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(revised version: June 2011)

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Preprint no.: 13 2011

FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR ELASTICITY PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We analyze variational inequalities related to problems in the theory of elasticity that involve unilateral boundary conditions with or without friction. We are focused on deriving upper a posteriori estimates of difference between exact solutions of such type variational inequalities and any functions lying in the admissible functional class of the considered problem. These estimates are obtained by a modification of duality technique earlier used for variational problems with uniformly convex functionals by S. Repin. We also present a simple two dimensional axially symmetric problem with a friction boundary condition and derive an analytical solution. Several numerical tests are performed to demonstrate the quality of our developed estimates.

1. Introduction

The problem of how to properly define boundary conditions in a mathematical model d is of utmost importance in continuum mechanics. Usually considered Dirichlet or Neumann boundary conditions should often be replaced by more sophisticated conditions that much better reflect real physical conditions (e.g., friction or unilateral contact). A wide variety of boundary conditions can be studied within the framework of the conception of boundary dissipative potential which lead to variational inequalities of a special type (see, e.g., [6, 14, 11, 10]).

In this paper, we are concerned with deriving guaranteed and computable upper bounds of the difference between exact solutions of the corresponding variational inequality and any function in the admissible (energy) space. The estimates are obtained with the help of variational (duality) method that was developed in [15, 16, 17] for convex variational problems. In [5, 18, 12, 3] this method was applied to "classical" models associated with variational inequalities (e.g., to problems with obstacles) and in [22, 21, 4, 7] to various models generated by plasticity theory. Estimates for fourth order elliptic problems has been derived in [13]. A consequent exposition of the general theory can be found in two books [12, 19].

We note that in [20] some simpler models (for scalar valued variational problems) that involve nonlinear boundary conditions has been considered. However, these simplified problems cannot be considered as adequate representatives of real life problems, which are formulated in terms of stress (strain) tensors and vector valued functions of displacements (velocities). In this paper, we analyze this case taking the linear elasticity operator as the paradigm (similar consideration can be applied to other operators in continuum mechanics, e.g., to models of deformation plasticity with linear or power hardening subject to friction/contact type boundary conditions).

Outline of the paper is as follows. In section 2, we give a concise overview of mathematical notion related to the boundary potential theory and formulate some results which are used in subsequent sections. General form of the a posteriori estimate is presented in section 3 and estimates for particular boundary conditions further explained in section 4. Section 5 provides some numerical tests for a case nonlinear model with derived analytical solution.

2. Statement of a problem with nonlinear boundary conditions

2.1. **Classical statement.** Let $\Omega \in \mathbb{R}^d$, $d = 2,3$ be an open bounded domain with Lipschitz continuous boundary Γ. We assume that the boundary is piecewise smooth, so that one can uniquely define the unit outward normal in almost all points of Γ. It is assumed that Γ consists of two disjoint measurable parts Γ_0 and Γ_1. The domain is occupied by an elastic body, whose displacements u and stresses σ^* are subject to the *generalized Hooke's law*

$$\sigma^* = L\varepsilon(u)\,, \tag{2.1}$$

and the *equilibrium equation*

$$\mathrm{div}\sigma^* + f \;=\; 0\,, \tag{2.2}$$

where $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the tensor of small strains and $L : \mathbb{M}_s^{d\times d} \to \mathbb{M}_s^{d\times d}$ is the tensor of elastic constants. We assume that its components are bounded measurable functions and that the usual conditions of coercivity and symmetry take place, i.e.

$$c_\ominus \left|\kappa\right|^2 \;\leq\; L\kappa : \kappa \;\leq c_\oplus \left|\kappa\right|^2 \quad \forall \kappa \in \mathbb{M}_s^{d\times d}, \tag{2.3}$$

$$L_{ijkl} \;=\; L_{jikl} \;=\; L_{klij}\,. \tag{2.4}$$

Here two dots denotes the scalar product in $\mathbb{M}_s^{d\times d}$ and $|\kappa| := \sqrt{\kappa : \kappa}$ represents a matrix Frobenius norm. It is assumed that displacements are given on the part Γ_0, i.e.

$$u(x) = u_0(x), \qquad\quad x \in \Gamma_0. \tag{2.5}$$

The boundary conditions on Γ_1 are more complicated. To present them normal and tangential components of u and σ^* on Γ. Let

$$u_n = u \cdot n = u_i n_i, \quad u_\tau = u - (u \cdot n)n,$$
$$\sigma_n^* = \sigma^* n, \quad \sigma_t^* = \sigma_n^* - \sigma_{nn}^* n, \quad \sigma_{nn}^* = \sigma_n^* \cdot n = \sigma_{ij}^* n_i n_j,$$

where the convention of summation from 1 to d over repeated indexes is adopted. The traces of vector–valued functions u and σ_n^* admit the following decomposition on Γ:

$$u = u_n n + u_\tau, \qquad \sigma_n^* = \sigma_{nn}^* n + \sigma_t^*\,.$$

On Γ_1 the body may be subject to the action of surface forces or contact an obstacle. In practice, one can meet a wide spectrum of conditions (some of them are considered in Section 4), which have one common form

$$-\sigma_n^*(x) \in \partial j(u(x)) \quad x \in \Gamma_1\,, \tag{2.6}$$

where $j : \mathbb{R}^d \to \mathbb{R}$ is a convex lower semicontinuous functional (this functional is called the "boundary dissipative potential" – see e.g. [14]). For example, free surface condition corresponds to the case $j(u) \equiv 0$ (which means that $\sigma_n^* = 0$ on Γ_1) and a surface contacting an absolutely rigid obstacle (which leads to the Signorini problem) ψ is described by the potential

$$j(u) = \begin{cases} 0, & u_n - \psi \leq 0 \\ +\infty & u_n - \psi > 0. \end{cases}$$

In mechanics, the condition (2.6) is usually understood in the pointwise sense. However, in the mathematical statement, it must be associated with properties of the energy space containing a weak solution of problem in question. Therefore, we begin with recalling some facts in the theory of functions and establishing auxiliary results that are further used in our analysis.

2.2. **Functional spaces of traces.** We denote the spaces of square summable vector- and tensor–valued functions defined on the set S by $L_2(S,\mathbb{R}^d)$ and $L_2(S,\mathbb{M}_s^{d\times d})$, respectively. Their norms are associated with natural scalar products

$$\int_S u\cdot v\,ds \text{ and } \int_S \sigma : \tau\,ds.$$

Since no confusion my arise, we use for these norms one common symbol $\|\ \|$. We shall use special notations Σ and Σ^* for the spaces of admissible strains and stresses, respectively. In the considered case, they coincide with the space $L_2(\Omega,\mathbb{M}_s^{d\times d})$. However, by reasons that will become clear later, we keep different notation for this pair of spaces. We shall also use the space

$$Q^*(\Omega) := \left\{\sigma^*\in\Sigma^* \ \ \| \ \operatorname{div}\sigma^*\in L_2(\Omega,\mathbb{R}^d\,)\right\}\,.$$

It is known that Q^* is a Hilbert space with respect to the norm

$$\|\,\sigma^*\,\|^2_{Q^*} := \int\limits_\Omega (|\sigma^*|^2+|\operatorname{div}\sigma^*|^2)\,dx$$

and that smooth functions $C^\infty(\overline{\Omega},\mathbb{M}_s^{d\times d})$ are dense in Q^*. The Sobolev space $V=H^1(\Omega,\mathbb{R}^d)$ is the space containing admissible displacements. Let $\gamma\in\mathcal{L}\left(H^1(\Omega),H^{1/2}(\Gamma)\right)$ be the operator defining traces of H^1–functions on the boundary Γ. It is well known that the space of traces $H^{1/2}(\Gamma)$ is continuously embedded in $L_2(\Gamma)$ and is dense in it. The space $H^1_0(\Omega)$ is the kernel of γ. For any function $\phi\in H^{1/2}(\Gamma)$, one can define the continuation operator $\mu\in\mathcal{L}(H^{1/2}(\Gamma),H^1(\Omega))$ such that $\mu\phi=w$, $\gamma w=\phi$ on Γ and

$$\|\,\phi\,\|_{1/2,\Gamma}\le c_\gamma\|\,w\,\|_{1,\Omega},\qquad \|\,w\,\|_{1,\Omega}\le c_\mu\|\,\phi\,\|_{1/2,\Gamma}\,, \tag{2.7}$$

where $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{1/2,\Gamma}$ are norms in H^1 and $H^{1/2}$, respectively. For any $v\in V$, the operator $\gamma^d v := (\gamma v_1,\gamma v_2,\ldots,\gamma v_d)$ defines traces of vector–valued functions. Analogously, one can define the continuation operator μ^d. The operators γ^d and μ^d inherits the continuity property of γ and μ and meet the inequalities (2.7) in take the respective vector–valued norms. Evidently, the constants in the above inequalities depend on Ω and Γ. Under the assumptions made above, one can also define the normal component of $\gamma^d v$ as $\gamma^d_n v := (\gamma^d v)\cdot n$. It is easy to check that for a smooth function v the quantity $\gamma^d_n v$ coincides with v_n. By means of the operator γ^d we define the space

$$V_0 := \{v\in V \ \ \| \ \gamma^d v = 0 \text{ a.e. on } \Gamma_0\}\,,$$

which is a subspace of V. The set $\gamma^d(V_0)$ is a subspace of $H^{1/2}(\Omega,\Gamma)$. Hereafter, we denote this set by Z and the respective dual space by Z^* (also called $H^{-1/2}$), which can be identified with the set of traces on Γ_1 of functions belonging to $Q^*(\Omega)$. Really, for any smooth τ^* and any $v\in V_0$, we have the classical relation

$$\int\limits_{\Gamma_1}\tau^*_n\cdot\gamma^d v\,dx \;=\; \int\limits_\Omega(\tau^*:\boldsymbol{\varepsilon}(v)+\operatorname{div}\tau^*\cdot v)\,dx\,. \tag{2.8}$$

For any $\tau^*\in Q^*(\Omega)$, the right–hand side of this identity is a linear continuous functional $\Lambda_{\tau^*}: V_0\to\mathbb{R}$.

Lemma 1. *The functional Λ_{τ^*} satisfies the following relations:*

$$\Lambda_{\tau^*}(v) \;=\; 0 \qquad \forall v\in H^1_0(\Omega,\mathbb{R}^d)\,, \tag{2.9}$$

$$|\Lambda_{\tau^*}(v)|\le c_\mu\|\,\tau^*\,\|_{Q^*}\|\,\gamma^d v\,\|_{1/2,\Gamma}\,. \tag{2.10}$$

Proof. Let $v \in H^1_0(\Omega, \mathbb{R}^d)$. Smooth functions with compact supports are dense in H^1_0. Therefore, there is a sequence of smooth functions $\{w_k\}$ such that $w_k \to v$ in H^1. For any w_k, we have

$$\int\limits_\Omega \tau^* : \boldsymbol{\varepsilon}(w_k)\, dx + \int\limits_\Omega \mathrm{div}\tau^* \cdot w_k\, dx = 0\,.$$

Passing to the limit, we arrive at (2.9). The inequality (2.10) follows directly from the definition of Λ_{τ^*}:

$$|\Lambda_{\tau^*}(v)| \leq \|\, \tau^* \,\|_{Q^*} \|\, v \,\|_{H^1(\Omega,\mathbb{R}^d)} \leq c_\mu \|\, \tau^* \,\|_{Q^*} \|\, \gamma^d v \,\|_{1/2,\Gamma}\,.$$

□

In essence, Lemma 1 shows that Λ_{τ^*}, is a linear continuous mapping defined on a factorspace of V_0. Indeed,

$$\Lambda_{\tau^*}(v_1) = \Lambda_{\tau^*}(v_2) \qquad \text{if } v_1, v_2 \in V_0 \text{ and } \gamma^d v_1 = \gamma^d v_2\,.$$

Thus, in this factorspace two functions belong to one class if they have the same trace on Γ_1. This means that Λ_{τ^*} is a mapping from Z to $\mathbb{R}$ and, consequently, can be identified with a certain element in Z^*, which we denote $\delta^d_n \tau^*$ and call the trace of τ^* on Γ_1. Hereafter, we follow the usual convention and denote the value of $\xi^* \in Z^*$ on $\xi \in Z$ by means of the duality pairing $\langle \xi^*, \xi \rangle_{\Gamma_1}$. Then, (2.8) can be rewritten in the form

$$\Lambda_{\tau^*}(\gamma^d v) = \langle \delta^d_n \tau^*, \gamma^d v \rangle_{\Gamma_1} = \int\limits_\Omega (\tau^* : \boldsymbol{\varepsilon}(v) + \mathrm{div}\tau^* \cdot v)\, dx\,. \tag{2.11}$$

The norm of Λ_{τ^*} is defined by the relation

$$\|\, \delta^d_n \tau^* \,\|_{Z^*} = \sup_{v \in V_0} \frac{\int\limits_\Omega (\tau^* : \boldsymbol{\varepsilon}(v) + \mathrm{div}\tau^* \cdot v)\, dx}{\|\, \gamma^d v \,\|_Z}. \tag{2.12}$$

In view of (2.10), it is bounded:

$$\|\, \delta^d_n \tau^* \,\|_{Z^*} \leq c_\mu \|\, \tau^* \,\|_{Q^*}\,. \tag{2.13}$$

2.3. **Conjugate functionals defined on spaces of traces.** For any $\xi \in Z$ we define the functional

$$\Upsilon(\xi) := \int\limits_{\Gamma_1} j(\xi)\, d\Gamma\,.$$

We assume that the integrand $j : \mathbb{R}^d \to \mathbb{R}^d$ is a nonnegative, convex, and lower semicontinuous (l.s.c.) function. In addition, we assume that $j(0) = 0$ and

$$\mathrm{dom}\, j := \{p \in \mathbb{R}^d \,\|\, j(p) < +\infty\} \neq \emptyset\,,$$

so that j belongs to the class of so–called *proper* convex functions. In this case, the functional $\Upsilon(\xi)$ is nonnegative, convex and l.s.c. on Z . Since γ^d is a bounded linear operator, the functional $\Upsilon(\gamma^d v)$ also possesses the above properties as the functional on V_0. Let us introduce a new functional

$$\Upsilon^*(\xi^*) := \sup_{\xi \in Z} \left\{ \langle \xi^*, \xi \rangle_{\Gamma_1} - \Upsilon(\xi) \right\}, \tag{2.14}$$

which we call *conjugate* (in the sense of Young–Fenchel) to the functional Υ. Under the above assumptions, the functional $\Upsilon : Z \to \mathbb{R}$ coincides with pointwise supremum of all its affine minorants. It is easy to see that

$$\Upsilon(\xi) \geq \langle \xi^*, \xi \rangle_{\Gamma_1} + \lambda \qquad \forall\, \lambda \leq -\Upsilon^*(\xi^*).$$

This effectively means that

$$\Upsilon(\xi) = \sup_{\xi^* \in Z^*} \left\{ \langle \xi^*, \xi \rangle_{\Gamma_1} - \Upsilon^*(\xi^*) \right\} \tag{2.15}$$

By recalling (2.11), we see that

$$\Upsilon(\gamma^d v) = \sup_{\tau^* \in Q^*} \left\{ \int\limits_\Omega (\tau^* : \boldsymbol{\varepsilon}(v) + \mathrm{div}\tau^* \cdot v)\ dx - \Upsilon^*(\delta^d_n \tau^*) \right\}, \tag{2.16}$$

$$\Upsilon^*(\delta^d_n \tau^*) = \sup_{v \in V_0} \left\{ \int\limits_\Omega (\tau^* : \boldsymbol{\varepsilon}(v) + \mathrm{div}\tau^* \cdot v)\ dx - \Upsilon(\gamma^d v) \right\}. \tag{2.17}$$

In what follows we use the functional

$$\begin{aligned} \mathrm{D}_{\Gamma_1}(\gamma^d v, \delta^d_n \tau^*) &:= \Upsilon(\gamma^d v + \Upsilon^*(\delta^d_n \tau^*) - \langle \gamma^d v, \delta^d_n \tau^* \rangle_{\Gamma_1} = \\ \sup_{w \in V_0} \Big[\int\limits_\Omega &(\tau^* : \boldsymbol{\varepsilon}(w - v) + \mathrm{div}\tau^* \cdot (w - v))\, dx + \int\limits_{\Gamma_1} (j(\gamma^d v) - j(\gamma^d w))\, d\Gamma \Big], \end{aligned} \tag{2.18}$$

which is called the *compound* functional. It is easy to see that

$$\mathrm{D}_{\Gamma_1}(\gamma^d v, \delta^d_n \tau^*) \geq 0. \tag{2.19}$$

Moreover,

$$\mathrm{D}_{\Gamma_1}(\gamma^d v, \delta^d_n \tau^*) = 0 \ \Rightarrow\ \delta^d_n \tau^* \in \partial \Upsilon(\gamma^d v).$$

In the majority of practically interesting cases, the conjugate functional can be constructed with the help of algebraic conjugate integrand (see, e.g., [8]). In particular, if $\delta^d_n \tau^* \in L_2(\Gamma_1, \mathbb{R}^d)$ then

$$\Upsilon^*(\delta^d_n \tau^*) = \int\limits_{\Gamma_1} j^*(\delta^d_n \tau^*)\, dx\,,$$

where $j^* : \mathbb{R}^d \to \mathbb{R}$ is the function conjugate to j, i.e.

$$j^*(q^*) = \sup_{q \in \mathbb{R}^d} \{ q^* \cdot q - j(q) \}\ .$$

2.4. **Generalized statement.** On $V \times V$ we define the bilinear form

$$a(u, v) := \int\limits_\Omega L\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(v)\, dx.$$

The action of external forces is described by the linear functional

$$\ell(v) := \int\limits_\Omega f \cdot v\, dx\,.$$

Henceforth, we assume that $f \in L_2(\Omega, \mathbb{R}^d)$ and $u_0 \in V(\Omega)$.

Generalized formulation of the problem is presented in terms of a variational inequality.

Problem $\mathcal{P}$. Find $u \in V_0 + u_0 := \{w \;\|\; w = w_0 + u_0,\ w_0 \in V_0\}$ such that

$$a(u, w-u) + \Upsilon(w) - \Upsilon(u) \geq \ell(w-u) \qquad \forall w \in V_0 + u_0\,. \tag{2.20}$$

In view of the Lions-Stampacchia Lemma, this problem is equivalent to the variational problem: find $u \in V_0 + u_0$ such that

$$J(u) = inf\mathcal{P} := \inf_{w \in V_0 + u_0} J(w), \quad J(w) = \frac{1}{2}\| w \|_a^2 + \Upsilon(w) - \ell(w)\,, \tag{2.21}$$

where $\| w \|_a := (a(w,w))^{1/2}$. Since the functional J is strictly convex, continuous, and coercive on V and the set $V_0 + u_0$ is a convex closed subset of V, we arrive at the conclusion that Problem $\mathcal{P}$ is uniquely solvable.

Assume that the data of Problem $\mathcal{P}$ are such that the minimizer u is smooth. It is easy to show that in this case the minimizer u and the corresponding stress $\sigma = L\boldsymbol{\varepsilon}(u)$ satisfy the relations (2.1)–(2.5). Indeed, the condition

$$J(u) \leq J(u + \lambda v) \qquad \forall \lambda > 0,\ v \in C_0^\infty(\Omega, \mathbb{R}^d)$$

leads to the relation

$$\mathrm{div} L\boldsymbol{\varepsilon}(u) + f = 0 \qquad \text{a.e. in } \Omega\,. \tag{2.22}$$

Let w be an arbitrary element of $V_0 + u_0$. Then

$$0 = \int_\Omega (\mathrm{div} L\boldsymbol{\varepsilon}(u) + f)(w-u)\, dx = \tag{2.23}$$

$$= \int_\Omega (f \cdot (w-u) - L\boldsymbol{\varepsilon}(u) : \boldsymbol{\varepsilon}(w-u))\, dx + \int_{\Gamma_1} \sigma_n \cdot (w-u)\, d\Gamma\,. \tag{2.24}$$

Now (2.20) and (2.24) imply the inequality

$$\Upsilon(w) - \Upsilon(u) + \int_{\Gamma_1} \sigma_n \cdot (w-u) d\Gamma \geq 0, \tag{2.25}$$

which means that

$$-\sigma_n \in \partial j(u) \qquad \text{a.e. on } \Gamma_1\,. \tag{2.26}$$

3. Estimates of deviations

3.1. **General estimate.** The minimizer u to problem $\mathcal{P}$ meets the variational inequality (2.20). This leads to the inequality

$$\begin{aligned} J(v) - J(u) = \frac{1}{2}a(v-u, v-u) + \\ + a(u, v-u) - \langle f, v-u\rangle + \Upsilon(v) - \Upsilon(u) \geq \\ \geq \frac{1}{2}a(v-u, v-u) \qquad \forall\, v \in V_0 + u_0\,, \end{aligned}$$

which implies the basic "deviation" estimate

$$\frac{1}{2}\| v - u \|_a^2 \leq J(v) - \inf \mathcal{P} \quad \forall v \in V_0 + u_0\,, \tag{3.1}$$

In general, the exact lower bound $\inf \mathcal{P}$ is unknown so that (3.1) has mainly a theoretical meaning and cannot be directly used as a tool of error estimation. Our aim is to show that the right–hand side of (3.1) can be estimated from above by a quantity which is practically computable, possesses necessary continuity properties and has clear physical motivation. For this purpose, we consider the following perturbed functional

$$J_{\xi^*}(v) = \frac{1}{2}a(v,v) - \ell(v) + \langle \xi^*, \gamma^d v\rangle_{\Gamma_1} - \Upsilon^*(\xi^*)\,, \tag{3.2}$$

where ξ^* is an element in Z^*. Now, we arrive at the following
Perturbed Problem $\mathcal{P}_{\xi^*}$: find $u_{\xi^*} \in V_0 + u_0$ such that

$$J_{\xi^*}(u_{\xi^*}) \;=\; \inf_{v\in V_0+u_0} J_{\xi^*}(v) \;=\; \inf \mathcal{P}_{\xi^*}\,.$$

For any $\xi^* \in Z^*$, Problem $\mathcal{P}_{\xi^*}$ is a simple quadratic problem, which has a unique solution u_{ξ^*}. It is easy to see that

$$\sup_{\xi^*\in Z^*} J_{\xi^*}(v) = J(v)$$

and, consequently, for any $\xi^* \in Z^*$

$$\inf_{v\in V_0+u_0} J_{\xi^*}(v) \;\le\; \inf_{v\in V_0+u_0} J(v) \;=\; \inf \mathcal{P}. \tag{3.3}$$

The perturbed problem has a dual counterpart.

Problem $\mathcal{P}^*_{\xi^*}$: Find $\tau^*_{\xi^*} \in Q^*_{\ell_{\xi^*}}$ such that

$$I^*_{\xi^*}(\tau^*_{\xi^*}) \;=\; \sup_{\eta^*\in Q^*_{\ell_{\xi^*}}} I^*_{\xi^*}(\eta^*)\,,$$

where

$$I^*_{\xi^*}(\eta^*) = \int_\Omega \boldsymbol{\varepsilon}(u_0) : \eta^*\,dx - \frac{1}{2}a^*(\eta^*,\eta^*) - \ell_{\xi^*}(u_0) - \Upsilon^*(\xi^*)\,,$$

a^* is the bilinear form conjugate to a,

$$\ell_{\xi^*}(\cdot) = \ell(\cdot) - \langle \xi^*, \cdot\rangle_{\Gamma_1}$$

is a perturbed linear functional and

$$Q^*_{\ell_{\xi^*}} := \left\{ \eta^* \in \Sigma^* \quad \Big\| \quad \int_\Omega \eta^* : \boldsymbol{\varepsilon}(v)\,dx = \ell_{\xi^*}(v)\,, \ \ \forall v \in V_0 \right\}.$$

This problem also has a unique solution. Moreover,

$$\inf \mathcal{P}_{\xi^*} = \sup \mathcal{P}^*_{\xi^*}\,.$$

In view of the above connection between lower and upper bounds in Problems $\mathcal{P}_{\xi^*}$ and $\mathcal{P}^*_{\xi^*}$, we obtain

$$\frac{1}{2}\| v - u \|^2_a \;\le\; J(v) \;-\; \sup \mathcal{P}^*_{\xi^*} \;\le\; J(v) - I^*_{\xi^*}(\eta^*) \quad \forall \eta^* \in Q^*_{\ell_{\xi^*}}\,. \tag{3.4}$$

The right–hand side of (3.4) can be estimated as follows

$$
\begin{aligned}
J(v) - I^*_{\xi^*}(\eta^*) \;=\; & \frac{1}{2}a(v,v) + \frac{1}{2}a^*(\tau^*,\tau^*) - \int\limits_\Omega \boldsymbol{\varepsilon}(v) : \tau^*\,dx + \\
& + \Upsilon(\gamma^d v) + \Upsilon^*(\xi^*) - \ell(v) - \int\limits_\Omega \boldsymbol{\varepsilon}(u_0) : \eta^*\,dx - \ell_{\xi^*}(\gamma^d u_0) + \\
& + \int\limits_\Omega \boldsymbol{\varepsilon}(v) : \tau^*\,dx + \frac{1}{2}a^*(\eta^*,\eta^*) - \frac{1}{2}a(\tau^*,\tau^*)\,, \qquad (3.5)
\end{aligned}
$$

where τ^* is an arbitrary element of Y^*. Since

$$
\ell(v-u_0) = \int\limits_\Omega \eta^* : \boldsymbol{\varepsilon}(v-u_0)\,dx + \langle \xi^*, \gamma^d(v-u_0)\rangle_{\Gamma_1}, \qquad (3.6)
$$

we obtain

$$
\begin{aligned}
J(v) - I^*_{\xi^*}(\eta^*) \;=\; & \frac{1}{2}a(v,v) + \frac{1}{2}a^*(\tau^*,\tau^*) - \int\limits_\Omega \boldsymbol{\varepsilon}(v) : \tau^*\,dx + \\
& + \Upsilon(\gamma^d v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma^d v\rangle_{\Gamma_1} + \\
& + \int\limits_\Omega \boldsymbol{\varepsilon}(v) : (\tau^* - \eta^*)\,dx + \frac{1}{2}a^*(\eta^*,\eta^*) - \frac{1}{2}a(\tau^*,\tau^*)\,. \qquad (3.7)
\end{aligned}
$$

This identity has an equivalent form

$$
\begin{aligned}
J(v) - I^*_{\xi^*}(\eta^*) \;=\; & \frac{1}{2}\int\limits_\Omega (L\boldsymbol{\varepsilon}(v) : \boldsymbol{\varepsilon}(v) + L^*\tau^* : \tau^* - \boldsymbol{\varepsilon}(v) : \tau^*)\,dx + \\
& + \Upsilon(\gamma^d v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma^d v\rangle_{\Gamma_1} + \int\limits_\Omega (\boldsymbol{\varepsilon}(v) - L^*\tau^*) : (\tau^* - \eta^*)\,dx + \\
& + \frac{1}{2}\int\limits_\Omega L^*(\eta^* - \tau^*)(\eta^* - \tau^*)\,dx\,. \qquad (3.8)
\end{aligned}
$$

Now we use the inequality

$$
\eta : \eta^* \;\le\; \frac{\beta}{2}L\eta : \eta + \frac{1}{2\beta}L^*\eta^* : \eta^*\,,
$$

which is valid for all symmetric matrixes η and η^* and any $\beta > 0$. We obtain the estimate

$$
\begin{aligned}
\int_\Omega (\boldsymbol{\varepsilon}(v) - L^*\tau^*) : (\tau^* - \eta^*)\,dx \quad \le\; & \frac{\beta}{2}\int\limits_\Omega L(\boldsymbol{\varepsilon}(v) - L^*\tau^*) : (\boldsymbol{\varepsilon}(v) - L^*\tau^*)\,dx + \\
& + \frac{1}{2\beta}\int\limits_\Omega L^*(\tau^* - \eta^*) : (\tau^* - \eta^*)\,dx\,,
\end{aligned}
$$

which gives the relation

$$J(v) - I^*_{\xi^*}(\eta^*) \;=\; \frac{1}{2}(1+\beta)\int\limits_\Omega (L\boldsymbol{\varepsilon}(v) : \boldsymbol{\varepsilon}(v) + L^*\tau^* : \tau^* - 2\boldsymbol{\varepsilon}(v) : \tau^*)\,dx \,+ \\ + \Upsilon(\gamma^d v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma^d v\rangle_{\Gamma_1} + \\ + \frac{1}{2}\left(1+\beta^{-1}\right)\int\limits_\Omega L^*(\eta^* - \tau^*) : (\eta^* - \tau^*)\,dx. \quad (3.9)$$

Let us introduce the following quantities

$$M_1(v,\tau^*) = \frac{1}{2}\int\limits_\Omega (L\boldsymbol{\varepsilon}(v) : \boldsymbol{\varepsilon}(v) + L^*\tau^* : \tau^* - 2\boldsymbol{\varepsilon}(v) : \tau^*)\,dx, \quad (3.10)$$

$$M_2(\gamma^d v, \xi^*) = \mathrm{D}_\Upsilon(\gamma^d v, \xi^*) = \Upsilon(\gamma^d v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma^d v\rangle_{\Gamma_1}, \quad (3.11)$$

$$M_3(\tau^*, \xi^*) = \frac{1}{2}\inf_{\eta^* \in Q^*_{\ell_{\xi^*}}} \int\limits_\Omega L^*(\eta^* - \tau^*) : (\eta^* - \tau^*)\,dx. \quad (3.12)$$

Then (3.4), (3.9)– (3.12) result in the estimate

$$\frac{1}{2}\| v - u \|_a^2 \le (1+\beta)M_1(v,\tau^*) + M_2(\gamma^d v, \xi^*) + \left(1+\beta^{-1}\right)M_3(\tau^*, \xi^*) \quad (3.13)$$

where τ^*, ξ^* and β are arbitrary elements of the sets Y^*, Z^* and $\mathbb{R}_+$, respectively.

Let us discuss the meaning of three quantities in the right–hand side of (3.13). In view of Young–Fenchel inequality, M_1 and M_2 are evidently nonnegative. Since L^* is positive definite, the same proposition is true for M_3.

The quantity $M_1(v,\tau^*)$ vanishes if and only if v and τ^* satisfy the relation (2.4). Therefore, this term *is a measure of the error in the generalized Hooke's law.*

It is easy to see that $M_2(\gamma^d v, \xi^*) = 0$ if and only if

$$\xi^* = \partial\Upsilon(\gamma^d v) \qquad \text{on } \Gamma_1,$$

so that M_2 *is a measure of the error in the boundary condition (2.3)*, which vanishes if the function $-\xi^* \in Z^*$ and the trace of v on Γ_1 satisfy the subdifferential relation. $M_3(\tau^*)$ vanishes if and only if $\tau^* \in Q^*_{\ell_{\xi^*}}$, i.e.,

$$\int\limits_\Omega \tau^* : \boldsymbol{\varepsilon}(v)dx = \int\limits_\Omega f\cdot v dx - \langle \xi^*, \gamma^d v\rangle_{\Gamma_1} \qquad \forall v \in V_0\,.$$

In view of (2.11), for any $v \in V_0$

$$\int\limits_\Omega \tau^* : \boldsymbol{\varepsilon}(v)dx = \langle \delta^d_n\tau^*, \gamma^d v\rangle_{\Gamma_1} - \int\limits_\Omega \mathrm{div}\tau^* \cdot v dx \qquad \forall v \in V_0\,.$$

Hence, for any $v \in V_0$ we have

$$\int\limits_\Omega (f + \mathrm{div}\tau^*)\cdot v dx - \langle \xi^* + \delta^d_n\tau^*, \gamma^d v\rangle_{\Gamma_1} = 0\,,$$

which means that

(i) the equilibrium equation (2.2) holds;
(ii) the relation $\delta_n^d \tau^* = -\xi^*$ on Γ_1 holds (in a generalized sense).

In what follows, we will assume that the exact solution of the considered problem belongs to the set

$$Q^*_{\Gamma_1} := \left\{ \tau^* \in \Sigma^* \;\; \| \;\; \mathrm{div}\tau^* \in L_2(\Omega, \mathbb{R}^d),\; \delta_n^d \tau^* \in L_2(\Gamma_1, \mathbb{R}^d) \right\} .$$

The condition $\sigma^* \in Q^*_{\Gamma_1}$ holds if $f \in L_2(\Omega, \mathbb{R}^d)$ and the trace of σ^* on G_1 is a square summable function (this assumption is not very restrictive and holds in the majority of practically interesting cases).

3.2. **Another form of the estimate.** Now we concentrate on finding another form of the term M_3.

For this purpose we consider an auxiliary elasticity problem in the domain Ω. This problem is to find $\widetilde{u}$ and $\widetilde{\sigma^*}$ that satisfy the relations (2.1)– (2.4) where $f = g \in L_2(\Omega, \mathbb{R}^d)$ and the boundary condition on Γ_1 is $\sigma^*_n = G \in L_2(\Gamma_1, \mathbb{R}^d)$. Then, in view of the complementary energy principle, (see e.g. [6])

$$\sup_{\eta^* \in Q^*_g} \left[-\frac{1}{2} \int\limits_\Omega L^* \eta^* : \eta^* \, dx \right] = \inf_{w \in V_0} \left[\int\limits_\Omega \left(\frac{1}{2} L \boldsymbol{\varepsilon}(w) : \boldsymbol{\varepsilon}(w) - g \cdot w \right) dx - \int\limits_{\Gamma_1} G \cdot \gamma^d w \, d\Gamma \right], \quad (3.14)$$

where

$$Q^*_g := \left\{ \eta^* \in \Sigma^* \;\; \| \int\limits_\Omega \eta^* : \boldsymbol{\varepsilon}(v) dx = \int\limits_\Omega g \cdot w dx + \int\limits_{\Gamma_1} G \cdot \gamma^d w d\Gamma \;\; \forall w \in V_0 \right\} .$$

Let $\tau^* \in Q^*_{\Gamma_1}$ and $\eta^* \in Q^*_g$. Then

$$\int\limits_\Omega (\eta^* - \tau^*) : \boldsymbol{\varepsilon}(w) \, dx = \int\limits_\Omega (\mathrm{div} + g) \cdot w \, dx + \int\limits_{\Gamma_1} (G - \delta_n^d \tau^*) \cdot \gamma^d w \, d\Gamma \quad \forall w \in V_0 . \quad (3.15)$$

Let us set $\widetilde{g} = \mathrm{div}\tau^* + g \in L_2(\Omega, \mathbb{R}^d)$ and $\widetilde{G} = G - \delta_n^d \tau^* \in L_2(\Gamma_1, \mathbb{R}^d)$. Then $\kappa^* = \eta^* - \tau^*$ belongs to the set Q^*_g with $g = \widetilde{g}$ and $G = \widetilde{G}$ (hereafter it is called $Q^*_{\widetilde{g}}$). By the equality (3.14), we see that

$$\sup_{\kappa^* \in Q^*_{\widetilde{g}}} \left[-\frac{1}{2} \int\limits_\Omega L^* \kappa^* : \kappa^* \, dx \right] = \inf_{w \in V_0} \left[\int\limits_\Omega \left(\frac{1}{2} L \boldsymbol{\varepsilon}(w) : \boldsymbol{\varepsilon}(w) - \widetilde{g} \cdot w \right) dx - \int\limits_{\Gamma_1} \widetilde{G} \cdot \gamma^d w \, d\Gamma \right] . \quad (3.16)$$

Note that

$$\sup_{\kappa^* \in Q^*_{\widetilde{g}}} \left[-\frac{1}{2} \int\limits_\Omega L^* \kappa^* : \kappa^* \, dx \right] = \sup_{\eta^* \in Q^*_g} \left[-\frac{1}{2} \int\limits_\Omega L^* (\eta^* - \tau^*) : (\eta^* - \tau^*) \, dx \right] . \quad (3.17)$$

Thus, (3.16) and (3.17) means that

$$\sup_{\eta^*\in Q_g^*}\left[-\frac{1}{2}\int\limits_\Omega L^*(\eta^*-\tau^*):(\eta^*-\tau^*)dx\right]= \tag{3.18}$$

$$=\inf_{w\in V_0}\left[\int\limits_\Omega (\frac{1}{2}L\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)-\widetilde{g}\cdot w)\,dx-\int\limits_{\Gamma_1}\widetilde{G}\cdot\gamma^d w\,d\Gamma\right]= \tag{3.19}$$

$$=\inf_{w\in V_0}\left[\int\limits_\Omega (\frac{1}{2}L\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)-(\mathrm{div}\tau^*+g)\cdot w)dx-\int\limits_{\Gamma_1}(G-\delta_n^d\tau^*)\cdot\gamma^d wd\Gamma\right] \tag{3.20}$$

what gives the relation

$$\inf_{\eta^*\in Q_g^*}\left[\frac{1}{2}\int\limits_\Omega L^*(\eta^*-\tau^*):(\eta^*-\tau^*)dx\right]= \tag{3.21}$$

$$=-\inf_{w\in V_0}\left[\int\limits_\Omega\left(\frac{1}{2}L\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)-(\mathrm{div}\tau^*+g)\cdot w\right)dx-\int\limits_{\Gamma_1}(G-\delta_n^d\tau^*)\cdot\gamma^d wd\Gamma\right]. \tag{3.22}$$

The set $Q^*_{\ell_{\xi^*}}$ coincides with Q^*_g if $g=f$ and $G=-\xi^*\in L_2(\Gamma_1,\mathbb{R}^d)$. By applying (3.22), we obtain

$$\inf_{\eta^*\in Q^*_{\ell_{\xi^*}}}\left[\frac{1}{2}\int\limits_\Omega L^*(\eta^*-\tau^*):(\eta^*-\tau^*)dx\right]= \tag{3.23}$$

$$=-\inf_{w\in V_0}\left[\int\limits_\Omega\left(\frac{1}{2}L\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)-(\mathrm{div}\tau^*+f)\cdot w\right)dx+\int\limits_{\Gamma_1}(\xi^*+\delta_n^d\tau^*)\cdot\gamma^d wd\Gamma\right]. \tag{3.24}$$

It is easy to see that

$$\inf_{w\in V_0}\left[\int\limits_\Omega\left(\frac{1}{2}L\boldsymbol{\varepsilon}(w):\boldsymbol{\varepsilon}(w)-(\mathrm{div}\tau^*+f)\cdot w\right)dx+\int\limits_{\Gamma_1}(\xi^*+\delta_n^d\tau^*)\cdot\gamma^d w\,d\Gamma\right]\geq$$

$$\geq\inf_{w\in V_0}\left[\frac{1}{2}a(w,w)-r_\Omega(\tau^*)\|\,w\,\|_\Omega-r_{\Gamma_1}(\tau^*,\xi^*)\|\,\gamma^d w\,\|_{\Gamma_1}\right], \tag{3.25}$$

where

$$r_\Omega(\tau^*)^2:=\int\limits_\Omega(\mathrm{div}\tau^*+f)^2dx,\quad r_{\Gamma_1}(\delta_n^d\tau^*,\xi^*)^2:=\int\limits_{\Gamma_1}(\xi^*+\delta_n^d\tau^*)^2d\Gamma. \tag{3.26}$$

In view of embedding theorems for functions and their traces, there exist constants $C_{(\Omega)}$, and $C_{(\Omega,\Gamma_1)}$ such that

$$\|\,w\,\|^2_\Omega\leq C^2_{(\Omega)}\,a(w,w)\,, \tag{3.27}$$

$$\|\,\gamma^d w\,\|^2_{\Gamma_1}\leq C^2_{(\Omega,\Gamma_1)}a(w,w) \tag{3.28}$$

for all $w \in V_0$. Then the right–hand side of (3.25) is bounded from below by the quantity

$$\inf_{z\in\mathbb{R}_+}\left\{\frac{z^2}{2}-\left(C_{(\Omega)}r_\Omega(\tau^*)+C_{(\Omega,\Gamma_1)}r_{\Gamma_1}(\delta_n^d\tau^*,\xi^*)\right)z\right\}=-\tfrac{1}{2}\left(C_{(\Omega)}r_\Omega(\tau^*)+C_{(\Omega,\Gamma_1)}r_{\Gamma_1}(\delta_n^d\tau^*,\xi^*)\right)^2 .$$

Thus, we have

$$\tfrac{1}{2}\|\,v-u\,\|_a^2\le(1+\beta)M_1(v,\tau^*)+M_2(\gamma^d v,\xi^*)+ \\ +\tfrac{1}{2}\left(1+\beta^{-1}\right)\left(C_{(\Omega)}r_\Omega(\tau^*)+C_{(\Omega,\Gamma_1)}r_{\Gamma_1}(\delta_n^d\tau^*,\xi^*)\right)^2 \quad (3.29)$$

This inequality has some special forms. The first form follows from (3.29) if set $\xi^*=-\delta_n^d\tau^*$. In this case, $r_{\Gamma_1}(\delta_n^d\tau^*\xi^*)=0$ and we have

$$\tfrac{1}{2}\|\,v-u\,\|_a^2\le(1+\beta)M_1(v,\tau^*)+M_2(\gamma^d v,\delta_n^d\tau^*)+\tfrac{1}{2}\left(1+\beta^{-1}\right)C_{(\Omega)}^2 r_\Omega^2(\tau^*). \quad (3.30)$$

The second one arises after estimating the last term of (3.29). Then, we obtain the inequality

$$\tfrac{1}{2}\|\,v-u\,\|_a^2\le(1+\beta)M_1(v,\tau^*)+M_2(\gamma^d v,\xi^*) \\ +\tfrac{1}{2}\left(1+\beta^{-1}\right)(1+\alpha)C_{(\Omega)}^2 r_\Omega^2(\tau^*)+\tfrac{1}{2}\left(1+\beta^{-1}\right)\left(1+\alpha^{-1}\right)C_{(\Omega,\Gamma_1)}^2 r_{\Gamma_1}^2(\delta_n^d\tau^*,\xi^*)\,, \quad (3.31)$$

which involves an (arbitrary) positive constant α. Let us gather all terms related to the boundary condition on Γ_1. They are

$$I_{\Gamma_1}(\gamma^d v,\delta_n^d\tau^*,\xi^*) = M_2(\gamma^d v,\xi^*)+\tfrac{1}{2}\left(1+\beta^{-1}\right)\left(1+\alpha^{-1}\right)C_{(\Omega,\Gamma_1)}^2 r_{\Gamma_1}^2(\delta_n^d\tau^*,\xi^*) \quad (3.32)$$

$$= \int_{\Gamma_1}\left(j(\gamma^d v)+j^*(\xi^*)-\xi^*\cdot\gamma^d v+\tfrac{\theta}{2}\left|\delta_n^d\tau^*+\xi^*\right|^2\right)d\Gamma, \quad (3.33)$$

where

$$\theta=\left(1+\beta^{-1}\right)\left(1+\alpha^{-1}\right)C_{(\Omega,\Gamma_1)}^2. \quad (3.34)$$

To minimize the right–hand side of (3.31) we should minimize I_{Γ_1} over all $\xi^*\in L_2(\Gamma_1,\mathbb{R}^d)$. The corresponding result is given by the following proposition.

Lemma 2. *Under the above made assumptions,*

$$\inf_{\xi^*\in L_2} I_{\Gamma_1}(\gamma^d v,\delta_n^d\tau^*,\xi^*)=M_{\Gamma_1}(\gamma^d v,\delta_n^d\tau^*,\theta):=\int_{\Gamma_1}\left(j(\gamma^d v)+\frac{\theta}{2}\left|\delta_n^d\tau^*\right|^2-\phi(\gamma^d v-\theta\delta_n^d\tau^*)\right)d\Gamma, \quad (3.35)$$

where $\phi:\mathbb{R}^d\to\mathbb{R}$ *is the function conjugate to* $j^*(\xi^*)+\frac{\theta}{2}|\xi^*|^2$.

Proof. The direct reformulation shows

$$\inf_{\xi^*\in L_2} I_{\Gamma_1}(\gamma^d v,\delta_n^d\tau^*,\xi^*)=-\sup_{\xi^*\in L_2}-I_{\Gamma_1}(\gamma^d v,\delta_n^d\tau^*,\xi^*)=$$

$$=\int_{\Gamma_1}\left(j(\gamma^d v)+\frac{\theta}{2}\left|\delta_n^d\tau^*\right|^2\right)d\Gamma-\sup_{\xi^*\in L_2}\int_{\Gamma_1}\left(\xi^*\cdot(\gamma^d v-\theta\delta_n^d\tau^*)-j^*(\xi^*)-\frac{\theta}{2}\left|\xi^{*2}\right|\right)d\Gamma.$$

□

Now the estimate (3.31) comes in a new form

$$\tfrac{1}{2}\|\,v-u\,\|_a^2\le(1+\beta)M_1(v,\tau^*)+\tfrac{1}{2}\left(1+\beta^{-1}\right)(1+\alpha)C_{(\Omega)}^2 r_\Omega^2(\tau^*)+M_{\Gamma_1}(\gamma^d v,\xi^*,\theta)\,. \quad (3.36)$$

3.3. **Convergence.** Assume that

$$\begin{aligned} v_k &\to u && \text{in } V, \\ \tau_k^* &\to \sigma && \text{in } Y^*, \\ {\xi^*}_k &\to -\sigma_n && \text{in } Z^*. \end{aligned}$$

Then traces of v_k tend to the trace of u in $H^{1/2}$, so that

$$\begin{aligned} \liminf_{k\to\infty} j(v_k) &\geq j(v), \\ \liminf_{k\to\infty} \Upsilon^*({\xi^*}_k) &\geq \Upsilon^*(-\sigma_n), \\ \lim_{k\to\infty} \langle {\xi^*}_k, v_k \rangle_{\Gamma_1} &= \langle -\sigma_n, u \rangle_{\Gamma_1}. \end{aligned}$$

4. Estimates in particular cases

4.1. **Neumann boundary condition.** This type boundary condition corresponds to the case, in which Υ is a linear functional, i.e.

$$\Upsilon(\xi) := \langle \eta^*, \xi \rangle_{\Gamma_1} \tag{4.1}$$

where $\eta^* \in Z^*$. In particular, if η^* is associated with square summable (on Γ_1) function $-F$, then one can set

$$j(v) = -F \cdot v \qquad \text{a.e. on } \Gamma_1 \tag{4.2}$$

which corresponds according to (2.6) to the Neumann boundary condition

$$\sigma_n^* = F \qquad \text{a.e. on } \Gamma_1 . \tag{4.3}$$

Then

$$\Upsilon(\xi) = -\int\limits_{\Gamma_1} F \cdot \xi d\Gamma,$$

$$\Upsilon^*(\xi^*) = \begin{cases} 0, & \text{if } \xi^* = -F \text{ a.e. on } \Gamma_1, \\ +\infty & \text{otherwise.} \end{cases}$$

In the case $\xi^* = -F$ a.e. on Γ_1, we obtain

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} \left(-F \cdot \gamma^d v + 0 - (-F) \cdot \gamma^d v + \tfrac{\theta}{2} \left| \delta_n^d \tau^* - F \right|^2 \right) dx = \tfrac{\theta}{2} \int\limits_{\Gamma_1} \left| \delta_n^d \tau^* - F \right|^2 d\Gamma$$

and the majorant estimate (3.36) comes in the form

$$\tfrac{1}{2} \| v - u \|_a^2 \leq (1+\beta) M_1(v, \tau^*) + \tfrac{1}{2} \left(1 + \beta^{-1}\right)(1+\alpha) C^2_{(\Omega)} r^2_\Omega(\tau^*) + \tfrac{\theta}{2} \int\limits_{\Gamma_1} \left| \delta_n^d \tau^* - F \right|^2 d\Gamma. \tag{4.4}$$

4.2. **Winkler boundary condition.** In this case, a body is connected on Γ_1 with an elastic foundation which provides a certain response to boundary deflections (such condition can be modelled by a large amount of springs connected with Γ_1). Let

$$j(v) = \frac{1}{2} \left(k_n \left| v_n \right|^2 + k_\tau \left| v_\tau \right|^2 \right), \tag{4.5}$$

where k_n and k_τ are the Winkler constants associated with normal and tangential deflections v_n and v_t, respectively. It corresponds to the boundary conditions

$$-\sigma_n^* = k_n u_n, \quad -\sigma_\tau^* = k_\tau u_\tau \qquad \text{a.e. on } \Gamma_1 .$$

The conjugate functional to (4.5) reads

$$j^*(q^*) = \sup_{q\in\mathbb{R}^d} \left\{ q_n^* \cdot q_n + q_\tau^* \cdot q_\tau - \frac{1}{2}(k_n\,|q_n|^2 - k_\tau\,|q_\tau|^2) \right\} = \frac{1}{2k_n}\,|q_n^*|^2 + \frac{1}{2k_\tau}\,|q_\tau^*|^2$$

and therefore

$$I_{\Gamma_1} = \tfrac{1}{2}\int\limits_{\Gamma_1} \Big(k_n\,|v_n|^2 + k_\tau\,|v_\tau|^2 + \tfrac{1}{k_n}\,|\xi_n^*|^2 + \tfrac{1}{k_\tau}\,|\xi_\tau^*|^2 - 2\xi^*\cdot v + \theta\,|\tau_n^* + \xi_n^*|^2 + \theta\,|\tau_\tau^* + \xi_\tau^*|^2 \Big) d\Gamma.$$

The minimization of this quantity over ξ^* leads to the conditions

$$\frac{1}{k_n}\xi_n^* - v_n + \theta(\tau_n^* + \xi_n^*) = 0, \quad \frac{1}{k_\tau}{\xi^*}_\tau - v_\tau + \theta(\tau_\tau^* + \xi_\tau^*) = 0$$

with the solution

$$\xi_n^* = \frac{k_n(v_n - \theta\tau_n^*)}{1 + k_n\theta}, \quad \xi_\tau^* = \frac{k_\tau(v_\tau - \theta\tau_\tau^*)}{1 + k_\tau\theta}.$$

This gives

$$I_{\Gamma_1} = \tfrac{1}{2}\int\limits_{\Gamma_1} \Big(\frac{\theta}{1 + k_n\theta}(k_n v_n + \tau_n^*)^2 + \frac{\theta}{1 + k_\tau\theta}(k_\tau v_\tau + \tau_\tau^*)^2 \Big) d\Gamma, \tag{4.6}$$

which can be further substituted into (3.36) to obtain a majorant estimate similar to (4.4).

4.3. **Friction boundary condition.** This condition is characterized by the dissipation functional

$$j(v) = k_\tau\,|v_\tau|\,, \tag{4.7}$$

and the non-penetration conditions through the boundary

$$v_n = 0 \qquad \text{a.e. on}\,\Gamma_1. \tag{4.8}$$

The conjugate functional to (4.7) reads

$$j^*(q^*) = \sup_{q_\tau\in\mathbb{R}} \{q_\tau^* \cdot q_\tau - k_\tau\,|q_\tau|\} = \begin{cases} 0, & \text{if } |q_\tau^*| \le k_\tau, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.9}$$

Therefore, under the condition $|\xi_\tau^*| \le k_\tau$, we have

$$I_{\Gamma_1} := \int\limits_{\Gamma_1} \Big(k_\tau\,|v_\tau| - \xi_\tau^* v_\tau + \theta\,|\tau_\tau^* + \xi_\tau^*|^2 \Big) d\Gamma \tag{4.10}$$

and the minimization of this quantity over ξ_τ^* leads to the condition $-v_\tau + 2\theta(\tau_\tau^* + \xi_\tau^*) = 0$ with the solution

$$\xi_\tau^* = \frac{v_\tau}{2\theta} - \tau_\tau^*. \tag{4.11}$$

If ξ_τ^* from (4.11) does not satisfy the condition $|\xi_\tau^*| \le k_\tau$, then ξ_τ^* is one of the values

$$\xi_\tau^* = -k_\tau \quad \text{or} \quad \xi_\tau^* = -k_\tau. \tag{4.12}$$

5. Example with known analytical solution

5.1. **Analytical solution.** We consider the axially symmetric problem for a ring domain given in the polar coordinates (r, ϕ) as

$$\Omega = \{(r, \phi) \ \ \| \ r \in (a, b), \ \phi \in [0, 2\pi)\} .$$

This two dimensional ring model represents a dimension reduction of a three dimensional infinitely long cylinder stretched in the third z direction. A boundary of Ω consists of two circular parts

$$\Gamma_0 = \{(r, \phi) \ \ \| \ r = a\} , \quad \Gamma_1 = \{(r, \phi) \ \ \| \ r = b\} .$$

We assume that the right-hand side term f is axially symmetric (i.e. it does not depend on ϕ),

$$f = (f_r(r), f_\phi(r)) , \tag{5.1}$$

and the boundary conditions at Γ_0 are given as

$$u_r(a) = U_r, \qquad u_\phi(a) = U_\phi. \tag{5.2}$$

Therefore, it is natural to assume the solution u also axially symmetric

$$u = (u_r(r), u_\phi(r)). \tag{5.3}$$

and the ring model even transforms to a one dimensional model. We consider a plane strain model (and not a plane stress model), which assumes no displacement in z direction. It means that the symmetric strain tensor $\boldsymbol{\varepsilon}(u)$ has only two-dimensional nonzero components and it holds

$$\boldsymbol{\varepsilon}_{rr}(u) \ = \ u_{r,r}, \tag{5.4}$$

$$\boldsymbol{\varepsilon}_{\phi\phi}(u) \ = \ \frac{u_r}{r}, \tag{5.5}$$

$$\boldsymbol{\varepsilon}_{r\phi}(u) \ = \ \tfrac{1}{2}(u_{\phi,r} - \tfrac{u_\phi}{r}) \tag{5.6}$$

in case of axially symmetric displacement (5.3). The stress tensor σ^* is related to the strain tensor $\boldsymbol{\varepsilon}$ via the Hook's law

$$\sigma^*_{rr} = s_1 \boldsymbol{\varepsilon}_{rr} + s_2 \boldsymbol{\varepsilon}_{\phi\phi}, \tag{5.7}$$

$$\sigma^*_{\phi\phi} = s_2 \boldsymbol{\varepsilon}_{rr} + s_1 \boldsymbol{\varepsilon}_{\phi\phi}, \tag{5.8}$$

$$\sigma^*_{r\phi} = s_3 \boldsymbol{\varepsilon}_{r\phi}, \tag{5.9}$$

where coefficients

$$s_1 := K + \frac{4G}{3}, \quad s_2 := K - \frac{2G}{3}, \quad s_3 := \frac{1}{2G} \tag{5.10}$$

only depend on material parameters, i.e., a bulk modulus K and a shear modulus G. Using a different known set of Lamé parameters λ, μ, the same coefficients reformulate as

$$s_1 := 2\mu + \lambda, \quad s_2 := \lambda, \quad s_3 := 2\mu. \tag{5.11}$$

The Hook's law (5.7)- (5.9) can be inverted to obtain relations

$$\boldsymbol{\varepsilon}_{rr} = t_1 \sigma^*_{rr} + t_2 \sigma^*_{\phi\phi}, \tag{5.12}$$

$$\boldsymbol{\varepsilon}_{\phi\phi} = t_2 \sigma^*_{rr} + t_1 \sigma^*_{\phi\phi}, \tag{5.13}$$

$$\boldsymbol{\varepsilon}_{r\phi} = t_3 \sigma^*_{r\phi}, \tag{5.14}$$

where

$$t_1 := \frac{3K + 4G}{4(3K + G)G}, \quad t_2 := \frac{-3K - 2G}{4(3K + G)G}, \quad t_3 = 2G \tag{5.15}$$

or

$$t_1 := \frac{\lambda + 2\mu}{4\mu(\lambda+\mu)}, \quad t_2 := -\frac{\lambda}{4\mu(\lambda+\mu)}, \quad t_3 = \frac{1}{2\mu}. \tag{5.16}$$

Further, we consider parameters K, G only but the whole analysis can be easily transformed to parameters λ, μ. It is easy to see from (5.7)- (5.8) that

$$\sigma^*_{rr} - \sigma^*_{\phi\phi} = 2G(\varepsilon_{rr} - \varepsilon_{\phi\phi}) = 2G(u_{r,r} - \frac{u_r}{r}) = 2Gr\frac{d}{dr}(\frac{u_r}{r}). \tag{5.17}$$

Finally, the equilibrium equation $\mathrm{div}\sigma + f = 0$ rewrites in polar coordinates for the axial symmetric case as

$$\sigma^*_{rr,r} + \frac{1}{r}(\sigma^*_{rr} - \sigma^*_{\phi\phi}) + f_r = 0, \tag{5.18}$$

$$\sigma^*_{r\phi,r} + \frac{2}{r}\sigma^*_{r\phi} + f_\phi = 0. \tag{5.19}$$

More details on simplification of elastic equations in polar coordinates can be found in [23]. The substitution of (5.17) in (5.18) yields

$$\sigma^*_{rr,r} + 2G\frac{d}{dr}(\frac{u_r}{r}) + f_r = 0 \tag{5.20}$$

and for the case of zero external radial forces, $f_r = 0$, the integration over r provides

$$\sigma^*_{rr} + 2G\frac{u_r}{r} = c_1 \tag{5.21}$$

for some constant $c_1 \in \mathbb{R}$. Using (5.4), (5.5), (5.7) and an obvious equality $s_2 + 2G = s_1$, the latest equation reformulates as an ordinary differential equation

$$s_1(u_{r,r} + \frac{u_r}{r}) = c_1 \tag{5.22}$$

for a function $u_r : (a,b) \to \mathbb{R}$. This equation has a solution (the factor $s_1/2$ is included in the constant c_1)

$$u_r(r) = c_1 r + c_2/r \tag{5.23}$$

for some constant $c_2 \in \mathbb{R}$. If we also neglect the axial component, $f_\phi = 0$, (5.19) can be integrated to obtain

$$u_\phi = c_3 r + c_4/r \tag{5.24}$$

for some constants $c_3, c_4 \in \mathbb{R}$. Consequently, components of the exact strains tensor ε read

$$\varepsilon_{rr} = c_1 - \frac{c_2}{r^2}, \quad \varepsilon_{\phi\phi} = c_1 + \frac{c_2}{r^2}, \quad \varepsilon_{r\phi} = -\frac{c_4}{r^2}. \tag{5.25}$$

The elastic energy part is given due to (5.7)- (5.9) by

$$\begin{aligned} a(u,u) &= \int_\Omega L\varepsilon(u) : \varepsilon(u)\,dx = \int_\Omega \sigma^*(u) : \varepsilon(u)\,dx = \\ &= \int_\Omega \Big(s_1[(\varepsilon_{rr})^2 + (\varepsilon_{\phi\phi})^2] + 2s_2\varepsilon_{rr}\varepsilon_{\phi\phi} + 2s_3(\varepsilon_{r\phi})^2\Big)\,dx = \\ &= 2\pi\int_b^a \Big(s_1[(\varepsilon_{rr})^2 + (\varepsilon_{\phi\phi})^2] + 2s_2\varepsilon_{rr}\varepsilon_{\phi\phi} + 2s_3(\varepsilon_{r\phi})^2\Big) r\,dr \end{aligned} \tag{5.26}$$

and the substitution of (5.25) gives an exact analytical formula

$$a(u,u) = \frac{2\pi\left(b^2-a^2\right)\left(b^2{c_1}^2\left(s_1+s_2\right)a^2+\left(s_1-s_2\right){c_2}^2+{c_4}^2 s_3\right)}{a^2b^2}. \tag{5.27}$$

Constants c_1, c_2, c_3, c_4 are derived from the minimal energy principle (2.21) with neglected external forces

$$\frac{1}{2}a(u,u) \,+\, \Upsilon(u) \to \min. \tag{5.28}$$

for various cases of boundary conditions. All supporting computations were done in Maple.

5.1.1. *Exact solution for Dirichlet-Dirichlet conditions.* In the case of the Dirichlet boundary conditions on both radiuses,

$$(u_r, u_\phi)_{r=a} = (U_{ra}, U_{\phi a}), \quad (u_r, u_\phi)_{r=b} = (U_{rb}, U_{\phi b}), \tag{5.29}$$

the direct substitution into (5.23)- (5.24) provides the coefficients

$$c_1 = \tfrac{U_{rb}b-U_{ra}a}{b^2-a^2}, \quad c_2 = \tfrac{(U_{ra}b-U_{rb}a)ab}{b^2-a^2}, \tag{5.30}$$

$$c_3 = \tfrac{U_{\phi b}b-U_{\phi a}a}{b^2-a^2}, \quad c_4 = \tfrac{(U_{\phi a}b-U_{\phi b}a)ab}{b^2-a^2}. \tag{5.31}$$

5.1.2. *Exact solution for Dirichlet-Neumann conditions.* Assume the Dirichlet boundary condition on Γ_0

$$(u_r, u_\phi)_{r=a} = (U_{ra}, U_{\phi a}) \tag{5.32}$$

and conditions on Γ_1 represented by the functional

$$\Upsilon(u) = -\int_{\Gamma_1} (F_{rb}, F_{\phi b}) \cdot (u_r, u_\phi)\, dx \tag{5.33}$$

for some $F_{rb}, F_{\phi b} \in \mathbb{R}$. These conditions alternatively represent the Neumann boundary condition

$$(\sigma^*_{rr}, \sigma^*_{r\phi})_{r=b} = (F_{rb}, F_{\phi b}). \tag{5.34}$$

Then, the minimal energy principle (5.28) yields the solution (5.23)- (5.24) with coefficients

$$c_1 = \tfrac{U_{ra}(s_1-s_2)a+F_{rb}b^2}{(s_1-s_2)a^2+(s_1+s_2)b^2}, \quad c_2 = \tfrac{(U_{ra}(s_1+s_2)a-F_{rb}a)ab^2}{(s_1-s_2)a^2+(s_1+s_2)b^2}, \tag{5.35}$$

$$c_3 = \tfrac{U_{\phi a}s_3a+F_{\phi b}b^2}{s_3a^2}, \quad c_4 = -\tfrac{F_{\phi b}b^2}{s_3}. \tag{5.36}$$

5.1.3. *Exact solution for Dirichlet-friction conditions.* Assume the Dirichlet boundary condition on Γ_0

$$(u_r, u_\phi)_{r=a} = (U_{ra}, U_{\phi a}). \tag{5.37}$$

and the friction boundary condition on Γ_1 represented by the functional

$$\Upsilon(u) = \frac{1}{2}\int_{\Gamma_1} \Big(k_{\phi b}\,|u_\phi|\Big)\, dx \tag{5.38}$$

for some $k_{\phi b} \geq 0$ together with the nonpenetration boundary condition

$$u_r(r=b) = 0. \tag{5.39}$$

The substitution of (5.39) in (5.23) implies

$$c_1 = -\tfrac{U_{ra}a}{b^2-a^2}, \quad c_2 = \tfrac{U_{ra}a^2b}{b^2-a^2}. \tag{5.40}$$

and the combination of (5.37) and (5.24) yields

$$c_3 = \frac{aU_{\phi a} - c_4}{a^2} \tag{5.41}$$

and (5.24) rewrites as

$$u_\phi(r = b) = \frac{U_{\phi a} ab^2 - c_4(b^2 - a^2)}{a^2 b}. \tag{5.42}$$

The nondifferentibiality of the term $\Upsilon(u)$ from (5.38) divides the analysis into three cases:

(1) Sliding in positive direction: $u_\phi(r = b) > 0$. This is equivalent to the condition

$$U_{\phi a} ab^2 > c_4(b^2 - a^2) \tag{5.43}$$

and the friction condition can be replaced by the Neumann condition for $F_{rb} = 0, F_{\phi b} = -k_{\phi b}$. It leads to the solution

$$c_3 = \frac{U_{\phi a} s_3 a - k_{\phi b} b^2}{s_3 a^2}, \quad c_4 = \frac{k_{\phi b} b^2}{s_3}. \tag{5.44}$$

The back substitution of the constant c_4 into (5.43) shows that $u_\phi(r = b)$ is indeed nonnegative for

$$U_{\phi a} > \frac{b^2 - a^2}{a} \frac{k_{\phi b}}{s_3}. \tag{5.45}$$

(2) Sliding in negative direction: $u_\phi(r = b) < 0$. Similarly as in the positive case, we obtain the solution

$$c_3 = \frac{U_{\phi a} s_3 a + k_{\phi b} b^2}{s_3 a^2}, \quad c_4 = -\frac{k_{\phi b} b^2}{s_3}, \tag{5.46}$$

which is valid under the condition

$$U_{\phi a} < -\frac{b^2 - a^2}{a} \frac{k_{\phi b}}{s_3}. \tag{5.47}$$

(3) No sliding: $u_\phi(r = b) = 0$. It lead to the solution

$$c_3 = -\frac{U_{\phi a} a}{b^2 - a^2}, \quad c_4 = \frac{U_{\phi a} ab^2}{b^2 - a^2} \tag{5.48}$$

and it only happens if

$$-\frac{b^2 - a^2}{a} \frac{k_{\phi b}}{s_3} \le U_{\phi a} \le \frac{b^2 - a^2}{a} \frac{k_{\phi b}}{s_3}. \tag{5.49}$$

5.2. **Estimate of constant C_Ω.** Under the axisymmetric assumptions (5.4)- (5.6), inequality (3.27) is reduced to

$$\int_a^b r\Big(u_r^2 + u_\phi^2\Big)\, dr \quad \le \quad C_\Omega^2 \int_b^a r\Big(s_1[(\boldsymbol{\varepsilon}_{rr})^2 + (\boldsymbol{\varepsilon}_{\phi\phi})^2] + 2s_2 \boldsymbol{\varepsilon}_{rr}\boldsymbol{\varepsilon}_{\phi\phi} + 2s_3(\boldsymbol{\varepsilon}_{r\phi})^2\Big)\, dr = \tag{5.50}$$

$$= \quad C_\Omega^2 \int_a^b \Big(s_1 r[u_{r,r}^2 + \frac{u_r^2}{r^2}] + 2s_2 u_{r,r} u_r + \tfrac{1}{2} s_3 r[u_{\phi,r} - \frac{u_\phi}{r}]^2\Big) dr \tag{5.51}$$

It is equivalent to finding the constant C_Ω such that inequalities

$$\int_a^b r u_r^2\,dr \quad \leq \quad C_\Omega^2 \int_a^b \Big(s_1 r[u_{r,r}^2 + \frac{u_r^2}{r^2}] + 2s_2 u_{r,r} u_r\Big) dr, \tag{5.52}$$

$$\int_a^b r u_\phi^2\,dr \quad \leq \quad C_\Omega^2 \int_a^b \tfrac{1}{2} s_3 r[u_{\phi,r} - \frac{u_\phi}{r}]^2 dr \tag{5.53}$$

are satisfied simultaneously for all functions

$$u_r \in H^1(a,b) : u_r|_a = 0, \tag{5.54}$$

$$u_\phi \in H^1(a,b) : u_\phi|_a = 0. \tag{5.55}$$

Trace values of u_r and u_ϕ in (5.54) and (5.55) correspond to Dirichlet boundary counditions on the boundary $r = a$. The Friedrichs inequality $\int_b^a u_r^2\,dr \leq (b-a)^2 \int_b^a u_{r,r}^2\,dr$ yields

$$\int_a^b r\,u_r^2\,dr \leq b\,(b-a)^2 \int_a^b u_{r,r}^2\,dr \leq \frac{b}{s_1\,a}(b-a)^2 \int_a^b s_1\,r\,u_{r,r}^2\,dr \tag{5.56}$$

and besides it holds

$$\int_a^b r\,u_r^2\,dr \leq \frac{b^2}{s_1} \int_a^b s_1 r\,\frac{u_r^2}{r^2}\,dr. \tag{5.57}$$

Since

$$\int_a^b 2s_2 u_{r,r} u_r\,dr = \int_a^b s_2 \frac{du_r^2}{dr}\,dr = s_2 u_r^2(b) \geq 0, \tag{5.58}$$

the sum of (5.56) and (5.57) yields

$$\int_a^b r u_r^2\,dr \quad \leq \quad \max\{\frac{b^2}{2s_1}, \frac{b^2}{2s_1}\frac{(b-a)^2}{ab}\} \int_a^b \Big(s_1 r[u_{r,r}^2 + \frac{u_r^2}{r^2}] + 2s_2 u_{r,r} u_r\Big) dr \tag{5.59}$$

and provides the upper bound of the inequality (5.52). By substituting $u_\phi = r\bar{u}_\phi$, we transform (5.53) into

$$\int_a^b r^3 \bar{u}_\phi^2\,dr \leq C_\Omega^2 \int_a^b \tfrac{1}{2} s_3 r^3 \bar{u}_{\phi,r}^2 dr \tag{5.60}$$

which must be valid for all functions $\bar{u}_\phi \in H^1(a,b) : \bar{u}_\phi|_a = 0$. Then, estimate similar to (5.56) implies

$$\int_a^b r^3 \bar{u}_\phi^2\,dr \quad \leq \quad \frac{2b^3}{s_3}\frac{(b-a)^2}{a^3} \int_a^b \tfrac{1}{2} s_3 r^3 \bar{u}_{\phi,r}^2 dr. \tag{5.61}$$

Comparison of (5.59) and (5.61) yields the final upper bound

$$C_\Omega^2 \leq \max\{\frac{b^2}{2s_1}, \frac{b^2}{2s_1}\frac{(b-a)^2}{ab}, \frac{2b^3}{s_3}\frac{(b-a)^2}{a^3}\}. \tag{5.62}$$

5.3. **Estimate of constant** $C_{(\Omega,\Gamma_1)}$**.** Under the axisymmetric assumptions (5.4)- (5.6), inequality (3.28) is reduced to

$$b\Big(u_r(b)^2 + u_\phi(b)^2\Big) \leq C^2_{(\Omega,\Gamma_1)} \int\limits_a^b \Big(s_1 r[u_{r,r}^2 + \frac{u_r^2}{r^2}] + 2s_2 u_{r,r}u_r + \tfrac{1}{2}s_3 r[u_{\phi,r} - \frac{u_\phi}{r}]^2\Big)dr. \tag{5.63}$$

Similarly as for C_Ω, it is equivalent to finding the constant $C_{(\Omega,\Gamma_1)}$ such that inequalities

$$b\,u_r(b)^2 \quad\leq\quad C^2_{(\Omega,\Gamma_1)} \int\limits_a^b \Big(s_1 r[u_{r,r}^2 + \frac{u_r^2}{r^2}] + 2s_2 u_{r,r}u_r\Big)dr, \tag{5.64}$$

$$b\,u_\phi(b)^2 \quad\leq\quad C^2_{(\Omega,\Gamma_1)} \int\limits_a^b \tfrac{1}{2}s_3 r[u_{\phi,r} - \frac{u_\phi}{r}]^2 dr \tag{5.65}$$

are valid simultaneously for all functions u_r, u_ϕ satisfying (5.54)- (5.55).

The Cauchy–Schwarz inequality deduces

$$u_r(b)^2 \leq \Big(\int\limits_a^b u_{r,r}dr\Big)^2 \leq \int\limits_a^b 1dr \cdot \int\limits_a^b u_{r,r}^2 dr \leq \frac{b-a}{s_1 a}\int\limits_a^b s_1 r u_{r,r}^2 dr \tag{5.66}$$

and (cf. (5.58))

$$s_2 \frac{b-a}{s_1 a} u_r(b)^2 = \frac{b-a}{s_1 a}\int\limits_a^b 2s_2 u_{r,r}u_r\,dr. \tag{5.67}$$

The sum of two latter expressions implies

$$u_r(b)^2 \leq \frac{b-a}{s_1 a + s_2(b-a)}\int\limits_a^b \Big(s_1 r u_{r,r}^2 + 2s_2 u_{r,r}u_r\Big)dr. \tag{5.68}$$

By substituting $u_\phi = r\bar{u}_\phi$, we transform (5.65) into

$$b^3\,\bar{u}_\phi(b)^2 \quad\leq\quad C^2_{(\Omega,\Gamma_1)}\int\limits_a^b \tfrac{1}{2}s_3 r^3 \bar{u}_{\phi,r}^2 dr \tag{5.69}$$

which must be valid for all functions $\bar{u}_\phi \in H^1(a,b) : \bar{u}_\phi|_a = 0$. Since

$$\bar{u}_\phi(b)^2 = \Big(\int\limits_a^b \bar{u}_{\phi,r}dr\Big)^2 \leq \int\limits_a^b 1dr \cdot \int\limits_a^b \bar{u}_{\phi,r}^2 dr \leq \frac{2(b-a)}{s_3 a^3}\int\limits_a^b \tfrac{1}{2}s_3 r^3 \bar{u}_{\phi,r}^2 dr, \tag{5.70}$$

the comparison of (5.68) and (5.70) implies the final upper bound

$$C^2_{(\Omega,\Gamma_1)} \leq \max\{\frac{b(b-a)}{s_1 a + s_2(b-a)}, \frac{2b^3(b-a)}{s_3 a^3}\}. \tag{5.71}$$

5.4. **Majorant term in axisymmetric case.** As for the exact displacement vector $u = u(r)$ and the corresponding stress tensor $\sigma^* = \sigma^*(r)$, we consider deviations v and τ^* also axially symmetric,

$$v = (v_r(r), v_\phi(r)), \qquad \tau^* = \tau^*(r).$$

The majorant part M_1 defined in (3.11) then reads using the substitution in polar coordinates

$$\begin{aligned} M_1(v,\tau^*) = \pi \int\limits_a^b \Big(& s_1[(\varepsilon_{rr})^2 + (\varepsilon_{\phi\phi})^2] + 2s_2\varepsilon_{rr}\varepsilon_{\phi\phi} + 2s_3(\varepsilon_{r\phi})^2 + \\ & + t_1[(\tau^*_{rr})^2 + (\tau^*_{\phi\phi})^2] + 2t_2\tau^*_{rr}\tau^*_{\phi\phi} + 2t_3(\tau^*_{r\phi})^2 - \\ & - 2\varepsilon_{rr}\tau^*_{rr} - 2\varepsilon_{\phi\phi}\tau^*_{\phi\phi} - 4\varepsilon_{r\phi}\tau^*_{r\phi}\Big) r\, dr. \end{aligned} \tag{5.72}$$

Since all terms from equilibrium equations (5.18)- (5.19) are also axisymmetric, we can similarly rewrite the majorant term $r_\Omega(\tau^*)^2$ defined in (3.26) as

$$r^2_\Omega(\tau^*) = 2\pi \int\limits_a^b \Big(\tau^*_{rr,r} + \frac{\tau^*_{rr} - \tau^*_{\phi\phi}}{r} - f_r)^2 + (\tau^*_{r\phi,r} + \frac{2\tau^*_{r\phi}}{r} - f_\phi)^2 \Big) r dr. \tag{5.73}$$

Finally, the boundary majorant term (4.10) in the friction case rewrites as

$$I_{\Gamma_1}(v,\tau^*,\xi,\theta) := \int\limits_{\Gamma_1} \Big(k_\phi\, |v_\phi| - \xi^*_\tau v_\phi + \theta \left| \tau^*_{r\phi} + \xi^*_\tau \right|^2 \Big) d\Gamma. \tag{5.74}$$

In view of (4.11) and (4.12), the function ξ^*_τ attains the pointwise value

$$\xi = \frac{v_\phi}{2\theta} - \tau^*_{r\phi} \quad \text{if } |\frac{v_\phi}{2\theta} - \tau^*_{r\phi}| \le k_\phi \tag{5.75}$$

or one of the values

$$\xi = -k_\phi \quad \text{or} \quad \xi = -k_\phi. \tag{5.76}$$

6. Numerical tests for example with known analytical solution

All numerical test are performed only for the case of Dirichlet-friction boundary conditions described in subsection 5.1.3. A developed MATLAB software is available for testing at Matlab Central at `http://www.mathworks.com/matlabcentral/fileexchange/authors/37756` as a package 'Rotating symetric elastic ring with a friction boundary condition'.
Our main interest is numerical testing of the majorant estimate (4.4),

$$\tfrac{1}{2}a(v-u,v-u) \le (1+\beta)M_1(v,\tau^*) + \tfrac{1}{2}\left(1+\beta^{-1}\right)(1+\alpha)C^2_{(\Omega)} r^2_\Omega(\tau^*) + I_{\Gamma_1}(v,\tau^*,\xi,\theta). \tag{6.1}$$

Note that

- $a(\cdot,\cdot)$ is defined in (5.26), $M_1(\cdot,\cdot)$ in (5.72), $r_\Omega(\cdot)$ in (5.73) and $I_{\Gamma_1}(\cdot,\cdot,\cdot,\cdot)$ in (5.74)
- the scalar parameters α, β must be positive but arbitrary
- the scalar parameter θ is defined in (3.34) and depends on α, β and the constant $C_{(\Omega,\Gamma_1)}$
- the optimal value of the scalar function ξ is given by (5.75) or by (5.76)

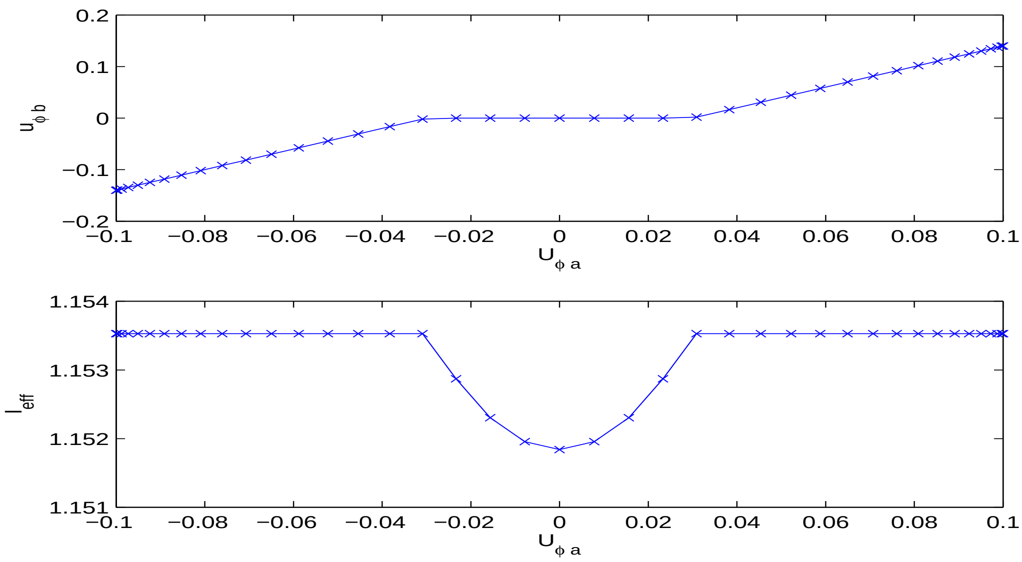

FIGURE 1. The value of the slip $u_{\phi b}$ on the friction boundary (upper picture) and computed indices of efficiency I_{eff} (lower picture) for various boundary conditions $U_{ra} = 0.1, U_{\phi a} = 0.1\cos(\frac{t \cdot \pi}{40}), k_{\phi b} = 0.02$, where $t = 0, \ldots 40$.

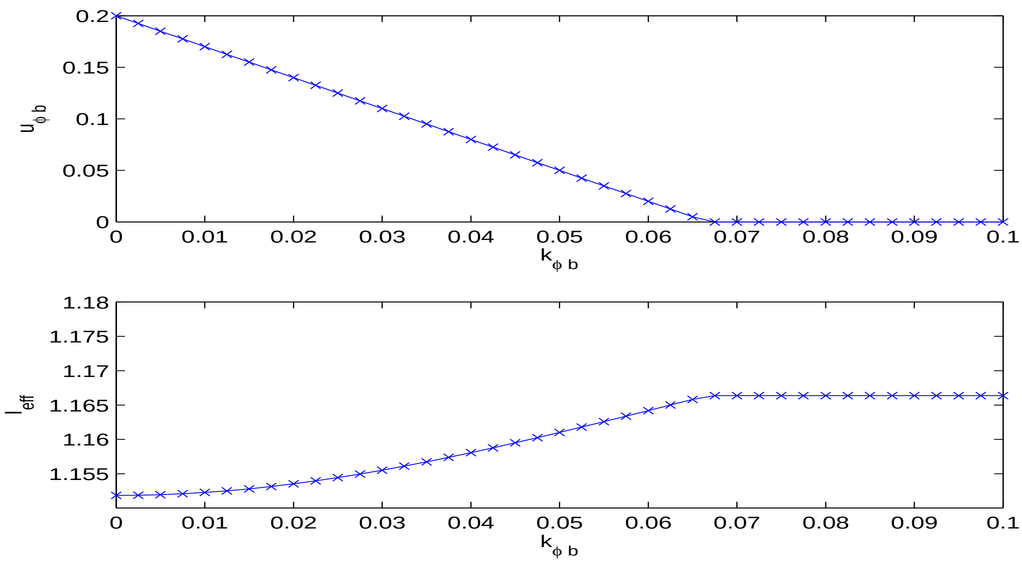

FIGURE 2. The value of the slip $u_{\phi b}$ on the friction boundary (upper picture) computed indices of efficiency I_{eff} (lower picture) for various boundary conditions $U_{ra} = 0.1, U_{\phi a} = 0.1, k_{\phi b} = 0, \frac{1}{400}, \frac{2}{400}, \ldots 0.1$.

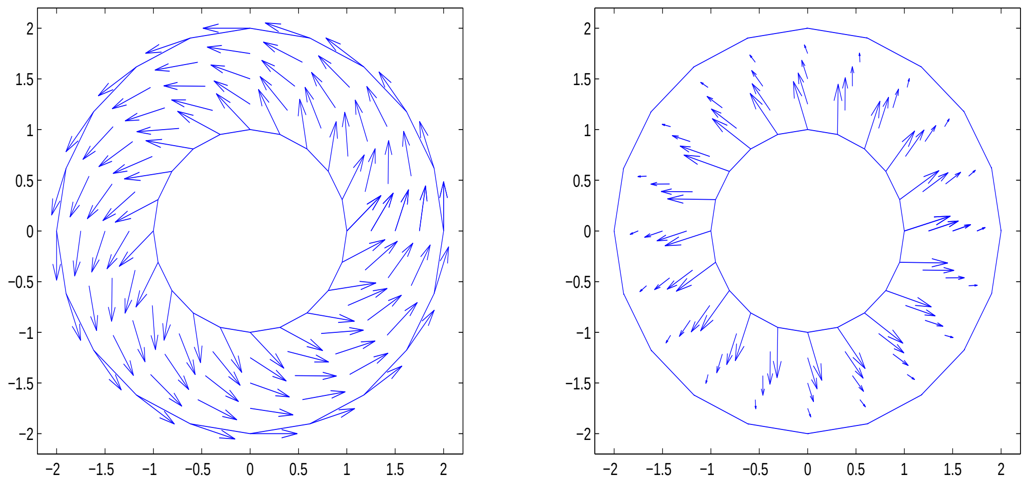

FIGURE 3. Axisymmetric displacement fields in cases of active slip (left) characterized by the condition $u_{\phi b} \neq 0$ and no slip (right) characterized by the condition $u_{\phi b} = 0$.

– values of constants $C_{(\Omega)}$ and $C_{(\Omega,\Gamma_1)}$ are replaced by their upper bounds (5.62) and (5.71)

– the exact displacement u is given due to (5.23) and (5.24) with constants c_1, c_2, c_3, c_4 given in Subsection 5.1.3 and can be used for computation of the exact error

– v can be any discrete approximation of u

We discretize the radial variable r by one dimensional equidistant mesh of form

$$T = a, a+h, a+2h, \ldots, b$$

where $h = \frac{b-a}{N-1}$ is the mesh-size parameter and N denotes the number of mesh nodes. In all numerical experiments, the discrete displacement v is considered as a piecewise linear function with nodal values identical to values of the exact displacement u. By this technique we avoid computation of v by eg. finite element method, it would be feasible but it is not our focus here.

The sharpness of the estimate (6.1) is significantly decided by the quality of τ^*. Let us assume that all of its components $\tau^*_{rr}, \tau^*_{\phi\phi}, \tau^*_{r\phi}$ are also piecewise linear function defined on the same mesh T. For given values of α, β, the miminization of the right-hand side of (6.1) with respect to τ^* reduces to a convex minimization problem of the size $3 \times N$. The minimization with respect to all variables τ^*, α, β results apparently in a challenging nonconvex optimization problem which is not studied here. The MATLAB blackbox tool 'fminunc' was used to the unconstrained minimization. After the optimal value of τ^* is found, it is substituted in the right-hand side of (6.1) is compared

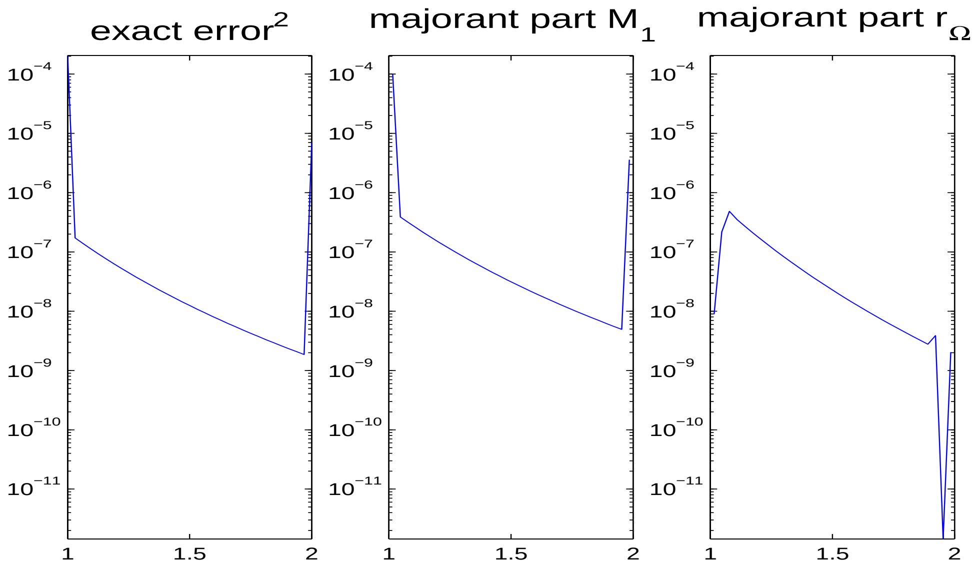

FIGURE 4. Space distribution of the square of exact error and majorant parts M_1 and r_Ω for $U_{ra} = 0.1, U_{\phi a} = 0.1, k_{\phi b} = 0.02$ and the uniform mesh with 33 nodes. The majorant part I_{Γ_1} is not visualized since it is only related to the right boundary $r = 2$.

number of mesh nodes	error^2	majorant	part M_1	part r_Ω	part I_{Γ_1}	I_{eff}
3	4.14e-03	4.52e-03	4.43e-03	7.20e-05	1.17e-05	1.04
5	6.72e-04	7.50e-04	7.42e-04	6.83e-06	7.30e-07	1.06
9	1.02e-04	1.22e-04	1.21e-04	5.33e-07	4.94e-08	1.09
17	1.43e-05	1.86e-05	1.86e-05	2.26e-08	3.51e-09	1.14
33	1.90e-06	2.53e-06	2.49e-06	4.13e-08	1.57e-12	1.15
65	2.46e-07	3.28e-07	3.25e-07	3.24e-09	1.47e-13	1.15
129	3.13e-08	4.17e-08	4.15e-08	2.26e-10	00	1.15
257	3.94e-09	5.26e-09	5.24e-09	1.41e-11	00	1.15
513	4.95e-10	6.60e-10	6.59e-10	8.82e-13	00	1.15
1025	6.20e-11	8.27e-11	8.26e-11	5.52e-14	00	1.15

TABLE 1. Values of the exact and majorant including its components for $U_{ra} = 0.1, U_{\phi a} = 0.1, k_{\phi b} = 0.02$ and various uniform meshes.

with the left-hand side representing the exact error of approximation. The square root of their ratio must be always greated than 1 and it is called the index of efficiency I_{eff}.

For numerical tests, we consider an elastic ring with inner and outer radiuses

$$a = 1, \quad b = 2$$

characterized by Lamé parameters

$$\lambda = \mu = 1.$$

In order to test various situations we consider sets of conditions:

(1) The friction boundary condition on Γ_1

$$k_{\phi b} = 0.02$$

and a series of Dirichlet boundary conditions on Γ_0

$$U_{ra} = 0.1, \quad U_{\phi a} = 0.1 \cos(\frac{t \cdot \pi}{20}),$$

where the integer parameter t changes from 0 to 40.

(2) A series of friction boundary conditions on Γ_1

$$k_{\phi b} = 0, \frac{1}{400}, \frac{2}{400}, \ldots, 0.1$$

and the Dirichlet boundary conditions on Γ_0

$$U_{ra} = 0.1, U_{\phi a} = 0.1.$$

Figures 1 and 2 report on values of the slip $u_{\phi b}$ on the friction boundary Γ_1 and values of index of efficiency I_{eff} computed for the choice of parameters $\alpha = \beta = 1$ and the mesh with $N = 33$ nodes in case of above mentioned sets of boundary conditions. The exact error $a(v - u, v - u)$ is computed numericaly using a trapezoidal rule as an integral of a difference of an exact solution u taken as an exact nodal linear interpolant on twice more uniformly refined mesh with 129 nodes and the discrete solution v.

Figure 3 displays axisymmetric displacements fields for two different values boundary leading to nozero or zero slip $u_{\phi b}$ on a friction boundary Γ_1. A full animation is available with our software mentioned above. Finally, Figure 4 compares distributions of the exact error and the majorant terms M_1 and r_Ω. The distribution of M_1 serves apparently as a good indication of the distribution of the exact error. Detailed information on all majorant terms for different uniform meshes can be found in Table (1). The index of efficiency I_{eff} is very stable with respect to the mesh size.

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