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On delay-induced stability in diffusively coupled discrete-time systems

by

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Abstract

The stability of networked systems is considered under time-delayed diffusive coupling. Necessary conditions for stability are given for general directed and weighted networks with both positive and negative weights. Exact stability conditions are obtained for undirected networks with nonnegative weights, and it is shown that the largest eigenvalue of the graph Laplacian determines the effect of the connection topology on stability. It is further shown that the stability region in the parameter space shrinks with increasing values of the largest eigenvalue, or of the time delay of the same parity. In particular, unstable fixed points of the individual maps can be stabilized for certain parameter ranges when they are coupled with an odd time delay, provided that the connection structure is not bipartite. Furthermore, signal propagation delays are compared to signal processing delays and it is shown that delay-induced stability cannot occur for the latter. Connections to continuous-time systems are indicated.

Keywords: Delay, Stability, Signed graph, Synchronization, Chaos

Mathematics Subject Classification (2000): 39A30, 94C15, 05C22

1 Introduction

Dynamical networks constitute a class of dynamical systems that has increasingly been attracting interest over the recent years. The underlying discrete space can often be described by a graph where each node corresponds to a dynamical system. Consequently, the analysis of the rich spectrum of spatio-temporal dynamics of the overall network necessitates a synthesis of knowledge from the fields of differential or difference equations and of graph theory. Additionally, modeling of real-world systems usually requires taking into account the inevitable presence of time delays in the information flow in the network, which can cause a range of interesting dynamical behavior [1]. One such phenomenon is delay-induced stability, sometimes called *amplitude death* or *oscillator death* in certain contexts, where oscillatory or even chaotic units exhibit a stable equilibrium solution when they are coupled to form a network. In this paper, we study the stability of discrete-time systems on networks where information transmission is subject to time delays. Our aim is to understand the role of both time delays and the graph structure on the stability of coupled systems.

We consider coupled identical maps of the form

$$x_i(t+1) = f(x_i(t)) + \frac{1}{d_i} \sum_{j=1}^{n} a_{ij} g(x_j(t-\tau), x_i(t)). \tag{1}$$

Here $x_i(t) \in \mathbb{R}^m$, $i = 1, \ldots, n$, denotes the state of the ith map at discrete time $t \in \mathbb{Z}$, $f : \mathbb{R}^m \to \mathbb{R}^m$ describes the individual dynamics of the maps in isolation, and $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ represents the pairwise interaction between the maps. Both f and g are assumed to be differentiable functions. We study two types of diffusive interaction, namely, linear diffusion where

$$g(x, y) = \kappa(x - y), \tag{2}$$

and the function

$$g(x, y) = \kappa(f(x) - f(y)) \tag{3}$$

which arises in the paradigm of coupled map lattices [2]. In both cases κ is a scalar representing a diffusion coefficient or coupling strength in the network. The interaction between the units can account for the finite speed of information transmission by allowing a time delay $\tau \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the nonnegative integers. The numbers $a_{ij} \in \mathbb{R}$ determine the connection structure, and $d_i = \sum_j a_{ij}$ is the *in-degree* of the ith node, which in this paper is assumed to be nonzero for all i. In the simplest case, a_{ij} takes on binary values depending on whether or not there is a connection between the nodes i and j. In other words, $a_{ij} = 1$ if there is a (directed) edge from j to i, and $a_{ij} = 0$ otherwise; thus $A = [a_{ij}]$ defines the transpose of the adjacency matrix of the underlying graph with n vertices. For simple graphs $a_{ii} = 0$ $\forall i$, and for undirected graphs $a_{ij} = a_{ji}$ $\forall i, j$. We will also consider more general cases: Allowing a_{ij} to be nonnegative real numbers yields a weighted graph, and if a_{ij} is not necessarily equal to a_{ji} then one obtains a directed graph. Furthermore, if a_{ij} are arbitrary real numbers with mixed signs we have the case of a weighted *signed graph*, which arises in some important applications, such as neuronal networks, where one needs to distinguish between excitatory and inhibitory connections.

Our interest in this paper is the stability of spatially uniform equilibrium solutions of (1). By the form of the interaction function (2) or (3), it is easy to see that $\mathbf{x}^* := (x^*, \ldots, x^*) \in \mathbb{R}^{nm}$ is a spatially uniform equilibrium solution of (1) if and only if $x^* \in \mathbb{R}^m$ is a fixed point of f. The main question is how the local stability of $\mathbf{x}^*$ for the coupled system is related to the stability of x^* for the individual maps.

The local stability of $\mathbf{x}^* = (x^*, \ldots, x^*)$ is determined by the linear variational equation

$$u_i(t+1) = Bu_i(t) + \frac{1}{d_i} C \sum_{j=1}^{n} a_{ij} [u_j(t-\tau) - u_i(t)], \quad i = 1, \ldots n, \tag{4}$$

where $u_i = x_i - x^* \in \mathbb{R}^m$, B is the Jacobian matrix of f at the fixed point x^*, and the matrix $C \in \mathbb{R}^{m \times m}$ is equal to either κI_m or κB depending on whether g has the form (2) or (3), respectively. Let $\mathbf{u} \in \mathbb{R}^{nm}$ denote the concatenation $(u_1, \dots, u_n)$ and $D = \operatorname{diag}\{d_1, \dots, d_n\}$ denote the diagonal matrix of vertex degrees. Then (4) can be written as

$$\mathbf{u}(t+1) = (I_n \otimes (B - C))\mathbf{u}(t) + (D^{-1}A \otimes C)\mathbf{u}(t - \tau), \tag{5}$$

with $\otimes$ denoting the Kronecker product. The asymptotic stability of the zero solution of the linear equation (4) or (5) is equivalent to the exponential stability of $\mathbf{x}^*$ in the nonlinear equation (1) (see e.g. [3]).

The present paper deals with the stability of (5), in particular stability induced by delays. To appreciate the use of the term "delay-induced stability", consider (5) with $\tau = 0$:

$$\begin{aligned} \mathbf{u}(t+1) &= [I_n \otimes B + (D^{-1}A - I_n) \otimes C]\mathbf{u}(t) \\ &:= [I_n \otimes B - L \otimes C]\mathbf{u}(t) \end{aligned} \tag{6}$$

where $L = I - D^{-1}A$ is the (normalized) Laplacian matrix, which arises as a natural consequence of the diffusive-type interaction (2)–(3)[1]. By its definition, L has zero row sums, so $L\mathbf{1} = 0$, where $\mathbf{1} = (1, 1, \dots, 1)^\top$ (see also Lemma 1 later in the text). Hence if $\mathbf{u}(0) = \mathbf{1} \otimes \mathbf{v}$ for some nonzero $\mathbf{v} \in \mathbb{R}^m$, then $\mathbf{u}(t) = \mathbf{1} \otimes (B^t \mathbf{v})$ by iterations of (6), which converges to zero if and only if B is a stable matrix (i.e., all its eigenvalues are inside the unit circle). Therefore, an unstable fixed point x^* of f cannot be stabilized in the coupled network (1) in the absence of delays. This observation holds even when the specific forms (2)–(3) are replaced by the more *general diffusive condition*, namely that

$$g(x, x) = 0 \quad \forall x \in \mathbb{R}^m, \tag{7}$$

or when the normalization terms $1/d_i$ are omitted in (1): In either case one arrives at the linear variational equation (6) with an appropriate Laplacian matrix having zero row sums. Therefore, time delays are necessary to stabilize an unstable fixed point of the map f in a diffusively coupled network.

In the following sections, we will mainly be interested in the case when the Jacobian B is unstable but the zero solution of the coupled system (5) is stable. Since delays are often known for their destabilizing effects, it can be expected that delay-induced stability occurs only for rather restricted parameter sets. Section 2 confirms this by proving several necessary conditions for stability in general signed networks, and shows how delay-induced stability can be ruled out in many cases, for instance in bipartite networks. For the more conventional undirected graphs with nonnegative weights, exact stability conditions are given in Section 3 for the case when the Jacobian has real eigenvalues, which generalize the existing results on scalar maps [5] to higher-dimensions. In particular, it will be seen that that the effect of the network structure on stability is completely determined by a single scalar quantity, namely the largest eigenvalue of the Laplacian. Moreover, the stability properties of the system depend monotonically on the largest eigenvalue, or on the delay magnitude for delays of the same parity. In view of the above-observed impossibility of stability for $\tau = 0$, the monotonicity property entails in particular that delay-induced stabilization is not possible for any even delay. The implication for the nonlinear system (1) is that (odd) delays may induce stability

[1] The normalized Laplacian also arises in other contexts, such as random walks on graphs [4].

for maps undergoing period-doubling bifurcations but not necessarily for other type of bifurcations. A numerical example in Section 3.3 demonstrates delay-induced stability in coupled chaotic Hénon maps. As opposed to the signal *transmission* delays modeled in (1), Section 4 considers a slightly different model treating signal *processing* delays, and shows that stabilization is not possible in this case. The paper is concluded in Section 5 with some remarks on continuous-time systems.

2 General signed graphs and bipartite structures

Let $A = [a_{ij}]$, $a_{ij} \in \mathbb{R}$, denote the transposed adjacency matrix of the graph describing the connection structure of the network. Thus, there is a (directed) link from vertex j to i if and only if $a_{ij} \neq 0$. For the purposes of this section, A need not be symmetric and the numbers a_{ij} can have arbitrary magnitudes and signs. The in-degrees $d_i = \sum_{j=1}^{n} a_{ij}$ of vertices can thus have different signs and magnitudes. We say that the vertex i is *quasi-isolated* if $d_i = 0$. We consider only graphs without quasi-isolated vertices, and define the normalized Laplacian by $L = I - D^{-1}A$. In general L can have complex eigenvalues, and its spectrum is not uniformly bounded since the vertex degrees can be arbitrarily close to zero. A signed graph is called *bipartite* if its vertex set can be divided into two parts such that $a_{ij} = 0$ whenever i and j belong to the same partition. The next lemma shows that some familiar spectral properties carry over to signed graphs.

Lemma 1 *For a signed graph $\mathcal{G}$ without quasi-isolated vertices, zero is always an eigenvalue of $L(\mathcal{G})$. If $\mathcal{G}$ is bipartite, then 2 is also an eigenvalue of $L(\mathcal{G})$.*

Proof. By definition, the row sums of L are 0, so 0 is an eigenvalue of L corresponding to the eigenvector $\mathbf{1} = (1, \ldots, 1)^\top$. On the other hand, if $\mathcal{G}$ is bipartite, then after possible relabeling of vertices, A can be written in the block form

$$A = \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix}.$$

where $A_1 \in \mathbb{R}^{p\times q}$ and $A_2 \in \mathbb{R}^{q\times p}$ for some positive integers p, q such that $p + q = n$. Then the matrix $D^{-1}A$ has the block form,

$$D^{-1}A = \begin{bmatrix} 0 & \tilde{A}_1 \\ \tilde{A}_2 & 0 \end{bmatrix}$$

with the same block sizes as A, where each row has the same sum 1. It can be checked that the vector $\mathbf{1}^{\pm} :=(1, \ldots, 1, -1, \ldots, -1)^\top$ with p positive and q negative entries is an eigenvector of $D^{-1}A$ corresponding to the eigenvalue -1. Hence $I - D^{-1}A$ has an eigenvalue equal to 2. $\square$

We next prove some necessary conditions for delay-induced stability.

Proposition 2 *Suppose that the Jacobian of f at the fixed point x^* has an eigenvalue β such that $|\beta| > 1$. Then the following statements are necessary for the stability of the equilibrium solution $\mathbf{x}^* = (x^*, \ldots, x^*)$ of the coupled system (1):*

(i) $\frac{1}{2}(|\beta| - 1) \leq |\kappa| \leq 1$ *when the interaction function is given by (2), and* $\dfrac{|\beta| - 1}{2|\beta|} \leq$

$|\kappa| \le \dfrac{1}{|\beta|}$ *when it is given by (3).*
(ii) $|\beta| \le 3$.
(iii) *If β is real, then $-3 \le \beta < -1$ and τ is a positive odd integer.*

Proof. By Lemma 1, zero is an eigenvalue of L, or equivalently, 1 is an eigenvalue of $D^{-1}A$, and $D^{-1}A\mathbf{1} = \mathbf{1}$, with $\mathbf{1} = (1, 1, \ldots 1)^\top$. Let $\mathbf{b}$ denote the eigenvector of B corresponding to the eigenvalue β. Note that $C\mathbf{b} = c\mathbf{b}$ where c equals κ or $\kappa\beta$, respectively, depending on whether the interaction has the form (2) or (3). It follows that the subspace spanned by the vector $\mathbf{1} \otimes \mathbf{b}$ is invariant under the dynamics of (5). That is, if $\mathbf{u}(s) = \alpha(s)\mathbf{1} \otimes \mathbf{b}$ for some scalars $\alpha(s)$, $s = t - \tau, \ldots, t$, then by (5),

$$\mathbf{u}(t+1) = [(\beta - c)\alpha(t) + c\alpha(t-\tau)]\mathbf{1} \otimes \mathbf{b}$$

Hence, the coefficients α obey the equation

$$\alpha(t+1) = (\beta - c)\alpha(t) + c\alpha(t-\tau) \tag{8}$$

and describe the dynamics on the subspace spanned by $\mathbf{1} \otimes \mathbf{b}$. The characteristic polynomial corresponding to (8) is

$$\chi(s) = s^{\tau+1} - (\beta - c)s^\tau - c. \tag{9}$$

Clearly, the condition

$$|c| \le 1 \tag{10}$$

is necessary for the stability of (8) since the product of the characteristic roots equals $(-1)^\tau c$. We further claim that the condition

$$2|c| \ge |\beta| - 1 \tag{11}$$

is necessary for stability. To see this, we write the characteristic polynomial as $\chi(s) = \chi_1(s) + \chi_2(s)$, where $\chi_1(s) = s^{\tau+1} - \beta s^\tau$ and $\chi_2(s) = cs^\tau - c$. Note that χ_1 has a root outside the unit circle by the assumption $|\beta| > 1$. If (11) does not hold, then for s on the unit circle,

$$|\chi_1(s)| \ge \big||\beta||s^\tau| - |s^{\tau+1}|\big| = |\beta| - 1 > 2|c| \ge |\chi_2(s)|$$

so that by Rouché's theorem χ_1 and $\chi_1 + \chi_2$ have the same number of roots inside the unit circle, i.e., χ has a root outside the unit circle. This proves the necessity of (11) for stability. Putting $c = \kappa$ or $c = \kappa\beta$ in the conditions (10) and (11) proves statement (i). Moreover, combining (10) and (11) gives $|\beta| - 1 \le 2$, which establishes (ii). To show (iii), consider the characteristic polynomial $\chi(s)$ as a real-valued function of the real argument s. Suppose $\beta > 1$. Then $\chi(1) = 1 - \beta < 0$. Since $\lim_{s\to\infty} \chi(s) = +\infty$, χ has a real root greater than 1. Hence, stability is not possible for $\beta > 1$ for any value of the delay. Suppose now $\beta < -1$ and τ is even. Then $\chi(-1) = -1 - \beta > 0$, and since $\lim_{s\to-\infty} \chi(s) = -\infty$, χ has a real root less than -1. Thus, in case β is real, stabilization is only possible if $\beta < -1$ and τ is odd. Combining with (ii) establishes (iii). □

Statement (iii) of Proposition 2 is particularly relevant for bifurcations of the nonlinear system (1). Hence, instabilities arising from a real eigenvalue β crossing the unit circle can be stabilized in the network only for flip bifurcations (β crossing -1) but not for others (β crossing $+1$); furthermore, stabilization is not possible for even delays. We next show that it is also not possible in bipartite topologies. Later in Section 3 we will give exact conditions for stabilization of eigenvalues $\beta < -1$ and demonstrate with a numerical example in Section 3.3 that odd delays can indeed stabilize flip bifurcations in appropriate coupling topologies.

Proposition 3 *Consider (1) where the coefficients a_{ij} define the transposed adjacency matrix of a signed bipartite graph. If the Jacobian of f at the fixed point x^* has a real eigenvalue $|\beta| > 1$, then $\mathbf{x}^* = (x^*, \ldots, x^*)$ is an unstable fixed point of (1).*

Proof. By Proposition 2 it suffices to consider the case when τ is odd and $\beta < -1$. By Proposition 1, the Laplacian L has an eigenvalue equal to 2, or equivalently, $D^{-1}A$ has an eigenvalue equal to -1, with eigenvector $\mathbf{1}^{\pm} := (1, \ldots, 1, -1, \ldots, -1)^{\top}$, as defined in the proof of Proposition 1. Similar to the proof of Proposition 2, we let $\mathbf{b}$ denote the eigenvector of B corresponding to the eigenvalue β, and observe that the subspace spanned by the vector $\mathbf{1}^{\pm} \otimes \mathbf{b}$ is invariant under the dynamics of (5). Writing $\mathbf{u}(t) = \alpha(t)\mathbf{1}^{\pm} \otimes \mathbf{b}$ and substituting into (5), it is seen that the scalars $\alpha(t)$ obey

$$\alpha(t+1) = (\beta - c)\alpha(t) - c\alpha(t - \tau)$$

whose characteristic polynomial is

$$\chi(s) = s^{\tau+1} - (\beta - c)s^{\tau} + c. \tag{12}$$

Consider χ as a mapping $\chi : \mathbb{R} \to \mathbb{R}$. We have $\chi(-1) = 1 + \beta < 0$, and $\lim_{s \to -\infty} \chi(s) = \infty$. So, χ has a real root less than -1, and hence is an unstable polynomial. $\square$

Proposition 3 hints at the important role played by the network structure in stability, which will be fully characterized in the next section for undirected graphs with nonnegative weights in terms of the largest Laplacian eigenvalue.

3 Undirected graphs with nonnegative weights

In this section we restrict ourselves to undirected graphs with nonnegative weights; $a_{ij} \geq 0$ and $a_{ij} = a_{ji}$ for all i, j. Note that even though $A = [a_{ij}]$ is a symmetric matrix, $L = I - D^{-1}A$ need not be symmetric. Nevertheless, the observation that $L = D^{-1/2}(I - D^{-1/2}AD^{-1/2})D^{1/2}$ shows that L is similar to the real symmetric matrix $I - D^{-1/2}AD^{1/2} = [\delta_{ij} - a_{ij}/\sqrt{d_i d_j}]$. Thus, the eigenvalues λ_k of L are real and the corresponding eigenvectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ form a complete basis for $\mathbb{R}^n$. An application of Gershgorin's theorem (e.g. [6]) shows that the eigenvalues are confined to the interval $[0, 2]$, and the smallest eigenvalue is always zero (e.g. Lemma 1). The largest eigenvalue will play a special role on the stability analysis, and it will be denoted $\lambda_{\max}$ in the sequel. It is easy to see that $n/(n-1) \leq \lambda_{\max} \leq 2$: The upper bound follows by the general bound on eigenvalues noted above, and is achieved by bipartite graphs (Lemma 1). The lower bound follows from the observation that the trace of the Laplacian, and hence the sum of the eigenvalues, equals n, and there is always a zero eigenvalue. It can be easily checked that $\lambda = n/(n-1)$ when $a_{ij} = 1$ for all $i \neq j$, i.e., for *complete graphs*.

We also assume in this section that the Jacobian B has a complete set of eigenvectors $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ and real eigenvalues $\{\beta_1, \dots, \beta_n\}$. This condition is clearly satisfied for scalar maps, but it also holds for some familiar higher-dimensional maps such as the Hénon map (see Section 3.3) and makes it possible to give a complete stability analysis of (4).

3.1 Exact stability conditions

Note that $\{\mathbf{v}_i \otimes \mathbf{b}_j : i = 1, \dots, n,\ j = 1, \dots, m\}$ is a basis for $\mathbb{R}^{nm}$; hence $\boldsymbol{u}(t)$ in (5) can be written as $\boldsymbol{u}(t) = \sum_{i,j} u_{ij}(t)\mathbf{v}_i \otimes \mathbf{b}_j$ for some scalars u_{ij}. Furthermore, $B\mathbf{b}_j = \beta_j \mathbf{b}_j$ and $D^{-1}A\mathbf{v}_i = (I - L)\mathbf{v}_i = (1 - \lambda_i)\mathbf{v}_i$, $\forall i, j$. It then follows from (5) that

$$u_{ij}(t+1) = (\beta_j - c_j)u_{ij}(t) + c_j(1 - \lambda_i)u_{ij}(t - \tau) \tag{13}$$

where $c_j = \kappa$ or $c_j{=}\kappa\beta_j$ depending on whether g is given by (2) or (3), respectively. The characteristic equation corresponding to (13) is

$$s^{\tau+1} - (\beta_j - c_j)s^{\tau} - c_j(1 - \lambda_i) = 0. \tag{14}$$

Hence, the spatially uniform equilibrium solution $\mathbf{x}^*$ of (1) is exponentially stable if and only if all roots of the equation (14) are inside the unit circle for $i = 1, \dots, n$, $j = 1, \dots, m$. The next theorem gives the precise conditions for stability.

Theorem 4 *Let τ be a positive integer. For $j = 1, \dots, m$, let Φ_j denote the unique number satisfying*

$$\frac{\sin((\tau+1)\Phi_j)}{\sin(\tau\Phi_j)} = |\beta_j - c_j|\,, \qquad \Phi_j \in \left(0, \frac{\pi}{\tau+1}\right). \tag{15}$$

Then the zero solution of (4) is asymptotically stable if and only if one of the following holds for both $\lambda = 0$ and $\lambda = \lambda_{\max}$ (i.e. for the smallest and the largest eigenvalues of the Laplacian) and all $j = 1, \dots, m$:

(i) τ is odd and

$$|\beta_j - c_j| - 1 < -c_j(1 - \lambda) < \sqrt{(\beta_j - c_j)^2 + 1 - 2\,|\beta_j - c_j|\cos\Phi_j}\,; \tag{16}$$

(ii) τ is even,

$$|\beta_j - c_j\lambda| < 1, \quad \textit{and} \tag{17}$$

$$|c_j| < \sqrt{(\beta_j - c_j)^2 + 1 - 2\,|\beta_j - c_j|\cos\Phi_j}\,. \tag{18}$$

On the other hand, for $\tau = 0$, the zero solution is asymptotically stable if and only if (17) holds for $\lambda = 0$ and $\lambda = \lambda_{\max}$ and all $j = 1, \dots, m$.

The proof is an extension of a similar result which was proved for scalar maps [5, Theorem 2], and will be omitted here.

We remark that the role of the coupling topology is completely determined by the largest eigenvalue $\lambda_{\max}$ of the graph Laplacian. Since $\lambda_{\max} = 2$ for bipartite graphs (Lemma 1), it follows that all bipartite graphs have the same stability properties when other parameters are kept constant. This is an analogue of a corresponding result given in [7] for continuous-time systems.

3.2 Monotonicity properties

We present two results that show how the stability is affected by delays and the connection topology. The first result shows that the stability domains in the parameter space are nested with respect to varying delays of the same parity, where a larger delay delays implies a smaller stability region. It should be noted that the statement does not necessarily hold when comparing delays of different parity.

Proposition 5 *Let τ_1 and τ_2 be positive integers, $\tau_1 < \tau_2$, which are both odd or both even. If the zero solution of (4) is asymptotically stable for $\tau = \tau_2$, then it is also asymptotically stable for $\tau = \tau_1$.*

Proof. Suppose τ_1 and τ_2 are both odd or both even, with $\tau_1 < \tau_2$. Fix j and let $\Phi_j = \Phi_j(\tau)$ be the solution of (15) belonging to the interval $(0, \pi/(\tau+1))$. By Lemma 5 in the Appendix of [5], Φ_j is a decreasing function of τ, so $\cos\Phi_j$ is increasing in τ. Therefore, the radicands in (16) and (18) are decreasing functions of τ, which implies that (16) or (18) is satisfied for $\Phi_j(\tau_2)$ whenever it is satisfied for $\Phi_j(\tau_1)$. □

A similar monotonicity holds with respect to the largest Laplacian eigenvalue $\lambda_{\max}$, a smaller value of $\lambda_{\max}$ implying a larger stability region in the parameter space.

Proposition 6 *Let $\mathcal{G}_a$ and $\mathcal{G}_b$ be two graphs with corresponding largest Laplacian eigenvalues $\lambda^a_{\max} \leq \lambda^b_{\max}$. If the zero solution of (4) is asymptotically stable under the connection topology of $\mathcal{G}_b$, then it is also asymptotically stable for $\mathcal{G}_a$.*

Proof. Suppose that the zero solution of (4) is asymptotically stable under the connection topology of $\mathcal{G}_b$. Then by Theorem 4, β_j and c_j satisfy either (16) or (17)–(18) depending on whether τ is odd or even, for both $\lambda = 0$ and $\lambda = \lambda^b_{\max}$. Because these inequalities are linear in λ, they also hold for each $\lambda \in (0, \lambda^b_{\max})$, and in particular for $\lambda = \lambda^a_{\max}$ (see Lemma 2 in [5]). Since the conditions (16) or (17)–(18) are also sufficient for stability, the theorem is proved. □

It follows by the above theorem that complete graphs have the best stability characteristics among all graphs of a given size, whereas bipartite graphs have the worst stability among all graphs of all sizes. Furthermore, as seen in Proposition 2, in case of real eigenvalues, stabilization can only be achieved by odd delays. Combining with Propositions 5 and 6, we see that the largest stability domain is obtained for complete graphs and $\tau = 1$.

3.3 Stabilizing chaotic maps: A numerical example

Consider the two-dimensional Hénon map

$$f(x, y) = (y + 1 - ax^2, bx).$$

For the choice of parameters $a = 1.4$ and $b = 0.3$, the Hénon map is known to be chaotic. It has two unstable fixed points: The first one is at $(-1.1314, -0.3394)$, where the Jacobian eigenvalues are $\beta_1 = 3.2598$ and $\beta_2 = -0.0920$, and the second one is at $(0.6314, 0.1894)$ with Jacobian eigenvalues $\beta_1 = -1.9238$ and $\beta_2 = 0.1894$. By the foregoing results, the first fixed point cannot be stabilized in a diffusively-coupled network, while second fixed point can be stabilized using an odd delay.

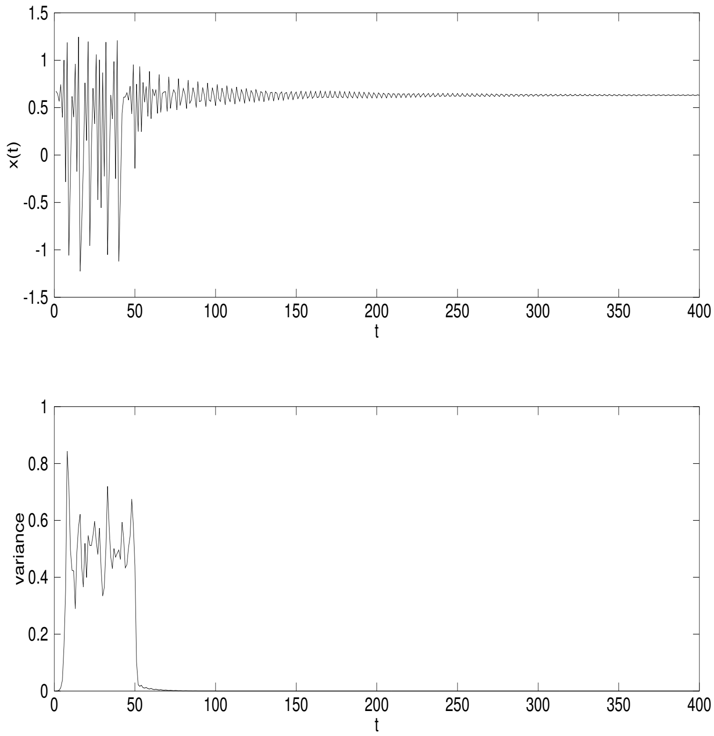

Figure 1: Stabilization in a network of coupled chaotic Hénon maps. The coupling is turned on at $t = 50$, after which the chaotic dynamics is replaced by a transient to a stable equilibrium solution. The first plot shows the time evolution of the x-component of a randomly selected unit, and the plot below shows the variance over the vertices of the network, indicating that each unit asymptotically approaches the same equilibrium value. The network is a complete graph on 40 vertices which are coupled through the interaction function (3) with $\kappa = 0.51$ and $\tau = 1$.

To demonstrate delay-induced stability, we consider identical Hénon maps coupled in a complete graph topology through the interaction function (3). Starting from random initial conditions, the maps are run without coupling ($\kappa = 0$) for 50 time steps, and afterwards the coupling is turned on with $\kappa = 0.51$ and $\tau = 1$. It can be checked from Theorem 4 that the chosen parameter values yield stability at the second fixed point of f. Figure 1 confirms that chaotic dynamics is indeed replaced by equilibrium behavior in the coupled network.

4 Signal processing delays

The way delay enters the coupled system (1) is motivated by the finite speed of signal transmission in spatially extended systems, where the information from vertex j reaches i after some time delay and processed together with the current state of vertex i. In a similar fashion, one can model *signal processing delays* in the form

$$x_i(t+1) = f(x_i(t)) + \frac{1}{d_i} \sum_{j=1}^{n} a_{ij} g(x_j(t-\tau), x_i(t-\tau)). \tag{19}$$

Here, the states of both the vertices i, j are instantly available, but the processing of the information takes some time τ. We show that delay-induced stability is not possible under this scheme for any diffusive-type interaction g satisfying (7). Hence, the system (1) considered in this paper is a relevant prototype for studying stability caused by delays.

Theorem 7 *Suppose g satisfies the general diffusive condition (7). If x^* is an unstable fixed point of f, then $\mathbf{x}^* = (x^*, \dots, x^*)$ is an unstable fixed point of (19).*

Proof. Since by assumption x^* is an unstable fixed point of f, there exists an $\varepsilon > 0$ such that for any $\delta > 0$ one can find $u \in \mathbb{R}^m$, $\|u\| \le \delta$, and $t \in \mathbb{Z}^+$ satisfying

$$\|f^{(t)}(x^* + u) - x^*\| > \varepsilon, \tag{20}$$

where $f^{(t)}$ denotes the t-th iterate of f. If in (19) we choose initial conditions $x_i(s) = x^* + u$ for all $i = 1, \dots, n$ and $s = -\tau, \dots, 0$, and use the fact that g satisfies (7), we obtain $x_i(t) = f^{(t)}(x^* + u)$. Hence by (20),

$$\|x_i(t) - x^*\| > \varepsilon \quad \forall i,$$

proving that $(x^*, \dots, x^*)$ is an unstable fixed point of (19). □

5 Remarks on continuous-time systems

The discrete-time systems considered in this paper are subject to integer-valued delays; hence, the characteristic equation responsible for stability is a polynomial, with a finite number of roots. Continuous-time systems with delays, on the other hand, have transcendental characteristic equations with an infinite number of roots, making their analysis generally more difficult. A particular case where an explicit analysis can be given arises when the individual units in the network are near a Hopf bifurcation [7]. In this case, the results obtained for undirected networks are in many ways similar to those given in Section 3. For instance, it is true also for continuous-time systems that the largest eigenvalue of the Laplacian determines the role of network topology on stability, and complete graphs have the best stability characteristics, whereas bipartite graphs have the worst [7]. In fact, Proposition 6 holds without change. The distinction between even and odd delays are of course absent when delays are allowed to be arbitrary nonnegative numbers; so Proposition 5 as such does not apply to the continuous-time case. The corresponding analogy for continuous-time systems is the so-called *stability switches* in linear systems [8] or *stability islands* in coupled nonlinear networks [9], where stability may be lost and regained several times as the value of delay is continuously increased. Finally, the impossibility of delay-induced stability under processing delays, as stated in Theorem 7, holds in continuous time as well, with an identical proof up to obvious modifications.

It is worth noting the similarity of the two settings where delays prove to be efficient stabilizing mechanisms, namely near flip bifurcations in discrete time and Hopf bifurcations in continuous time, which are both linked to instabilities leading to the birth of periodic solutions in their respective cases. Hence, under appropriate conditions, the well-known oscillatory instabilities typically associated with time delays can be used to advantage to counteract the intrinsic instabilities in the system, thereby yielding overall stability. The effects of delays on instabilities resulting from other types of local and global bifurcations of spatially uniform and non-uniform fixed points, or of non-constant solutions, remain problems for future investigations.

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