposition to prove the conjecture in the seminormal case. We will again use this idea to give a combinatorial proof of the Eisenbud-Goto conjecture for monomial curves in Theorem 4.14; unfortunately our proof does not yield the L'vovsky bound (see Remark 4.15). In Section 3 we will prove the conjecture in case that all monomial ideals in the decomposition are generated by at most two elements for arbitrary d. In Section 2 we will again recall the construction of the decomposition of the ring $K[S]$, moreover, we will develop the main tools which are needed to prove the assertions in Section 3 and in Section 4. For unspecified notation we refer to [1, 12].

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2 Basics

Let $G := G(S)$ be the group generated by S in $\mathbb{Z}^d$. By $x_{[i]}$ we denote the i-th component of x and $\deg x := (\sum_{j=1}^{d} x_{[j]})/\alpha$, for $x \in G$. We set $B_S := \{x \in S \mid x - e_j \notin S, \forall j = 1, \dots, d\}$. We note that if $x \notin B_S$, then $x + y \notin B_S$ for all $x, y \in S$. We define $x \sim y$ if $x - y \in \alpha\mathbb{Z}^d$, hence $\sim$ is an equivalence relation on G. It is obvious that every element in G is equivalent to an element in $G \cap D$, where $D := \{(x_{[1]}, \dots, x_{[d]}) \in \mathbb{Q}^d \mid 0 \leq x_{[i]} < \alpha, \forall i\}$ and for all $x, y \in G \cap D$ with $x \neq y$ we have $x \not\sim y$. Hence the number of equivalence classes $f := \#(G \cap D)$ in G is finite. One can show that there are exactly f equivalence classes in G, $G \cap D$, S, and in B_S. By $\Gamma_1, \dots, \Gamma_f$ we denote the equivalence classes on B_S. For $t = 1, \dots, f$ we define

$$h_t := (\min\{m_{[1]} \mid m \in \Gamma_t\}, \min\{m_{[2]} \mid m \in \Gamma_t\}, \dots, \min\{m_{[d]} \mid m \in \Gamma_t\}).$$

Let $T := K[y_1, \dots, y_d]$ be the polynomial ring graded by $\deg y_i = 1$ for all $i = 1, \dots, d$. We set $\tilde{\Gamma}_t := \{y^{(x-h_t)/\alpha} \mid x \in \Gamma_t\}$, where $y^{(a_{[1]}, \dots, a_{[d]})} := y_1^{a_{[1]}} \cdots y_d^{a_{[d]}}$, for $(a_{[1]}, \dots, a_{[d]}) \in \mathbb{N}^d$. By construction $I_t := \tilde{\Gamma}_t T$ are monomial ideals in T, since $h_t \sim x$ for all $x \in \Gamma_t$. We note that $\mathrm{height} I_t \geq 2$, since $\gcd \tilde{\Gamma}_t = 1$, for all $t = 1, \dots, f$. We define m_T as the homogeneous maximal ideal of T and m_S as the homogenous maximal ideal of $K[S]$ (see [8, Section 2]).

Proposition 2.1 ([8, Proposition 2.2])**.** *There are isomorphisms of $\mathbb{Z}$-graded T-modules:*

1.) $K[S] \cong \bigoplus_{t=1}^{f} I_t(-\deg h_t)$.

2.) $H^i_{m_S}(K[S]) \cong \bigoplus_{t=1}^{f} H^i_{m_T}(I_t)(-\deg h_t)$ *for all* $i \geq 0$.

Applying the fact $H^i_{m_R}(K[S]) \cong H^i_{m_S}(K[S])$ we have:

$$\mathrm{reg} K[S] = \max\{\mathrm{reg} I_t + \deg h_t \mid t = 1, \dots, f\}, \tag{1}$$

where $\mathrm{reg} I_t$ is the regularity of I_t considered as a $\mathbb{Z}$-graded T-module.

Remark 2.2. After a talk of the author given in Berkeley, David Eisenbud and Janko Böhm have written the Macaulay2 package MonomialAlgebras.m2. In this package they consider the case of arbitrary affine semigroups $Q' \subseteq Q \subseteq \mathbb{N}^d$ such that $K[Q]$ is finite over $K[Q']$; the package is able to decompose the ring $K[Q]$ as a direct sum of monomial ideals in $K[Q']$ (see [8, Proposition 2.2] and [15, Proposition 4.1] for results in the simplicial case). We refer to the Macaulay2 homepage [4], where the package should appear soon.

Definition 2.3. Let $x, y \in S$. We define $x \geq y$ if $x_{[k]} \geq y_{[k]}$ for all $k = 1, \dots, d$. Moreover, we say that $x > y$ if $x \geq y$ and there is at least one $k \in \{1, \dots, d\}$ such that $x_{[k]} > y_{[k]}$.

Remark 2.4. By Proposition 2.1 it follows that $\deg K[S] = f$. Since $\Gamma_t \subset B_S$, we have $\Gamma_t \subset \langle a_1, \dots, a_c\rangle$ for all $t = 1, \dots, f$. Moreover, it is clear that $\{0, a_1, \dots, a_c\} \subseteq B_S$. Consider an element $x \in \{0, a_1, \dots, a_c\}$ and an element $y \in B_S$ with $x \neq y$. Suppose that $x \sim y$. Since $0 \leq x_{[i]} < \alpha$ for all $i = 1, \dots, d$ we have $y \geq x$ and therefore $y \notin B_S$. This shows that $x \not\sim y$. Without loss of generality we therefore may assume that $\Gamma_1 = \{0\}, \Gamma_2 = \{a_1\}, \dots, \Gamma_{c+1} = \{a_c\}$.

Definition 2.5. For an element $x \in S$ we say that a sequence $\lambda = (b_1, \dots, b_n)$ has $*$-property if $b_1, \dots, b_n \in \{a_1, \dots, a_c\}$ and $x - b_1 \in S, x - b_1 - b_2 \in S, \dots, x - (\sum_{j=1}^{n} b_j) \in S$; we say that the length of λ is n. Let $\lambda = (b_1, \dots, b_n)$ be a sequence with $*$-property; we define $x(\lambda, i) := x - (\sum_{j=1}^{i} b_j)$ for $i = 1, \dots, n$ and $x(\lambda, 0) := x$. By Λ_x we denote the set of all sequences with $*$-property of x with length $\deg x$.

Remark 2.6. Assume that $x \in S$ has a sequence $\lambda = (b_1, \ldots, b_n)$ with $*$-property. Then we get $\deg x(\lambda, i) = \deg x - i$ for $i = 0, \ldots, n$ and therefore $x(\lambda, \deg x) = 0$ for $n = \deg x$. Hence the length of a sequence with $*$-property of x is bounded by $\deg x$. Moreover, for $0 \le i \le j \le n$, we have $x(\lambda, i) \ge x(\lambda, j)$. There are elements in S with no sequence with $*$-property, e. g., $\Lambda_{e_j} = \emptyset$. We note that the set Λ_x is always finite.

Proposition 2.7 ([14, Proposition 2.5])**.** *Let $x \in B_S \setminus \{0\}$.*

1) $\Lambda_x \neq \emptyset$.
2) *Let $(b_1, \ldots, b_n)$ be a sequence with $*$-property of x. Then there exists a sequence with $*$-property $(b_1, \ldots, b_n, b_{n+1}, \ldots, b_{\deg x}) \in \Lambda_x$.*

Definition 2.8. Let $\lambda = (b_1, b_2, \ldots, b_n)$ be a sequence with $*$-property of x. We define $\lambda^* := (b_n, b_{n-1}, \ldots, b_1)$ as the trivial permutation of λ.

Proposition 2.9 ([14, Proposition 2.6])**.** *Let $x \in S$ and $\lambda = (b_1, \ldots, b_n)$ be a sequence with $*$-property of x. Let $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijection.*

1) *$(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$ is a sequence with $*$-property of x, in particular λ^* has $*$-property.*
2) *$(b_1, \ldots, b_m)$ is a sequence with $*$-property of x for all $1 \le m \le n$.*

Lemma 2.10. *Let $x \in B_S \setminus \{0\}$ and $\lambda = (b_1, \ldots, b_{\deg x}) \in \Lambda_x$. Then*

1) *$x(\lambda, i) \in B_S$ for all $i = 0, \ldots, \deg x$.*
2) *We have $x(\lambda, i) \not\sim x(\lambda, j)$ for all $0 \le i < j \le \deg x$.*
3) *$x - x(\lambda, i) = x(\lambda^*, \deg x - i)$ for all $i = 0, \ldots, \deg x$.*

Proof. 1) and 2) can be found in [14, Lemma 2.7]. We have

$$x - x(\lambda, i) = x - (x - \sum_{j=1}^{i} b_j) = \sum_{j=1}^{i} b_j = x - \sum_{j=1}^{\deg x - i} b_{\deg x + 1 - j} = x(\lambda^*, \deg x - i).$$

□

Theorem 2.11 ([8, Theorem 1.1])**.** *We have $\deg x \le \deg K[S] - \operatorname{codim} K[S]$ for all $x \in B_S$.*

We also refer to [14, Corollary 2.8] for a proof of Theorem 2.11 in our notation.

Definition 2.12. Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. We define

1. $\Delta(\lambda, \nu) := \{(i, j) \in \mathbb{N}^2 \mid x(\lambda, i) \sim y(\nu, j), 0 \le i \le \deg x, 0 \le j \le \deg y\}$ and
2. $\delta(\lambda, \nu) := \#\Delta(\lambda, \nu) - 2$.

Definition 2.13. Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, we define the number $\delta(x, y)$ by:

$$\delta(x, y) := \min_{\lambda \in \Lambda_x, \nu \in \Lambda_y} \delta(\lambda, \nu).$$

Definition 2.14. Let $x, y \in S$ with $x \sim y$. We define $h(x, y) \in G$ by:

$$h(x, y) := (\min\{x_{[1]}, y_{[1]}\}, \min\{x_{[2]}, y_{[2]}\}, \ldots, \min\{x_{[d]}, y_{[d]}\}).$$

Remark 2.15. Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. We always have $(0,0), (\deg x, \deg y) \in \Delta(\lambda, \nu)$, since $x(\lambda, 0) \sim y(\nu, 0)$ and $x(\lambda, \deg x) \sim y(\nu, \deg y)$. Hence $\delta(\lambda, \nu) \geq 0$ and $\delta(x, y) \geq 0$. Moreover, if $(i, j) \in \Delta(\lambda, \nu)$, then $(i, k) \notin \Delta(\lambda, \nu)$ for all $k \in \{0, \ldots, \deg y\} \setminus \{j\}$ by Lemma 2.10, since otherwise $y(\nu, k) \sim y(\nu, j)$ for $k \neq j$. This argument shows that $\#\Delta(\lambda, \nu) \leq \min\{\deg x, \deg y\} + 1$.

Conjecture 2.16. *Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$. Then $\delta(x, y) \leq \deg h(x, y) - 1$.*

Example 2.17. Consider the semigroup $S = \langle (30, 0), (0, 30), (3, 27), (23, 7) \rangle$. We have $x = (27, 243), y = (207, 63) \in B_S$ and $x - y = (-180, 180) \in 30\mathbb{Z}^2$, hence $x \sim y$. Clearly $\Lambda_x = \{((3, 27), \ldots, (3, 27))\} = \{\lambda\}$ and $\Lambda_y = \{((23, 7), \ldots, (23, 7))\} = \{\nu\}$. Moreover, we have $\delta(x, y) = 2$, since $\Delta(\lambda, \nu) = \{(0, 0), (3, 3), (6, 6), (9, 9)\}$ and $\#\Lambda_x = \#\Lambda_y = 1$. Moreover, $h(x, y) = (27, 63)$, hence $\deg h(x, y) = 3$. In this case $\delta(x, y) = 2 = 3 - 1 = \deg h(x, y) - 1$, i. e., Conjecture 2.16 holds and is sharp.

Remark 2.18. Let $x \in B_S \setminus \{0\}$. It is often useful to illustrate a sequence with $*$-property $\lambda \in \Lambda_x$ as a graph, where the set of vertices is a subset of $\{x(\lambda, i) \mid i \in \{0, \ldots, \deg x\}\}$. Let $x(\lambda, i)$ and $x(\lambda, j)$ be two vertices; there is an edge between $x(\lambda, i)$ and $x(\lambda, j)$ if $j > i$ and there is no vertex $x(\lambda, k)$ with $j > k > i$. Moreover, x and 0 will always be vertices. So Example 2.17 can be illustrated by the graph

$$x \;\text{———}\; x(\lambda, 3) \;\text{———}\; x(\lambda, 6) \;\text{———}\; x(\lambda, 9) = 0,$$

and by the graph

$$y \;\text{———}\; y(\nu, 3) \;\text{———}\; y(\nu, 6) \;\text{———}\; y(\nu, 9) = 0.$$

To get a better understanding and to avoid extensive writing we will illustrate these situations by:

$$\begin{array}{ccccccc} x & \text{———} & x(\lambda, 3) & \text{———} & x(\lambda, 6) & \text{———} & x(\lambda, 9) = 0 \\ \wr & & \wr & & \wr & & \wr \\ y & \text{———} & y(\nu, 3) & \text{———} & y(\nu, 6) & \text{———} & y(\nu, 9) = 0, \end{array}$$

where the sidled lines denote equivalent elements. **Sidled lines always denote equivalent elements, though equivalent elements may not be illustrated in such a picture.**

Definition 2.19. Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$.

1. Let $(i, j), (i', j') \in \Delta(\lambda, \nu)$. We define a partial order $\leq$ on $\Delta(\lambda, \nu)$ by $(i, j) \leq (i', j')$ if $i \leq i'$ and $j \leq j'$.
2. We say that λ and ν are *crossless* if $(\Delta(\lambda, \nu), \leq)$ is a totally ordered set, meaning for all $(i, j), (i', j') \in \Delta(\lambda, \nu)$ we have $(i, j) \leq (i', j')$ or $(i, j) \geq (i', j')$.
3. We say that x and y are *crossless* if there exist sequences with $*$-property $\lambda' \in \Lambda_x$ and $\nu' \in \Lambda_y$ which are crossless.

Remark 2.20. We note that x and x are crossless, since we may choose the same $\lambda \in \Lambda_x$, in particular $\Delta(\lambda, \lambda) = \{(0, 0), (1, 1), \ldots, (\deg x, \deg x)\}$, i. e., $\#\Delta(\lambda, \lambda) = \deg x + 1$.

Example 2.21. Note that x and y in Example 2.17 are crossless. Unfortunately this property does not hold in general. Consider the semigroup $S = \langle (79,0),(0,79),(77,2),(34,45) \rangle$. For $x = (1232,32), y = (442,585) \in B_S$ with $x \sim y$, $\Lambda_x = \{((77,2),\ldots,(77,2))\} = \{\lambda\}$, and $\Lambda_y = \{((34,45),\ldots,(34,45))\} = \{\nu\}$. We have $\Delta(\lambda,\nu) = \{(0,0),(5,9),(11,4),(16,13)\}$. This situation can be illustrated by:

$$\begin{array}{ccccccc} x & \text{———} & x(\lambda,5) & \text{———} & x(\lambda,11) & \text{———} & 0 \\ \wr & & & \times & & & \wr \\ y & \text{———} & y(\nu,4) & \text{———} & y(\nu,9) & \text{———} & 0, \end{array}$$

i. e., λ and ν are not crossless and therefore x and y are not crossless, since $\#\Lambda_x = \#\Lambda_y = 1$. Moreover, we have $\delta(\lambda,\nu) = \delta(x,y) = 2$ and $\deg h(x,y) = \deg(442,32) = 6$, i. e., Conjecture 2.16 holds.

Remark 2.22. Let $x \in B_S \setminus \{0\}, \lambda = (b_1,\ldots,b_{\deg x}) \in \Lambda_x$, and $i \in \{1,\ldots,\deg x - 1\}$, i. e.,

$$x \text{ ——— } x(\lambda,i) \text{ ——— } 0.$$

Then $(b_1,\ldots,b_i) \in \Lambda_{x(\lambda^*,\deg x - i)}$, since $x(\lambda^*,\deg x - i) = \sum_{j=1}^{i} b_j$; moreover, we have $(b_{i+1},\ldots,b_{\deg x}) \in \Lambda_{x(\lambda,i)}$, since $x(\lambda,i) = \sum_{j=1}^{\deg x - i} b_{i+j}$. Let $B, C \subseteq \mathbb{N}^d$. We define the set $B + C := \{b + c \mid b \in B, c \in C\} \subseteq \mathbb{N}^d$ with the usual addition of tuples.

Lemma 2.23. *Let $x,y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda = (b_1,\ldots,b_{\deg x}) \in \Lambda_x$, and $\nu = (g_1,\ldots,g_{\deg y}) \in \Lambda_y$ with $\delta(\lambda,\nu) > 0$, i. e.,*

$$\begin{array}{ccccc} x & \text{———} & x(\lambda,i) & \text{———} & 0 \\ \wr & & \wr & & \wr \\ y & \text{———} & y(\nu,k) & \text{———} & 0, \end{array}$$

for some $i \in \{1,\ldots,\deg x - 1\}$ and some $k \in \{1,\ldots,\deg y - 1\}$. Let $x' = x(\lambda^*,\deg x - i)$, $x'' = x(\lambda,i)$, $y' = y(\nu^*,\deg y - k)$, and $y'' = y(\nu,k)$. Moreover, let $\lambda' = (b_1,\ldots,b_i) \in \Lambda_{x'}$, $\lambda'' = (b_{i+1},\ldots,b_{\deg x}) \in \Lambda_{x''}$, $\nu' = (g_1,\ldots,g_k) \in \Lambda_{y'}$, and $\nu'' = (g_{k+1},\ldots,g_{\deg y}) \in \Lambda_{y''}$. We have:

1) $x(\lambda^*,\deg x - i) \sim y(\nu^*,\deg y - k)$.
2) $\Delta(\lambda',\nu') = \{(m,n) \in \Delta(\lambda,\nu) \mid (m,n) \leq (i,k)\}$.
3) $\{(i,k)\} + \Delta(\lambda'',\nu'') = \{(m,n) \in \Delta(\lambda,\nu) \mid (m,n) \geq (i,k)\}$.
4) *If λ and ν are crossless, then λ' and ν' are crossless.*
5) *If λ and ν are crossless, then λ'' and ν'' are crossless.*
6) $\delta(\lambda',\nu') + \delta(\lambda'',\nu'') \leq \delta(\lambda,\nu) - 1$. *Equality holds, if λ and ν are crossless.*

Proof. 1) This follows from $x - y, x(\lambda,i) - y(\nu,k) \in \alpha\mathbb{Z}^d$.

2) Let $m,n \in \mathbb{N}$ with $m \leq i$ and $n \leq k$. We have $x(\lambda,m) - x'(\lambda',m) = x(\lambda,i)$ and $y(\nu,n) - y'(\nu',n) = y(\nu,k)$. Hence

$$x(\lambda,m) - y(\nu,n) + y'(\nu',n) - x'(\lambda',m) \in \alpha\mathbb{Z}^d,$$

which proves 2).

3) Let $m, n \in \mathbb{N}$ with $m \leq \deg x - i$ and $n \leq \deg y - k$. The assertion follows from

$$x''(\lambda'', m) = x(\lambda, m+i) \quad \text{and} \quad y''(\nu'', n) = y(\nu, n+k).$$

4), 5) This follows from 2) and 3).

6) Since $(i,k) \in \Delta(\lambda', \nu'), (0,0) \in \Delta(\lambda'', \nu'')$ and $\Delta(\lambda', \nu') \subseteq \{0, \ldots, i\} \times \{0, \ldots, k\}$, we have

$$\#\left(\Delta(\lambda', \nu') \cap (\{(i,k)\} + \Delta(\lambda'', \nu''))\right) = 1.$$

Hence

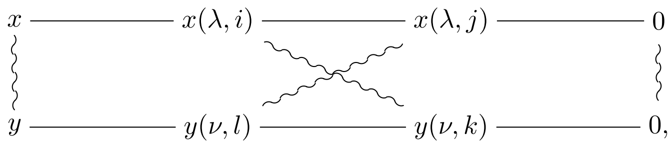
$$\#\Delta(\lambda', \nu') + \#\Delta(\lambda'', \nu'') - 1 = \#\left(\Delta(\lambda', \nu') \cup (\{(i,k)\} + \Delta(\lambda'', \nu''))\right) \overset{2),3)}{\leq} \#\Delta(\lambda, \nu). \quad (2)$$

By this we get

$$\delta(\lambda', \nu') + \delta(\lambda'', \nu'') = \#\Delta(\lambda', \nu') + \#\Delta(\lambda'', \nu'') - 1 - 3 \overset{(2)}{\leq} \#\Delta(\lambda, \nu) - 2 - 1 = \delta(\lambda, \nu) - 1. \quad (3)$$

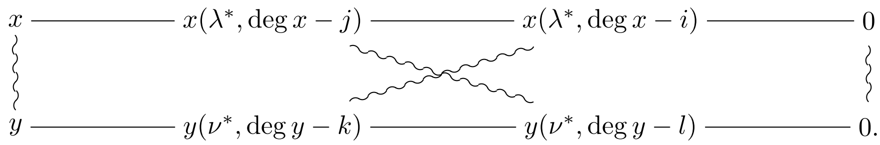
If λ and ν are crossless we have equality in (2), by 2) and 3). Hence we have equality in (3). □

Lemma 2.24. *Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda = (b_1, \ldots, b_{\deg x}) \in \Lambda_x$, and $\nu = (g_1, \ldots, g_{\deg y}) \in \Lambda_y$. If λ and ν are not crossless, i. e.,*



for some $i, j, l, k \in \mathbb{N}$ with $i < j \leq \deg x$ and $l < k \leq \deg y$, then

1) *λ^* and ν^* are not crossless, in particular:*



2) *$i, l \geq 2$ and $j \leq \deg x - 2$, $k \leq \deg y - 2$.*
3) *$x(\lambda, i) \neq y(\nu, k)$ and $x(\lambda, j) \neq y(\nu, l)$.*
4) *$y(\nu, k)_{[n]} > x(\lambda, i)_{[n]}$ and $x(\lambda, j)_{[m]} > y(\nu, l)_{[m]}$ for some $n, m \in \{1, \ldots, d\}$ with $n \neq m$.*
5) *$y(\nu, k)_{[n']} < x(\lambda, i)_{[n']}$ and $x(\lambda, j)_{[m']} < y(\nu, l)_{[m']}$ for some $n', m' \in \{1, \ldots, d\}$.*

Proof. 1) By Lemma 2.23 1) we get $x(\lambda^*, \deg x - i) \sim y(\nu^*, \deg y - k)$ and $x(\lambda^*, \deg x - j) \sim y(\nu^*, \deg y - l)$ with $\deg x - i > \deg x - j$ and $\deg y - k < \deg y - l$. Hence λ^* and ν^* are not crossless.

2) By Lemma 2.10 we have $i, l \neq 0$, $j \neq \deg x$, $k \neq \deg y$. Suppose $j = \deg x - 1$, i. e., $\deg x(\lambda, j) = 1$; which contradicts $\deg y(\nu, l) \geq 2$, since $l < k < \deg y$ (see also Remark 2.4). The claim follows by symmetry and 1).

3) By symmetry we only need to show that $x(\lambda, i) \neq y(\nu, k)$. Suppose to the contrary that $x(\lambda, i) = y(\nu, k)$. Then $\nu' = (g_1, \ldots, g_k, b_{i+1}, \ldots, b_{\deg x}) \in \Lambda_y$. By this we get

$y(\nu', k+j-i) = x(\lambda, j) \sim y(\nu, l) = y(\nu', l)$. Which contradicts Lemma 2.10, since $k+j-i > l$.

4), 5) Since $x(\lambda, i) \neq y(\nu, k)$ and $x(\lambda, i), y(\nu, k) \in B_S \setminus \{0\}$ with $x(\lambda, i) \sim y(\nu, k)$ we have $y(\nu, k)_{[n]} > x(\lambda, i)_{[n]}$ and $y(\nu, k)_{[n']} < x(\lambda, i)_{[n']}$ for some $n, n' \in \{1, \ldots, d\}$. Analogous $x(\lambda, j)_{[m]} > y(\nu, l)_{[m]}$ and $x(\lambda, j)_{[m']} < y(\nu, l)_{[m']}$ for some $m, m' \in \{1, \ldots, d\}$. Suppose that $m = n$, then $x(\lambda, j)_{[m]} > y(\nu, l)_{[m]} \geq y(\nu, k)_{[m]} > x(\lambda, i)_{[m]} \geq x(\lambda, j)_{[m]}$, a contradiction. □

Lemma 2.25. *Consider the same situation as in Lemma 2.24. Let $n, m \in \{1, \ldots, d\}$ such that $y(\nu, k)_{[n]} > x(\lambda, i)_{[n]}$ and $x(\lambda, j)_{[m]} > y(\nu, l)_{[m]}$. Then*

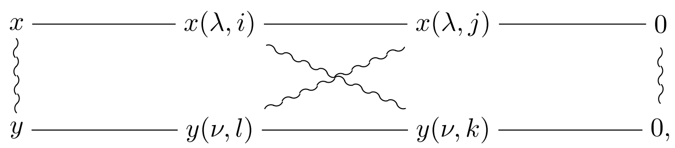
1) $y(\nu, l)_{[n]} > x(\lambda, j)_{[n]}$.
2) $x(\lambda, i)_{[m]} > y(\nu, k)_{[m]}$.

Proof. 1) We have $y(\nu, l)_{[n]} \geq y(\nu, k)_{[n]} > x(\lambda, i)_{[n]} \geq x(\lambda, j)_{[n]}$.

2) We have $x(\lambda, i)_{[m]} \geq x(\lambda, j)_{[m]} > y(\nu, l)_{[m]} \geq y(\nu, k)_{[m]}$. □

Proposition 2.26. *Let $x, y \in \Gamma_t \subseteq B_S \setminus \{0\}$ for some $t \in \{1, \ldots, f\}$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. If λ and ν are not crossless, then there is some $z \in \Gamma_t$ with $z \neq x, y$.*

Proof. We have



for some $i, j, l, k \in \mathbb{N}$ with $0 < i < j < \deg x$ and $0 < l < k < \deg y$. We set

$$z' := x(\lambda, j) + y - y(\nu, l) = x(\lambda, j) + y(\nu^*, \deg y - l).$$

By Lemma 2.24 5) we have:

$$x(\lambda, j)_{[h]} < y(\nu, l)_{[h]}$$

for some $h \in \{1, \ldots, d\}$. By this we get $z'_{[h]} < y_{[h]}$. By Lemma 2.24 1) and 5) we get

$$y(\nu^*, \deg y - l)_{[g]} < x(\lambda^*, \deg x - j)_{[g]}$$

for some $g \in \{1, \ldots, d\}$. By this we get $z'_{[g]} < x_{[g]}$. By construction $z' \in S$. Consider an element $z := z' - \sum_{u=1}^{d} n_u e_u \in S$ such that $\sum_{u=1}^{d} n_u$ is maximal. This means $z \in B_S$, in particular $z \leq z'$. By this we have $z \neq x, y$. Moreover, $z \sim z'$ and by Lemma 2.23 1):

$$z' - x = x(\lambda, j) + y(\nu^*, \deg y - l) - x = y(\nu^*, \deg y - l) - x(\lambda^*, \deg x - j) \in \alpha\mathbb{Z}^d,$$

hence $z' \sim x$, i. e., $z \in \Gamma_t$. □

Corollary 2.27. *Let $\#\Gamma_t = 2$ for some $t \in \{1, \ldots, f\}$, say $\Gamma_t = \{x, y\}$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. Then λ and ν are crossless, in particular x and y are crossless.*

Proof. Suppose that λ and ν are not crossless. Then by Proposition 2.26 we get $z \in \Gamma_t$ with $z \neq x, y$, which contradicts $\#\Gamma_t = 2$. Hence x and y are crossless as well. □

Lemma 2.28. *Let $x', x'', y', y'' \in S$ such that $x' \sim y', x'' \sim y''$. Moreover, let $x = x' + x''$ and $y = y' + y''$. Then*

$$h(x', y') + h(x'', y'') \leq h(x, y).$$

Proof. Let $i \in \{1, \ldots, d\}$, we have $x \sim y$ and

$$2 \min \left\{ x_{[i]}, y_{[i]} \right\} = x_{[i]} + y_{[i]} - |x_{[i]} - y_{[i]}| = x'_{[i]} + y'_{[i]} + x''_{[i]} + y''_{[i]} - |x'_{[i]} - y'_{[i]} + x''_{[i]} - y''_{[i]}|$$

$$\geq x'_{[i]} + y'_{[i]} - |x'_{[i]} - y'_{[i]}| + x''_{[i]} + y''_{[i]} - |x''_{[i]} - y''_{[i]}| = 2 \min \left\{ x'_{[i]}, y'_{[i]} \right\} + 2 \min \left\{ x''_{[i]}, y''_{[i]} \right\}.$$

Hence $h(x', y') + h(x'', y'') \leq h(x, y)$. □

Proposition 2.29. *Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. If λ and ν are crossless, then $\delta(\lambda, \nu) \leq \deg h(x, y) - 1$.*

Proof. We show this by induction on $\delta(\lambda, \nu) \in \mathbb{N}$. Let $\delta(\lambda, \nu) = 0$, i. e., we need to show that $\deg h(x, y) \geq 1$. Suppose to the contrary that $\deg h(x, y) = 0$, hence $h(x, y) = 0$. Thus $x, y \sim 0$, which contradicts $x, y \neq 0$.

Let $\delta(\lambda, \nu) = n + 1 > 0$. Fix an $i \in \{1, \ldots, \deg x - 1\}$ such that $x(\lambda, i) \sim y(\nu, k)$ for some $k \in \{1, \ldots \deg y - 1\}$. With the notation of Lemma 2.23 $x', x'', y', y'' \in B_S \setminus \{0\}$ (see Lemma 2.10) with $x' \sim y'$ and $x'' \sim y''$. Since λ and ν are crossless we get by Lemma 2.23 that $\lambda' \in \Lambda_{x'}$ and $\nu' \in \Lambda_{y'}$ are crossless and also that $\lambda'' \in \Lambda_{x''}$ and $\nu'' \in \Lambda_{y''}$ are crossless. Hence by induction

$$\delta(\lambda, \nu) \overset{2.23}{=} \delta(\lambda', \nu') + \delta(\lambda'', \nu'') + 1 \leq \deg h(x', y') + \deg h(x'', y'') - 1 \overset{2.28}{\leq} \deg h(x, y) - 1.$$

□

Corollary 2.30. *Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$. If x and y are crossless, then $\delta(x, y) \leq \deg h(x, y) - 1$.*

Proof. Since x and y are crossless, there are some sequences with $*$-property $\lambda \in \Lambda_x$ and $\nu \in \Lambda_y$ which are crossless. Hence by Proposition 2.29

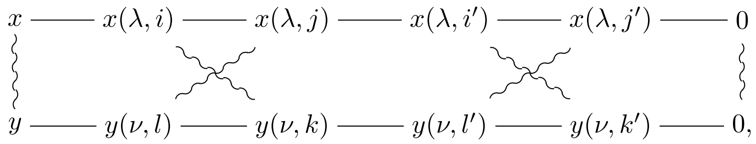
$$\delta(x, y) \leq \delta(\lambda, \nu) \leq \deg h(x, y) - 1.$$

□

Definition 2.31. Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$.

1. By a cross we mean a tuple $(\lambda, \nu, i, j, l, k) \in \Lambda_x \times \Lambda_y \times \mathbb{N}^4$ with $i < j \leq \deg x$ and $l < k \leq \deg y$ such that $x(\lambda, i) \sim y(\nu, k)$ and $x(\lambda, j) \sim y(\nu, l)$. We say that λ and ν have a cross.
2. Let $\lambda \in \Lambda_x$ and $\nu \in \Lambda_y$. We say that two crosses $(\lambda, \nu, i, j, l, k)$ and $(\lambda, \nu, i', j', l', k')$ are disjoint if $j < i'$ and $k < l'$ or if $j' < i$ and $k' < l$.
3. The height of a cross $(\lambda, \nu, i, j, l, k)$ is defined to be $(j - i, k - l) \in \mathbb{N}^2$.

Lemma 2.32. *Let $x, y \in B_S \setminus \{0\}$ with $x \sim y$, $\lambda \in \Lambda_x$, and $\nu \in \Lambda_y$. If we have two disjoint crosses $(\lambda, \nu, i, j, l, k)$ and $(\lambda, \nu, i', j', l', k')$ of height $(j-i, k-l)$ and of height $(j'-i', k'-l')$, i. e.,*



with $0 < i < j < i' < j' < \deg x$ and $0 < l < k < l' < k' < \deg y$, then there are elements $\lambda' \in \Lambda_x$ and $\nu' \in \Lambda_y$ with a cross of height $(j-i+j'-i', k-l+k'-l')$.

Proof. Let $\lambda = (b_1, \ldots, b_{\deg x})$ and $\nu = (g_1, \ldots, g_{\deg y})$. Set

$$\lambda' = (b_{j+1}, \ldots, b_{j'}, b_{i+1}, \ldots, b_j, b_1, \ldots, b_i, b_{j'+1}, \ldots, b_{\deg x})$$

and

$$\nu' = (g_{k+1}, \ldots, g_{k'}, g_{l+1}, \ldots, g_k, g_1, \ldots, g_l, g_{k'+1}, \ldots, g_{\deg y}).$$

By construction and Proposition 2.9, $\lambda' \in \Lambda_x$ and $\nu' \in \Lambda_y$. We claim that $x(\lambda', i'-j) \sim y(\nu', k'-l)$ and $x(\lambda', j'-i) \sim y(\nu', l'-k)$. Note that $i'-j < j'-i$ and $k'-l > l'-k$; therefore $(\lambda', \nu', i'-j, j'-i, l'-k, k'-l)$ is a cross of height $(j-i+j'-i', k-l+k'-l')$. To verify the claim, note that

$$x(\lambda', i'-j) = x - \sum_{t=1}^{i'-j} b_{j+t} = x - (x(\lambda, j) - x(\lambda, i')) \sim y - (y(\nu, l) - y(\nu, k'))$$

$$= y - \sum_{t=1}^{k'-l} g_{l+t} = y - \sum_{t=1}^{k'-k} g_{k+t} - \sum_{u=1}^{k-l} g_{l+u} = y(\nu', k'-l),$$

and

$$y(\nu', l'-k) = y - \sum_{t=1}^{l'-k} g_{k+t} = y - (y(\nu, k) - y(\nu, l')) \sim x - (x(\lambda, i) - x(\lambda, j'))$$

$$= x - \sum_{t=1}^{j'-i} b_{i+t} = x - \sum_{t=1}^{j'-j} b_{j+t} - \sum_{u=1}^{j-i} b_{i+u} = x(\lambda', j'-i).$$

□

3 The case of at most two elements

Definition 3.1. For a monomial $m = y_1^{b_1} \cdots y_d^{b_d} \in T$ we define $\deg m = \sum_{j=1}^{d} b_j$.

Definition 3.2. We define the set $\Gamma(S) \subseteq \{\Gamma_1, \ldots, \Gamma_f\}$ by: $\Gamma_t \in \Gamma(S)$ for $t \in \{1, \ldots, f\}$ if $\mathrm{reg} K[S] = \mathrm{reg} I_t + \deg h_t$.

Theorem 3.3. *Let $\Gamma_t \in \Gamma(S)$ for some $t \in \{1, \ldots, f\}$. If $\#\Gamma_t \leq 2$, then*

$$\mathrm{reg} K[S] \leq \deg K[S] - \mathrm{codim} K[S].$$

Proof. If $\#\Gamma_t = 1$, then the assertion follows from Theorem 2.11. So we only have to consider the case $\#\Gamma_t = 2$. Let $x, x' \in \Gamma_t$ with $x \neq x'$, $m = y^{(x-h_t)/\alpha}$ and $n = y^{(x'-h_t)/\alpha}$. By construction m, n are a regular sequence on T. Using the Koszul Complex (e. g., see [1, Section 17.1]) we get

$$\mathrm{reg} K[S] = \mathrm{reg} I_t + \deg h_t = \deg x + \deg x' - \deg h_t - 1. \quad (4)$$

Let $\lambda \in \Lambda_x$ and $\nu \in \Lambda_{x'}$. By Corollary 2.27, λ and ν are crossless. Consider the set L in B_S:

$$L = \{x(\lambda, 0), \ldots, x(\lambda, \deg x-2), x(\lambda, \deg x)\} \cup \{x'(\nu, 0), \ldots, x'(\nu, \deg x'-2), x'(\nu, \deg x')\}.$$

By construction, every element in L is not equivalent to an element in $\{a_1, \ldots, a_c\}$, since for all $z \in L$ we have $\deg z \neq 1$ (see Remark 2.4). By $\Gamma'_1, \ldots, \Gamma'_g$ we denote the equivalence classes on L. Hence

$$\begin{aligned} g = \deg x + \deg x' - \#\left(\Delta(\lambda, \nu) \setminus \{(\deg x - 1, \deg x' - 1)\}\right) &\geq \deg x + \deg x' - \#\Delta(\lambda, \nu) \\ = \deg x + \deg x' - \delta(\lambda, \nu) - 2 &\overset{2.29}{\geq} \deg x + \deg x' - \deg h_t - 1, \quad (5) \end{aligned}$$

since $h(x, x') = h_t$. Hence

$$\deg K[S] \geq g + c \overset{(5)}{\geq} \deg x + \deg x' - \deg h_t - 1 + c \overset{(4)}{=} \mathrm{reg} K[S] + c.$$

□

Corollary 3.4. *If $\#\Gamma_t \leq 2$ for all $t = 1, \ldots, f$, then*

$$\mathrm{reg} K[S] \leq \deg K[S] - \mathrm{codim} K[S].$$

Proof. Follows from Theorem 3.3. □

Example 3.5. Consider the following semigroup in $\mathbb{N}^4$ with $\alpha = 6$:

$$S = \langle e_1, \ldots, e_4, (0, 2, 0, 4), (3, 0, 2, 1), (0, 2, 2, 2) \rangle.$$

We define the reduction number $\mathrm{r}(K[S]) := \max\{\deg x \mid x \in B_S\}$ (see [8]), by Theorem 2.11 the Eisenbud-Goto conjecture holds for the reduction number. Using Macaulay2 [4] we get $\mathrm{reg} K[S] = 6 > \mathrm{r}(K[S]) = 5$. Moreover, we have

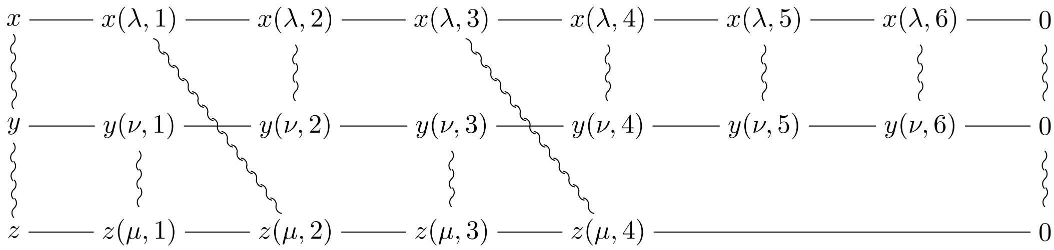
$$\Gamma_t = \{(3, 6, 4, 11), (15, 0, 10, 5)\} \in \Gamma(S),$$

for some $t \in \{1, \ldots, f\}$, since $\mathrm{reg} I_t + \deg h_t = \mathrm{reg}\langle y_2 y_4, y_1^2 y_3 \rangle + 2 = 6$ and therefore Eisenbud-Goto holds by Theorem 3.3. We note that S is not seminormal by [11, Theorem 4.1.1] and not Buchsbaum, since $(3, 6, 10, 5) + 2e_1, (3, 6, 10, 5) + e_4 \in S$, but $(3, 6, 10, 5) + e_1 = (9, 6, 10, 5) \notin S$ (see [19, Lemma 3]).

Example 3.6. Let $\Gamma_t \in \Gamma(S)$ for some $t \in \{1, \ldots, f\}$ with $\#\Gamma_t > 2$. Unfortunately this case is much more complicated. Consider the following situation, let $\alpha = 20$ and

$$\Gamma_t = \{x = (44, 104, 12), y = (104, 44, 12), z = (24, 24, 72)\}.$$

We get $h(x,y) = (44,44,12)$, $h(x,z) = (24,24,12)$ and $h(y,z) = (24,24,12)$. Assume that Conjecture 2.16 holds, so x and y could have 4 non-trivial pairwise equivalent elements, x and z could have 2, as well as y and z. Let us consider a worst case scenario:



for some $\lambda \in \Lambda_x, \nu \in \Lambda_y$, and $\mu \in \Lambda_z$. Note that no element in the picture has degree 1. If we follow the proof of Theorem 3.3 we would get $g = 10$. So we want the ideal plus the shift to be smaller or equal to 10. But this is not the case since $\deg h_t = (24+24+12)/20 = 3$ and $\mathrm{reg} I_t = \mathrm{reg}\langle y_1 y_2^4, y_1^4 y_2, y_3^3\rangle = 9$.

4 Monomial curves

In this section we will assume that $\dim K[S] = 2$, i.e., $d = 2$. Thus, we consider the case of monomial curves, i.e.,

$$S = \{e_1, e_2, a_1, \ldots, a_c\} \subseteq \mathbb{N}^2.$$

We have $f = \alpha$, i.e., $\deg K[S] = \alpha$. Moreover, $T = K[y_1, y_2]$ and every monomial ideal I in T can be uniquely written as:

$$I = < m_1, \ldots, m_r >, \text{with } m_i = y_1^{b_i} y_2^{c_i}, i = 1, \ldots, r,$$

where $b_1 > \ldots > b_r \geq 0$ and $0 \leq c_1 < \ldots < c_r$ (see [12, Section 3.1]). The case $r = 1$ is not relevant in our context. Let us assume that $r \geq 2$; it is a well known fact that the regularity of I can be computed by:

Proposition 4.1.

$$\mathrm{reg} I = \max_{i=1,\ldots,r-1} \{b_i + c_{i+1}\} - 1$$

Proof. By [12, Proposition 3.1] the kernel of $g : T^r \to I$, $\hat{e}_i \mapsto m_i$ is minimally generated by $y_2^{c_{i+1}-c_i}\hat{e}_i - y_1^{b_i - b_{i+1}}\hat{e}_{i+1}$, $i = 1, \ldots, r-1$. Hence the minimal free graded resolution of I has the following form

$$0 \longrightarrow \bigoplus_{l=1}^{r-1} T(-(b_l + c_{l+1})) \longrightarrow \bigoplus_{j=1}^{r} T(-(b_j + c_j)) \longrightarrow I \longrightarrow 0,$$

since $y_2^{c_{i+1}-c_i} \in T(-(b_i + c_i))_{b_i + c_{i+1}}$ and $y_1^{b_i - b_{i+1}} \in T(-(b_{i+1} + c_{i+1}))_{b_i + c_{i+1}}$. By assumption $c_{i+1} > c_i$ and $b_i > b_{i+1}$, thus $b_i + c_{i+1} > \max\{b_i + c_i, b_{i+1} + c_{i+1}\}$ and therefore

$$\mathrm{reg} I = \max\{b_1 + c_1, \ldots, b_r + c_r, b_1 + c_2 - 1, \ldots, b_{r-1} + c_r - 1\} = \max_{i=1,\ldots,r-1}\{b_i + c_{i+1}\} - 1.$$

□

Remark 4.2. Let $\#\Gamma_t \geq 2$ for some $t \in \{1, \ldots, \alpha\}$. Consider two elements $x, y \in \Gamma_t$ with $x \neq y$. Suppose $x_{[i]} = y_{[i]}$ for some $i \in \{1, 2\}$, then $x > y$ or $x < y$, a contradiction. Without loss of generality we may assume that $x_{[i]} < y_{[i]}$ for some $i \in \{1, 2\}$, then $x_{[j]} > y_{[j]}$ for $j \in \{1, 2\} \setminus \{i\}$, since otherwise $x < y$. This shows that $\tilde{\Gamma}_t$ is a minimal generating set of I_t. We note that this holds for arbitrary d. By construction and the above argument

$$I_t =< m_1, \ldots, m_{\#\Gamma_t} >, \text{with } m_i \in \tilde{\Gamma}_t, m_i = y_1^{b_i} y_2^{c_i}, i = 1, \ldots, \#\Gamma_t,$$

where $b_1 > \ldots > b_{\#\Gamma_t} = 0$ and $0 = c_1 < \ldots < c_{\#\Gamma_t}$.

Definition 4.3. Let $x, y \in \Gamma_t$ for some $t \in \{1, \ldots, \alpha\}$ with $x \neq y$, i. e., $x_{[i]} > y_{[i]}$ and $x_{[j]} < y_{[j]}$ for $i, j \in \{1, 2\}$ with $i \neq j$. We say that x and y are close if there is no element $z \in \Gamma_t$ with $x_{[i]} > z_{[i]} > y_{[i]}$ and $x_{[j]} < z_{[j]} < y_{[j]}$.

Example 4.4. Consider the following smooth monomial curve in $\mathbb{P}^5$ given by

$$S = \langle (12, 0), (0, 12), (11, 1), (9, 3), (4, 8), (1, 11) \rangle.$$

Then by [13, Corollary 3.9] we get $\text{reg}K[S] = 4$. Moreover, we have:

$$K[S] \cong T \oplus T(-1)^4 \oplus \langle y_1, y_2 \rangle(-1)^2 \oplus \langle y_1, y_2^2 \rangle(-1)^2 \oplus \langle y_1^2, y_2 \rangle(-1)^2 \oplus \underbrace{\langle y_1^2, y_1 y_2, y_2^3 \rangle}_{=I_{12}}(-1).$$

By Proposition 4.1 we have $\Gamma(S) = \{\Gamma_{12}\}$, where $\Gamma_{12} = \{(31, 5), (19, 17), (7, 41)\}$. We note that $(31, 5)$ and $(19, 17)$ are close, as well as $(19, 17)$ and $(7, 41)$.

Remark 4.5. Let us consider the case of smooth monomial curves, i. e., we assume that $a_1 = (\alpha - 1, 1)$ and $a_c = (1, \alpha - 1)$. In this case there is still a much better combinatorial bound than the one given by L'vovsky in [9]; namely $\text{reg}K[S] \leq \#L + 1$, where $\#L$ is the maximal number of consecutive integer points on the line $[(\alpha, 0), (0, \alpha)]$ not belonging to S (see [6]). Anyway, even this bound is not sharp, see [13, Introduction]. We will now give a short proof of the Eisenbud-Goto conjecture for smooth monomial curves. Let $\Gamma_t \in \Gamma(S)$ for some $t \in \{1, \ldots, \alpha\}$. By Theorem 2.11 we may assume that $\#\Gamma_t \geq 2$. Since $(\alpha - 1, 1), (1, \alpha - 1) \in S$ we have $(k\alpha - l, l), (\alpha - l, k'\alpha + l) \in \Gamma_t$ for some $l, k, k' \in \mathbb{N}$ with $0 < l < \alpha$. Set $x = (k\alpha - l, l)$ and $x' = (\alpha - l, k'\alpha + l)$; since $0 < l < \alpha$ we have $I_t = \langle y_1^{\deg x - 1}, \ldots, y_2^{\deg x' - 1} \rangle$ and $h_t = (\alpha - l, l)$ and by construction

$$\text{reg}K[S] = \text{reg}I_t + \deg h_t = \text{reg}\langle y_1^{\deg x - 1}, \ldots, y_2^{\deg x' - 1} \rangle + 1 \leq \deg x + \deg x' - 2.$$

Let $\Gamma_1 = \{0\}$. By a similar argument, one can show that $\deg h_{t'} = 1$ for all $t' = 2, \ldots, \alpha$. Let $\lambda \in \Lambda_x$ and $\nu \in \Lambda_{x'}$. Suppose that $x(\lambda, m) \sim x'(\nu, n)$ for some $m \in \{1, \ldots, \deg x - 1\}$ and some $n \in \{1, \ldots, \deg x' - 1\}$, then by Lemma 2.23 1) and 2.28 we have $\deg h(x, x') \geq 2$; since $\deg h(z, z') \geq 1$ for all $z, z' \in B_S \setminus \{0\}$ with $z \sim z'$. Hence $\#\Delta(\lambda, \nu) = 2$. By a similar argument as in Theorem 3.3 we get:

$$\text{deg}K[S] \geq \deg x + \deg x' - 2 + c \geq \text{reg}K[S] + c.$$

Let us consider the Macaulay curves, i. e., $S = \langle (\alpha, 0), (0, \alpha), (\alpha - 1, 1), (1, \alpha - 1) \rangle$. We have $(\alpha - 1, 1) + (1, \alpha - 1) \notin B_S$, hence $B_S = \{i(1, \alpha - 1), j(\alpha - 1, 1)\} \mid 0 \leq i, j \leq \alpha - 2\}$, i. e.,

$$B_S = \{0, (1, \alpha - 1), (2, 2\alpha - 2), \ldots, (\alpha - 2, \underbrace{(\alpha - 3)\alpha + 2}_{=(\alpha-2)\alpha-\alpha+2}), ((\alpha - 3)\alpha + 2, \alpha - 2), \ldots, (\alpha - 1, 1)\}.$$

We have:

$\Gamma_1 = \{0\}, \Gamma_2 = \{(1, \alpha-1)\}, \Gamma_3 = \{(\alpha-1, 1)\}, \Gamma_4 = \{(2, 2\alpha-2), ((\alpha-3)\alpha+2, \alpha-2)\},$

$\Gamma_5 = \{(3, 3\alpha-3), ((\alpha-4)\alpha+3, \alpha-3)\}, \ldots, \Gamma_\alpha = \{(\alpha-2, (\alpha-3)\alpha+2), (2\alpha-2, 2)\}.$

Hence

$$K[S] \cong T \oplus T(-1)^2 \oplus \langle y_1^{\alpha-3}, y_2\rangle(-1) \oplus \langle y_1^{\alpha-4}, y_2^2\rangle(-1) \oplus \ldots \oplus \langle y_1, y_2^{\alpha-3}\rangle(-1),$$

meaning each T-module of the form $\langle y_1^\beta, y_2^\gamma\rangle(-1)$, $1 \le \beta, \gamma \le \alpha-3$ with $\beta+\gamma = \alpha-2$ appears exactly once in the decomposition. We have $\mathrm{reg} K[S] = \alpha - 2 = \deg K[S] - \mathrm{codim} K[S]$, i. e., the Eisenbud-Goto conjecture is sharp in this case.

Definition 4.6. Let $\#\Gamma_t \ge 2$ for some $t \in \{1, \ldots, \alpha\}$. With the notation of Proposition 4.1 and Remark 4.2 we get $\mathrm{reg} I_t = b_k + c_{k+1} - 1$ for some $k \in \{1, \ldots, \#\Gamma_t - 1\}$; fix such an integer k. Let $x, x' \in \Gamma_t$ such that $m_k = y^{(x-h_t)/\alpha}$ and $m_{k+1} = y^{(x'-h_t)/\alpha}$. We define the set $\bar{\Gamma}_t := \{x, x'\} \subseteq \Gamma_t$.

Remark 4.7. Consider Example 4.4, then $\bar{\Gamma}_{12} = \{(19, 17), (7, 41)\}$. Whenever $\#\Gamma_t = 2$ for some $t \in \{1, \ldots, \alpha\}$, then $\Gamma_t = \bar{\Gamma}_t$.

Proposition 4.8. *Let $\Gamma_t \in \Gamma(S)$ for some $t \in \{1, \ldots, \alpha\}$ with $\#\Gamma_t \ge 2$ and $\bar{\Gamma}_t = \{x, x'\}$. If Conjecture 2.16 holds for x and x', then*

$$\mathrm{reg} K[S] \le \deg K[S] - \mathrm{codim} K[S].$$

In particular this holds, if x and x' are crossless.

Proof. Assume that $x_{[1]} > x'_{[1]}$ and $x_{[2]} < x'_{[2]}$. Let $m_k = y^{(x-h_t)/\alpha} = y_1^{b_k} y_2^{c_k}$ and $m_{k+1} = y^{(x'-h_t)/\alpha} = y_1^{b_{k+1}} y_2^{c_{k+1}}$. By construction,

$$\begin{aligned} \mathrm{reg} K[S] &= \mathrm{reg} I_t + \deg h_t \stackrel{\text{Def.}}{=} b_k + c_{k+1} - 1 + \deg h_t \\ &= ((x-h_t)/\alpha)_{[1]} + ((x'-h_t)/\alpha)_{[2]} - 1 + \deg h_t = \deg(x_{[1]}, x'_{[2]}) - 1. \end{aligned} \tag{6}$$

Fix $\lambda \in \Lambda_x$ and $\nu \in \Lambda_{x'}$ such that $\delta(x, x') = \delta(\lambda, \nu)$ and consider the set L in B_S:

$$L = \{x(\lambda, 0), \ldots, x(\lambda, \deg x - 2), x(\lambda, \deg x)\} \cup \{x'(\nu, 0), \ldots, x'(\nu, \deg x' - 2), x'(\nu, \deg x')\}.$$

By construction, every element in L is not equivalent to an element in $\{a_1, \ldots, a_c\}$, since for all $z \in L$ we have $\deg z \ne 1$ (see Remark 2.4). By $\Gamma'_1, \ldots, \Gamma'_g$ we denote the equivalence classes on L. Hence

$$\begin{aligned} g &= \deg x + \deg x' - \#\left(\Delta(\lambda, \nu) \setminus \{(\deg x - 1, \deg x' - 1)\}\right) \ge \deg x + \deg x' - \#\Delta(\lambda, \nu) \\ &= \deg(x_{[1]}, x'_{[2]}) + \deg(x'_{[1]}, x_{[2]}) - \delta(x, x') - 2 \stackrel{2.16}{\ge} \deg(x_{[1]}, x'_{[2]}) - 1, \end{aligned} \tag{7}$$

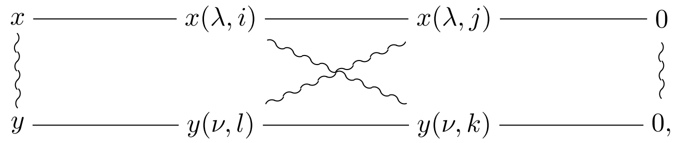
since $h(x, x') = (x'_{[1]}, x_{[2]})$ and therefore

$$\deg K[S] \ge g + c \stackrel{(7)}{\ge} \deg(x_{[1]}, x'_{[2]}) - 1 + c \stackrel{(6)}{=} \mathrm{reg} K[S] + c.$$

If x and x' are crossless, then Conjecture 2.16 holds by Corollary 2.30. $\square$

Remark 4.9. Let $\#\Gamma_t \geq 2$ for some $t \in \{1, \ldots, \alpha\}$. If $\bar{\Gamma}_t = \{x, x'\}$, then x and x' are close. Thus, by proving Conjecture 2.16 for close elements in B_S we would immediately get a combinatorial proof of the Eisenbud-Goto conjecture for monomial curves.

Remark 4.10. Let $x, y \in \Gamma_t$ for some $t \in \{1, \ldots, \alpha\}$ with $x \neq y$. Moreover, we assume that $x_{[1]} > y_{[1]}$ and $x_{[2]} < y_{[2]}$. Let $\lambda \in \Lambda_x$ and $\nu \in \Lambda_y$ be not crossless, i. e.,



for some $i, j, l, k \in \mathbb{N}$ with $0 < i < j < \deg x$ and $0 < l < k < \deg y$. Fix i, k (we could also fix l, j), then we have one of the following two cases:

1. $x(\lambda, i)_{[1]} > y(\nu, k)_{[1]}$ and $x(\lambda, i)_{[2]} < y(\nu, k)_{[2]}$,
2. $x(\lambda, i)_{[1]} < y(\nu, k)_{[1]}$ and $x(\lambda, i)_{[2]} > y(\nu, k)_{[2]}$,

by Lemma 2.24. The first case is what you normally would expect, since the first coordinate of x is bigger than the first coordinate of y. The second case looks a little strange, but still possible. Keep in mind that $x(\lambda^*, \deg x - i) \sim y(\nu^*, \deg y - k)$ by Lemma 2.23, $x(\lambda^*, \deg x - i), y(\nu^*, \deg y - k) \in B_S$ by Lemma 2.10, and $x(\lambda^*, \deg x - i) \neq y(\nu^*, \deg y - k)$ by Lemma 2.24. Moreover, by construction, $x(\lambda^*, \deg x - i) + x(\lambda, i) = x$ and $y(\nu^*, \deg y - k) + y(\nu, k) = y$; see Lemma 2.10.

Lemma 4.11. *Consider the same situation as in Remark 4.10. Moreover, let x and y be close. If $x(\lambda, i)_{[1]} > y(\nu, k)_{[1]}$ and $x(\lambda, i)_{[2]} < y(\nu, k)_{[2]}$, then*

$$x(\lambda^*, \deg x - i)_{[1]} < y(\nu^*, \deg y - k)_{[1]} \text{ and } x(\lambda^*, \deg x - i)_{[2]} > y(\nu^*, \deg y - k)_{[2]}.$$

Proof. Suppose to the contrary that $x(\lambda^*, \deg x - i)_{[1]} > y(\nu^*, \deg y - k)_{[1]}$ and $x(\lambda^*, \deg x - i)_{[2]} < y(\nu^*, \deg y - k)_{[2]}$; see Lemma 2.24. Define $z := y(\nu, k) + x(\lambda^*, \deg x - i)$, by construction $z \sim x, y$. Moreover, we have $x_{[1]} > z_{[1]}, x_{[2]} < z_{[2]}$ and $z_{[1]} > y_{[1]}, z_{[2]} < y_{[2]}$, i. e.,

$$x_{[1]} > z_{[1]} > y_{[1]}, x_{[2]} < z_{[2]} < y_{[2]}.$$

Consider an element $z' := z - n_1 e_1 - n_2 e_2 \in S$ such that $n_1 + n_2$ is maximal. We have $z' \in B_S$, $z' \neq x, y$, $z' \leq z$, and $z' \sim z \sim x, y$. Suppose $z'_{[1]} \leq y_{[1]}$, then $z' < y$, a contradiction. Suppose $z'_{[2]} \leq x_{[2]}$, then $z' < x$, a contradiction. Hence

$$x_{[1]} > z'_{[1]} > y_{[1]}, x_{[2]} < z'_{[2]} < y_{[2]},$$

and therefore x and y are not close, which is a contradiction. □

Remark 4.12. With the notation of Remark 4.10 and the assumption that x and y are close we get by Remark 4.10 and Lemma 4.11 one of the following two cases:

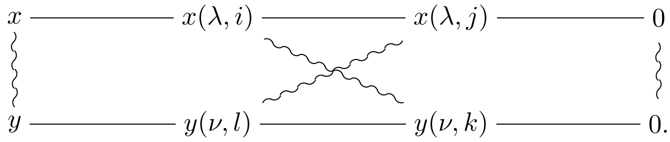
1. $x(\lambda^*, \deg x - i)_{[1]} < y(\nu^*, \deg y - k)_{[1]}$ and $x(\lambda^*, \deg x - i)_{[2]} > y(\nu^*, \deg y - k)_{[2]}$.
2. $x(\lambda, i)_{[1]} < y(\nu, k)_{[1]}$ and $x(\lambda, i)_{[2]} > y(\nu, k)_{[2]}$.

Proposition 4.13. *Let $x, y \in \Gamma_t$ for some $t \in \{1, \ldots, \alpha\}$ with $x \neq y$. If x and y are close, then*

$$\delta(x, y) \leq \deg h(x, y) - 1,$$

i. e., Conjecture 2.16 holds for x and y.

Proof. By Corollary 2.30 we may assume that x and y are not crossless. Moreover, we may assume that $x_{[1]} > y_{[1]}$ and $x_{[2]} < y_{[2]}$. Let us fix a maximal cross in the following sense, let $(\lambda, \nu, i, j, l, k) \in \Lambda_x \times \Lambda_y \times \mathbb{N}^4$ be a cross such that $j - i$ is maximal among all crosses; say $\lambda = (b_1, \ldots, b_{\deg x})$ and $\nu = (g_1, \ldots, g_{\deg y})$. This can be illustrated by the picture:



Without loss of generality, we may assume that for all $j', k' \in \mathbb{N}$ with $j < j' < \deg x$ and $k < k' < \deg y$ we have $x(\lambda, j') \not\sim y(\nu, k')$, since otherwise we consider the following sequences with $*$-property:

$$\lambda' = (b_{j'+1}, \ldots, b_{\deg x}, b_1, \ldots, b_{j'}) \in \Lambda_x \quad \text{and} \quad \nu' = (g_{k'+1}, \ldots, g_{\deg y}, g_1, \ldots, g_{k'}) \in \Lambda_y,$$

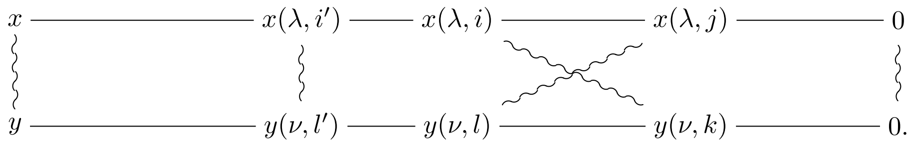
by this we would get a cross $(\lambda', \nu', \deg x - j' + i, \deg x - j' + j, \deg y - k' + l, \deg y - k' + k)$. Let $i' \in \mathbb{N}$ be maximal with $0 \leq i' < i$ and $x(\lambda, i') \sim y(\nu, l')$ for some $l' \in \{0, \ldots, \deg y\}$. Let $x' = x(\lambda^*, \deg x - i')$, $y' = y(\nu^*, \deg y - l')$, $x'' = x(\lambda, i')$, $y'' = y(\nu, l')$, $\lambda' = (b_1, \ldots, b_{i'}) \in \Lambda_{x'}$, and $\nu' = (g_1, \ldots, g_{l'}) \in \Lambda_{y'}$ (see Remark 2.22). So $x = x' + x''$ and $y = y' + y''$. We claim that:

$$\#\Delta(\lambda, \nu) \leq \#\Delta(\lambda', \nu') + \deg\left(x(\lambda, i) - x(\lambda, j)\right) + 2. \tag{8}$$

In case that $i' = 0$ we set $\#\Delta(\lambda', \nu') = 1$. Consider an element $j' \in \mathbb{N}$ with $j < j' < \deg x$ and suppose to the contrary that $x(\lambda, j') \sim y(\nu, k')$ for some $k' \in \{0, \ldots, \deg y\}$. By construction we have $k' < k$. Hence we get a cross $(\lambda, \nu, i, j', k', k)$ with height $(j' - i, k - k')$ which is a contradiction, since $j - i$ is assumed to be maximal. By this we have (see Remark 2.15)

$$\#\Delta(\lambda, \nu) \leq \#\left(\Delta(\lambda, \nu) \cap (\{0, \ldots, i'\} \times \mathbb{N})\right) + \deg\left(x(\lambda, i) - x(\lambda, j)\right) + 2,$$

i. e., we need to show that $(\Delta(\lambda, \nu) \cap (\{0, \ldots, i'\} \times \mathbb{N})) \subseteq \Delta(\lambda', \nu')$. In case that $i' = 0$ we have $\#\left(\Delta(\lambda, \nu) \cap (\{0, \ldots, i'\} \times \mathbb{N})\right) = 1$. Suppose to the contrary that $l' > l$, by this we get a cross $(\lambda, \nu, i', j, l, l')$ of height $(j - i', l' - l)$, which contradicts the maximality of $j - i$. That means $l' < l$, since $l' \neq l$, i. e., (assume for the picture $i' > 0$)



Let $(m, n) \in (\Delta(\lambda, \nu) \cap (\{0, \ldots, i'\} \times \mathbb{N}))$ and assume that $m \notin \{0, i'\}$. Suppose to the contrary that $x(\lambda, m) \sim y(\nu, n)$ with $n \geq l'$. By a similar argument as above, we get $n < l$ and clearly $n \neq l'$, i.e., we suppose that $l' < n < l$. Hence $(\lambda, \nu, m, i', l', n)$ and $(\lambda, \nu, i, j, l, k)$ are two disjoint crosses, which contradicts Lemma 2.32, since $j - i$ is maximal. That means $n < l'$ and therefore $(m, n) \in \Delta(\lambda', \nu')$ by Lemma 2.23 2), which proves (8).

Since x and y are close, we get by Remark 4.12 one of the following two cases:

1. $x(\lambda^*, \deg x - i)_{[1]} < y(\nu^*, \deg y - k)_{[1]}$ and $x(\lambda^*, \deg x - i)_{[2]} > y(\nu^*, \deg y - k)_{[2]}$.
2. $x(\lambda, i)_{[1]} < y(\nu, k)_{[1]}$ and $x(\lambda, i)_{[2]} > y(\nu, k)_{[2]}$.

Case 1:

Applying Lemma 2.25 to Lemma 2.24 1) we get $x(\lambda^*, \deg x - j)_{[2]} > y(\nu^*, \deg y - l)_{[2]}$ and therefore $x(\lambda^*, \deg x - j)_{[1]} < y(\nu^*, \deg y - l)_{[1]}$. Keep in mind that by construction $h(x, y) = (y_{[1]}, x_{[2]})$. Hence

$$h(x, y)_{[1]} = y_{[1]} \geq y(\nu^*, \deg y - l)_{[1]} > x(\lambda^*, \deg x - j)_{[1]}$$

and

$$h(x, y)_{[2]} = x_{[2]} \geq x(\lambda^*, \deg x - j)_{[2]}.$$

Thus

$$\deg x(\lambda^*, \deg x - j) + 1 \leq \deg h(x, y). \tag{9}$$

Moreover, we have $\Delta(\lambda', \nu') \subseteq (\{0, \ldots, i'\} \times \{0, \ldots, l'\})$, i. e., $\#\Delta(\lambda', \nu') \leq i' + 1$ (see Remark 2.15) and $i' + 1 \leq i$. By this we get

$$\begin{aligned} \#\Delta(\lambda', \nu') + \deg\left(x(\lambda, i) - x(\lambda, j)\right) &\leq i' + 1 + \deg x - i - (\deg x - j) \\ &= j + i' + 1 - i \leq j = \deg x(\lambda^*, \deg x - j), \end{aligned} \tag{10}$$

and therefore

$$\delta(x, y) \leq \delta(\lambda, \nu) = \#\Delta(\lambda, \nu) - 2 \overset{(8)}{\leq} \#\Delta(\lambda', \nu') + \deg\left(x(\lambda, i) - x(\lambda, j)\right) \overset{(9),(10)}{\leq} \deg h(x, y) - 1.$$

Case 2:

By Lemma 2.23 2) and 2.32 λ' and ν' are crossless, since $(j - i)$ is assumed to be maximal. Hence by Proposition 2.29 we get:

$$\#\Delta(\lambda', \nu') - 2 \leq \deg h(x', y') - 1. \tag{11}$$

In case that $i' = 0$ we have $\#\Delta(\lambda', \nu') = 1$ and $\deg h(x', y') = 0$, i. e., equation (11) holds. We get $x''_{[2]} \geq x(\lambda, i)_{[2]}$, and $y''_{[1]} \geq y(\nu, k)_{[1]} > x(\lambda, i)_{[1]}$ and therefore $\deg(y''_{[1]}, x''_{[2]}) \geq \deg x(\lambda, i) + 1$. Hence

$$\begin{aligned} &\deg h(x, y) - 1 = \deg(y_{[1]}, x_{[2]}) - 1 = \deg(y'_{[1]}, x'_{[2]}) + \deg(y''_{[1]}, x''_{[2]}) - 1 \\ &\geq \deg h(x', y') - 1 + \deg(y''_{[1]}, x''_{[2]}) \overset{(11)}{\geq} \#\Delta(\lambda', \nu') - 2 + \deg x(\lambda, i) + 1 \\ &\geq \#\Delta(\lambda', \nu') - 2 + \deg\left(x(\lambda, i) - x(\lambda, j)\right) + 1 + 1 \overset{(8)}{\geq} \#\Delta(\lambda, \nu) - 2 = \delta(\lambda, \nu) \geq \delta(x, y). \end{aligned}$$

□

Theorem 4.14. *We have:*

$$\mathrm{reg} K[S] \leq \deg K[S] - \mathrm{codim} K[S].$$

Proof. Let $\Gamma_t \in \Gamma(S)$ for some $t \in \{1, \ldots, \alpha\}$. If $\#\Gamma_t = 1$, then the assertion follows from Theorem 3.3. If $\#\Gamma_t \geq 2$, then the assertion follows from Proposition 4.8 and 4.13. □

Remark 4.15. This proof is a new combinatorial proof of the Eisenbud-Goto conjecture for monomial curves; unfortunately this proof does not yield the L'vovsky bound (see [9]). So it would be nice to prove Conjecture 2.16 to get better combinatorial bounds for the regularity of $K[S]$.

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