

Max-Planck-Institut  
für Mathematik  
in den Naturwissenschaften  
Leipzig

Topological bifurcations of minimal invariant sets  
for set-valued dynamical systems

by

*Jeroen S. W. Lamb, Martin Rasmussen, and Christian S. Rodrigues*

Preprint no.: 26

2011





# TOPOLOGICAL BIFURCATIONS OF MINIMAL INVARIANT SETS FOR SET-VALUED DYNAMICAL SYSTEMS

JEROEN S. W. LAMB, MARTIN RASMUSSEN, AND CHRISTIAN S. RODRIGUES

ABSTRACT. We discuss the dependence of set-valued dynamical systems on parameters. Under mild assumptions which are often satisfied for random dynamical systems with bounded noise and control systems, we establish the fact that topological bifurcations of minimal invariant sets are discontinuous with respect to the Hausdorff metric, taking the form of lower semi-continuous explosions and instantaneous appearances. We also characterise these transitions by properties of Morse-like decompositions.

## 1. INTRODUCTION

Dynamical systems usually refer to time evolutions of states, where each initial condition leads to a unique state of the system in the future. Set-valued dynamical systems allow a multi-valued future, motivated, for instance, by impreciseness or uncertainty. In particular, set-valued dynamical systems naturally arise in the context of random and control systems.

The main motivation for the work in this paper is the study of random dynamical systems represented by a mapping  $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  with a bounded noise of size  $\varepsilon > 0$ ,

$$x_{n+1} = f(x_n) + \xi_n,$$

where the sequence  $(\xi_n)_{n \in \mathbb{N}}$  is a random variable with values in  $\overline{B_\varepsilon(0)} := \{x \in \mathbb{R}^d : \|x\| \leq \varepsilon\}$ . The collective behavior of all future trajectories is then represented by a set-valued mapping  $F : \mathcal{K}(\mathbb{R}^d) \rightarrow \mathcal{K}(\mathbb{R}^d)$ , defined by

$$F(M) := \overline{B_\varepsilon(f(M))} \quad \text{for all } M \in \mathcal{K}(\mathbb{R}^d),$$

where  $\mathcal{K}(\mathbb{R}^d)$  is the set of all nonempty compact subsets of  $\mathbb{R}^d$ .

Under the natural assumption that the probability distribution on  $\overline{B_\varepsilon(0)}$  has a non-vanishing Lebesgue density, it turns out that the supports of stationary measures of the random dynamical system are minimal invariant sets of the set-valued mapping  $F$  [ZH07]. A minimal invariant set is a compact set  $M \subset \mathbb{R}^d$  that is invariant (i.e.  $F(M) = M$ ) and contains no proper invariant subset.

In this paper, we are mainly interested in topological bifurcations of minimal invariant sets, while considering a parameterized family of set-valued mappings

---

*Date:* May 25, 2011.

1991 *Mathematics Subject Classification.* 37G35, 37H20, 37C70, 49K21 (primary), 37B25, 34A60 (secondary).

*Key words and phrases.* Set-valued dynamical systems, random dynamical systems, control.

The first author was supported by a FAPESP-Brazil Visiting Professorship (2009-18338-2). The first and second author gratefully acknowledge partial support by EU IRSES project DynEur-Braz and the warm hospitality of IMECC UNICAMP during the final write-up of this paper. The second author was supported by an EPSRC Career Acceleration Fellowship and a Marie Curie Intra European Fellowship of the European Community.

$(F_\lambda)_{\lambda \in \Lambda}$ , where  $\Lambda$  is a metric space. These bifurcations involve discontinuous changes as well as disappearances of minimal invariant sets under variation of  $\lambda$ .

**Definition 1** (Topological bifurcation of minimal invariant sets). *Let  $\mathcal{M}_\lambda$  denote the union of minimal invariant sets of  $F_\lambda$ ,  $\lambda \in \Lambda$ . We say that  $F_{\lambda_*}$  admits a topological bifurcation of minimal invariant sets if for any neighbourhood  $V$  of  $\lambda_*$ , there does not exist a family of homeomorphisms  $(h_\lambda)_{\lambda \in V}$ ,  $h_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , depending continuously on  $\lambda$ , with the property that*

$$h_\lambda(\mathcal{M}_\lambda) = \mathcal{M}_{\lambda_*} \quad \text{for all } \lambda \in V.$$

The main result concerns the necessity of discontinuous changes of minimal invariant sets at bifurcation points with two possible local scenarios.

**Theorem 1.1.** *Suppose that the family  $(F_\lambda)_{\lambda \in \Lambda}$  admits a bifurcation at  $\lambda_*$ . Then a minimal invariant set changes discontinuously at  $\lambda = \lambda_*$  in one of the following ways:*

- (i) *it explodes lower semi-continuously at  $\lambda_*$ , or*
- (ii) *it disappears instantaneously at  $\lambda_*$ .*

A more technical formulation of this result can be found in Theorem 5.1. In fact, the setting of set-valued dynamical systems in this paper is slightly more general than presented above, including also upper semi-continuous and continuous-time systems. For a simple one-dimensional example illustrating this theorem, see Section 7.

Another focus of this paper lies in extending Morse decomposition theory to study bifurcation problems in our context. Recently, Morse decompositions have been discussed for set-valued dynamical systems [BBS, Li07, McG92], and we generalize certain fundamental results for attractors and repellers to complementary invariant sets. The second main result of this paper (Theorem 6.1) asserts that at a bifurcation point, these complementary invariant sets must touch.

In the context of the presented motivation above, we note that the study of random dynamical systems with bounded noise can be separated into a topological part (involving the mapping  $F$ ) and the evolution of measures. In contrast, the topological part is redundant in the case of unbounded noise (modelled for instance by Brownian motions), where there is only one minimal invariant set, given by the whole space and supporting a unique stationary measure.

Initial research on bifurcations in random dynamical systems with unbounded noise started in the 1980s, mainly by Ludwig Arnold and co-workers [Arn98, Bax94, ASNSH96, JKP02]. Two types of bifurcation have been distinguished so far: the *phenomenological bifurcation* (P-bifurcation), concerning qualitative changes in stationary densities, and the *dynamical bifurcation* (D-bifurcation), concerning the sign change of a Lyapunov exponent, cf. also [Ash99]. To a large extent, however, bifurcations in random dynamical systems remain unexplored.

In modelling, bounded noise is often approximated by unbounded noise with highly localized densities in order to enable the use of stochastic analysis. In this approximation, topological tools to identify bifurcations are inaccessible, leaving the manifestation of a topological bifurcation as a cumbersome quantitative and qualitative change of properties of invariant measures.

Our work contributes to the abstract theory of set-valued dynamical systems dating back to the 1960s. Early contributions were motivated mainly by control

theory [Rox65, Klo78], and later developments include stability and attractor theory [Aki93, Ara00, GK01, Grü02, KMR, McG92, Rox97], Morse decompositions [BBS, Li07, McG92] and ergodic theory [Art00, MA99].

Our results build upon initial piloting studies concerning bifurcations in random dynamical systems with bounded noise [BHY, CGK08, CHK10, HY06, HY10, ZH07, ZH08] and control systems [CK03, CMKS08, CW09, Gay04, Gay05]. In particular, Theorem 1.1 unifies and generalises observations in [BHY, HY06, ZH07] to higher dimensions and non-invertible (set-valued) systems, while the bifurcation analysis in terms of Morse-like decompositions is a new perspective.

We finally remark that set-valued dynamical systems appear in the literature also as *closed relations*, *general dynamical systems*, *dispersive systems* or *families of semi-groups*.

## 2. SET-VALUED DYNAMICAL SYSTEMS

Throughout this paper, we consider the phase space of our set-valued dynamical systems to be a compact metric space  $(X, d)$ . To aid the presentation, we restrict to the setting of a compact phase space, although our results extend naturally to noncompact complete phase spaces.

We write  $B_\varepsilon(x_0) = \{x \in X : d(x, x_0) < \varepsilon\}$  for the  $\varepsilon$ -neighbourhood of a point  $x_0 \in X$ . For arbitrary nonempty sets  $A, B \subset X$  and  $x \in X$ , let  $\text{dist}(x, A) := \inf \{d(x, y) : y \in A\}$  be the *distance* of  $x$  to  $A$  and  $\text{dist}(A, B) := \sup \{ \text{dist}(x, B) : x \in A \}$  be the *Hausdorff semi-distance* of  $A$  and  $B$ . The *Hausdorff distance* of  $A$  and  $B$  is then defined by  $h(A, B) := \max \{ \text{dist}(A, B), \text{dist}(B, A) \}$ .

The set of all nonempty compact subsets of  $X$  will be denoted by  $\mathcal{K}(X)$ . Equipped with the Hausdorff distance  $h$ ,  $\mathcal{K}(X)$  is also a metric space  $(\mathcal{K}(X), h)$ . It is well-known that if  $X$  is complete or compact, then  $\mathcal{K}(X)$  is also complete or compact, respectively.

Define for a sequence  $(M_n)_{n \in \mathbb{N}}$  of bounded subsets of  $X$ ,

$$\limsup_{n \rightarrow \infty} M_n := \left\{ x \in X : \liminf_{n \rightarrow \infty} \text{dist}(x, M_n) = 0 \right\}$$

and

$$\liminf_{n \rightarrow \infty} M_n := \left\{ x \in X : \limsup_{n \rightarrow \infty} \text{dist}(x, M_n) = 0 \right\}$$

(see [Aki93, p. 125–126] and [AF90, Definition 1.1.1]).

In this paper, a *set-valued dynamical system* is understood as a mapping  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  with time set  $\mathbb{T} = \mathbb{N}_0$  (discrete) or  $\mathbb{T} = \mathbb{R}_0^+$  (continuous), which fulfills the following properties:

- (H1)  $\Phi(0, \xi) = \{\xi\}$  for all  $\xi \in X$ ,
- (H2)  $\Phi(t + \tau, \xi) = \Phi(t, \Phi(\tau, \xi))$  for all  $t, \tau \geq 0$  and  $\xi \in X$ ,
- (H3)  $\Phi$  is *upper semi-continuous*, i.e.

$$\Phi(\tau, \xi) \supset \limsup_{(t, x) \rightarrow (\tau, \xi)} \Phi(t, x) \quad \text{for all } (\tau, \xi) \in \mathbb{T} \times X.$$

- (H4)  $t \mapsto \Phi(t, \xi)$  is continuous with respect to the Hausdorff metric for all  $\xi \in X$ .

Note that in (H2), the extension  $\Phi(t, M) := \bigcup_{x \in M} \Phi(t, x)$  for  $M \subset X$  was used.

There is a one-to-one correspondence between discrete set-valued dynamical systems and upper semi-continuous mappings  $f : X \rightarrow \mathcal{K}(X)$ . On the other hand,

continuous set-valued dynamical systems arise in the context of differential inclusions, which canonically generalize ordinary differential equations to multi-valued vector fields [AC84, Dei92].

The  $\varepsilon$ -perturbation of a discrete mapping as discussed in the Introduction yields a set-valued dynamical system with *continuous* dependence on  $x$ . Our setting also includes *upper semi-continuous* set-valued dynamical systems as set out in (H3), motivated by differential equations with discontinuous right hand side and problems from control theory [AC84].

Associated to every set-valued dynamical system is a so-called dual set-valued dynamical system.

**Definition 2** (Dual set-valued dynamical system). *Let  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  be a set-valued dynamical system. Then the dual set-valued dynamical system is defined by  $\Phi^* : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$ , where*

$$\Phi^*(t, \xi) := \{x \in X : \xi \in \Phi(t, x)\} \quad \text{for all } (t, \xi) \in \mathbb{T} \times X.$$

Note that in case of an invertible (single-valued) dynamical system,  $\Phi^*$  coincides with the system under time reversal.

To see that  $\Phi^*$  is well-defined, i.e.  $\Phi^*(t, \xi) \in \mathcal{K}(X)$ , consider for given  $(t, \xi) \in \mathbb{T} \times X$  a sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\Phi^*(t, \xi)$  converging to  $x \in X$ . This means that  $\xi \in \Phi(t, x_n)$  for all  $n \in \mathbb{N}$ , and hence,  $\xi \in \limsup_{n \rightarrow \infty} \Phi(t, x_n) \subset \Phi(t, x)$  by the upper semi-continuity of  $\Phi$ . Thus,  $x \in \Phi^*(t, \xi)$ , which proves that this set belongs to  $\mathcal{K}(X)$ .

The dual  $\Phi^*$  was introduced already in [McG92] without formalising its properties. The following proposition shows that indeed  $\Phi^*$  defines a set-valued dynamical system.

**Proposition 2.1.**  *$\Phi^*$  is a set-valued dynamical system.*

*Proof.* The conditions (H1)–(H4) will be checked in the following.

(H1) One has  $\Phi^*(0, \xi) = \{x \in X : \xi \in \Phi(0, x)\} = \{x \in X : \xi \in \{x\}\} = \{\xi\}$  for all  $\xi \in X$ .

(H2) It follows that

$$\begin{aligned} & \Phi^*(t + \tau, \xi) \\ &= \{x \in X : \xi \in \Phi(t + \tau, x)\} = \{x \in X : \xi \in \Phi(\tau, \Phi(t, x))\} \\ &= \{x \in X : \exists y \in \Phi(t, x) : \xi \in \Phi(\tau, y)\} = \{x \in X : \exists y \in \Phi(t, x) : y \in \Phi^*(\tau, \xi)\} \\ &= \{x \in X : \Phi(t, x) \cap \Phi^*(\tau, \xi) \neq \emptyset\} = \{x \in X : \exists y \in \Phi^*(\tau, \xi) : y \in \Phi(t, x)\} \\ &= \{x \in X : \exists y \in \Phi^*(\tau, \xi) : x \in \Phi^*(t, y)\} = \{x \in X : x \in \Phi^*(t, \Phi^*(\tau, \xi))\} \\ &= \Phi^*(t, \Phi^*(\tau, \xi)). \end{aligned}$$

(H3) Let  $(\tau, \xi) \in \mathbb{T} \times X$ , and consider a sequence  $(t_n, x_n)_{n \in \mathbb{N}}$  converging to  $(\tau, \xi)$  as  $n \rightarrow \infty$ . To prove upper semi-continuity, one needs to clarify that

$$\limsup_{n \rightarrow \infty} \Phi^*(t_n, x_n) = \limsup_{n \rightarrow \infty} \{x \in X : x_n \in \Phi(t_n, x)\} \subset \Phi^*(\tau, \xi).$$

There to, choose  $y \in \limsup_{n \rightarrow \infty} \{x \in X : x_n \in \Phi(t_n, x)\}$ . Hence, there exists a subsequence  $(x_{n_j})_{j \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  and a sequence  $(y_j)_{j \in \mathbb{N}}$  such that  $\lim_{j \rightarrow \infty} y_j = y$  and  $x_{n_j} \in \Phi(t_{n_j}, y_j)$ . The upper semi-continuity of  $\Phi$  then implies that  $\xi \in \Phi(\tau, y)$ , which in turn means that  $y \in \Phi^*(\tau, \xi)$ . This proves upper semi-continuity of  $\Phi^*$ .

(H4) Let  $(\tau, \xi) \in \mathbb{T} \times X$ , and consider a sequence  $(t_n)_{n \in \mathbb{N}}$  converging to  $\tau$  as  $n \rightarrow \infty$ . From (H3), it follows that

$$\limsup_{n \rightarrow \infty} \Phi^*(t_n, \xi) \subset \Phi^*(\tau, \xi).$$

The proof is thus finished if we show that

$$\liminf_{n \rightarrow \infty} \Phi^*(t_n, \xi) \supset \Phi^*(\tau, \xi).$$

Let  $x \in \Phi^*(\tau, \xi)$ . Then  $\xi \in \Phi(\tau, x) = \liminf_{n \rightarrow \infty} \Phi(t_n, x)$ , i.e. there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  with  $x_n \in \Phi(t_n, x)$  and  $x_n \rightarrow \xi$  as  $n \rightarrow \infty$ . Hence,  $x \in \Phi^*(t_n, x_n)$  for all  $n \in \mathbb{N}$ , which implies that

$$\begin{aligned} x \in \liminf_{n \rightarrow \infty} \Phi^*(t_n, x_n) &= \{y \in X : \limsup_{n \rightarrow \infty} \text{dist}(y, \Phi^*(t_n, x_n)) = 0\} \\ &\subset \{y \in X : \limsup_{n \rightarrow \infty} \text{dist}(y, \Phi^*(t_n, \xi)) = 0\} \\ &= \liminf_{n \rightarrow \infty} \Phi^*(t_n, \xi), \end{aligned}$$

where “ $\subset$ ” follows from the upper semi-continuity proved in (H3). This finishes the proof of this proposition.  $\square$

### 3. MINIMAL INVARIANT SETS

In the following, the focus lies on the determination and bifurcation of so-called *minimal invariant sets* of a set-valued dynamical system  $\Phi$ . Such sets have been discussed, e.g., in [HY06, HY10, BHY] in the continuous case of random differential equations and in [ZH07] for random maps. In the context of control theory, minimal invariant sets are denoted as *invariant control sets* (see the monograph [CK00]).

Given a set-valued dynamical system  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$ . A nonempty and compact set  $M \subset X$  is called  $\Phi$ -*invariant* if

$$\Phi(t, M) = M \quad \text{for all } t \geq 0.$$

A  $\Phi$ -invariant set is called *minimal* if it does not contain a proper  $\Phi$ -invariant set.

Minimal  $\Phi$ -invariant sets are pairwise disjoint, and under the assumption that  $\Phi(t, x)$  contains at least one ball for all  $t > 0$  and  $x \in X$ , there are only finitely many of such sets, since  $X$  is compact.

Minimal  $\Phi$ -invariant sets are important, because they are exactly the supports of stationary measures of a random dynamical system, whenever  $\Phi$  describes the topological part of the random system [HY06, ZH07]. Moreover, in case  $\Phi$  describes a control system, minimal  $\Phi$ -invariant sets coincide with invariant control sets [CK00].

**Proposition 3.1.** *Let  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  be a set-valued dynamical system and let  $M \subset X$  be compact with*

$$\Phi(t, M) \subset M \quad \text{for all } t \geq 0,$$

*and suppose that no proper subset of  $M$  fulfills this property. Then  $M$  is  $\Phi$ -invariant.*

*Proof.* Standard arguments lead to the fact that the  $\omega$ -limit set

$$\limsup_{t \rightarrow \infty} \Phi(t, M) = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \Phi(s, M)}$$

is a nonempty and compact  $\Phi$ -invariant set [AF90]. Since  $\limsup_{t \rightarrow \infty} \Phi(t, M) \subset M$ , it follows that  $\limsup_{t \rightarrow \infty} \Phi(t, M) = M$ .  $\square$

The existence of minimal  $\Phi$ -invariant sets follows from Zorn's Lemma.

**Proposition 3.2** (Existence of minimal invariant sets). *Let  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  be a set-valued dynamical system and  $M \subset X$  be compact such that*

$$\Phi(t, M) \subset M \quad \text{for all } t \geq 0.$$

*Then there exists at least one subset of  $M$  which is minimal  $\Phi$ -invariant.*

*Proof.* Consider the collection

$$\mathcal{C} := \{A \subset \mathcal{K}(M) : \Phi(t, A) \subset A \text{ for all } t \geq 0\}.$$

$\mathcal{C}$  is partially ordered with respect to set inclusion, and let  $\mathcal{C}'$  be a totally ordered subset of  $\mathcal{C}$ . It is obvious that  $\bigcap_{A \in \mathcal{C}'} A$  is nonempty, compact and lies in  $\mathcal{C}$ . Thus, Zorn's Lemma implies that there exists at least one minimal element in  $\mathcal{C}$  which is a minimal  $\Phi$ -invariant set.  $\square$

While minimal  $\Phi$ -invariant sets always exists, they are typically non-unique. Uniqueness directly follows for set-valued dynamical systems which are contractions in the Hausdorff metric. Such contractions can be identified as follows.

**Lemma 3.3.** *Consider the set-valued dynamical system  $\Phi : \mathbb{T} \times \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ , defined by*

$$\Phi(1, x) := U(f(x)) \quad \text{for all } x \in X,$$

*where  $f : X \rightarrow X$  is a contraction on the compact metric space  $(X, d)$ , i.e. one has*

$$d(f(x), f(y)) \leq Ld(x, y) \quad \text{for all } x, y \in X$$

*with some Lipschitz constant  $L < 1$ , and  $U : X \rightarrow \mathcal{K}(X)$  is a function such that  $U(x)$  is a neighbourhood of  $x$  for all  $x \in X$ . Assume that  $U$  is globally Lipschitz continuous (but not necessarily a contraction) with Lipschitz constant  $M > 0$  such that  $ML < 1$ . The mapping  $\Phi(1, \cdot)$  then is a contraction in  $(\mathcal{K}(X), h)$ . The unique fixed point of  $\Phi(1, \cdot)$  is the unique minimal  $\Phi$ -invariant set, which is also globally attractive.*

*Proof.* First prove that the extension  $U : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ , defined by  $U(A) := \bigcup_{a \in A} U(a)$ , is Lipschitz continuous. Given  $A, B \in \mathcal{K}(X)$ , we have both

$$\begin{aligned} \sup_{x \in A} \inf_{y \in B} h(U(x), U(y)) &\leq L \sup_{x \in A} \inf_{y \in B} d(x, y) \\ &\leq Lh(A, B) \end{aligned}$$

and

$$\begin{aligned} \sup_{x \in A} \inf_{y \in B} h(U(x), U(y)) &\geq \sup_{x \in A} \inf_{y \in B} \text{dist}(U(x), U(y)) \\ &= \sup_{x \in A} \inf_{y \in B} \sup_{\tilde{x} \in U(x)} \inf_{\tilde{y} \in U(y)} d(\tilde{x}, \tilde{y}) \\ &\geq \sup_{x \in A} \sup_{\tilde{x} \in U(x)} \inf_{y \in B} \inf_{\tilde{y} \in U(y)} d(\tilde{x}, \tilde{y}) \\ &= \sup_{x \in U(A)} \inf_{y \in U(B)} d(x, y) \\ &= \text{dist}(U(A), U(B)). \end{aligned}$$



This means that

$$\text{dist}(U(A), U(B)) \leq Lh(A, B) \quad \text{for all } A, B \in \mathcal{K}(X),$$

which finally implies

$$h(U(A), U(B)) \leq Lh(A, B) \quad \text{for all } A, B \in \mathcal{K}(X).$$

The fact that  $\Phi(1, \cdot)$  is a contraction then follows, since it is essentially the composition of two Lipschitz continuous mappings, where the product of the respective Lipschitz constants is less than 1. Application of the contraction mapping theorem finishes the proof of this lemma.  $\square$

The above lemma applies in particular to the motivating example presented in the Introduction. In this case,  $U(x) := \overline{B_\varepsilon(x)}$  with Lipschitz constant 1. Hence, if  $f$  is a contraction, then the set-valued mapping  $F$  has a globally attractive unique minimal invariant set.

#### 4. GENERALISATION OF ATTRACTOR-REPELLER DECOMPOSITION

The purpose of this section is to provide generalisations of attractor-repeller decompositions which have been introduced in [MW06, Li07] for the study of Morse decompositions of set-valued dynamical systems. These generalisations are necessary for our purpose, because we deal with invariant sets rather than attractors, and they will be applied in Section 7 in the context of bifurcation theory.

Fundamental for the definition of Morse decompositions are domains of attraction (of attractors), because complementary repellers are then given by the complements of these sets. For a given  $\Phi$ -invariant set  $M$ , the *domain of attraction* is defined by

$$\mathcal{A}(M) = \left\{ x \in X : \lim_{t \rightarrow \infty} \text{dist}(\Phi(t, x), M) = 0 \right\}.$$

If  $M$  is an attractor, that is a  $\Phi$ -invariant set such that there exists an  $\eta > 0$  with

$$\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, B_\eta(M)), M) = 0,$$

then the complementary set  $X \setminus \mathcal{A}(M)$  is a  $\Phi^*$ -invariant set, which has the interpretation of a repeller, because all points outside of this set converge to the attractor in forward time. It is worth to note that this repeller is not necessarily  $\Phi$ -invariant (which is a difference from the classical Morse decomposition theory).

For a  $\Phi$ -invariant set  $M$  which is not an attractor, the complementary set  $X \setminus \mathcal{A}(M)$  is not necessarily  $\Phi^*$ -invariant, but this property can be attained when  $\mathcal{A}(M)$  is replaced by a slightly smaller set.

**Proposition 4.1.** *Let  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  be a set-valued dynamical system, and let  $M \subset X$  be  $\Phi$ -invariant such that  $\mathcal{A}(M) \neq X$ , i.e.  $M$  is not globally attractive. Then the complement of the set*

$$\mathcal{A}_-(M) := \mathcal{A}(M) \setminus \left\{ x \in \mathcal{A}(M) : \text{there exist } t \geq 0 \text{ with } \Phi(t, x) \cap \partial \mathcal{A}(M) \neq \emptyset, \text{ or for all } \gamma > 0, \text{ one has } \limsup_{t \rightarrow \infty} \text{dist}(\Phi(t, B_\gamma(x)), M) > 0 \right\},$$

i.e. the set  $M^* := X \setminus \mathcal{A}_-(M)$ , is  $\Phi^*$ -invariant.

The set  $M^*$  is called the *dual* of  $M$ . Under the additional assumption that  $M$  is an attractor in Proposition 4.1, i.e.  $\mathcal{A}(M)$  is a neighbourhood of  $M$ , the pair  $(M, M^*)$  is an attractor-repeller pair as discussed in [MW06]. This pair can be extended to obtain Morse decompositions, see [Li07].

Before proving this proposition, we will derive an alternative characterization of the set  $\mathcal{A}_-(M)$ .

**Lemma 4.2.** *Let  $\Phi : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$  be a set-valued dynamical system and  $M \subset X$  be  $\Phi$ -invariant. Then the set  $\mathcal{A}_-(M)$  admits the representation*

$$\mathcal{A}_-(M) = \left\{ x \in X : \text{for all } T \geq 0, \text{ there exists a neighbourhood } V \text{ of } \Phi(T, x) \right. \\ \left. \text{with } \lim_{t \rightarrow \infty} \text{dist}(\Phi(t, V), M) = 0 \right\}.$$

*Proof.* First, note that compact subsets  $K$  of  $\mathcal{A}_-(M)$  are attracted by  $M$ , i.e.  $\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, K), M) = 0$ . We have to show two set inclusions.

( $\subset$ ) Let  $x \in \mathcal{A}_-(M)$  and  $T > 0$ . Since  $\Phi(T, x)$  lies in the interior of  $\mathcal{A}(M)$ , there exists a compact neighbourhood  $V$  of  $\Phi(T, x)$  that is contained in  $\mathcal{A}(M)$ . This proves that  $\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, V), M) = 0$ , and hence,  $x$  is contained in the right hand side.

( $\supset$ ) Let  $x \in X$  such that for all  $T \geq 0$ , there exists a neighbourhood  $V$  of  $\Phi(T, x)$  with  $\lim_{t \rightarrow \infty} \text{dist}(\Phi(t, V), M) = 0$ . This implies that for all  $T \geq 0$ , one has  $\Phi(t, x) \cap \partial\mathcal{A}(M) = \emptyset$ , which finishes the proof of this lemma.  $\square$

The set  $\mathcal{A}_-(M)$  thus describes all trajectories in the domain of attraction which are attracted also under perturbation.

*of Proposition 4.1.* It will be shown that  $\Phi^*(t, M^*) = M^*$  for all  $t \geq 0$ .

( $\subset$ ) Assume that there exist  $t \geq 0$  and  $x \in \Phi^*(t, M^*) \setminus M^* = \Phi^*(t, M^*) \cap \mathcal{A}(M)$ . This implies that  $\Phi(t, x) \cap M^* \neq \emptyset$  and  $x \in \mathcal{A}(M)$ , which contradicts the fact that  $\mathcal{A}(M)$  fulfills  $\Phi(t, \mathcal{A}(M)) \subset \mathcal{A}(M)$  for all  $t \geq 0$ .

( $\supset$ ) Assume that there exist  $t \geq 0$  and  $x \in M^* \setminus \Phi^*(t, M^*)$ . This means that  $\Phi(t, x) \cap M^* = \emptyset$ , and hence,  $\Phi(t, x) \subset \mathcal{A}(M)$ . We will show that this implies that  $x \in \mathcal{A}(M)$ , which is a contradiction. Let  $T \geq 0$ , and consider first the case that  $T \leq t$ . The fact that  $\mathcal{A}(M)$  is open and  $\overline{\Phi(t, x)} \subset \mathcal{A}(M)$  is compact implies that there exists a  $\gamma > 0$  such that  $\overline{B_\gamma(\Phi(t, x))} \subset \mathcal{A}(M)$ . Moreover, the upper semi-continuity of  $\Phi$  and the relation  $\overline{\Phi(t-T, \Phi(T, x))} = \overline{\Phi(t, x)}$  yield the existence of  $\delta > 0$  such that  $\overline{\Phi(t-T, \overline{B_\delta(\Phi(T, x))})} \subset \overline{B_\gamma(\Phi(t, x))} \subset \mathcal{A}(M)$ . Since compact subsets of  $\mathcal{A}(M)$  are attracted to  $M$ , the assertion follows. Consider now the case  $T \geq t$ . Since  $\mathcal{A}(M)$  is invariant and  $\Phi(t, x) \subset \mathcal{A}(M)$ ,  $\Phi(T, x)$  is a compact subset of  $\mathcal{A}(M)$ .  $\mathcal{A}(M)$  is open, so there exists a compact neighbourhood of  $\Phi(T, x)$  which is attracted by  $M$ . This finishes the proof of this proposition.  $\square$

## 5. DEPENDENCE OF MINIMAL INVARIANT SETS ON PARAMETERS

The main goal of this section is to describe how minimal invariant sets depend on parameters. We consider a family  $(\Phi_\lambda)_{\lambda \in \Lambda}$  of set-valued dynamical systems  $\Phi_\lambda : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$ , where  $(\Lambda, d_\Lambda)$  is a metric space and

(H5)  $(\lambda, t) \mapsto \Phi_\lambda(t, x)$  is continuous in  $(\lambda, t) \in \Lambda \times \mathbb{T}$  uniformly in  $x$ .

Motivated by the setting of set-valued dynamical systems in the Introduction, we exclude single-valued dynamical systems in the following and assume

(H6)  $\Phi_\lambda(t, x)$  contains a ball of positive radius for all  $(t, x) \in \mathbb{T} \times X$  with  $t > 0$  and  $\lambda \in \Lambda$ , and moreover, there exist  $T > 0$  and  $\varepsilon > 0$  such that  $\Phi_\lambda(T, x)$  contains a ball of size  $\varepsilon$  for all  $x \in X$ .

The union of all minimal  $\Phi_\lambda$ -invariant sets in  $X$  will be denoted by  $\mathcal{M}_\lambda$ . The following theorem describes how  $\mathcal{M}_\lambda$  depends on the parameter.

**Theorem 5.1** (Dependence of minimal invariant sets on parameters). *Given  $(\Phi_\lambda)_{\lambda \in \Lambda}$  a family of set-valued dynamical systems satisfying (H1)–(H6), let  $M_{\lambda_\infty} \subset \mathcal{M}_{\lambda_\infty}$  be a minimal  $\Phi_{\lambda_\infty}$ -invariant set for some  $\lambda_\infty \in \Lambda$ . Then for each sequence  $(\lambda_n)_{n \in \mathbb{N}}$  converging to  $\lambda_\infty$ , there exist a subsequence  $(\lambda_{n_k})_{k \in \mathbb{N}}$  and a  $\delta > 0$  such that exactly one of the following statements holds.*

(i) Lower semi-continuous dependence:

$$M_{\lambda_\infty} \subset \liminf_{k \rightarrow \infty} (\mathcal{M}_{\lambda_{n_k}} \cap B_\delta(M_{\lambda_\infty})).$$

(ii) Instantaneous appearance:

$$\emptyset = \limsup_{k \rightarrow \infty} (\mathcal{M}_{\lambda_{n_k}} \cap B_\delta(M_{\lambda_\infty})).$$

*Proof.* Let  $(\lambda_n)_{n \in \mathbb{N}}$  be a sequence with  $\lambda_n \rightarrow \lambda_\infty$  as  $n \rightarrow \infty$ . Define the sequence  $(c_n)_{n \in \mathbb{N}}$  by

$$c_n := \begin{cases} 1 & : \mathcal{M}_{\lambda_n} \cap M_{\lambda_\infty} \neq \emptyset \\ 2 & : \mathcal{M}_{\lambda_n} \cap M_{\lambda_\infty} = \emptyset \end{cases} \quad \text{for all } n \in \mathbb{N},$$

and choose  $\tilde{\delta} > 0$  such that  $B_{\tilde{\delta}}(M_{\lambda_\infty}) \cap \mathcal{M}_{\lambda_\infty} = M_{\lambda_\infty}$ . Since  $\{1, 2\}$  is finite, there exists a constant subsequence  $(c_{n_k})_{k \in \mathbb{N}}$ .

If  $c_{n_k} \equiv 2$ , assume to the contrary that for all  $k \in \mathbb{N}$ , there exist  $m \geq k$  and  $a_k \in \mathcal{M}_{\lambda_{n_m}} \cap B_{1/k}(M_{\lambda_\infty})$ . The sequence  $(a_k)_{k \in \mathbb{N}}$  has a convergent subsequence with limit  $a_\infty \in M_{\lambda_\infty}$ . Now  $\Phi_{\lambda_\infty}(T, a_\infty) \subset M_{\lambda_\infty}$ , and the upper semi-continuity of  $\Phi$  and (H6) imply that there exists  $\gamma > 0$  such that  $\Phi_{\lambda_\infty}(T, x) \cap \text{int } M_{\lambda_\infty} \neq \emptyset$  for all  $x \in B_\gamma(a_\infty)$ . This is a contradiction to the definition of the sequence  $c_{n_k}$ , because of the continuous dependence of  $\Phi$  on  $\lambda$ , and this proves that there exists  $\delta \in (0, \tilde{\delta})$  with  $\mathcal{M}_{\lambda_{n_k}} \cap B_\delta(M_{\lambda_\infty}) = \emptyset$  whenever  $\frac{1}{k} < \delta$ . Hence, (ii) holds.

If  $c_{n_k} \equiv 1$ , define  $\delta := \tilde{\delta}$ . Choose minimal  $\Phi_{\lambda_{n_k}}$ -invariant sets  $M_{\lambda_{n_k}} \subset \mathcal{M}_{\lambda_{n_k}}$  such that  $M_{\lambda_{n_k}} \cap M_{\lambda_\infty} \neq \emptyset$  for  $k \in \mathbb{N}$ . First note that (H6) yields that the set  $\Phi_{\lambda_\infty}(T, M_{\lambda_{n_k}} \cap M_{\lambda_\infty})$  is contained in  $M_{\lambda_\infty}$  and contains an  $\varepsilon$ -ball. Having this in mind, (H5) implies that there exists a  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$ , the set  $\Phi_{\lambda_{n_k}}(T, M_{\lambda_{n_k}} \cap M_{\lambda_\infty}) \subset M_{\lambda_{n_k}}$  contains an  $\varepsilon/2$ -ball which completely lies in  $M_{\lambda_\infty}$ . Let  $B_{\varepsilon/2}(d_1), \dots, B_{\varepsilon/2}(d_r)$  be finitely many  $\varepsilon/2$ -balls covering the compact set  $M_{\lambda_\infty}$ . In particular, each of the sets  $M_{\lambda_{n_k}}$  contains (at least) one of the points  $d_1, \dots, d_r$ . We can thus assign the sets  $M_{\lambda_{n_k}}$  to  $r$  different categories, with the benefit that the sets in each category intersect in at least one point (given by the center of the balls). We show now that the asserted limit relation in (i) holds when restricting to a subsequence corresponding to each category, from which the assertion follows, since there are only finitely many categories. For simplicity, assume now that there is only one category. It will be shown now that  $\liminf_{k \rightarrow \infty} M_{\lambda_{n_k}}$  cannot be left in forward time, i.e. fulfills the conditions of Proposition 3.1. Since this set is nonempty and intersects  $M_{\lambda_\infty}$ , minimality of  $M_{\lambda_\infty}$  then implies the assertion. Assume to the contrary that there exists an  $\tilde{x} \in \liminf_{k \rightarrow \infty} M_{\lambda_{n_k}}$  such that  $\Phi_{\lambda_\infty}(\tau, \tilde{x}) \setminus \liminf_{k \rightarrow \infty} M_{\lambda_{n_k}} \neq \emptyset$  for some  $\tau > 0$ , i.e. there exists a

$\xi \in \Phi_{\lambda_\infty}(\tau, \tilde{x})$  such that  $\xi \notin \liminf_{k \rightarrow \infty} M_{\lambda_{n_k}}$ . We can choose  $\xi$  to be even in the interior  $\text{int } \Phi_{\lambda_\infty}(\tau, \tilde{x})$ , which is possible, since  $\liminf_{k \rightarrow \infty} M_{\lambda_{n_k}}$  is closed. In addition, the closedness of  $\liminf_{k \rightarrow \infty} M_{\lambda_{n_k}}$  and the continuous dependence of  $\Phi_\lambda$  on  $\lambda$  implies that there exists an  $k_0 \in \mathbb{N}$  and  $\zeta > 0$  such that

$$B_\zeta(\xi) \subset \Phi_{\lambda_{n_k}}(\tau, \tilde{x}) \quad \text{for all } k \geq k_0 \quad \text{and} \quad B_\zeta(\xi) \subset \Phi_{\lambda_\infty}(\tau, \tilde{x})$$

and

$$B_\zeta(\xi) \cap \liminf_{k \rightarrow \infty} M_{\lambda_{n_k}} = \emptyset. \quad (1)$$

Since there exists  $k_1 \geq k_0$  such that  $\tilde{x} \in M_{\lambda_{n_k}}$  for  $k \geq k_1$ , the invariance of  $M_{\lambda_{n_k}}$  implies that  $B_\zeta(\xi) \subset M_{\lambda_{n_k}}$  for all  $k \geq k_1$ . This contradicts (1) and finishes the proof of this theorem.  $\square$

The above theorem asserts that discontinuous changes in minimal invariant sets occur either as explosions or as instantaneous appearances. We are left to address the question if a continuous merging process of two minimal invariant sets is possible (note that this is not ruled out by (i) of Theorem 5.1). The following proposition shows that the answer to this question is negative if the set-valued dynamical system is continuous rather than only upper semi-continuous.

**Proposition 5.2.** *Let  $(\Phi_\lambda)_{\lambda \in \Lambda}$  be a family of continuous set-valued dynamical systems fulfilling (H1)–(H6), and let  $M_\lambda^1$  and  $M_\lambda^2$  be two different minimal  $\Phi_\lambda$ -invariant sets. Then for all  $\lambda^* \in \Lambda$ , one has*

$$\liminf_{\lambda \rightarrow \lambda^*} \inf_{(x,y) \in M_\lambda^1 \times M_\lambda^2} d(x,y) > 0,$$

i.e. the sets  $M_\lambda^1$  and  $M_\lambda^2$  cannot collide under variation of  $\lambda$ .

*Proof.* Suppose the contrary, which means that there exist an  $x^* \in X$  and a sequence  $\lambda_n \rightarrow \lambda^*$  as  $n \rightarrow \infty$  with

$$\lim_{n \rightarrow \infty} \text{dist}(x^*, M_{\lambda_n}^1) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{dist}(x^*, M_{\lambda_n}^2) = 0.$$

Due to (H5) and (H6), for  $t > 0$ , the set  $\Phi_{\lambda^*}(t, x^*)$  intersects the interior of both  $M_{\lambda_n}^1$  and  $M_{\lambda_n}^2$  when  $n$  is large enough. This, however, contradicts the fact that  $M_{\lambda_n}^1$  and  $M_{\lambda_n}^2$  are  $\Phi$ -invariant and finishes the proof of this proposition.  $\square$

The above proposition cannot be extended to upper semi-continuous set-valued dynamical systems as is illustrated by the following the example.

**Example 1.** *Let  $X = [-4, 4]$  and  $\Lambda = [0, 1]$ , and consider the discrete set-valued dynamical systems  $\Phi_\lambda : \mathbb{N}_0 \times X \rightarrow \mathcal{K}(X)$ ,  $\lambda \in \Lambda$ , generated by the time-1 mappings*

$$\Phi_\lambda(1, x) := \begin{cases} \left[ \frac{x}{2} - \lambda - 1, \frac{x}{2} - \lambda \right] & : x < 0 \\ [-2, 2] & : x = 0 \\ \left[ \frac{x}{2} + \lambda, \frac{x}{2} + \lambda + 1 \right] & : x > 0 \end{cases} \quad \text{for all } \lambda \in \Lambda.$$

*Obviously, the set-valued system is not continuous, but only upper semi-continuous at  $x = 0$ . For  $\lambda > 0$ , there are exactly two minimal  $\Phi_\lambda$ -invariant sets, given by*

$$M_\lambda^1 := [-2\lambda - 2, -2\lambda] \quad \text{and} \quad M_\lambda^2 := [2\lambda, 2\lambda + 2].$$

*In the limit  $\lambda \rightarrow 0$ , these two sets collide, yielding the minimal  $\Phi$ -invariant set  $M_0 := [-2, 2]$  at  $\lambda = 0$  (see Figure 1). Note that the singleton  $\{0\}$  is  $\Phi^*$ -invariant, so this bifurcation can be seen as a collision process of  $\Phi$ -invariant and  $\Phi^*$ -invariant*

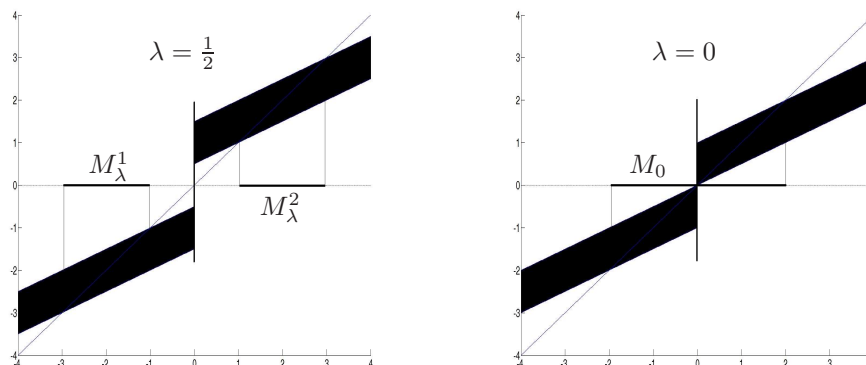


FIGURE 1. Graph of  $\Phi_\lambda(1, \cdot)$  with minimal invariant sets  $M_\lambda^1$  and  $M_\lambda^2$  for  $\lambda = \frac{1}{2}$  (left), and  $M_0$  for  $\lambda = 0$  (right), illustrating the continuous merging process of the minimal invariant sets in Example 1

sets. We will see in the next section that also in the case of discontinuous bifurcations, these complementary invariant sets must touch.

The above proposition and example show that for discontinuous set-valued dynamical systems, one can have continuous bifurcations in the sense that minimal invariant sets converge to each other. On the other hand, only discontinuous bifurcations can occur for continuous set-valued dynamical systems.

## 6. A NECESSARY CONDITION FOR BIFURCATION

Consider a family  $(\Phi_\lambda)_{\lambda \in \Lambda}$  of set-valued dynamical systems  $\Phi_\lambda : \mathbb{T} \times X \rightarrow \mathcal{K}(X)$ , where  $(\Lambda, d_\Lambda)$  is a metric space, and suppose that (H1)–(H6) hold. Motivated by Proposition 5.2, we assume that  $\Phi_\lambda$  is continuous rather than upper semi-continuous.

Recall the definition of a topological bifurcation (Definition 1) and the fact that  $\mathcal{M}_\lambda$  denotes the union of all minimal  $\Phi_\lambda$ -invariant sets. As a direct consequence of Theorem 5.1 and Proposition 5.2, for continuous set-valued dynamical systems, a topological bifurcation of  $\mathcal{M}_\lambda$  is characterised by a minimal  $\Phi_{\lambda_\infty}$ -invariant set  $M_{\lambda_\infty}$ , a sequence  $\lambda_n \rightarrow \lambda_\infty$  as  $n \rightarrow \infty$  and  $\delta > 0$  such that

$$M_{\lambda_\infty} \subsetneq \liminf_{n \rightarrow \infty} (\mathcal{M}_{\lambda_n} \cap B_\delta(M_{\lambda_\infty})) \quad \text{or} \quad \emptyset = \limsup_{n \rightarrow \infty} (\mathcal{M}_{\lambda_n} \cap B_\delta(M_{\lambda_\infty})). \quad (2)$$

The following theorem provides a necessary condition for a topological bifurcation of minimal invariant sets involving the dual  $M_{\lambda_\infty}^*$  of  $M_{\lambda_\infty}$  as introduced in Section 4.

**Theorem 6.1** (Necessary condition for bifurcation). *Let  $(\Phi_\lambda)_{\lambda \in \Lambda}$  be a family of continuous set-valued dynamical systems fulfilling (H1)–(H6), and assume that  $(\Phi_\lambda)_{\lambda \in \Lambda}$  admits a topological bifurcation such that (2) holds for a minimal invariant set  $M_{\lambda_\infty}$ . Then  $M_{\lambda_\infty}^*$  has nonempty intersection with  $M_{\lambda_\infty}$ .*

*Proof.* Consider the sequence  $\lambda_n \rightarrow \lambda_\infty$  as defined before the statement of the theorem. Assume to the contrary that there exists a  $\gamma > 0$  such that  $B_\gamma(M_{\lambda_\infty}) \subset$

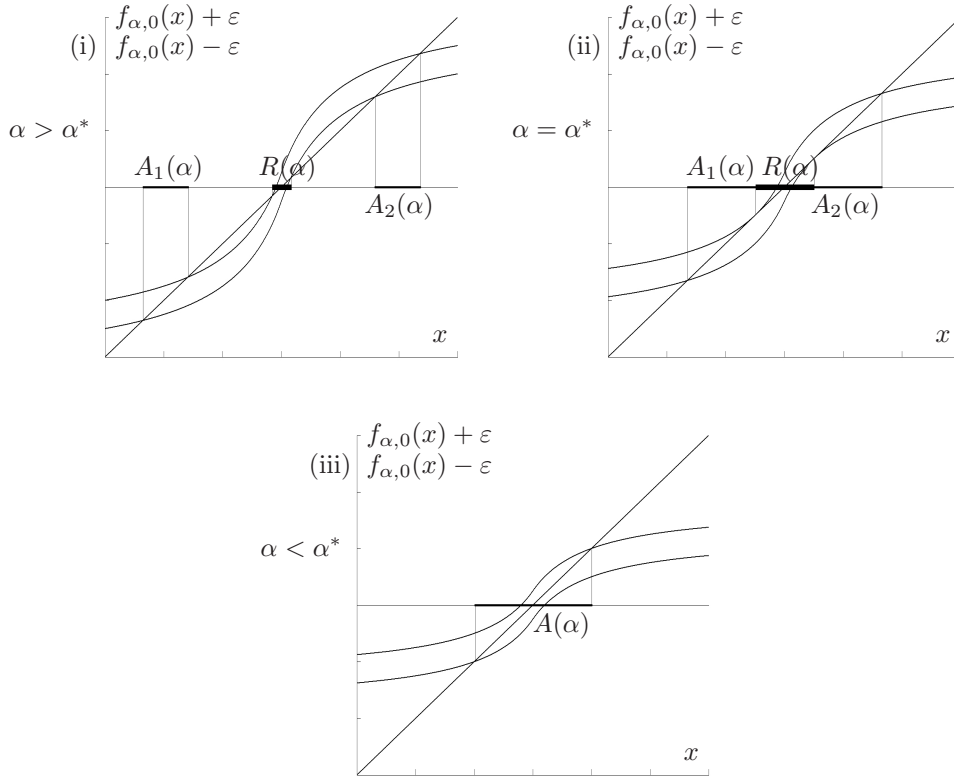


FIGURE 2. Graphs of the extremal functions  $f_{\alpha,0} \pm \varepsilon$ , (i) before the bifurcation, (ii) at the bifurcation point, and (iii) after the bifurcation.

$\mathcal{A}_-(M_{\lambda_\infty})$ . Then for each  $\delta > 0$ , there exists a compact absorbing set  $B$  such that  $M_{\lambda_\infty} \subset B \subset B_\delta(M_{\lambda_\infty})$ , that is,  $\Phi_{\lambda_\infty}(t, B) \subset \text{int } B$  for  $t > 0$  [Aki93, Theorem 3, p. 43]. Due to continuous dependence on  $\lambda$ , there exists an  $n_0 \in \mathbb{N}$  such that  $\Phi_{\lambda_n}(t, B) \subset \text{int } B$  for all  $n \geq n_0$  and  $t > 0$ . This means that there exists a minimal  $\Phi_{\lambda_n}$ -invariant set in  $B$  for all  $n \geq n_0$ . Note that  $n_0$  depends on  $\delta$ , and in the limit  $\delta \rightarrow 0$ , this minimal invariant set converges to  $M_{\lambda_\infty}$ , because of Theorem 5.1. Hence, there is no bifurcation, which shows that  $X \setminus \mathcal{A}_-(M_{\lambda_\infty}) \cap M_{\lambda_\infty} \neq \emptyset$ .  $\square$

## 7. A ONE-DIMENSIONAL ILLUSTRATION

This section is devoted to the illustration of bifurcations characterised by discontinuous explosions and instantaneous appearances of minimal invariant sets in the one-dimensional example

$$F_{\alpha,\beta}(x) := \overline{B_\varepsilon(f_{\alpha,\beta}(x))},$$

where

$$f_{\alpha,\beta}(x) := \frac{\alpha x}{1 + |x|} + \beta$$

and  $\alpha, \beta$  are real parameters. Although similar examples have been discussed already in the literature, see e.g. [HY06], we judge this context best suited to explain the essence of our main theorems.

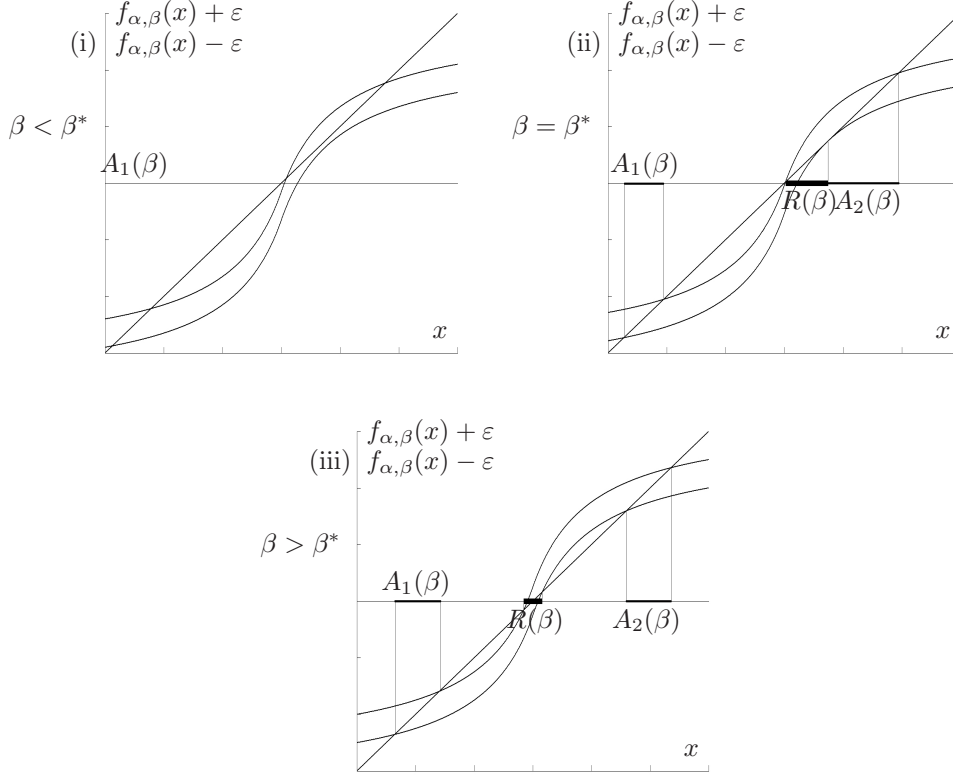


FIGURE 3. Graphs of the extremal functions  $f_{\alpha,0} \pm \varepsilon$ , (i) before the bifurcation, (ii) at the bifurcation point, and (iii) after the bifurcation.

The set-valued map  $F_{\alpha,0}$  admits a discontinuous explosion at  $\alpha^* := 1 + \varepsilon + 2\sqrt{\varepsilon}$ . When  $\alpha > \alpha^*$ , the mapping  $F_{\alpha,0}$  admits two minimal invariant sets, given by the attractors  $A_1(\alpha)$  and  $A_2(\alpha)$  (see Figure 2 (i)). These attractors are bounded by fixed points of the extremal maps  $f_{\alpha,0} - \varepsilon$  and  $f_{\alpha,0} + \varepsilon$ . Between the two attractors, we identify a unique minimal  $F_{\alpha,0}^*$ -invariant set  $R(\alpha) = [r_-(\alpha), r_+(\alpha)]$ . This set is the intersection of the two complementary  $F_{\alpha,0}^*$ -invariant sets  $A_1^*(\alpha) = [r_-(\alpha), \infty)$  and  $A_2^*(\alpha) = (-\infty, r_+(\alpha)]$  (note that due to noncompactness of the phase space, these sets are only closed rather than compact). When decreasing  $\alpha$ , the two attractors approach each other until they collide with  $R(\alpha)$  at the bifurcation point  $\alpha^*$  (see Figure 2 (ii)). At  $\alpha = \alpha^*$ , the two separate attractors still exist, but they explode lower semi-continuously to form a unique minimal invariant set  $A(\alpha)$  as soon as  $\alpha < \alpha^*$  (see Figure 2 (iii)). This scenario illustrates both Theorem 1.1 (i) (cf. Theorem 5.1) and Theorem 6.1. Note that the simultaneous collision of  $R(\alpha)$  with  $A_1(\alpha)$  and  $A_2(\alpha)$  is due to a symmetry of the set-valued mapping  $F_{\alpha,\beta}$ .

Next we show that this mapping also admits an instantaneous appearance of a minimal invariant set. Fix  $\alpha > \alpha^*$ . For  $\beta < \beta^* := -(\alpha + 1 - 2\sqrt{\alpha}) + \varepsilon$ , there exists exactly one minimal invariant set, given by the attractor  $A_1(\beta)$  (see Figure 3 (i)). At  $\beta = \beta^*$ , a new minimal invariant set  $A_2(\beta)$  appears, and alongside also a minimal  $F_{\alpha,\beta}^*$ -invariant set  $R(\beta)$  (see Figure 3 (ii)). As before,  $R(\beta)$  is the intersection of the complementary  $F_{\alpha,\beta}^*$ -invariant sets  $A_1^*(\beta)$  and  $A_2^*(\beta)$ , detaching from  $A_2(\beta)$  as soon as  $\beta > \beta^*$  (see Figure 3 (iii)). This scenario illustrates both Theorem 1.1 (ii) (cf. Theorem 5.1) and Theorem 6.1.

## REFERENCES

- [AC84] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Grundlehren der Mathematischen Wissenschaften, vol. 264, Springer, Berlin, 1984.
- [AF90] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Systems and Control: Foundations and Applications, vol. 2, Birkhäuser, Boston, 1990.
- [Aki93] E. Akin, *The General Topology of Dynamical Systems*, Graduate Studies in Mathematics, no. 1, American Mathematical Society, Providence, Rhode Island, 1993.
- [Ara00] V. Araújo, *Attractors and time averages for random maps*, Annales de l'Institut Henri Poincaré Analyse Non Linéaire **17** (2000), no. 3, 307–369.
- [Arn98] L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998.
- [Art00] Z. Artstein, *Invariant measures of set-valued maps*, Journal of Mathematical Analysis and Applications **252** (2000), no. 2, 696–709.
- [Ash99] P. Ashwin, *Minimal attractors and bifurcations of random dynamical systems*, The Royal Society of London. Proceedings. Series A. Mathematical, Physical and Engineering Sciences **455** (1999), no. 1987, 2615–2634.
- [ASNSH96] L. Arnold, N. Sri Namachchivaya, and K.R. Schenk-Hoppé, *Toward an understanding of stochastic Hopf bifurcation: a case study*, International Journal of Bifurcation and Chaos **6** (1996), no. 11, 1947–1975.
- [Bax94] P.H. Baxendale, *A stochastic Hopf bifurcation*, Probability Theory and Related Fields **99** (1994), no. 4, 581–616.
- [BBS] C.J. Braga Barros and J.A. Souza, *Attractors and chain recurrence for semigroup actions*, to appear in: Journal of Dynamics and Differential Equations.
- [BHY] R.T. Botts, A.J. Homburg, and T.R. Young, *The hopf bifurcation with bounded noise*, submitted.
- [CGK08] F. Colonius, T. Gayer, and W. Kliemann, *Near invariance for Markov diffusion systems*, SIAM Journal on Applied Dynamical Systems **7** (2008), no. 1, 79–107.
- [CHK10] F. Colonius, A.J. Homburg, and W. Kliemann, *Near invariance and local transience for random diffeomorphisms*, Journal of Difference Equations and Applications **16** (2010), no. 2–3, 127–141.
- [CK00] F. Colonius and W. Kliemann, *The Dynamics of Control*, Birkhäuser, Boston, 2000.
- [CK03] F. Colonius and W. Kliemann, *Limits of input-to-state stability*, Systems & Control Letters **49** (2003), no. 2, 111–120.
- [CMKS08] F. Colonius, A. Marquardt, E. Kreuzer, and W. Sichertmann, *A numerical study of capsizing: comparing control set analysis and Melnikov's method*, International Journal of Bifurcation and Chaos **18** (2008), no. 5, 1503–1514.
- [CW09] F. Colonius and T. Wichtrey, *Control systems with almost periodic excitations*, SIAM Journal on Control and Optimization **48** (2009), no. 2, 1055–1079.
- [Dei92] Klaus Deimling, *Multivalued Differential Equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 1, Walter de Gruyter & Co., Berlin, 1992.
- [Gay04] T. Gayer, *Control sets and their boundaries under parameter variation*, Journal of Differential Equations **201** (2004), no. 1, 177–200.
- [Gay05] ———, *Controllability and invariance properties of time-periodic systems*, International Journal of Bifurcation and Chaos **15** (2005), no. 4, 1361–1375.
- [GK01] L. Grüne and P.E. Kloeden, *Discretization, inflation and perturbation of attractors*, Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems (B. Fiedler, ed.), Springer, Berlin, 2001, pp. 399–416.



- [Grü02] L. Grüne, *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization*, Springer Lecture Notes in Mathematics, vol. 1783, Springer, Berlin, Heidelberg, 2002.
- [HY06] A.J. Homburg and T. Young, *Hard bifurcations in dynamical systems with bounded random perturbations*, Regular & Chaotic Dynamics **11** (2006), no. 2, 247–258.
- [HY10] ———, *Bifurcations for random differential equations with bounded noise on surfaces*, Topological Methods in Nonlinear Analysis **35** (2010), no. 1, 77–98.
- [JKP02] R.A. Johnson, P.E. Kloeden, and R. Pavani, *Two-step transition in nonautonomous bifurcations: An explanation*, Stochastics and Dynamics **2** (2002), no. 1, 67–92.
- [Klo78] P.E. Kloeden, *General control systems*, Mathematical Control Theory (Proc. Conf., Australian Nat. Univ., Canberra, 1977), Lecture Notes in Mathematics, vol. 680, Springer, Berlin, 1978, pp. 119–137.
- [KMR] P.E. Kloeden and P. Marín-Rubio, *Negatively invariant sets and entire trajectories of set-valued dynamical systems*, to appear in: Set-Valued and Variational Analysis.
- [Li07] D. Li, *Morse decompositions for general dynamical systems and differential inclusions with applications to control systems*, SIAM Journal on Control and Optimization **46** (2007), no. 1, 35–60.
- [MA99] W. Miller and E. Akin, *Invariant measures for set-valued dynamical systems*, Transactions of the American Mathematical Society **351** (1999), no. 3, 1203–1225.
- [McG92] R.P. McGehee, *Attractors for closed relations on compact Hausdorff spaces*, Indiana University Mathematics Journal **41** (1992), no. 4, 1165–1209.
- [MW06] R.P. McGehee and T. Wiandt, *Conley decomposition for closed relations*, Journal of Difference Equations and Applications **12** (2006), no. 1, 1–47.
- [Rox65] E.O. Roxin, *On generalized dynamical systems defined by contingent equations*, Journal of Differential Equations **1** (1965), 188–205.
- [Rox97] ———, *Control Theory and Its Applications*, Stability and Control: Theory, Methods and Applications, vol. 4, Gordon and Breach Science Publishers, Amsterdam, 1997.
- [ZH07] H. Zmarrou and A.J. Homburg, *Bifurcations of stationary measures of random diffeomorphisms*, Ergodic Theory and Dynamical Systems **27** (2007), no. 5, 1651–1692.
- [ZH08] ———, *Dynamics and bifurcations of random circle diffeomorphisms*, Discrete and Continuous Dynamical Systems B **10** (2008), no. 2–3, 719–731.

JEROEN S. W. LAMB, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW72AZ, UNITED KINGDOM, AND IMECC-UNICAMP, CEP 13081-970, CAMPINAS, SÃO PAULO, BRAZIL  
*E-mail address:* jeroen.lamb@imperial.ac.uk

MARTIN RASMUSSEN, DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW72AZ, UNITED KINGDOM  
*E-mail address:* m.rasmussen@imperial.ac.uk

CHRISTIAN S. RODRIGUES, MAX-PLANCK-INSTITUTE FOR MATHEMATICS IN THE SCIENCES, INSELSTR. 22, 04103 LEIPZIG, GERMANY  
*E-mail address:* christian.rodrigues@mis.mpg.de