

**Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig**

**Universally Typical Sets for Ergodic Sources of
Multidimensional Data**

by

*Tyll Krüger, Guido Montúfar, Ruedi Seiler, and Rainer
Siegmond-Schultze*

Preprint no.: 20

2011



Universally Typical Sets for Ergodic Sources of Multidimensional Data

Tyll Krüger^{1,4} Guido Montufar² Ruedi Seiler³
Rainer Siegmund-Schultze¹

¹Universität Bielefeld Fakultät für Physik, Universitätsstraße 25, 33501 Bielefeld, Germany

²Max Planck Institute for Mathematics in the Sciences, Inselstraße 22, 04103 Leipzig, Germany

³Technische Universität Berlin Institut für Mathematik MA 7-2, Straße des 17. Juni 136, 10623 Berlin, Germany

⁴Universität Bielefeld, CITEC Center of Excellence Cognitive Interaction Technology

May 2, 2011

Abstract

We lift important results of the theory of samples of discrete ergodic information sources to the multidimensional setting. We use the technique of packings and coverings with multidimensional windows in entropy estimation and universal lossless compression. In particular, we construct sequences of multidimensional array sets which, in the limit, build the generated samples of any ergodic source of entropy rate below an h_0 with probability 1 and whose cardinality grows at most at exponential rate h_0 . Thereby we extrapolate mathematical framework relevant for universal source coding of multi-dimensionally correlated data.

Keywords: Universal codes, ergodic theory, typical sets, discrete samplings.

1 Introduction

The purpose of this paper is to lift results about universally typical sets, typically sampled sets and empirical entropy estimation from the usual 1-dimensional (discrete time) setting to a multidimensional setting. We start with a short description of these concepts and a very brief review of related literature.

An entropy-typical set is defined as a set of nearly full measure consisting of output sequences the negative log-probability of which is close to the entropy of the source distribution. The scope of this definition is revealed by the asymptotic equipartition property (AEP), which is present for a large class of processes [9, 3, 1, 10, 2]. The AEP was introduced by McMillan [9] as the convergence in probability of the sequence $-\frac{1}{n} \log \mu(x_1^n)$ to a constant h , namely the entropy rate of the process as introduced by Shannon [12]. Roughly speaking it implies that the output sequences of a random process are typically confined

to a ‘small’ set T_n of events which have all approximately the same probability of being realized, in contrast to the much larger set of all possible output sequences. This means that individual outcomes with much higher or smaller probability than e^{-nh} will rarely be observed. In particular, for stationary ergodic processes the AEP is guaranteed by the Shannon-McMillan-Breiman (SMB) theorem [9, 3]. By the AEP, the entropy-typical sets have total probability close to one, and their cardinality is fairly minimal among all sets with the latter property. This way, entropy-typical sets provide an important theoretical framework for communication theory. Lossless source coding is a type of algorithm which performs data compression while ensuring that the exact reconstruction of the original data is possible from the compressed data. Lossless data compression can be achieved by encoding the typical set of a stochastic source with fixed length block codes of length nh . By the AEP, this length nh is also the average length needed, cf. [13]. Hence compression at an asymptotic rate given by the entropy rate is possible. This is optimal in view of Shannon’s source coding theorem [12, 4].

The extension of the SMB theorem (and the AEP) from discrete time processes \mathbb{Z} to amenable groups including the multidimensional setting \mathbb{Z}^d , by Ornstein and Weiss [10] represented an important progress. It amounts to the theory of encoding multidimensional sources. The relation is rather obvious: In fact, any (asymptotically) optimal universal compression scheme defines sequences of universally typical sets: for given ε , the set of all n^d -blocks such that their comprimate needs at most $(h + \varepsilon)n^d$ bits, is universally typical for all sources with entropy rate h or less. Vice versa, any *constructive* solution to the problem of finding universally typical sets yields an universal compression scheme, since the index in the universally typical set is an optimal code for the block. As will turn out, our approach is constructive. But one has to admit that such an *ad hoc* algorithm is –generally speaking– not very useful in practice because determining the index should be very time consuming.

In universal source coding, the aim is to find codes which efficiently compress down to the theoretical limit, i.e. the entropy rate, for any ergodic source without a need to be adapted to the specific source. We emphasize here that codes of that type are optimal data compressors for any *stationary* source, since by the ergodic decomposition theorem (see e.g. [14]) any stationary source is a convex mixture of ergodic sources. Many prominent examples of formats for lossless data compression, like ZIP, are based on the implementation of the algorithms proposed by Lempel and Ziv (LZ) LZ77 [5] and LZ78 [6], or variants of them, like the Welch modification [15]. The LZ algorithms constitute a universal means of constructing libraries. Yet, the LZ algorithms are designed as text compression schemes, i.e. for 1-dimensional data sources.

For multidimensional data, Lempel and Ziv showed [7] that universal coding of images is possible by first transforming the image to a 1-dimensional stream (scanning the image with a Peano-Hilbert curve, a special type of Hamilton path), and then applying the 1-dimensional algorithm LZ78. The idea behind that approach is that the Peano-Hilbert curve scans hierarchically complete blocks before leaving them, maintaining most of local correlations that way. In contrast, a simple row-by-row scan only preserves horizontal correlations.

But with the Peano curve approach, while preserving local correlations in any non-horizontal direction, too, these correlations are much encrypted due to the inevitably fractal nature of that space-filling curve.

We take the point of view that the techniques of packing and counting can be better exploited in data compression with priorly unknown distributions if, instead of transforming the ‘image’ into a dim-1-stream by scanning it with a curve, the multi-dimensional block structure is left untouched. This will allow to take more advantage of multidimensional correlations between neighboring parts of data, speed up the convergence of the counting statistics, and in turn fasten estimation and data compression tasks. This approach will be carried out in a forthcoming paper. The idea of the present paper is to extend relevant theoretical results about typical sets and universally typical sets to a truly multi-dimensional sampling window setting. The proofs of these extensions are guided by the discussion of the 1-dimensional situation in Shield’s monograph [13].

2 Settings

We consider the d -dimensional lattice \mathbb{Z}^d and the quadrant \mathbb{Z}_+^d . Consider a finite alphabet \mathcal{A} , $|\mathcal{A}| < \infty$ and the set of arrays with that alphabet: $\Sigma = \mathcal{A}^{\mathbb{Z}^d}$, $\Sigma_+ = \mathcal{A}^{\mathbb{Z}_+^d}$. We define the set of n -words as the set of $n \times \dots \times n$ arrays $\Sigma^n := \mathcal{A}^{\Lambda_n}$ for the n -box $\Lambda_n := \{(i_1, \dots, i_d) \in \mathbb{Z}_+^d : 0 \leq i_j \leq n-1, j \in \{1, \dots, d\}\}$. An element $x^n \in \Sigma^n$ has elements $x^n(\mathbf{i}) \in \mathcal{A}$ for $\mathbf{i} \in \Lambda_n$.

Let $\mathfrak{A}^{\mathbb{Z}^d}$ denote the σ -algebra of subsets of Σ generated by cylinder sets, i.e. sets of the following kind:

$$[y] := \{x \in \Sigma : x(\mathbf{i}) = y(\mathbf{i}), \mathbf{i} \in \Lambda\}, \quad y \in \mathcal{A}^\Lambda, \Lambda \text{ finite.}$$

If C is a subset of \mathcal{A}^Λ , we will use the notation $[C]$ for $\cup_{y \in C} [y]$.

We denote by $\sigma_{\mathbf{r}}$ the natural lattice translation by the vector $\mathbf{r} \in \mathbb{Z}^d$ acting on Σ by $\sigma_{\mathbf{r}}x(\mathbf{i}) := x(\mathbf{i} + \mathbf{r})$. We use the same notation $\sigma_{\mathbf{r}}$ to denote the induced action on the set \mathbb{P} of probability measures ν over $(\Sigma, \mathfrak{A}^{\mathbb{Z}^d})$: $\sigma_{\mathbf{r}}\nu(E) := \nu(\sigma_{\mathbf{r}}^{-1}E)$. The set of all stationary (translation-invariant) elements of \mathbb{P} is denoted by \mathbb{P}_{stat} , i.e. $\nu \in \mathbb{P}_{\text{stat}}$ if $\sigma_{\mathbf{r}}\nu = \nu$ for each $\mathbf{r} \in \mathbb{Z}^d$. Those $\nu \in \mathbb{P}_{\text{stat}}$ which cannot be decomposed as a proper convex combination $\nu = \lambda_1\nu_1 + \lambda_2\nu_2$, with $\nu_1 \neq \nu \neq \nu_2$ and $\nu_1, \nu_2 \in \mathbb{P}_{\text{stat}}$ are called *ergodic*. The corresponding subset of \mathbb{P}_{stat} is denoted by \mathbb{P}_{erg} . Throughout this paper μ will denote an ergodic \mathcal{A} -process on Σ . By ν^n we denote the restriction of the measure ν to the block Λ_n , obtained by the projection $\Pi_n : x \in \Sigma \rightarrow x^n \in \Sigma^n$ with $x^n(\mathbf{i}) = x(\mathbf{i}), \mathbf{i} \in \Lambda_n$. We use the same notation Π_k to denote the projections from Σ^n to $\Sigma^k, n \geq k$, defined in the same obvious way. The measurable map Π_n transforms the given probability measure ν to the probability measure denoted by ν^n .

The entropy rate of a stationary probability measure ν is defined as limit of

the scaled n -word entropies:

$$H(\nu^n) := - \sum_{x \in \Sigma^n} \nu^n(\{x\}) \log \nu^n(\{x\})$$

$$h(\nu) := \lim_{n \rightarrow \infty} \frac{1}{n^d} H(\nu^n).$$

Here and in the following we write \log for the dyadic logarithm \log_2 .

For a *shift* $\mathbf{p} \in \Lambda_k$ we consider the following partition of \mathbb{Z}^d into k -blocks:

$$\mathbb{Z}^d = \bigcup_{\mathbf{r} \in k \cdot \mathbb{Z}^d} (\Lambda_k + \mathbf{r} + \mathbf{p}),$$

and in general we use the following notation:

The *regular k -block partitions* of a subset $M \subset \mathbb{Z}^d$ are the families of sets defined by

$$\mathcal{R}_{M,k} := \{R_{M,k}(\mathbf{p}) : \mathbf{p} \in \Lambda_k\}, \quad R_{M,k}(\mathbf{p}) := \{(\Lambda_k + \mathbf{p} + \mathbf{r}) \cap M\}_{\mathbf{r} \in k \cdot \mathbb{Z}^d}.$$

Clearly, for any \mathbf{p} the elements of $R_{M,k}(\mathbf{p})$ are disjoint and their union gives M .

In the case $M = \Lambda_n$, given a sample $x^n \in \Sigma^n$, such a partition yields a *parsing* of x^n in elements of $\mathcal{A}^{(\Lambda_k + \mathbf{r} + \mathbf{p}) \cap \Lambda_n}$, $\mathbf{r} \in k \cdot \mathbb{Z}^d$. We call those elements the *words* of the parsing of x^n induced by the partition $R_{\Lambda_n,k}(\mathbf{p})$. With exception of those \mathbf{r} , for which $\Lambda_k + \mathbf{r} + \mathbf{p}$ crosses the boundary of Λ_n , these are cubic k -words. Forgetting about their \mathbf{r} -position, we may identify $\Pi_{\Lambda_k} x \sim \Pi_{\Lambda_k + \mathbf{r}} \sigma_{-\mathbf{r}} x \in \mathcal{A}^{\Lambda_k + \mathbf{r}} \cong \mathcal{A}^{\Lambda_k}$.

For $k, n \in \mathbb{N}$, $k < n$, any element $x \in \Sigma$ gives rise to a probability distribution, defined by the relative frequency of the different k -words in a given parsing of x_n . Let us introduce the following expression:

$$Z_x^{\mathbf{p},k,n}(a) \quad : \quad = \quad \sum_{\mathbf{r} \in \times_{i=1}^d \{0, \dots, \lfloor (n-p_i)/k \rfloor - 1\}} \mathbf{1}_{[a]}(\sigma_{k \cdot \mathbf{r} + \mathbf{p}} x),$$

$$n \in \mathbb{N}, k \leq n, a \in \mathcal{A}^{\Lambda_k}, \mathbf{p} = (p_1, \dots, p_d) \in \Lambda_k.$$

For regular, k -block parsings, the *non-overlapping empirical k -block distribution* generated by $x \in \Sigma$ in the box Λ_n is defined as the probability distribution on Σ^k given by:

$$\tilde{\mu}_x^{k,n}(\{a\}) := \frac{1}{\lfloor n/k \rfloor^d} Z_x^{\mathbf{0},k,n}(a) \quad \text{for } a \in \mathcal{A}^{\Lambda_k}. \quad (1)$$

Similarly, for any $\mathbf{p} = (p_1, \dots, p_d) \in \Lambda_k$ the shifted regular k -block partition gives a non-overlapping empirical k -block distribution:

$$\tilde{\mu}_x^{\mathbf{p},k,n}(\{a\}) := \frac{1}{\prod_{i=1}^d \lfloor (n-p_i)/k \rfloor} Z_x^{\mathbf{p},k,n}(a). \quad (2)$$

Furthermore, we define the *overlapping empirical k -block distribution*, in which all k -words present in x are considered:

$$\tilde{\mu}_{x,overl}^{k,n}(\{a\}) := \frac{1}{(n-k+1)^d} \sum_{\mathbf{r} \in \Lambda_{n-k+1}} \mathbf{1}_{[a]}(\sigma_{\mathbf{r}} x) \quad \text{for } a \in \mathcal{A}^{\Lambda_k}. \quad (3)$$

Remember here the definition of $[a]$. Observe that all three empirical distributions only depend on the values of x in the positions Λ_n , i.e. on $x^n := \prod_n x \in \mathcal{A}^{\Lambda_n}$.

3 Results

The main contribution of this paper is the following:

Theorem 1 (Universally typical sets) *For any given $h_0 > 0$ there exists a sequence of subsets $\{\mathcal{T}_n(h_0) \subset \Sigma^n\}_n$ such that for all $\mu \in \mathbb{P}_{erg}$ with $h(\mu) < h_0$ the following holds:*

$$a) \lim_{n \rightarrow \infty} \mu_n(\mathcal{T}_n(h_0)) = 1,$$

$$b) \lim_{n \rightarrow \infty} \frac{\log |\mathcal{T}_n(h_0)|}{n^d} = h_0.$$

Furthermore, for any sequence $\{\mathcal{U}_n \subset \Sigma^n\}_n$ with $\liminf_{n \rightarrow \infty} \frac{1}{n^d} \log |\mathcal{U}_n| < h_0$ there exists a $\mu \in \mathbb{P}_{erg}$ with $h(\mu) < h_0$ such that:

$$c) \liminf_{n \rightarrow \infty} \mu(\mathcal{U}_n) = 0.$$

The proof of this result is based on other assertions following now. We start lifting the packing lemma from [13], which will allow us to use the proof's strategy of the 1-dimensional statement. The packing lemma states that if a set of words $C \subset \Sigma^m$ is *typical* among all m -blocks present in a sample $x^k \in \Sigma^k$, $k \geq m$, i.e., C has large probability with respect to the *overlapping empirical m -block distribution*, then, the sample x^k can be parsed into non-overlapping blocks in such a way, that nearly all words belong to C .

While in the $d = 1$ setting the statement is rather evident, for $d \geq 2$ it is not immediately clear how a parsing can be chosen, such that it yields many matchings with C , and few 'holes'. Our lemma asserts that this parsing can be realized through a regular partition. I.e. C receives large probability in the non-overlapping empirical distribution of some shift of x .

Lemma 2 (Packing Lemma) *Consider for any fixed $0 < \delta \leq 1$ integers k and m related through $k \geq d \cdot m / \delta$. Let $C \subset \Sigma^m$ and $x \in \Sigma$ with the property that $\tilde{\mu}_{x,overl}^{m,k}(C) \geq 1 - \delta$. Then, there exists a $\mathbf{p} \in \Lambda_m$ such that: a) $\tilde{\mu}_x^{\mathbf{p},m,k}(C) \geq 1 - 2\delta$, and also b) $|Z_x^{\mathbf{p},m,k}(C)| \geq (1 - 4\delta)(\lfloor \frac{k}{m} \rfloor + 2)^d$.*

Recall the definition of the overlapping empirical m -block distribution. The condition $\tilde{\mu}_{x,overl}^{m,k}(C) \geq 1 - \delta$ means $\sum_{\mathbf{r} \in \Lambda_{k-m+1}} \mathbf{1}_{[C]}(\sigma_{\mathbf{r}} x) \geq (1 - \delta)(k - m + 1)^d$. The result a) $\exists \mathbf{p} \in \Lambda_m : \tilde{\mu}_x^{\mathbf{p},m,k}(C) \geq 1 - 2\delta$ means that there exists a regular m -block partition $R_{\Lambda_k,m}(\mathbf{p}) \in \mathcal{R}_{\Lambda_k,m}$ that parses x^k in such a way that at least a $(1 - 2\delta)$ -fraction of the m -words are elements of C . The result b) implies that at least a $(1 - 4\delta)$ -fraction of the total number of words (this total number including non-cubic words at the boundary), are elements of C . This is the case

because the boundary non-cubic elements cover only a small volume. For $\delta = 0$ and any $k \geq m$ the result a) is trivial, since in that case all m -words in x^k are in C .

Proof of Lemma 2. Denote by Ξ the set of vectors $\{\mathbf{r} \in \Lambda_{k-m+1} : \sigma_{\mathbf{r}}x \text{ is in } [C]\}$. For any $\mathbf{p} \in \Lambda_m$ denote by $\lambda(\mathbf{p})$ the number of those $\mathbf{r} \in \Xi$ satisfying $\mathbf{r} = \mathbf{p} \bmod(m)$. Clearly, $\lambda(\mathbf{p}) = |Z_x^{\mathbf{p},m,k}(C)|$ is the number of cubic blocks in the p -shifted regular m -block partition of Λ_k which belong to C . Then we have $\sum_{\mathbf{r} \in \Lambda_{k-m+1}} \mathbf{1}_{[C]}(\sigma_{\mathbf{r}}x) = \sum_{\mathbf{p} \in \Lambda_m} \lambda(\mathbf{p}) \geq (1-\delta)(k-m+1)^d$, by assumption. Hence, there is at least one $\mathbf{p}' \in \Lambda_m$ for which $\lambda(\mathbf{p}') \geq \frac{(1-\delta)(k-m+1)^d}{m^d}$. It is easy to see that $(1-\delta)\frac{(k-m+1)^d}{m^d} \geq (1-\delta)\frac{k^d-dmk^{d-1}}{m^d} \geq (1-\delta)^2\frac{k^d}{m^d} \geq (1-2\delta)\frac{k^d}{m^d}$. Since the maximal number of m -blocks that can occur in $R_{\Lambda_k,m}(\mathbf{p}')$ is $(\frac{k}{m})^d$, this completes the proof of a). For b) observe that the total number of partition elements of the regular partition (including the non-cubic at the boundary) is upper bounded by $(\lfloor \frac{k}{m} \rfloor + 2)^d \leq \frac{1}{m^d}(k+2m)^d \leq \frac{1}{m^d}(k^d + (k+2m)^{d-1}2dm) \leq \frac{1}{m^d} \sum_{j=0}^d k^{d-j}(2dm)^j \leq \frac{k^d}{m^d} \frac{1-(2\delta)^{d+1}}{1-2\delta}$. Here for the second inequality we used the estimate $1 - (d-1)y \leq 1/(1+y)^{d-1}$, $y \geq 0$ and for the third one the estimate $\binom{d-1}{j} \leq d^j$. On the other hand, from the first part we have $\lambda(\mathbf{p}') = |Z_x^{\mathbf{p}',m,k}(C)| \geq (1-2\delta)\frac{k^d}{m^d}$ and $1-2\delta \geq \frac{1-4\delta}{1-2\delta} \geq (1-4\delta)\frac{1-(2\delta)^{d+1}}{1-2\delta}$, which completes the proof. \blacksquare

Before we continue formulating the results, we give the definitions of entropy-typical sets and of typical sampling sets. The latter name is motivated by the properties guaranteed by Theorem 5 below.

Definition 3 (Entropy-typical sets) Let $\delta < \frac{1}{2}$. For some μ with entropy rate $h(\mu)$ the entropy-typical sets are defined as:

$$C_m(\delta) := \left\{ x \in \Sigma^m : 2^{-m^d(h(\mu)+\delta)} \leq \mu^m(\{x\}) \leq 2^{-m^d(h(\mu)-\delta)} \right\}. \quad (4)$$

We will use these sets as basic sets for the typical-sampling-sets defined below, see Figure 1.

Definition 4 (Typical-sampling sets) Consider some μ and $\delta < \frac{1}{2}$. For $k \geq m$, we define a typical-sampling set $\mathcal{T}_k(\delta, m)$ as the set of elements in Σ^k that have a regular m -block partition such that the resulting words belonging to the μ -entropy typical-set $C_m = C_m(\delta)$ contribute at least a $(1-\delta)$ -fraction to the (slightly modified) number of partition elements in that regular m -block partition.

$$\mathcal{T}_k(\delta, m) := \left\{ x \in \Sigma^k : \sum_{\substack{\mathbf{r} \in m \cdot \mathbb{Z}^d \\ (\Lambda_m + \mathbf{r} + \mathbf{p}) \subseteq \Lambda_k}} \mathbf{1}_{[C_m]}(\sigma_{\mathbf{r}+\mathbf{p}}x) \geq (1-\delta) \left(\frac{k}{m}\right)^d \text{ for some } \mathbf{p} \in \Lambda_m \right\}.$$

We fix some $\alpha > 0$ and assume $\delta < \frac{\alpha}{\log|\mathcal{A}|+1}$. Also, in the following we will choose m depending on k such that $m \xrightarrow{k \rightarrow \infty} \infty$, and $\lim_{k \rightarrow \infty} \frac{m}{k} = 0$. As we will see, a sequence of sets $\mathcal{T}_k(\delta, m)$, $k > 0$ with parameters fulfilling these

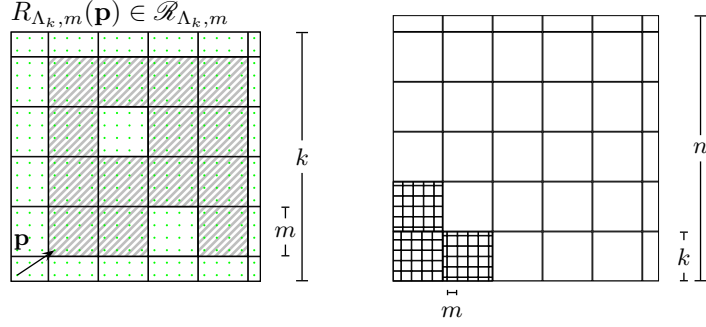


Figure 1: **Left:** This is an example of a regular m -block parsing of an element in $\mathcal{T}_k(\delta, m)$ for $d = 2$. The shaded blocks contain elements of C_m , and fill at least a $(1 - \delta)$ -fraction of the total volume k^2 . For $k \gg m$ the boundary (non-cubic) blocks comprise a neglectable volume. **Right:** Here we visualize for $d = 2$ and some $x^n \in \Sigma^n$ the parsing which is used for the empirical k -block distribution $\tilde{\mu}_x^{k, n}$. A k -block of x^n belongs to $\mathcal{T}_k(\delta, m)$, if it can be parsed by some (possibly shifted) regular m -block partition in such a way that the resulting (non-overlapping) m -words belonging to $C_m(\delta)$ cover a $(1 - \delta)$ -fraction of all the k^2 sites of that k -block. The non-cubic boundary blocks resulting from the k -block partition do not affect the empirical k -block distribution $\tilde{\mu}_x^{k, n}$.

conditions constitutes a sequence of typical-sampling sets $\mathcal{T}_k(\alpha)$ (Theorem 5 a)).

The following theorem is a generalization to $d \geq 1$ of a result by Ornstein and Weiss in [11], (Theorem II.3.1 in the monograph of Shields [13]). It ensures the existence of ‘small’ libraries from which asymptotically almost surely the realization of an ergodic process can be constructed, i.e., parsed as words belonging to that library. The library is given by the typical-sampling sets of Definition 4. Furthermore, it states that smaller libraries do not suffice.

Theorem 5 *Given any $\mu \in \mathbb{P}_{erg}$ and any $\alpha \in (0, \frac{1}{2})$ we have the following:*

- a) *For all k larger than some $k_0 = k_0(\alpha)$ there is a set $\mathcal{T}_k(\alpha) \subset \Sigma^k$ satisfying*

$$\frac{\log |\mathcal{T}_k(\alpha)|}{k^d} \leq h(\mu) + \alpha,$$

and such that for μ -a.e. x the following holds:

$$\tilde{\mu}_x^{k, n}(\mathcal{T}_k(\alpha)) > 1 - \alpha,$$

for all n and k such that $\frac{k}{n} < \varepsilon$ for some $\varepsilon = \varepsilon(\alpha) > 0$ and n larger than some $n_0(x)$.

- b) *Let $\{\tilde{\mathcal{T}}_{k, n}(x)\}_{k, n > 0}$ be a family of double-sequences of subsets of Σ^k depending measurably on $x \in \Sigma$, such that $|\tilde{\mathcal{T}}_{k, n}(x)| \leq 2^{k^d(h(\mu) - \alpha)}$. Then there exists a $k_1(\alpha) \geq k_0(\alpha)$ and for μ -a.e. x there exists an $n_0(x)$ such that*

$$\tilde{\mu}_x^{k, n}(\tilde{\mathcal{T}}_{k, n}(x)) \leq \alpha,$$

for any indices k, n fulfilling $k > k_1(\alpha), n > n_0(x)$ and $2^{k^d(h(\mu) + \alpha)} \leq n^d$.

The above result is closely related to the so called *typical sequence theorem*, (cf. Theorem I.4.1 in [13]), a consequence of the individual ergodic theorem, which says that for an ergodic μ the following limit exists and satisfies the equation for almost every x : $\lim_{n \rightarrow \infty} \tilde{\mu}_{x, \text{overl}}^{k,n}(a_k) = \mu([a_k])$ for any k and any $a_k \in \mathcal{A}^k$.

The following theorem states that the entropy of the empirical distribution of a sample almost surely converges to the true entropy of the process. This is an important component of the proof of the existence of universally typical libraries of small cardinality, Theorem 1a), b).

Theorem 6 (Empirical entropy theorem) *Let $\mu \in \mathbb{P}_{\text{erg}}$. Then for any sequence $\{k_n\}$ with $k_n \xrightarrow{n \rightarrow \infty} \infty$ and $k_n^d(h(\mu) + \alpha) \leq \log n^d$ (for some $\alpha > 0$) we have*

$$\lim_{n \rightarrow \infty} \frac{1}{k_n^d} H(\tilde{\mu}_x^{k_n, n}) = h(\mu), \quad \mu\text{-a.s.}$$

This concludes the section of results. Below we provide the proofs.

Proofs

Proof of Theorem 5a). We show that the claim holds choosing $\mathcal{T}_k(\alpha)$ as typical sampling sets $\mathcal{T}_k(\delta, m)$ from Definition 4 with $\delta < \frac{\alpha}{\log |\mathcal{A}| + 1}$, $m \xrightarrow{k \rightarrow \infty} \infty$ and $\lim_{k \rightarrow \infty} \frac{m}{k} = 0$.

Cardinality. We estimate the cardinality of the sets $\mathcal{T}_k(\delta, m)$. For a given m , there are m^d possible values of \mathbf{p} . There are at most $\left(\frac{k}{m}\right)^d$ cubic boxes in any m -block partition of Λ_k . Therefore, the number of choices for the contents of all blocks which belong to C_m is at most $|C_m|^{\left(\frac{k}{m}\right)^d}$. By the definition of $\mathcal{T}_k(\delta, m)$ the number of lattice sites not belonging to the regular partition being referred in this definition, is at most δk^d . There are at most $|\mathcal{A}|^{\delta k^d}$ possible choices for the contents of those array sites. Set $K = \lfloor \frac{k}{m} \rfloor + 2$. The maximal number of blocks occurring in the partition (including non cubic ones) is K^d . For $\frac{m}{k}$ small enough, not more than a $2\delta \leq \alpha < \frac{1}{2}$ fraction of all these blocks have contents not in C_m . Taking into account that the binomial coefficients $\binom{K}{l}$ do not decrease in l while $l \leq \frac{1}{2}K$, we get the following bound:

$$\begin{aligned} |\mathcal{T}_k(\delta, m)| &\leq m^d \sum_{0 \leq l \leq 2\delta K^d} \binom{K^d}{l} |\mathcal{A}|^{\delta k^d} |C_m|^{\left(\frac{k}{m}\right)^d} \\ &\leq m^d K^d \binom{K^d}{\lfloor \frac{1}{2}K^d \rfloor} |\mathcal{A}|^{\delta k^d} |C_m|^{\left(\frac{k}{m}\right)^d}. \end{aligned}$$

We apply Stirling's formula $N! \simeq \sqrt{2\pi N} \left(\frac{N}{e}\right)^N$, taking into account that the multiplicative error for positive N is uniformly bounded from below and above. A coarse bound will suffice. In the following estimate we make use of the relation $|C_m| \leq 2^{m^d(h(\mu) + \delta)}$, following immediately from the definition of C_m . For some

positive constants c, c' , and c'' we have

$$\begin{aligned}
\log |\mathcal{T}_k(\delta, m)| &\leq \log cm^d K^d \left(\frac{K^d}{\lfloor \frac{1}{2} K^d \rfloor} \right)^{K^d} \sqrt{\frac{K^d}{\lfloor \frac{1}{2} K^d \rfloor^2}} |\mathcal{A}|^{\delta k^d} |C_m|^{\left(\frac{k}{m}\right)^d} \\
&\leq \log c' m^d 3^{K^d} K^{d/2} |\mathcal{A}|^{\delta k^d} |C_m|^{\left(\frac{k}{m}\right)^d} \\
&\leq \log c'' k^d 3^{\left(\frac{k}{m} + 2\right)^d} 2^{(h(\mu) + \delta + \delta \log |\mathcal{A}|) k^d} \\
&\leq k^d \left(h(\mu) + \delta(\log |\mathcal{A}| + 1) + \frac{2^d}{m^d} \log 3 + \frac{\log k^d + \log c''}{k^d} \right).
\end{aligned}$$

In the last line we used $1/m + 2/k \leq 2/m$, which is fulfilled if k/m is large enough.

Whenever $\delta < \frac{\alpha}{\log |\mathcal{A}| + 1}$ and m as well as k are large enough (depending on α) this yields $\log |\mathcal{T}_k(\alpha)| \leq k^d (h(\mu) + \alpha)$.

Probability bound. The Ornstein-Weiss extension for amenable groups [10] of the Shannon-McMillan-Breiman-theorem yields¹:

$$\lim_{m \rightarrow \infty} -\frac{1}{m^d} \log \mu(\Pi_m x) = h(\mu) \quad \mu\text{-a.s.}$$

Thus, in view of the definition of C_m (Definition 3), there exists an $m_0(\delta)$ such that $\mu^m(C_m) \geq 1 - \delta^2/5$ for all $m \geq m_0(\delta)$. We fix such an m . The individual ergodic theorem [8] asserts that the following limit exists for μ -a.e. $x \in \Sigma$: $\lim_{n \rightarrow \infty} \frac{1}{n^d} \sum_{r \in \Lambda_n} \mathbf{1}_{[C_m]}(\sigma_r x) = \int \mathbf{1}_{[C_m]}(x) d\mu(x) = \mu^m(C_m)$, and therefore

$$\sum_{r \in \Lambda_{n-m+1}} \mathbf{1}_{[C_m]}(\sigma_r x) \geq (1 - \delta^2/4)(n - m + 1)^d > (1 - \delta^2/3)n^d \quad (5)$$

holds eventually almost surely, i.e. for μ -almost every x , and choosing n large enough depending on x , $n \geq n_0(x)$.

Take an $x \in \Sigma$ and an $n \in \mathbb{Z}_+$ for which this is the case. Choose a k with $m < k < n$. Consider the unshifted regular k -block partition of the n -block Λ_n :

$$\Lambda_n = \bigcup_{\mathbf{r} \in k \cdot \mathbb{Z}^d} (\Lambda_k + \mathbf{r}) \cap \Lambda_n.$$

Now, from equation (5) we deduce, that, if k/m and n/k are large enough, at least a $(1 - 2\delta)$ -fraction of the elements of this regular k -block parsing of $\Pi_n x$ which do not cross the boundary of Λ_n (those which count for the empirical distribution $\tilde{\mu}_x^{k,n}$, i.e. $\Pi_k \sigma_{\mathbf{r}} x$ with $\mathbf{r} \in k \cdot \mathbb{Z}^d \cap \Lambda_{n-k+1}$) satisfy

$$\frac{1}{(k - m + 1)^d} \sum_{\mathbf{s} \in \Lambda_{k-m+1}} \mathbf{1}_{[C_m]}(\sigma_{\mathbf{s} + \mathbf{r}} x) \geq (1 - \delta/4). \quad (6)$$

This is because if more than the specified 2δ -fraction of the k -blocks had more than a $\delta/4$ -fraction of ‘bad’ m -blocks, then the total number of ‘bad’ m -blocks

¹In fact we only need the convergence in probability, which ensures $\mu(C_m) \xrightarrow{m \rightarrow \infty} 1$.

in $\Pi_n x$ would be larger than

$$\begin{aligned} 2\delta \left\lfloor \frac{n}{k} \right\rfloor^d \cdot \frac{\delta}{4}(k-m+1)^d &\geq \frac{\delta^2}{2} \left(\left(1 - \frac{k}{n}\right) \left(1 - \frac{m}{k}\right) \right)^d n^d \\ &> \frac{\delta^2}{3} n^d, \end{aligned}$$

for $\frac{k}{n}$ and $\frac{m}{k}$ small enough, in contradiction to equation (5). While n had to be chosen large enough depending on x , we see that k needs to be chosen such that $\frac{k}{n}$ and $\frac{m}{k}$ are both small enough.

By Lemma 2 if $k \geq 4dm/\delta$, the k -blocks which satisfy equation (6) have a regular m -block partition with at least a $(1-\delta)$ -fraction of all partition members in C_m . Hence, at least a $(1-2\delta)$ -fraction of all k -blocks in Λ_n counting for the empirical distribution belong to $\mathcal{T}_k(\delta, m)$. For $2\delta \leq \alpha$ we get the probability bound:

$$\tilde{\mu}_x^{k,n}(\mathcal{T}_k(\delta, m)) \geq 1 - \alpha. \quad (7)$$

This completes the proof of Theorem 5a). \blacksquare

Proof of Theorem 5b). The statement is trivial for $h(\mu) = 0$. Let $h(\mu) > 0$.

For fixed $\delta < \alpha$, consider the sets $E_n(\delta)$ of all x in Σ with the property

$$\tilde{\mu}_x^{k,n}(\mathcal{T}_k(\delta)) \geq 1 - \delta \text{ for all } k \geq k_0(\delta), 2^{k^d(h(\mu)+\alpha)} \leq n^d$$

where $k_0 = k_0(\delta)$ is chosen large enough corresponding to the first part of the theorem.

Consider the sets $D_n(\alpha, \delta)$ of all x in Σ with the property

$$\tilde{\mu}_x^{k,n}(\tilde{\mathcal{T}}_{k,n}(x)) > \alpha \text{ for some } k \text{ with } k \geq k_0(\delta), 2^{k^d(h(\mu)+\alpha)} \leq n^d.$$

Remember the definition of entropy-typical sets:

$$C_n(\delta) \equiv \left\{ a \in \Sigma^n : 2^{-n^d(h(\mu)+\delta)} \leq \mu^n(\{a\}) \leq 2^{-n^d(h(\mu)-\delta)} \right\}.$$

Finally, set $F_n(\delta, \alpha) = [C_n(\delta)] \cap D_n(\alpha, \delta) \cap E_n(\delta)$.

The restriction of any x in $D_n(\alpha, \delta) \cap E_n(\delta)$ to Λ_n , i.e. $a := \Pi_n x$ can be described as follows.

1. First we specify a k with $k \geq k_0(\delta)$, $2^{k^d(h(\mu)+\alpha)} \leq n^d$ as in the definition of $D_n(\alpha, \delta)$.

2. Next, for each of the $\lfloor \frac{n}{k} \rfloor^d$ blocks counting for the empirical distribution, we specify whether this block belongs to $\tilde{\mathcal{T}}_{k,n}(x)$, to $\mathcal{T}_k(\delta) \setminus \tilde{\mathcal{T}}_{k,n}(x)$ or to $\Sigma^k \setminus (\mathcal{T}_k(\delta) \cup \tilde{\mathcal{T}}_{k,n}(x))$.

3. Then we specify for each such block its contents, pointing either to a list containing all elements of $\tilde{\mathcal{T}}_{k,n}(x)$, or to a list containing $\mathcal{T}_k(\delta) \setminus \tilde{\mathcal{T}}_{k,n}(x)$ or, in the last case, listing all elements of that block.

4. Finally, we list all elements (at the boundary) not covered by the empirical distribution.

In order to specify k we need at most $\log n$ bits (in fact, much less, due to the bound on k). We need at most $2 \lfloor \frac{n}{k} \rfloor^d$ bits to say which of the cases under

2. is valid for each of the blocks. For 3. we need the two lists for the given k . This needs at most $\left(2^{k^d(h(\mu)+\delta)} + 2^{k^d(h(\mu)-\alpha)}\right) k^d(\log |\mathcal{A}| + 1)$ bits. According to the definitions of $D_n(\alpha, \delta)$ and $E_n(\delta)$, to specify the contents of all k -blocks, we need at most

$$\left(\frac{n}{k} + 1\right)^d k^d (\alpha(h(\mu) - \alpha) + (1 - \alpha)(h(\mu) + \delta) + \delta(\log |\mathcal{A}| + 1))$$

bits. For 4. we need at most $(n^d - \lfloor \frac{n}{k} \rfloor^d k^d)(\log |\mathcal{A}| + 1)$ bits. Hence the cardinality of $\Pi_n F_n(\delta, \alpha)$ can be estimated by

$$\begin{aligned} & \log |\Pi_n F_n(\delta, \alpha)| \\ & \leq \log n + 2 \frac{n^d}{k_1^d(\alpha)} \\ & \quad + n^d \left(n^{-d(1 - \frac{h(\mu)+\delta}{h(\mu)+\alpha})} + n^{-d(1 - \frac{h(\mu)-\alpha}{h(\mu)+\alpha})} \right) \frac{d \log n}{h(\mu) + \alpha} (\log |\mathcal{A}| + 1) \\ & \quad + n^d \left(1 + \frac{1}{n} \sqrt[d]{\frac{d \log n}{h(\mu) + \alpha}} \right)^d (h(\mu) - \alpha^2 + \delta(\log |\mathcal{A}| + 2)) \\ & \quad + n^d \left(1 - \left(1 - \frac{1}{n} \sqrt[d]{\frac{d \log n}{h(\mu) + \alpha}} \right)^d \right) (\log |\mathcal{A}| + 1) \\ & \leq n^d (h(\mu) - \alpha^2/2 + \delta(\log |\mathcal{A}| + 2)) \end{aligned}$$

bits, supposed n is large enough and $k_1(\alpha)$ is chosen sufficiently large. Now, due to $\Pi_n F_n(\delta, \alpha) \subset C_n(\delta)$, we get

$$\mu(F_n(\delta, \alpha)) = \mu^n(\Pi_n F_n(\delta, \alpha)) \leq 2^{-n^d(\alpha^2/2 - \delta(\log |\mathcal{A}| + 3))}.$$

Making δ small enough from the beginning, the exponent here is negative. Hence, by the Borel-Cantelli lemma, only finitely many of the events $x \in F_n(\delta, \alpha)$ may occur, almost surely. But we know from the first part of the theorem that $x \in E_n(\delta)$ eventually a.s. (observe that the condition $2^{k^d(h(\mu)+\alpha)} \leq n^d$ implies $\frac{k}{n} < \varepsilon(\delta)$ as supposed there, for n large enough). And we know from the Ornstein-Weiss-Theorem that $\Pi_n x \in C_n(\delta)$ eventually a.s. Hence $x \in (\Sigma \setminus F_n(\delta, \alpha)) \cap E_n(\delta) \cap [C_n(\delta)] \subset \Sigma \setminus D_n(\delta, \alpha)$ eventually a.s.

This is the assertion *b*) of the theorem. \blacksquare

Proof of Theorem 6. The proof follows the ideas of the proof of the one-dimensional statement Theorem II.3.5 in [13].

Let $\alpha < \frac{1}{4}$ and consider the sets $\mathcal{T}_k(\alpha)$ given in theorem 5. Define $U_{k,n}(x) := \{a \in \mathcal{T}_k(\alpha) : \tilde{\mu}_x^{k,n}(a) < 2^{-k^d(h(\mu)+2\alpha)}\}$. We have $|\mathcal{T}_k(\alpha)| \leq 2^{k^d(h(\mu)+\alpha)}$. From this we deduce $\tilde{\mu}_x^{k,n}(U_{k,n}(x)) \leq 2^{-k^d\alpha}$ for any x .

Consider also the sets $V_{k,n}(x) := \{a \in \mathcal{T}_k(\alpha) : \tilde{\mu}_x^{k,n}(a) > 2^{-k^d(h(\mu)-2\alpha)}\}$. Then obviously $|V_{k,n}(x)| \leq 2^{k^d(h(\mu)-2\alpha)}$. Now the second part of Th. 5 states that for μ -a.e. x there exists an $n_0(x)$, such that $\tilde{\mu}_x^{k,n}(V_{k,n}(x)) \leq 2\alpha$, supposed $n > n_0(x)$, $k > k_1(2\alpha)$ and $2^{k^d(h(\mu)+2\alpha)} \leq n^d$.

We define $M_{k,n}(x) := \mathcal{T}_k(\alpha) \setminus (U_{k,n}(x) \cup V_{k,n}(x))$, and conclude that for μ -a.e. x the following holds

$$\tilde{\mu}_x^{k,n}(M_{k,n}(x)) \geq 1 - 4\alpha,$$

supposed $n > n_0(x)$, $k > k_2(2\alpha)$ and $2^{k^d(h(\mu)+2\alpha)} \leq n^d$, where $k_2(\alpha) \geq k_1(\alpha)$ is chosen such that $2^{-k_2(\alpha)^d \alpha} < \alpha$.

Consider now the definition of the Shannon entropy of the empirical distribution

$$\begin{aligned} H(\tilde{\mu}_x^{k,n}) &= - \sum_{a \in \Sigma^k} \tilde{\mu}_x^{k,n}(a) \log \tilde{\mu}_x^{k,n}(a) \\ &= - \underbrace{\sum_{\Sigma^k \setminus M_{k,n}} \dots}_{\Xi_{k,n}} - \underbrace{\sum_{M_{k,n}} \dots}_{\chi_{k,n}}. \end{aligned}$$

We write $B_{k,n}(x) := \Sigma^k \setminus M_{k,n}(x)$. For the first sum an upper bound is given by²

$$\Xi_{k,n} \leq \tilde{\mu}_x^{k,n}(B_{k,n}(x)) k^d \log |\mathcal{A}| - \tilde{\mu}_x^{k,n}(B_{k,n}(x)) \log \tilde{\mu}_x^{k,n}(B_{k,n}(x)).$$

Hence $\limsup_{n \rightarrow \infty} \frac{1}{k_n^d} \Xi_{k(n),n} \leq 4\alpha \log |\mathcal{A}|$ holds μ -almost surely under the assumptions of the theorem.

As for the second sum, bear in mind that the elements a in $M_{k,n}(x)$ have the property

$$k^d(h(\mu) - 2\alpha) \leq -\log \tilde{\mu}_x^{k,n}(a) \leq k^d(h(\mu) + 2\alpha)$$

and thus

$$\begin{aligned} \frac{1}{k_n^d} \chi_{k,n} &\geq \sum_{a \in M_{k,n}(x)} \tilde{\mu}_x^{k,n}(a)(h(\mu) - 2\alpha) \geq (1 - 4\alpha)(h(\mu) - 2\alpha) \\ \frac{1}{k_n^d} \chi_{k,n} &\leq \sum_{a \in M_{k,n}(x)} \tilde{\mu}_x^{k,n}(a)(h(\mu) + 2\alpha) \leq h(\mu) + 2\alpha. \end{aligned}$$

Therefore we have

$$\begin{aligned} &(1 - 4\alpha)(h(\mu) - 2\alpha) \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{k_n^d} H(\tilde{\mu}_x^{k(n),n}) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{k_n^d} H(\tilde{\mu}_x^{k(n),n}) \\ &\leq h(\mu) + \alpha(2 + 4 \log |\mathcal{A}|) \end{aligned}$$

holding μ -a.s.

Finally, observe that a sequence k_n fulfilling the two assumptions of the theorem for some $\alpha > 0$ in fact fulfils them for any smaller α too. This proves the result. \blacksquare

Proof of Theorem 1. 1. Each $x \in \Sigma$ gives rise to a family of empirical distributions $\{\tilde{\mu}_x^{k,n}\}_{k \leq n}$.

²Observe that $\sum_{a \in B} p(a) \log p(a) \leq p(B) \log |B| - p(B) \log p(B)$.

We define for each n the set $\mathcal{T}_n(h_0)$ as the set of elements in Σ^n having empirical k -block entropy per symbol not greater than h_0 :

$$\mathcal{T}_n(h_0) := \Pi_n \{x \in \Sigma : H(\tilde{\mu}_x^{k,n}) \leq k^d h_0\}.$$

Here we have to choose k depending on n , (how exactly will be specified later).

The number of all (non-overlapping) empirical k -block distributions in Σ^n is upper bounded by $\left(\left(\frac{n}{k}\right)^d\right)^{|\mathcal{A}|^{k^d}}$, since $\lfloor \frac{n}{k} \rfloor^d$ is the maximum number of occurrences of any particular k -block in the parsing of an element of Σ^n , and $|\mathcal{A}|^{k^d}$ is the number of elements in Σ^k .

For the number of elements $x^n \in \Sigma^n$ which give rise to the same empirical distribution $(\tilde{\mu}_x^{k,n})$ we find an upper bound which depends only on the entropy of that empirical distribution:

For a given n such that $\lfloor n/k \rfloor = n/k$ we consider the product measure $P = (\tilde{\mu}_x^{k,n})^{\otimes (n/k)^d}$ on Σ^n : $P(y^n) = \prod_{\substack{\mathbf{r} \in k \cdot \mathbb{Z}^d \\ \Lambda_k + \mathbf{r} \subset \Lambda_n}} \tilde{\mu}_x^{k,n}(\Pi_k(\sigma_{\mathbf{r}} y))$, which yields

$$P(y^n) = \prod_{a \in \Sigma^k} (\tilde{\mu}_x^{k,n}(a))^{(n/k)^d \tilde{\mu}_x^{k,n}(a)} = 2^{-(n/k)^d H(\tilde{\mu}_x^{k,n})}, \quad \forall y : \tilde{\mu}_y^{k,n} = \tilde{\mu}_x^{k,n}, \quad (8)$$

and thus $|\{y \in \Sigma^n : \tilde{\mu}_y^{k,n} = \tilde{\mu}_x^{k,n}\}| \leq 2^{(n/k)H(\tilde{\mu}_x^{k,n})}$.

For a general $n : \lfloor n/k \rfloor \neq n/k$, the entries in the positions $\Lambda_n \setminus \Lambda_{k \cdot \lfloor n/k \rfloor}$ of an $y \in \Sigma^n$ may be occupied arbitrarily, giving the following bound:

$$|\{y \in \Sigma^n : \tilde{\mu}_y^{k,n} = \tilde{\mu}_x^{k,n}\}| \leq 2^{\lfloor n/k \rfloor^d H(\tilde{\mu}_x^{k,n})} \cdot |\mathcal{A}|^{n^d - (n-k)^d}. \quad (9)$$

Now we are able to give an upper estimate for the number $|\mathcal{T}_n(h_0)|$ of all configurations in Λ_n which produce an empirical distribution with entropy not larger than $k^d h_0$:

$$\begin{aligned} |\mathcal{T}_n(h_0)| &\leq 2^{h_0 k^d \left(\frac{n}{k}\right)^d} |\mathcal{A}|^{n^d - (n-k)^d} \left(\left(\frac{n}{k}\right)^d\right)^{|\mathcal{A}|^{k^d}}, \\ \log |\mathcal{T}_n(h_0)| &\leq n^d h_0 + (n^d - (n-k)^d) \log |\mathcal{A}| + |\mathcal{A}|^{k^d} d \log \frac{n}{k}. \end{aligned}$$

Introducing the restriction $k^d \leq \frac{1}{1+\varepsilon} \log_{|\mathcal{A}|} n^d = \frac{\log n^d}{(1+\varepsilon) \log |\mathcal{A}|}$, $\varepsilon > 0$ arbitrary, we conclude that $|\mathcal{T}_n(h_0)| \leq 2^{n^d h_0 + o(n^d)}$ (uniformly in k under the restriction). This yields $\limsup_{n \rightarrow \infty} \frac{\log |\mathcal{T}_n(h_0)|}{n^d} = h_0$.

2. Next we have to prove that such a sequence of sets, with $k = k(n)$ suitably specified, is asymptotically typical for all $\mu \in \mathbb{P}_{\text{erg}}$ with $h(\mu) < h_0$. Given any μ with $h(\mu) < h_0$, Theorem 6 states that for μ -a.e. x the $k(n)$ -block empirical entropy of $\tilde{\mu}_x^{k,n}$ converges to $h(\mu)$, provided that $k(n)$ is a sequence with $k(n) \rightarrow \infty$ and $k(n)^d \leq \frac{\log n^d}{h(\mu) + \alpha}$, where $\alpha > 0$ can be chosen arbitrarily. Since $h(\mu) \leq \log |\mathcal{A}|$, choosing $k^d(n) \leq \frac{\log n^d}{(1+\varepsilon) \log |\mathcal{A}|}$, $\varepsilon > 0$ arbitrary, we get assertion a) from the definition of $\mathcal{T}_n(h_0)$.

3. For a sequence $\{\mathcal{U}_n \subset \Sigma^n\}_n$ with $\liminf_{n \rightarrow \infty} \frac{1}{n^d} \log |\mathcal{U}_n| = h_1 < h_0$ we find a μ with $h(\mu) = h_2, h_1 < h_2 < h_0$. We know, that μ^n is asymptotically confined to the entropy typical subsets:

$$C_n(\delta) := \left\{ a \in \Sigma^n : 2^{-n^d(h_2+\delta)} \leq \mu^n(\{a\}) \leq 2^{-n^d(h_2-\delta)} \right\}.$$

Hence, we get the following:

$$\liminf_{n \rightarrow \infty} \mu(\mathcal{U}_n) = \liminf_{n \rightarrow \infty} \mu(\mathcal{U}_n \cap C_n(\delta)) \leq \liminf_{n \rightarrow \infty} |\mathcal{U}_n| 2^{-n^d(h_2-\delta)} = \lim_{n \rightarrow \infty} 2^{n^d(h_1-h_2+\delta)}.$$

Choosing δ small enough, this is zero. This proves *c)*. Also, combining *c)* with *a)*, we get $\liminf_{n \rightarrow \infty} \frac{1}{n^d} \log |\mathcal{T}_n(h_0)| \geq h_0$. In 1. we proved $\limsup_{n \rightarrow \infty} \frac{1}{n^d} \log |\mathcal{T}_n(h_0)| = h_0$, thus *b)* is verified. ■

4 Conclusions

We have formulated and shown multidimensional extensions of important theoretical results about samplings of ergodic sources. Since these results give a mathematical basis for the design of universal source coding schemes, we herewith provide a truly multidimensional mathematical framework for the optimal compression of multidimensional data.

We have shown that the set of $n \times \dots \times n$ -arrays which have empirical k -block distributions of per site entropy not larger than h_0 is asymptotically typical for all ergodic \mathcal{A} -processes of entropy rate smaller than h_0 , where $k = \left\lceil \sqrt[d]{c \log_{|\mathcal{A}|} n^d} \right\rceil$, $0 < c < 1$. In other words, for all \mathcal{A} -processes of entropy rate smaller than h_0 the probability of the corresponding cylinder set tends to 1 as $n \rightarrow \infty$. These sets have a log cardinality of order $n^d h_0$.

References

- [1] Paul H. Algoet and Thomas M. Cover. A sandwich proof of the Shannon-McMillan-Breiman theorem. *The Annals of Probability*, 16(2):899–909, 1988.
- [2] Igor Bjelaković, Tyll Krüger, Rainer Siegmund-Schultze, and Arleta Szkoła. The Shannon-McMillan theorem for ergodic quantum lattice systems. *Inventiones Mathematicae*, 155(1):203 – 222, 2004.
- [3] Leo Breiman. The individual ergodic theorem of information theory. *Ann. Math. Statist.*, 28:809–811, 1957.
- [4] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory*. Wiley-Interscience, 99th edition, 1991.
- [5] A Lempel and J Ziv. A universal algorithm for sequential data compression. *IEEE Transactions on Information Theory*, 23(3):337–343, May 1977.
- [6] A Lempel and J Ziv. Compression of individual sequences via variable-rate coding. *IEEE Transactions on Information Theory*, vol. 24, no. 5, 1978.
- [7] A Lempel and J Ziv. Compression of two-dimensional data. *IEEE Trans. Inf. Theor.*, 32(1):2–8, 1986.
- [8] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Inventiones Mathematicae*, 146(2):259–295, November 2001.
- [9] Brockway McMillan. The basic theorems of information theory. *The Annals of Mathematical Statistics*, 24(2):196–219, 1953.
- [10] Donald S. Ornstein and Benjamin Weiss. The Shannon-McMillan-Breiman theorem for a class of amenable groups. *Isr. J. Math.*, 44, 1983.
- [11] Donald S. Ornstein and Benjamin Weiss. How sampling reveals a process. *The Annals of Probability*, 18(3):905–930, 1990.
- [12] C. E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27(1):379–423, 623–656, 1948.
- [13] P. Shields. *The Ergodic Theory of Discrete Sample Paths*, volume 13 of *Graduate Studies in Mathematics*. American Mathematical Society, 1996.
- [14] K. Schmidt. A Probabilistic Proof of Ergodic Decomposition. *Sankhya: The Indian Journal of Statistics, Series A*, 40(1):10–18, 1978.
- [15] T. A. Welch. A technique for high-performance data compression. *Computer*, 17(6):8–19, 1984.