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curvature dimension inequalities on graphs

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OLLIVIER'S RICCI CURVATURE, LOCAL CLUSTERING AND CURVATURE DIMENSION INEQUALITIES ON GRAPHS

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ABSTRACT. In Riemannian geometry, Ricci curvature controls how fast geodesics emanating from a common source are diverging on average, or equivalently, how fast the volume of distance balls grows as a function of the radius. Recently, such ideas have been extended to Markov processes and metric spaces. Employing a definition of generalized Ricci curvature proposed by Ollivier and applied in graph theory by Lin-Yau, we derive lower Ricci curvature bounds on graphs in terms of local clustering coefficients, that is, the relative proportion of connected neighbors among all the neighbors of a vertex. This translates the above Riemannian ideas into a combinatorial setting. We also study curvature dimension inequalities on graphs, building upon previous work of several authors.

1. INTRODUCTION

Ricci curvature is a fundamental concept in Riemannian geometry. It is a quantity computed from second derivatives of the metric tensor. It controls how fast geodesics starting at the same point diverge on average. Equivalently, it controls how fast the volume of distance balls grows as a function of the radius. It also controls the amount of overlap of two distance balls in terms of their radii and the distance between their centers. In fact, such upper bounds follow from a lower bound on the Ricci curvature. It was then natural to look for generalizations of such phenomena on metric spaces more general than Riemannian manifolds. That is, the question to find substitute for the lower bounds on the above mentioned second derivative combinations of the metric tensor that yield the same geometric control on a general metric space. By now, there exist several beautiful works on defining a synthetic Ricci curvature on general metric measure spaces, see Sturm [20, 21], Lott-Villani [15], Ohta [16] etc.

In this paper, we want to explore the implications of such ideas in graph theory. There is already important work of Lin-Yau [14, 12] in this direction, utilizing a definition of Ricci curvature developed by Ollivier [17] and obtaining analogues of Bochner type inequalities and eigenvalue estimates as known from Riemannian geometry. Here, we want to explore the geometric idea that a lower Ricci curvature bound prevents geodesics from diverging too fast and balls from growing too fast in volume. On a graph, the analogue of geodesics starting in different directions, but eventually approaching each other again, would be a triangle. Therefore, it is natural that the Ricci curvature on a graph should be related to the relative abundance of triangles. The latter is captured by the local clustering coefficient introduced by Watts-Strogatz [24]. Thus, in the present paper, we refine the Lin-Yau inequality in terms of the local clustering coefficient. We believe that this really captures the intuition of Ricci curvature on a graph. Compared with other definitions of generalized Ricci curvature, the one of Ollivier works particularly on discrete spaces like graphs. It is formulated in terms of the transportation distance between local measures. When two balls strongly overlap, as is the case in Riemannian geometry when the Ricci curvature has a large lower bound, then it is easier to transport the mass of one to the other. This is the

idea of Ollivier's definition as we see it and explore in this paper. We shall obtain both upper and lower bounds for Ollivier's Ricci curvature on graphs in Section 3, which are optimal on many graphs.

In Riemannian geometry, the Bochner formula encodes deep analytic properties of Ricci curvature. It is a key ingredient in proving many results, e.g. the spectral gap of the Laplace-Beltrami operator. A lower bound of the Ricci curvature implies a curvature dimension inequality involving the Laplace-Beltrami operator through the Bochner formula. In an important work, Bakry and Émery [2] generalize this inequality to generators of Markov semigroups, which works on measure spaces. Their inequality contains plentiful information and implies a lot of functional inequalities including spectral gap inequalities, Sobolev inequalities, and logarithmic Sobolev inequalities and many celebrated geometric theorems (see [1] and the references therein). In their recent paper, Lin-Yau [14] study such inequalities on locally finite graphs.

In the present paper, we also want to find relations on locally finite graphs between Ollivier's Ricci curvature and Bakry-Émery's curvature dimension inequalities, which represent the geometric and analytic aspects of graphs respectively. Again, this is inspired by Riemannian geometry where one may attach a Brownian motion with a drift to a Riemannian metric [17]. We also mention that the definitions given by Sturm and Lott-Villani are also consistent with that of Bakry-Émery [20, 21, 15]. So exploring the relations on nonsmooth spaces may provide a good point of view to connect Ollivier's definition to Sturm and Lott-Villani's. In Section 4, we use the local clustering coefficient again to establish more precise curvature dimension inequalities than those of Lin-Yau [14]. And with this in hand, we prove curvature dimension inequalities under the condition that Ollivier's Ricci curvature of the graph is positive.

Further analytical results following from curvature dimension inequalities on finite graphs have been described in [12], and Lin-Lu-Yau [13] study a modified definition of Ollivier's Ricci curvature on graphs in an upcoming paper. For other works of synthetic Ricci curvatures on discrete spaces, see Dodziuk-Karp [8], Chung-Yau [7], Bonciocat-Sturm [5], and on cell complexes see Forman [11], Stone [19] etc.

In this paper, $G = (V, E)$ will denote an undirected connected simple graph without loops, where V is the set of vertices and E is the set of edges. V could be an infinite set. But we require that G is locally finite, i.e., for every $x \in V$, the number of edges connected to x is finite. If $x, y \in V$ are connected by an edge, then we say x and y are neighbors and denote this by $x \sim y$. For simplicity and in order to see more geometry, we mainly work on unweighted graphs. But we will also derive similar results on weighted graphs. In that case, we denote by w_{xy} the weight associated to $x, y \in V$, where $x \sim y$ (we may simply put $w_{xy} = 0$ if x and y are not neighbors, to simplify the notation). The unweighted case corresponds to $w_{xy} = 1$ whenever $x \sim y$. The degree of $x \in V$ is $d_x = \sum_{y, y \sim x} w_{xy}$.

2. OLLIVIER'S RICCI CURVATURE AND BAKRY-ÉMERY'S CALCULUS

In this section, we present some basic facts about Ollivier's Ricci curvature and Bakry-Emery's Γ_2 calculus, in particular on graphs.

2.1. Ollivier's Ricci curvature. Ollivier's Ricci curvature works on a general metric space (X, d) , on which we attach to each point $x \in X$ a probability measure $m_x(\cdot)$. We denote this structure by (X, d, m) .

For a locally finite unweighted graph $G = (V, E)$, we define the metric d as follows. For neighbors x, y , $d(x, y) = 1$. For general distinct vertices x, y , $d(x, y)$ is the length of the shortest path connecting x and y , i.e. the number of edges of the path. We attach to each vertices $x \in V$ a probability measure

$$m_x(y) = \begin{cases} \frac{1}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

An intuitive illustration of this is a random walker that sits at x and then chooses amongst the neighbors of x with equal probability $\frac{1}{d_x}$.

Definition 1 (Ollivier). *On (X, d, m) , for any two distinct points $x, y \in X$, the (Ollivier-) Ricci curvature of (X, d, m) along (xy) is defined as*

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)}. \quad (2.2)$$

Here, $W_1(m_x, m_y)$ is the optimal transportation distance between the two probability measures m_x and m_y , defined as follows (cf. Villani [22, 23], Evans [10]).

Definition 2. *For two probability measures μ_1, μ_2 on a metric space (X, d) , the transportation distance between them is defined as*

$$W_1(\mu_1, \mu_2) := \inf_{\xi \in \Pi(\mu_1, \mu_2)} \int_{X \times X} d(x, y) d\xi(x, y), \quad (2.3)$$

where $\Pi(\mu_1, \mu_2)$ is the set of probability measures on $X \times X$ projecting to μ_1 and μ_2 .

In other words, ξ satisfies

$$\xi(A \times X) = \mu_1(A), \quad \xi(X \times B) = \mu_2(B), \quad \forall A, B \subset X.$$

Remark 1. *Intuitively, this distance measures the optimal cost to move one pile of sand to another one with the same mass. For case of a graph $G = (G, d, m)$, the supports of m_x and m_y are finite discrete sets, and thus, ξ is just a matrix with terms $\xi(x', y')$ representing the mass moving from $x' \in \text{support of } m_x$ to $y' \in \text{support of } m_y$. That is, in this case,*

$$W_1(m_x, m_y) = \inf_{\xi} \sum_{x', x' \sim x} \sum_{y', y' \sim y} d(x', y') \xi(x', y'),$$

where the infimum is taken over all matrices ξ which satisfy

$$\sum_{x', x' \sim x} \xi(x', y') = \frac{w_{yy'}}{d_y}, \quad \sum_{y', y' \sim y} \xi(x', y') = \frac{w_{xx'}}{d_x}.$$

We also call ξ a transfer plan. If we can find a particular transfer plan, we then get an upper bound for W_1 and therefore a lower bound for κ .

A very important property of transportation distance is the Kantorovich duality (see, e.g. Theorem 1.14 in Villani [22]). We state it here in our particular graph setting

Proposition 1 (Kantorovich duality).

$$W_1(m_x, m_y) = \sup_{f, 1\text{-Lip}} \left[\sum_{z, z \sim x} f(z) m_x(z) - \sum_{z, z \sim y} f(z) m_y(z) \right],$$

where the supremum is taken over all functions on G that satisfy

$$|f(x) - f(y)| \leq d(x, y),$$

for any $x, y \in V$, $x \neq y$.

From this property, a good choice of a 1-Lipschitz function f will yield a lower bound for W_1 and therefore an upper bound for κ .

Remark 2. *We list some basic first observations about this curvature concept:*

- $\kappa(x, y) \leq 1$.
- *Rewriting (2.2) gives $W_1(m_x, m_y) = d(x, y)(1 - \kappa(x, y))$, which is analogous to the expansion in the Riemannian case.*
- *A lower bound $\kappa(x, y) \geq k$ for any $x, y \in X$ implies*

$$W_1(m_x, m_y) \leq (1 - k)d(x, y), \quad (2.4)$$

which can be seen as some kind of Lipschitz continuity of measures.

2.2. Bakry-Émery's curvature-dimension inequality.

2.2.1. *Laplace operator.* We will study the following operator which is an analogue of the Laplace-Beltrami operator in Riemannian geometry.

Definition 3. *The Laplace operator on (X, d, m) is defined as follows*

$$\Delta f(x) = \int_X f(y) dm_x(y) - f(x), \text{ for functions } f : X \longrightarrow R. \quad (2.5)$$

For our choice of $\{m_x(\cdot)\}$, this is the graph Laplacian studied by many authors, see e.g. [6], [3], [4], [8], [14].

2.2.2. *Bochner formula and curvature-dimension inequality.* In the Riemannian case, many analytical consequences of a lower bound of the Ricci curvature are obtained through the well-known Bochner formula,

$$\frac{1}{2}\Delta(|\nabla f|^2) = |\text{Hess } f|^2 + \langle \nabla(\Delta f), \nabla f \rangle + \text{Ric}(\nabla f, \nabla f).$$

Analytically, $|\text{Hess } f|^2$ is difficult to be defined on a nonsmooth space. But using Schwarz's inequality, we have

$$|\text{Hess } f|^2 \geq \frac{(\Delta f)^2}{m},$$

where m is the dimension constant. So we can use

$$\frac{1}{2}\Delta(|\nabla f|^2) \geq \frac{(\Delta f)^2}{m} + \langle \nabla(\Delta f), \nabla f \rangle + K|\nabla f|^2 \quad (2.6)$$

to characterize $\text{Ric} \geq K$.

Bakry-Émery [1, 2] take this inequality as the starting point and directly use the operators to define curvature bounds. Starting from an operator Δ , they define iteratively,

$$\begin{aligned} \Gamma_0(f, g) &= fg, \\ \Gamma(f, g) &= \frac{1}{2}\{\Delta\Gamma_0(f, g) - \Gamma_0(f, \Delta g) - \Gamma_0(\Delta f, g)\}, \\ \Gamma_2(f, g) &= \frac{1}{2}\{\Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g)\}. \end{aligned}$$

In fact, $\Gamma(f, f)$ is an analogue of $|\nabla f|^2$, and $\Gamma_2(f, f)$ is an analogue of $\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla(\Delta f), \nabla f \rangle$ in (2.6).

Definition 4. We say an operator Δ satisfies a curvature-dimension inequality $CD(m, K)$ if for all functions f in the domain of the operator

$$\Gamma_2(f, f)(x) \geq \frac{1}{m}(\Delta f(x))^2 + K(x)\Gamma(f, f)(x), \quad \forall x \in X, \quad (2.7)$$

where $m \in [1, +\infty]$ is the dimension parameter, $K(x)$ is the curvature function.

Applying this construction to the operator (2.5) we have

$$\Gamma(f, f)(x) = \frac{1}{2} \int_X (f(y) - f(x))^2 dm_x(y), \quad (2.8)$$

In fact generally

$$\Gamma(f, g)(x) = \frac{1}{2} \int_X (f(y) - f(x))(g(y) - g(x)) dm_x(y).$$

For the sake of convenience, we will denote

$$Hf(x) := \frac{1}{4} \int_X \int_X (f(x) - 2f(y) + f(z))^2 dm_y(z) dm_x(y).$$

A similar calculation as in Lin-Yau [14] gives

$$\begin{aligned} \Delta\Gamma(f, f)(x) &= 2Hf(x) - \int_X \int_X (f(x) - 2f(y) + f(z))(f(x) - f(y)) dm_y(z) dm_x(y), \\ 2\Gamma(f, \Delta f)(x) &= -(\Delta f(x))^2 - \int_X \int_X (f(z) - f(y))(f(x) - f(y)) dm_y(z) dm_x(y). \end{aligned}$$

Then by definition,

$$\Gamma_2(f, f) = Hf(x) - \Gamma(f, f)(x) + \frac{1}{2}(\Delta f(x))^2. \quad (2.9)$$

3. OLLIVIER'S RICCI CURVATURE AND TRIANGLES

In this section, we mainly prove lower bounds for Ollivier's Ricci curvature on locally finite graphs. In particular we shall explore the implication between lower bounds of the curvature and the number of triangles including neighboring vertices; the latter is encoded in the local clustering coefficient. We remark that we only need to bound $\kappa(x, y)$ from below for neighboring x, y , since by the triangle inequality of W_1 , this will also be a lower bound for $\kappa(x, y)$ of any pair of x, y . (See Proposition 19 in Ollivier [17].)

3.1. Unweighted graphs. In this subsection, we only consider unweighted graphs.

In Lin-Yau [14], they prove a lower bound of Ollivier's Ricci curvature on locally finite graphs G . Here, for later purposes, we include the case where G may have vertices of degree 1 and get the following modified result.

Theorem 1. *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y ,*

$$\kappa(x, y) \geq -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+ = \begin{cases} -2 + \frac{2}{d_x} + \frac{2}{d_y}, & \text{if } d_x > 1 \text{ and } d_y > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Remark 3. *Notice that if $d_x = 1$, then we can calculate $\kappa(x, y) = 0$ exactly. So, even though in this case $-2 + \frac{2}{d_x} = 0$, $\kappa(x, y) \geq \frac{2}{d_y}$ doesn't hold.*

For completeness, we state the proof of Theorem 1 here. It is essentially the one in Lin-Yau [14] with a small modification.

Proof of Theorem 1: Since $d(x, y) = 1$ for $x \sim y$, we have

$$\kappa(x, y) = 1 - W_1(m_x, m_y). \quad (3.1)$$

Using Kantorovich duality, we get

$$\begin{aligned} W_1(m_x, m_y) &= \sup_{f, 1\text{-Lip}} \left(\frac{1}{d_x} \sum_{z, z \sim x} f(z) - \frac{1}{d_y} \sum_{z', z' \sim y} f(z') \right) \\ &= \sup_{f, 1\text{-Lip}} \left(\frac{1}{d_x} \sum_{z, z \sim x, z \neq y} (f(z) - f(x)) - \frac{1}{d_y} \sum_{z', z' \sim y, z' \neq x} (f(z') - f(y)) \right. \\ &\quad \left. + \frac{1}{d_x} (f(y) - f(x)) - \frac{1}{d_y} (f(x) - f(y)) + (f(x) - f(y)) \right) \\ &\leq \frac{d_x - 1}{d_x} + \frac{d_y - 1}{d_y} + \left| 1 - \frac{1}{d_x} - \frac{1}{d_y} \right| \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} + \left| 1 - \frac{1}{d_x} - \frac{1}{d_y} \right| \\ &= 1 + 2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+. \end{aligned} \quad (3.2)$$

Inserting the above estimate into (3.1) gives

$$\kappa(x, y) \geq -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+.$$

□

Note that trees attain this lower bound. This coincides with the geometric intuition of curvature. Since trees have the fastest volume growth rate, it is plausible that they have the smallest negative curvature.

Proposition 2. *We consider a tree $T = (V, E)$. Then for any neighboring x, y , we have*

$$\kappa(x, y) = -2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+. \quad (3.3)$$

Proof: In fact with Theorem 1 in hand, we only need to prove that $1 + 2 \left(1 - \frac{1}{d_x} - \frac{1}{d_y} \right)_+$ is also a lower bound of W_1 . If one of x, y is a vertex of degree 1, say $d_x = 1$, it is obvious that $W_1(m_x, m_y) = 1$. So we only need to deal with the case $1 - \frac{1}{d_x} - \frac{1}{d_y} \geq 0$.

We can find a 1-Lipschitz function f on a tree as follows.

$$f(z) = \begin{cases} 0, & \text{if } z \sim y, z \neq x; \\ 1, & \text{if } z = y; \\ 2, & \text{if } z = x; \\ 3, & \text{if } z \sim x, z \neq x. \end{cases} \quad (3.4)$$

Since on a tree, the path joining two vertices are unique, there is no further path between neighbors of x and y . So this can be easily extended to a 1-Lipschitz function

on the whole graph. Then by Kantorovich duality, we have

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x}(3(d_x - 1) + 1) - \frac{1}{d_y} \cdot 2 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y}. \end{aligned} \quad (3.5)$$

This completes the proof. \square

In order to make clear the geometric meaning of the term $\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)_+$, and also to prepare the idea used in the next theorem, we give another method to get the upper bound of W_1 . That works through a particular transfer plan. If

$$1 - \frac{1}{d_x} - \frac{1}{d_y} \geq 0, \text{ or } 1 - \frac{1}{d_y} \geq \frac{1}{d_x},$$

then for m_y , the mass at all z such that $z \sim y$, $z \neq x$ is larger than that of m_x at y . So we can move the mass $\frac{1}{d_x}$ at y to z , $z \sim y$, $z \neq x$ for distance 1. Symmetrically, we can move a mass of $\frac{1}{d_y}$ at the vertices z which satisfy $z \sim x$, $z \neq y$ to x for distance 1. The remaining mass of $\left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right)$ needs to be moved for distance 3. This gives

$$\begin{aligned} W_1(m_x, m_y) &\leq \left(\frac{1}{d_x} + \frac{1}{d_y}\right) \times 1 + \left(1 - \frac{1}{d_x} - \frac{1}{d_y}\right) \times 3 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y}. \end{aligned} \quad (3.6)$$

If

$$1 - \frac{1}{d_x} - \frac{1}{d_y} \leq 0,$$

we only need to move the mass of m_x for distance 1 to the support of m_y . So we have in this case $W_1(m_x, m_y) = 1$. This gives the same upper bound as in (3.2).

From the view of transfer plans, the existence of triangles including neighboring vertices would save a lot of transport costs and therefore affect the curvature heavily. We denote for $x \sim y$,

$$\sharp(x, y) := \text{number of triangles which include } x, y \text{ as vertices} = \sum_{x_1, x_1 \sim x, x_1 \sim y} 1.$$

Remark 4. This quantity $\sharp(x, y)$ is related to the local clustering coefficient introduced by Watts-Strogatz [24],

$$c(x) := \frac{\text{number of edges between neighbors of } x}{\text{number of possible existing edges between neighbors of } x},$$

which measures the extent to which neighbors of x are directly connected. In fact, we have the relation

$$c(x) = \frac{1}{d_x(d_x - 1)} \sum_{y, y \sim x} \sharp(x, y). \quad (3.7)$$

We will explore the relation between the curvature $\kappa(x, y)$ and the number of triangles $\sharp(x, y)$. A critical observation is that $\kappa(x, y)$ is symmetric w.r.t. x and y . So we try to express the curvature through symmetric quantities

$$d_x \wedge d_y := \min\{d_x, d_y\}, \quad d_x \vee d_y := \max\{d_x, d_y\}.$$

Theorem 2. *On a locally finite graph $G = (V, E)$, we have for any pair of neighboring vertices x, y ,*

$$\kappa(x, y) \geq - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y} \right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \vee d_y} \right)_+ + \frac{\sharp(x, y)}{d_x \vee d_y}.$$

Moreover, this inequality is sharp.

Remark 5. *If $\sharp(x, y) = 0$, then this lower bound reduces to the one in Theorem 1.*

Example 1. *On a complete graph \mathcal{K}_n ($n \geq 2$) with n vertices, $\sharp(x, y) = n - 2$ for any x, y . So Theorem 2 implies*

$$\kappa(x, y) \geq \frac{n - 2}{n - 1}.$$

In fact, we can easily check that the above inequality is an equality. Also notice that on those graphs, the local clustering coefficient $c(x) = 1$ attains the largest value.

Before carrying out the proof of Theorem 2, we fix some notations. The vertices z that are adjacent to x or y , where $x \sim y$, are divided into three classes.

- common neighbors of x, y : $z \sim x$ and $z \sim y$;
- x 's own neighbors: $z \sim x, z \not\sim y, z \neq y$;
- y 's own neighbors: $z \sim y, z \not\sim x, z \neq x$.

Proof of Theorem 2: We suppose w.l.o.g.,

$$d_x = d_x \vee d_y, \quad d_y = d_x \wedge d_y.$$

In principle, our transfer plan moving m_x to m_y should be as follows.

- (1) Move the mass of $\frac{1}{d_x}$ from y to y 's own neighbors;
- (2) Move a mass of $\frac{1}{d_y}$ from x 's own neighbors to x ;
- (3) Fill gaps using the mass at x 's own neighbors. Filling the gaps at common neighbors costs 2 and the one at y 's own neighbors costs 3.

A critical point will be whether (1) and (2) can be realized or not. It is easy to see that we can realize step (1) if and only if

$$1 - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_y} \geq \frac{1}{d_x}, \quad \text{or} \quad A := 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \wedge d_y} \geq 0. \quad (3.8)$$

That is, after taking off the mass at x and common neighbors, m_y still has at least a mass of $\frac{1}{d_x}$. Step (2) can be realized if and only if

$$1 - \frac{1}{d_x} - \frac{\sharp(x, y)}{d_x} \geq \frac{1}{d_y}, \quad \text{or} \quad B := 1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x, y)}{d_x \vee d_y} \geq 0. \quad (3.9)$$

That is, after taking off the mass at y and common neighbors, m_x still has enough mass to fill $\frac{1}{d_y}$. Obviously, $A \leq B$.

We will divide the discussion into 3 cases according to whether the first two steps can be realized or not.

- $0 \leq A \leq B$. This means we can adopt the above transfer plan. By definition of $W_1(m_x, m_y)$, we get

$$\begin{aligned} W_1(m_x, m_y) &\leq \frac{1}{d_x} \times 1 + \frac{1}{d_y} \times 1 + \left(\frac{1}{d_y} - \frac{1}{d_x} \right) \times \#(x, y) \times 2 \\ &\quad + \left[1 - \frac{1}{d_x} - \frac{1}{d_y} - \left(\frac{1}{d_y} - \frac{1}{d_x} \right) \times \#(x, y) - \frac{1}{d_x} \#(x, y) \right] \times 3 \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y} - \frac{\#(x, y)}{d_y} - \frac{2\#(x, y)}{d_x}. \end{aligned}$$

Or in a symmetric way,

$$W_1(m_x, m_y) \leq 3 - \frac{2}{d_x \vee d_y} - \frac{2}{d_x \wedge d_y} - \frac{\#(x, y)}{d_x \wedge d_y} - \frac{2\#(x, y)}{d_x \vee d_y}. \quad (3.10)$$

Moreover, in this case the following function f (as shown in Figure 1) can be extended as a 1-Lipschitz function,

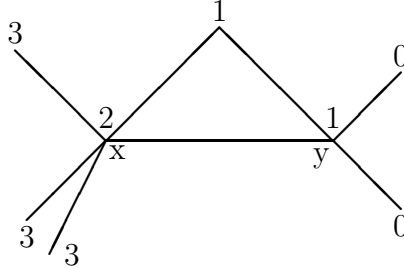


Figure 1. Mass moved from vertices with larger value to those with smaller ones.

$$f(z) = \begin{cases} 0, & \text{at } y\text{'s own neighbors;} \\ 1, & \text{at } y \text{ or common neighbors;} \\ 2, & \text{at } x; \\ 3, & \text{at } x\text{'s own neighbor,} \end{cases}$$

(that is, if there are no paths of length 1 between common neighbors and x 's own neighbors, nor paths of length 1 or 2 between x 's own neighbors and y 's own ones,) we have by Kantorovich duality,

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x} [f(y) + 3(d_x - 1 - \#(x, y)) + \#(x, y)] - \frac{1}{d_y} (f(x) + \#(x, y)) \\ &= 3 - \frac{2}{d_x} - \frac{2}{d_y} - \frac{\#(x, y)}{d_y} - \frac{2\#(x, y)}{d_x}. \end{aligned}$$

That is, in this case, (3.6) should be an equality. In conclusion,

$$\kappa(x, y) \geq -2 + \frac{2}{d_x} + \frac{2}{d_y} + \frac{\#(x, y)}{d_x \wedge d_y} + \frac{2\#(x, y)}{d_x \vee d_y},$$

and the "=" can be attained.

Remark 6. $A \geq 0$ is equivalent to

$$d_x \wedge d_y > 1, \text{ and } \#(x, y) \leq d_x \wedge d_y - 1 - \frac{d_x \wedge d_y}{d_x \vee d_y}.$$

Since $\#(x, y) \in \mathbf{Z}$, we know that $d_x \wedge d_y \geq 2$ and $\#(x, y) \leq d_x \wedge d_y - 2$. This means both x and y have at least one own neighbor.

If $A < 0$, we get

$$d_x \wedge d_y - 1 - \frac{d_x \wedge d_y}{d_x \vee d_y} < \#(x, y) \leq d_x \wedge d_y - 1.$$

I.e., $\#(x, y) = d_x \wedge d_y - 1$. This means the vertex with smaller degree has no own neighbors.

- $A < 0 \leq B$. In this case we cannot realize step (1) but step (2) can be realized. By the above remark, $A < 0$ implies that y has no own neighbors. Our transfer plan should be step (2) at first. Since $B \geq 0$ also implies

$$1 - \frac{1}{d_y} - \frac{\#(x, y)}{d_x} \geq \frac{1}{d_x}, \quad (3.11)$$

so we can move the mass of $\frac{1}{d_x}$ at y for distance 1 to common neighbors. Finally, we fill the gap at common neighbors for distance 2. In a formula,

$$\begin{aligned} W_1(m_x, m_y) &\leq \frac{1}{d_x} \times 1 + \frac{1}{d_y} \times 1 + \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\#(x, y)}{d_x}\right) \times 2 \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{2\#(x, y)}{d_x}. \end{aligned}$$

Or in a symmetric manner,

$$W_1(m_x, m_y) \leq 2 - \frac{1}{d_x \vee d_y} - \frac{1}{d_x \wedge d_y} - \frac{2\#(x, y)}{d_x \vee d_y}. \quad (3.12)$$

Moreover, in case the following function f can be extended as a 1-Lipschitz one,

$$f(z) = \begin{cases} 0, & \text{at common neighbors;} \\ 1, & \text{at } x \text{ and } y; \\ 2, & \text{at } x\text{'s own neighbor,} \end{cases}$$

(that is, if there are no paths of length 1 between common neighbors and x 's own neighbors,) we have by Kantorovich duality,

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x} [f(y) + 2(d_x - 1 - \#(x, y))] - \frac{1}{d_y} f(x) \\ &= 2 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{2\#(x, y)}{d_x}. \end{aligned}$$

In conclusion,

$$\kappa(x, y) \geq -1 + \frac{1}{d_x} + \frac{1}{d_y} + \frac{2\#(x, y)}{d_x \vee d_y},$$

and the "=" can be attained.

Remark 7. Noting that if $\#(x, y) = d_x \wedge d_y - 1$ then $B \geq 0$ is equivalent to

$$d_x \vee d_y \geq \frac{d_x \wedge d_y}{d_x \wedge d_y - 1} d_x \wedge d_y. \quad (3.13)$$

In this case, one of d_x, d_y has no own neighbors, and if the other one has sufficiently many own neighbors, $B \geq 0$ will be satisfied.

- $A \leq B < 0$. In this case, neither step (1) nor (2) is applicable. Also, y has no own neighbor, and $B < 0$ implies that we can move all the mass at x 's own neighbors to x at first. And then we move the mass of $\frac{1}{d_x}$ at y for distance 1 to fill the gaps at x and the common neighbors. In a formula,

$$W_1(m_x, m_y) \leq \left(1 - \frac{\sharp(x, y)}{d_x}\right) \times 1 = 1 - \frac{\sharp(x, y)}{d_x}.$$

Or in a symmetric way,

$$W_1(m_x, m_y) \leq 1 - \frac{\sharp(x, y)}{d_x \vee d_y}. \quad (3.14)$$

We can find a 1-Lipschitz function

$$f(z) = \begin{cases} 0, & \text{at } x \text{ and common neighbors;} \\ 1, & \text{at } y \text{ and } x\text{'s own neighbors,} \end{cases}$$

Then by Kantorovich duality,

$$\begin{aligned} W_1(m_x, m_y) &\geq \frac{1}{d_x}(f(y) + d_x - 1 - \sharp(x, y)) - \frac{1}{d_y} \times 0 \\ &= 1 - \frac{\sharp(x, y)}{d_x}. \end{aligned}$$

In this case f can be extended to a 1-Lipschitz function on the graph, so we get finally,

$$\kappa(x, y) = \frac{\sharp(x, y)}{d_x}.$$

Luckily, we can write those three cases in a uniform formula. \square

Remark 8. From extending f to a 1-Lipschitz function, we see that the paths of length 1 or 2 between neighbors of x and y have an important effect on the curvature. That is, in addition to triangles, quadrangles and pentagons are also related to Ollivier's Ricci curvature. But polygons with more than 5 edges would not impact it.

Remark 9. If we see the graph $G = (V, E)$ as a metric measure space (G, d, m) , then the term $\sharp(x, y)/d_x \vee d_y$ is exactly $m_x \wedge m_y(G) := m_x(G) - (m_x - m_y)_+(G)$, i.e. the intersection measure of m_x and m_y . From a metric view, the vertices x_1 that satisfy $x_1 \sim x$, $x_1 \sim y$ constitute the intersection of the unit metric spheres $S_x(1)$ and $S_y(1)$.

From Theorem 2, we can force the curvature $\kappa(x, y)$ to be positive by increasing the number $\sharp(x, y)$.

Theorem 3. On a locally finite graph $G = (V, E)$, for any neighboring x, y , we have

$$\kappa(x, y) \leq \frac{\sharp(x, y)}{d_x \vee d_y}. \quad (3.15)$$

Proof: Since except for the mass at common neighbors which we need not move, the others have to be moved for a distance at least 1, we have

$$W_1(m_x, m_y) \geq \left(1 - \frac{\sharp(x, y)}{d_x \vee d_y}\right) \times 1.$$

\square

So if $\kappa(x, y) > 0$, then $\sharp(x, y)$ is at least 1. Moreover, if $\kappa(x, y) \geq k > 0$, we have

$$\sharp(x, y) \geq \lceil kd_x \vee d_y \rceil, \quad (3.16)$$

where $\lceil a \rceil := \min\{A \in \mathbf{Z} \mid A \geq a\}$, for $a \in \mathbf{R}$.

We will denote $D(x) := \max_{y, y \sim x} d_y$. By the relation (3.7), we can get immediately

Corollary 1. *The average value of the curvature around x , which may be seen as a synthetic scalar curvature (see Problem Q in [18]), can be controlled by the local clustering coefficient at x ,*

$$\frac{d_x - 1}{d_x} c(x) \geq \frac{1}{d_x} \sum_{y, y \sim x} \kappa(x, y) \geq -2 + \frac{d_x - 1}{d_x \vee D(x)} c(x).$$

Remark 10. *In fact in some special cases, we can get more precise lower bounds*

$$\frac{1}{d_x} \sum_{y, y \sim x} \kappa(x, y) \geq \begin{cases} -2 + \frac{2}{d_x} + \frac{2}{D(x)} + \left[\frac{(d_x - 1)}{d_x} + \frac{2(d_x - 1)}{d_x \vee D(x)} \right] c(x), & \text{if } A \geq 0 \text{ for all } y; \\ -1 + \frac{1}{d_x} + \frac{1}{D(x)} + \frac{2(d_x - 1)}{d_x \vee D(x)} c(x), & \text{if } A < 0 \leq B \text{ for all } y; \\ \frac{d_x - 1}{d_x \vee D(x)} c(x), & \text{if } B < 0 \text{ for all } y. \end{cases}$$

3.2. Weighted graphs. The preceding considerations readily extend to weighted graphs.

Theorem 4. *On a weighted locally finite graph $G = (V, E)$, we have*

$$\kappa(x, y) \geq -2 \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} \right)_+. \quad (3.17)$$

Moreover, weighted trees attain this lower bound.

Theorem 5. *On a weighted locally finite graph $G = (V, E)$, we have*

$$\begin{aligned} \kappa(x, y) \geq & - \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \vee \frac{w_{x_1 y}}{d_y} \right)_+ \\ & - \left(1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y} \right)_+ + \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}. \end{aligned}$$

The inequality is sharp.

Remark 11. *Notice that the term replacing the number of triangles here satisfies*

$$\sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y} = m_x \wedge m_y(G).$$

Proof: Similar to the proof of Theorem 2, we need to understand the following two terms,

$$\begin{aligned} A_w & := 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \vee \frac{w_{x_1 y}}{d_y}, \\ B_w & := 1 - \frac{w_{xy}}{d_x} - \frac{w_{xy}}{d_y} - \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}. \end{aligned}$$

Only the transfer plan in the case $A_w < 0 \leq B_w$ needs a more careful discussion. \square

Theorem 6. *On a weighted locally finite graph $G = (V, E)$, we have for any neighboring x, y ,*

$$\kappa(x, y) \leq \sum_{x_1, x_1 \sim x, x_1 \sim y} \frac{w_{x_1 x}}{d_x} \wedge \frac{w_{x_1 y}}{d_y}. \quad (3.18)$$

4. CURVATURE DIMENSION INEQUALITIES

In this section, we establish curvature dimension inequalities on locally finite graphs. A very interesting one is the inequality under the condition $\kappa \geq k > 0$. Curvature dimension inequalities on locally finite graphs are studied in Lin-Yau [14]. We first state a detailed version of their results. Let's denote $D_w(x) := \max_{y, y \sim x} \frac{d_y}{w_{yx}}$. Notice that on an unweighted graph, this is the $D(x)$ we used in Section 3.

Theorem 7. *On a weighted locally finite graph $G = (V, E)$, the Laplace operator Δ satisfies*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{2}{D_w(x)} - 1 \right) \Gamma(f, f)(x). \quad (4.1)$$

Remark 12. *Since in this case we attach the weighted version of measure (2.1), we get*

$$Hf(x) = \frac{1}{4} \frac{1}{d_x} \sum_{y, y \sim x} \frac{w_{xy}}{d_y} \sum_{z, z \sim y} w_{yz} (f(x) - 2f(y) + f(z))^2.$$

We only need to choose special $z = x$ in the second sum and then (2.8) and (2.9) imply the theorem.

4.1. Unweighted graphs. We again restrict ourselves to unweighted graphs.

We observe the existence of triangles causes cancellations in calculating the term $Hf(x)$. This gives

Theorem 8. *On a locally finite graph $G = (V, E)$, the Laplace operator satisfies*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}t(x) - 1 \right) \Gamma(f, f)(x), \quad (4.2)$$

where

$$t(x) := \min_{y, y \sim x} \left(\frac{4}{d_y} + \frac{1}{D(x)} \sharp(x, y) \right).$$

Remark 13. *Notice that if there is a vertex y , $y \sim x$, such that $\sharp(x, y) = 0$, this will reduce to (4.1).*

Proof: Starting from (2.9), the main work is to compare $Hf(x)$ with

$$\Gamma(f, f)(x) = \frac{1}{2} \frac{1}{d_x} \sum_{y, y \sim x} (f(y) - f(x))^2.$$

First we try to write out $Hf(x)$ as

$$Hf(x) = \frac{1}{4} \frac{1}{d_x} \sum_{y, y \sim x} \left[\frac{4}{d_y} (f(x) - f(y))^2 + \frac{1}{d_y} \sum_{z, z \sim y, z \neq x} (f(x) - 2f(y) + f(z))^2 \right].$$

If there is a vertex x_1 which satisfies $x_1 \sim x$, $x_1 \sim y$, we have

$$\begin{aligned}
& \frac{1}{d_y}(f(x) - 2f(y) + f(x_1))^2 + \frac{1}{d_{x_1}}(f(x) - 2f(x_1) + f(y))^2 \\
& \geq \frac{1}{D(x)}[(f(x) - f(y))^2 + (f(y) - f(x_1))^2 + 2(f(x) - f(y))(f(x_1) - f(y)) \\
& \quad + (f(x) - f(x_1))^2 + (f(y) - f(x_1))^2 + 2(f(y) - f(x_1))(f(x) - f(x_1))] \\
& = \frac{1}{D(x)}[(f(x) - f(y))^2 + 4(f(y) - f(x_1))^2 + (f(x) - f(x_1))^2]. \\
& \geq \frac{1}{D(x)}(f(x) - f(y))^2.
\end{aligned} \tag{4.3}$$

So the existence of a triangle which includes x and y will give another term

$$\frac{1}{D(x)}(f(y) - f(x))^2$$

to the sum in $Hf(x)$. Since this effect is symmetric to y and x_1 , we can get

$$\begin{aligned}
Hf(x) & \geq \frac{1}{4} \frac{1}{d_x} \sum_{y, y \sim x} \left(\frac{4}{d_y} + \frac{1}{D(x)} \sharp(x, y) \right) (f(y) - f(x))^2 \\
& \geq t(x) \frac{1}{4} \frac{1}{d_x} \sum_{y, y \sim x} (f(y) - f(x))^2 \\
& = t(x) \cdot \frac{1}{2} \Gamma(f, f)(x).
\end{aligned}$$

Inserting this into (2.9) completes the proof. \square

Recalling Theorem 3 and the subsequent discussion, we get the following curvature dimension inequalities on graphs with positive Ollivier-Ricci curvature.

Corollary 2. *On a locally finite graph $G = (V, E)$, if $\kappa(x, y) > 0$, then we have*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{5}{2D(x)} - 1 \right) \Gamma(f, f)(x). \tag{4.4}$$

Corollary 3. *On a locally finite graph $G = (V, E)$, if $\kappa(x, y) \geq k > 0$, then we have*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2} \min_{y, y \sim x} \left\{ \frac{4}{d_y} + \frac{\lceil kd_x \vee d_y \rceil}{D(x)} \right\} - 1 \right) \Gamma(f, f)(x). \tag{4.5}$$

Remark 14. *Observe that a rough inequality in this case is*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{2}{D(x)} + \frac{kd_x}{2D(x)} - 1 \right) \Gamma(f, f)(x).$$

Comparing this one with (4.1), we see that positive κ increases the curvature function here.

Remark 15. *We point out that the condition $\kappa(x, y) \geq k > 0$ implies that the diameter of the graph is bounded by $\frac{2}{k}$ (see Proposition 23 in Ollivier [17]). So in this case the graph is a finite one.*

Let us revisit the example of a complete graph \mathcal{K}_n ($n \geq 2$) with n vertices. Recall in Example 1, we know

$$\kappa(x, y) = \frac{n-2}{n-1}, \quad \forall x, y.$$

For the curvature dimension inequality on \mathcal{K}_n , Theorem 8 or Corollary 3 using the above κ implies

$$\begin{aligned}\Gamma_2(f, f) &\geq \frac{1}{2}(\Delta f)^2 + \left(\frac{2}{n-1} - 1 + \frac{1}{2} \frac{n-2}{n-1}\right) \Gamma(f, f) \\ &= \frac{1}{2}(\Delta f)^2 + \frac{4-n}{2(n-1)} \Gamma(f, f).\end{aligned}\quad (4.6)$$

Moreover, the curvature term in the above inequality cannot be larger. To see this, we calculate, using the same trick as in (4.3),

$$\begin{aligned}Hf(x) &= \frac{1}{4(n-1)^2} \sum_{y, y \sim x} \sum_{z, z \sim x} (f(x) - 2f(y) + f(z))^2 \\ &= \frac{n+2}{2(n-1)} \Gamma(f, f)(x) + \frac{1}{(n-1)^2} \sum_{(x_1, x_2)} (f(x_1) - f(x_2))^2,\end{aligned}$$

where $\sum_{(x_1, x_2)}$ means the sum over all unordered pairs of neighbors of x . Recalling (2.9), we get

$$\Gamma_2(f, f)(x) = \frac{1}{2}(\Delta f)^2(x) + \frac{4-n}{2(n-1)} \Gamma(f, f)(x) + \frac{1}{(n-1)^2} \sum_{(x_1, x_2)} (f(x_1) - f(x_2))^2. \quad (4.7)$$

For any vertex x , we can find a particular function \bar{f} ,

$$\bar{f}(z) = \begin{cases} 2, & \text{when } z = x; \\ 1, & \text{when } z \sim x, \end{cases} \quad (4.8)$$

such that the last term in (4.7) vanishes, and $\Gamma(\bar{f}, \bar{f}) \neq 0$. This means the curvature term in (4.6) is optimal for dimension parameter 2.

But the curvature term $\frac{4-n}{2(n-1)}$ behaves very differently from κ . In fact as $n \rightarrow +\infty$,

$$\frac{4-n}{2(n-1)} \searrow -\frac{1}{2} \quad \text{whereas} \quad \kappa \nearrow 1.$$

To get a curvature dimension inequality with a curvature term which behaves like κ , it seems that we should adjust the dimension parameter. In fact, we have

Proposition 3. *On a complete graph \mathcal{K}_n ($n \geq 2$) with n vertices, the Laplace operator Δ satisfies for $m \in [1, +\infty]$,*

$$\Gamma_2(f, f)(x) \geq \frac{1}{m}(\Delta f(x))^2 + \left(\frac{4-n}{2(n-1)} + \frac{m-2}{m}\right) \Gamma(f, f)(x). \quad (4.9)$$

Moreover, for every fixed dimension parameter m , the curvature term is optimal.

Proof: We have from (4.7)

$$\begin{aligned}\Gamma_2(f, f)(x) &= \frac{1}{m}(\Delta f)^2(x) + \frac{4-n}{2(n-1)} \Gamma(f, f)(x) \\ &\quad + \frac{1}{(n-1)^2} \sum_{(x_1, x_2)} (f(x_1) - f(x_2))^2 + \left(\frac{1}{2} - \frac{1}{m}\right) (\Delta f)^2.\end{aligned}$$

Let us denote the sum of the last two terms by I . Then we have

$$\begin{aligned}
I &= \frac{1}{(n-1)^2} \left\{ \left(\frac{1}{2} - \frac{1}{m} \right) \sum_{y, y \sim x} (f(y) - f(x))^2 + \sum_{(x_1, x_2)} \left[(f(x_1) - f(x))^2 + (f(x_2) - f(x))^2 \right. \right. \\
&\quad \left. \left. + \left(2 \left(\frac{1}{2} - \frac{1}{m} \right) - 2 \right) (f(x_1) - f(x))(f(x_2) - f(x)) \right] \right\} \\
&= \frac{1}{(n-1)^2} \left[\left(\frac{1}{2} - \frac{1}{m} \right) \sum_{y, y \sim x} (f(y) - f(x))^2 + \left(1 - \frac{m+2}{2m} \right) (n-2) \sum_{y, y \sim x} (f(y) - f(x))^2 \right. \\
&\quad \left. + \sum_{(x_1, x_2)} \frac{m+2}{2m} (f(x_1) - f(x_2))^2 \right] \\
&= \frac{m-2}{m} \Gamma(f, f)(x) + \frac{m+2}{2m(n-1)^2} \sum_{(x_1, x_2)} (f(x_1) - f(x_2))^2.
\end{aligned}$$

This finishes the proof. \square

An interesting point appears when we choose the dimension parameter m of \mathcal{K}_n as $n-1$. Then we have

$$\Gamma_2(f, f) \geq \frac{1}{n-1} (\Delta f)^2 + \frac{1}{2} \frac{n-2}{n-1} \Gamma(f, f),$$

where the curvature term is exactly $\frac{1}{2}\kappa$. From the fact that \mathcal{K}_n could be considered as the boundary of a $(n-1)$ dimensional simplex, the m we choose here seems also natural.

Remark 16. *We point out another similar fact here. On a locally finite graph with maximal degree D and minimal degree larger than 1, Theorem 1 and Theorem 7 imply that*

$$\kappa(x, y) \geq 2 \left(\frac{2}{D} - 1 \right), \quad \forall x, y, \quad (4.10)$$

and

$$\Gamma_2(f, f) \geq \frac{1}{2} (\Delta f)^2 + \left(\frac{2}{D} - 1 \right) \Gamma(f, f), \quad (4.11)$$

respectively. It is not difficult to see that for regular trees with degree larger than 1, the curvature term in (4.11) is optimal. (Just consider the extension of the function (4.8), taking values 0 on vertices which are not x and neighbors of x there.) So on regular trees, the curvature term is also exactly $\frac{1}{2}\kappa$.

Remark 17. *In Erdős-Harary-Tutte [9], they define the dimension of a graph G as the minimum number n such that G can be embedded into a n dimensional Euclidean space with every edge of G having length 1. It is interesting that by their definition, the dimension of \mathcal{K}_n is also $n-1$ and the dimension of any tree is at most 2.*

From the above observations, it seems natural to expect stronger relations between the lower bound of κ and the curvature term in the curvature dimension inequality if one chooses proper dimension parameters.

4.2. Weighted graphs. We have similar results on weighted graphs here, with similar proofs.

Theorem 9. *On a weighted locally finite graph $G = (V, E)$, the Laplace operator satisfies*

$$\Gamma_2(f, f)(x) \geq \frac{1}{2}(\Delta f(x))^2 + \left(\frac{1}{2}t_w(x) - 1\right) \Gamma(f, f)(x), \quad (4.12)$$

where

$$t_w(x) := \min_{y, y \sim x} \left\{ \frac{4w_{xy}}{d_y} + \sum_{x_1, x_1 \sim x, x_1 \sim y} \left(\frac{w_{xy}}{d_y} \wedge \frac{w_{xx_1}}{d_{x_1}} \right) \frac{w_{x_1y}}{w_{xy}} \right\}.$$

REFERENCES

- [1] D. Bakry, *Functional inequalities for Markov semigroups*, Probability measures on groups: recent directions and trends, 91-147, Tata Inst. Fund. Res., Mumbai, 2006.
- [2] D. Bakry and M. Émery, *Diffusions hypercontractives. (French) [Hypercontractive diffusions]* Séminaire de probabilités, XIX, 1983/84, 177-206, Lecture Notes in Math., **1123**, Springer, Berlin, 1985.
- [3] A. Banerjee and J. Jost, *On the spectrum of the normalized graph Laplacian*, Linear Algebra Appl. **428** (2008), no. 11-12, 3015-3022.
- [4] F. Bauer and J. Jost, *Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian*, <http://arxiv.org/abs/0910.3118v3>.
- [5] A.-I. Bonciocat and K.-T. Sturm, *Mass transportation and rough curvature bounds for discrete spaces*, J. Funct. Anal. **256** (2009), no. 9, 2944-2966.
- [6] F. R. K. Chung, *Spectral graph theory*, CBMS Regional Conference Series in Mathematics, **92**, 1997.
- [7] F. R. K. Chung and S. T. Yau, *Logarithmic Harnack inequalities*, Math. Res. Lett. **3** (1996), no. 6, 793-812.
- [8] J. Dodziuk and L. Karp, *Spectral and function theory for combinatorial Laplacians*, Contemp. Math., **73**, Amer. Math. Soc., Providence, RI, 1988.
- [9] P. Erdős, F. Harary and W. T. Tutte, *On the dimension of a graph*, Mathematika **12** (1965), 118-122.
- [10] L. C. Evans, *Partial differential equations and Monge-Kantorovich mass transfer*, Current developments in mathematics, 1997 (Cambridge, MA), 65-126, Int. Press, Boston, MA, 1999.
- [11] R. Forman, *Bochner's method for cell complexes and combinatorial Ricci curvature*, Discrete Comput. Geom. **29** (2003), no. 3, 323-374.
- [12] Y. Lin, *Ricci curvature on graphs*, 2010 John H. Barrett Memorial Lecture, <http://www.math.utk.edu/barrett/2010/talks/YongLinRicci.pdf>.
- [13] Y. Lin, L. Lu and S. T. Yau, *Ricci Curvature of graphs*, to appear in Tohoku Math. J..
- [14] Y. Lin and S. T. Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. **17** (2010), no. 2, 343-356.
- [15] J. Lott and C. Villani, *Ricci curvature for metric measure spaces via optimal transport*, Ann. of Math. **169** (2009), no. 3, 903-991.
- [16] S.-I. Ohta, *On the measure contraction property of metric measure spaces*, Comment. Math. Helv. **82** (2007), no. 4, 805-828.
- [17] Y. Ollivier, *Ricci Curvature of Markov Chains on Metric Spaces*, J. Funct. Anal. **256** (2009), no. 3, 810-864.
- [18] Y. Ollivier, *A survey of Ricci curvature for metric spaces and Markov chains*, Probabilistic approach to geometry, 343-381, Adv. Stud. Pure Math., **57**, Math. Soc. Japan, Tokyo, 2010.
- [19] D. A. Stone, *A combinatorial analogue of a theorem of Myers*, Illinois J. Math. **20** (1976), no. 1, 12-21. Correction, Illinois J. Math. **20** (1976), no. 3, 551-554.
- [20] K.-T. Sturm, *On the geometry of metric measure spaces. I.*, Acta Math. **196** (2006), no. 1, 65-131.
- [21] K.-T. Sturm, *On the geometry of metric measure spaces. II.*, Acta Math. **196** (2006), no. 1, 133-177.
- [22] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, **58**. American Mathematical Society, Providence, RI, 2003.

- [23] C. Villani, *Optimal transport, Old and new*, Grundlehren der Mathematischen Wissenschaften, **338**. Springer-Verlag, Berlin, 2009.
- [24] D. J. Watts and S. H. Strogatz, *Collective dynamics of 'small-world' networks*. Nature **393** (1998), 440-442.

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