Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

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by

Danijela Horak

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COMBINATORIAL LAPLACE OPERATORS: A UNIFYING APPROACH, NORMALIZATION AND SPECTRA

DANIJELA HORAK

Abstract. In this paper we present a general framework for the systematic study of all known types of combinatorial Laplace operators i.e. the graph Laplacian, the combinatorial Laplacian on simplicial complexes, the weighted Laplacian, the normalized graph Laplacian. Furthermore, we define normalized Laplace operator Δ_i^{up} on simplicial complexes and present its basic properties. The effects of a wedge sum, a join and a duplication of a motif on the spectrum of normalized Laplace operator are investigated, and some of the combinatorial features of a simplicial complex that are encoded in its spectrum are identified.

1. Introduction

The study of graph Laplacian has a long and prolific history. It first appeared in a paper by Kirchhoff [21], where he analysed electrical networks and stated the celebrated matrix tree theorem. The Laplace operator L considered in [21] is

$$Lf(v_i) = \deg v_i f(v_i) - \sum_{v_i \sim v_j} f(v_j), \tag{1.1}$$

where f is a function on the vertices of a graph. In spite of its rather early beginnings this topic did not gain much attention among scientists until the early 1970's and the work of Fiedler [12], where he found a correlation among the smallest non-zero eigenvalue and the connectivity of a graph. Up until then it was common to characterize graphs by means of the spectrum of its adjacency matrix. However, after the ground-breaking work of Fiedler, there has been a number of papers (e.g. [17]) arguing in favour of the graph Laplacian and its spectrum. For a good survey articles on the graph Laplacian the reader is referred to [23] or [24].

The generalization of the graph Laplacian to simplicial complexes has first been carried out by Eckmann [11], who formulated and proved the discrete version of Hodge theorem , i.e.

$$\ker(\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*) \cong \tilde{H}^i(K, \mathbb{R})$$

and defined the higher order combinatorial Laplacian as

$$L_i = \delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*.$$

This has led to a further development of the area, which resulted in a substantial amount of work on properties of the *higher order combinatorial Laplacian* (see [8],[13],[9]), which build up on properties of the graph Laplacian. This operator has

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been employed extensively in investigating the features of networks related to dynamics and coverings (see [25],[26]). Recently the monograph [19] appeared, where the combinatorial Laplacian is systematically studied in a context of a discrete exterior calculus.

Almost a century after Kirchhoff's work a Norwegian scientist Bottema [4] studied a transition probability operator on graphs which is equivalent to the following version of the graph Laplace operator

$$\Delta f(v_i) = f(v_i) - \frac{1}{\deg v_i} \sum_{v_i \sim v_j} f(v_j). \tag{1.2}$$

This operator was a mean to investigate random walks on graphs. It took another one hundred years, until there has been a significant advance in the study of operator Δ, which got to be known by the name *normalized graph Laplacian* to distinguish it from the graph Laplacian L. The main advantage of Δ is to address the problems related to random walks on graphs and graph expanders, which the graph Laplacian was unable to tackle. For a good introduction to this topic the reader can consult [6] or [15].

The main goals of this paper are to develop a general and fully established theory, which can be used as a starting point for a study of any of the above mentioned versions of the Laplace operator and to define the normalized Laplacian on simplicial complexes. The latter is based on a simple observation that the form of the combinatorial Laplacian is tightly connected to the choice of the scalar product on the coboundary vector spaces. On the other hand, the scalar products can be viewed in terms of weight functions. Thus, by controlling these weights, we control the range of the eigenvalues of the Laplace operator. However, we will concentrate on the analysis of the combinatorial Laplacian whose eigenvalues are in range $[0, i+2]$, where i is the order of the Laplacian. This is a generalization of the normalized graph Laplacian Δ. Apart from describing the features of its spectrum and its connection with the combinatorial structure of simplicial complexes, we will emphasize how important this approach is to gain new insights on the already extensively studied normalized graph Laplacian. In the past, there have been few attempts towards the normalization of the combinatorial Laplace operator, in particular [5] and recently [27], but unlike the normalized Laplace operator proposed in this work, they fail to fit into general theory.

This paper is organized as follows. In Section 2 we give the basic definitions regarding simplicial complexes and recall Eckmann's discrete version of Hodge theorem. We define the combinatorial Laplace operator in its full generality and give explicit expressions for it. In Section 3 we state and prove the theorem about the number of zeros in the spectrum of the various versions of the combinatorial Laplace operators. Furthermore, we discuss the effect of the scalar products on the spectrum and give the upper bound of the spectrum. Finally, we state the definition of the *normalized combinatorial Laplace operator*, which will be the main object of the investigation in the remainder of the paper. We calculate spectra of the normalized combinatorial Laplacian for some special classes of simplicial complexes in Section 4. In particular, we discuss the spectrum of an i-simplex, of an orientable and a non-orientable circuit, of a path and of a star. In Section 5 we discuss the effect of wedges, joins and duplication of motifs on the spectrum of the normalized combinatorial Laplace operator. In Section 6 we identify the combinatorial features of simplicial complexes which cause the appearance of certain integer eigenvalues in the spectrum. We discuss the occurrence of eigenvalue $i+2$ in the spectrum of Δ_i^{up} and its connection to colorability of the underlying graph of a complex. The relation among eigenvalue $i+1$ and the duplication of vertices is established.

2. Notation, definitions and the combinatorial Laplace operator

An *abstract simplicial complex* K on a finite set V is a collection of subsets of V, which is closed under inclusion. An i-face or i-simplex of K is its element of cardinality $i+1$. 0-faces are usually called *vertices* and 1-faces *edges*. The collection of all i-faces of simplicial complex K is denoted by $S_i(K)$. The *dimension* of an i-face is i and the dimension of a complex K is the maximum dimension of a face in K. The faces which are maximal under inclusion are called *facets*. We say that a simplicial complex K is *pure* if all facets have the same dimension. For two $(i+1)$- simplices sharing an i-face we use the term i*-down neighbours* and for two i-simplices which are faces of an $(i+1)$- simplex, we say that they are $(i+1)$*-up neighbours*. When there is no danger of ambiguity, we will drop the terms *up* and *down*. We say that a face F is *oriented* if we chose an ordering on its vertices and write $[F]$. Two orderings of the vertices are said to determine *the same orientation* if there is an even permutation transforming one ordering into the other. If the permutation is odd, then the orientations are opposite.

In the remainder, K will be an abstract simplicial complex on a vertex set $[n]=\{1,2,\dots,n\}$, when not stated otherwise. The i-th chain group $C_i(K,\mathbb{R})$ of a complex K with coefficients in $\mathbb{R}$ is a vector space over field $\mathbb{R}$ with the basis $B_i(K,\mathbb{R})=\{[F]\mid F\in S_i(K)\}$. The *augmented cochain complex* of K with coefficients in $\mathbb{R}$ is a sequence of vector spaces and linear transformations

$$C^d(K,\mathbb{R})\leftarrow\dots\xleftarrow{\delta_{i+1}}C^{i+1}(K,\mathbb{R})\xleftarrow{\delta_i}C^i(K,\mathbb{R})\xleftarrow{\delta_{i-1}}\dots\leftarrow C^{-1}(K,\mathbb{R})\leftarrow 0. \tag{2.1}$$

The cochain groups[1] $C^i(K,\mathbb{R})$ are defined as duals of the chain groups, i.e. $C^i(K,\mathbb{R}):=\ \hom(K,\mathbb{R})$. The basis of $C^i(K,\mathbb{R})$ is given by the set of functions $\{e_{[F]}\mid [F]\in B_i(K,\mathbb{R}))\}$ such that

$$e_{[F]}([F'])=\begin{cases}1 & \text{if } [F']=[F]\\ 0 & \text{otherwise}\end{cases}.$$

The functions $e_{[F]}$ are also known as *elementary cochains*. Note that a one-dimensional vector space $C^{-1}(K,\mathbb{R})$ is generated by the function which is identity on the empty simplex. For the systematic treatment of the simplicial homology and cohomology the reader is referred to [18]. The connecting maps δ_i in the cochain complex (2.1) given by

$$(\delta_i f)([v_0,\dots,v_{i+1}])=\sum_{j=0}^{i+1}(-1)^j f([v_0,\dots,\hat{v}_j\dots v_{i+1}]),$$

where $\hat{v}_j$ denotes that the vertex v_j has been omitted, are called *simplicial coboundary maps*. Alternatively, δ_i can be viewed as the dual of the boundary map ∂_{i+1}, for details see [18]. It is trivial to check that $\delta_i\delta_{i-1}=0$, ergo the image of δ_{i-1} is

[1] Traditionally, $C^i(K,G)$ for arbitrary group G, are called cochain groups. Influenced by this naming, we will sometimes reffer to $C^i(K,\mathbb{R})$ as cochain groups, although we always keep in mind that $C^i(K,\mathbb{R})$ have the structure of a vector space.

contained in the kernel of δ_i and we define the reduced cohomology group for every $i \geq 0$

$$\tilde{H}^i(K,\mathbb{R}) := \ker \delta_i \diagup \operatorname{im} \delta_{i-1}.$$

After choosing inner products $(\ ,\ )_{C^i}$ and $(\ ,\ )_{C^{i+1}}$ on $C^i(K,\mathbb{R})$ and $C^{i+1}(K,\mathbb{R})$, respectively, we define the adjoint δ_i^* of the coboundary operator δ_i as a map

$$\delta_i^* : C^{i+1}(K,\mathbb{R}) \to\ C^i(K,\mathbb{R}),$$

which satisfies the following equality

$$(\delta_i f_1, f_2)_{C^{i+1}} = (f_1, \delta_i^* f_2)_{C^i},$$

for every $f_1 \in C^i(K,\mathbb{R})$ and $f_2 \in C^{i+1}(K,\mathbb{R})$.

Definition 2.1. We define the following three operators on $C^i(K,\mathbb{R})$:

(i) *i-dimensional combinatorial up Laplace operator* or simply i-up Laplace operator

$$\mathcal{L}_i^{up}(K) := \delta_i^* \delta_i,$$

(ii) *i-dimensional combinatorial down Laplace operator* or i-down Laplace operator

$$\mathcal{L}_i^{down}(K) := \delta_{i-1} \delta_{i-1}^*,$$

(iii) *i-dimensional combinatorial Laplace operator* or i-Laplace operator

$$\mathcal{L}_i(K) := \mathcal{L}_i^{up}(K) + \mathcal{L}_i^{down}(K) = \delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*.$$

Since

$$C^{i+1}(K,\mathbb{R}) \underset{\delta_i^*}{\overset{\delta_i}{\leftrightarrows}} C^i(K,\mathbb{R}) \underset{\delta_{i-1}^*}{\overset{\delta_{i-1}}{\leftrightarrows}} C^{i-1}(K,\mathbb{R}),$$

all three operators are well defined. Moreover, directly from the definition follows that $\mathcal{L}_i^{up}(K)$, $\mathcal{L}_i^{down}(K)$ and $\mathcal{L}_i(K)$ are self-adjoint and non-negative. Hence their eigenvalues are real and non-negative.

For any operator A acting on a Hilbert space, we denote the weakly increasing rearrangement of its eigenvalues by $\mathbf{s}(A) = (\lambda_0, \ldots, \lambda_m)$ and write $\mathbf{s}(A) \overset{\circ}{=} \mathbf{s}(B)$, when multisets $\mathbf{s}(A)$ and $\mathbf{s}(B)$ differ only in their multiplicities of zero. We denote the union of multisets by $\overset{\circ}{\cup}$.

Remark 2.1. $\overset{\circ}{=}$ is an equivalence relation.

The combinatorial Laplace operator first appeared in a study of the discrete version of Hodge theorem [11]. Here, we formulate the theorem and give its proof for the sake of completeness.

Theorem 2.1 (Eckmann 1944)**.** *For a given abstract simplicial complex K, the following holds*

$$\ker \mathcal{L}_i(K) \cong \tilde{H}^i(K,\mathbb{R})$$

regardless of the choice of inner products on cochain vector spaces.

Proof. Since $\delta_i \delta_{i-1} = 0$ and $\delta_{i-1}^* \delta_i^* = 0$[2] , then

$$\operatorname{im} \mathcal{L}_i^{down}(K) \subset \ker \mathcal{L}_i^{up}(K), \tag{2.2}$$

$$\operatorname{im} \mathcal{L}_i^{up}(K) \subset \ker \mathcal{L}_i^{down}(K). \tag{2.3}$$

[2]This is due to $(\delta_{i-1}^* \delta_i^* u, v) = (u, \delta_i \delta_{i-1} v) = 0$ and holds for any choice of scalar products.

Thus,

$$\begin{aligned}\ker \mathcal{L}_i(K) &= \ker \delta_i^* \delta_i \cap \ker \delta_{i-1}\delta_{i-1}^* \\ &= \ker \delta_i \cap \ker \delta_{i-1}^* \\ &= \ker \delta_i \cap (\operatorname{im} \delta_{i-1})^{\perp} \\ &\cong \tilde{H}^i(K, \mathbb{R}).\end{aligned}$$

□

Due to (2.2) and (2.3) λ is a non-zero eigenvalue of $\mathcal{L}_i(K)$ if and only if it is an eigenvalue of $\mathcal{L}_i^{up}(K)$ or $\mathcal{L}_i^{down}(K)$. Therefore,

$$\mathbf{s}(\mathcal{L}_i(K)) \stackrel{\circ}{=} \mathbf{s}(\mathcal{L}_i^{up}(K)) \stackrel{\circ}{\cup} \mathbf{s}(\mathcal{L}_i^{down}(K)). \tag{2.4}$$

Furthermore, as a direct consequence of the fact that $\mathbf{s}(AB) \stackrel{\circ}{=} \mathbf{s}(BA)$, for operators A and B on suitably chosen Hilbert spaces, we get the following equality

$$\mathbf{s}(\mathcal{L}_i^{up}(K)) \stackrel{\circ}{=} \mathbf{s}(\mathcal{L}_{i+1}^{down}(K)). \tag{2.5}$$

Based on (2.4) and (2.5) we conclude that each of the three families of multisets

$$\{\mathbf{s}(\mathcal{L}_i(K)) \mid -1 \le i \le d\},\ \{\mathbf{s}(\mathcal{L}_i^{up}(K)) \mid -1 \le i \le d\} \text{ or } \{\mathbf{s}(\mathcal{L}_i^{down}(K)) \mid 0 \le i \le d\}$$

determines the other two. Therefore, it suffices to observe only one. In the remainder of the paper, we will omit argument K in $\mathbf{s}(\mathcal{L}_i(K))$, $\mathbf{s}(\mathcal{L}_i^{up}(K))$, $\mathcal{L}_i^{up}(K)$, $\mathcal{L}_i^{down}(K)$, $S_i(K)$ etc when it is clear which simplicial complex we investigate or when we state our results for a general simplicial complex K.

In order to write down explicit expressions for up and down Laplacians, it is necessary to fix scalar products on the cochain groups. To that end, we introduce the weight function and additional notation.

Definition 2.2. The *weight function* w is an evaluation function on the set of all faces of K

$$w : \bigcup_{i=-1}^{\dim K} S_i(K) \to \mathbb{R}^+.$$

The *weight of a face* F is $w(F)$.

For any choice of the inner product on space $C^i(K, \mathbb{R})$, there exist a weight function w, such that

$$(f, g)_{C^i} = \sum_{F \in S_i(K)} w(F) f([F]) g([F]).$$

Furthermore, there is a one-to-one correspondence between weight functions and possible scalar products on cochain groups $C^i(K, \mathbb{R})$. In the remainder we will interchangeably use the terms weights, weight function and scalar product.

Definition 2.3. Let $\bar{F} = \{v_0, \dots, v_{i+1}\}$ be an $(i+1)$-face of a complex K and let F be an i face of $\bar{F}$. Let $\mid p \mid$ denote the parity of a permutation, which transforms the ordering of the vertices of $[\bar{F}]$ to the $[v_0, \dots, v_{i+1}]$, where $v_0 < \ldots < v_{i+2}$ in the usual ordering of $[n]$. Then *the boundary of the oriented face* $[\bar{F}]$ is

$$\partial[\bar{F}] = (-1)^{|p|} \sum_j (-1)^j [v_0, \dots, \hat{v_k}, \dots, v_{i+1}],$$

and the sign of $[F]$ in the boundary of $[\bar{F}]$ is denoted by $\operatorname{sgn}([F], \partial[\bar{F}])$.

By abuse of notation, we write $\partial\bar{F}$ to denote the set of all i-faces of $\bar{F}$ in the remainder. Finally, the explicit expression for the i-up Laplace operator is given by

$$(\mathcal{L}_i^{up} f)([F]) = \sum_{\substack{\bar{F}\in S_{i+1}:\\ F\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)} f([F]) + \sum_{\substack{F'\in S_i: F\neq F',\\ F,F'\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)} \operatorname{sgn}([F],\partial[\bar{F}]) \operatorname{sgn}([F'],\partial[\bar{F}]) f([F']),$$

and the expression for the i-down Laplace operator is given by

$$(\mathcal{L}_i^{down} f)([F]) = \sum_{E\in\partial F} \frac{w(F)}{w(E)} f([F]) + \sum_{F': F\cap F' = E} \frac{w(F')}{w(E)} \operatorname{sgn}([E],\partial[F]) \operatorname{sgn}([E],\partial[F']) f([F']).$$

When dealing with linear operators it is often more convenient to study their matrix form. Hence we give the following expressions for the $(e_{[F]}, e_{[F']})$-th and the $(e_{[F]}, e_{[F]})$-th entry of $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$, where $F \neq F'$

$$\begin{aligned}
(\mathcal{L}_i^{up})_{(e_{[F]}, e_{[F']})} &= \operatorname{sgn}([F],\partial[\bar{F}]) \operatorname{sgn}([F'],\partial[\bar{F}]) \frac{w(\bar{F})}{w(F)}, \\
(\mathcal{L}_i^{up})_{(e_{[F]}, e_{[F]})} &= \sum_{\substack{\bar{F}\in S_{i+1},\\ F\in\partial\bar{F}}} \frac{w(\bar{F})}{w(F)}, \\
(\mathcal{L}_i^{down})_{(e_{[F]}, e_{[F']})} &= \operatorname{sgn}([E],\partial[F]) \operatorname{sgn}([E],\partial[F']) \frac{w(F')}{w(E)}, \\
(\mathcal{L}_i^{down})_{(e_{[F]}, e_{[F]})} &= \sum_{E\in\partial F} \frac{w(F)}{w(E)}.
\end{aligned}$$

Let D_i be the matrix corresponding to operator δ_i, D_i^T its transpose and W_i matrix representing scalar product on C^i, then the $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$ operators are expressed as

$$\mathcal{L}_i^{up} = W_i^{-1} D_i^T W_{i+1} D_i$$

and

$$\mathcal{L}_i^{down} = D_{i-1} W_{i-1}^{-1} D_{i-1}^T W_i,$$

respectively. Now it becomes clear that the combinatorial Laplace operator analysed by Duval, Reiner [10], Friedmann [13] and others [25],[8],etc is combinatorial Laplace operator $\mathcal{L}_i$, where the weight matrices W_i $(-1 \leq i \leq \dim K)$ are chosen to be the identity matrices. In the remainder of the paper, this version of the Laplace operator will be denoted by L. The graph Laplacian (1.1) studied by Kirchhoff [21], Fiedler [12], Grone and Merris [17] and many others is a special case of L, in fact it is equal to L_0^{up}. The *normalized graph Laplace operator* (1.2) investigated by Jost [2], Chung[6] is equal to $\mathcal{L}_0^{up}$, when W_1 is chosen to be the identity matrix and W_0 diagonal degree matrix. Therefore, combinatorial Laplacian $\mathcal{L}$, as defined here, unifies all Laplace operators studied so far and gives the general framework for a systematic study of different versions of Laplacians.

Our goal in this paper is to define higher dimensional analogue of normalized graph Laplacian and investigate its properties. However, we will (whenever possible) state our results in the full generality and emphasize which results do not depend on the choice of the scalar products and which are the consequence of suitably chosen weights.

3. The normalized combinatorial Laplacian: definition and its basic properties

In this section we derive an upper bound on the maximum eigenvalue of $\mathcal{L}_i^{up}$, introduce the normalized combinatorial Laplacian Δ_i^{up}, and state and prove its basic properties. We emphasize its advantages compared to the other choices of weights.

Let λ_m and λ_0 be the maximum and the minimum eigenvalue of $\mathcal{L}_i^{up}(K)$, respectively. Due to the positive definiteness of the Laplace operator, λ_0 is always larger or equal to zero. The exact number of zero eigenvalues in the spectrum of $\mathcal{L}_i^{up}$ and $\mathcal{L}_i^{down}$ is given in the following theorem.

Theorem 3.1. *The multiplicity of the eigenvalue zero in*

(i) $\mathbf{s}(\mathcal{L}_i^{up})$ *is*

$$\dim C^i - \sum_{j=-1}^{i} (-1)^{i+j}(\dim C^j - \dim \tilde{H}^j),$$

or equivalently

$$\dim C^i + \sum_{j=1}^{d-i}(-1)^j(\dim C^{i+j} - \dim \tilde{H}^{i+j}).$$

(ii) $\mathbf{s}(\mathcal{L}_i^{down})$ *is*

$$\dim \tilde{H}^i - \sum_{j=0}^{i-1}(-1)^{i+j-1}(\dim C^j - \dim \tilde{H}^j).$$

Proof. The following are short exact sequences that split

$$0 \to \ker \delta_i \to C^i \to \operatorname{im} \delta_i \to 0,$$
$$0 \to \operatorname{im} \delta_{i-1} \to \ker \delta_i \to \tilde{H}^i \to 0.$$

This is a direct consequence of the fact that $\operatorname{im} \delta_i$ and $\tilde{H}^i$ are projective modules[3]. Therefore,

$$\dim C^i = \dim \ker \delta_i + \dim \operatorname{im} \delta_i \tag{3.1}$$

and

$$\dim \ker \delta_i = \dim \tilde{H}^i + \dim \operatorname{im} \delta_{i-1}. \tag{3.2}$$

From (3.1) and (3.2) follows

$$\dim \operatorname{im} \delta_i = \sum_{j=0}^{i}(-1)^{i+j}(\dim C^j - \dim \tilde{H}^j).$$

The number of zeros in the spectrum of $\mathcal{L}_i^{up}$ is equal to the dimension of its kernel, thus

$$\begin{aligned}\dim \ker \mathcal{L}_i^{up} &= \dim \ker \delta_i \\ &= \dim C^i - \sum_{j=0}^{i}(-1)^{j+i}(\dim C^j - \dim \tilde{H}^j).\end{aligned}$$

[3] For details on projective modules and splitting exact sequences the reader is referred to [7].

The expression (3.1) for the number of zeros in $\mathbf{s}(\mathcal{L}_i^{up})$ is easily obtainable by using Euler characteristic and equality $\chi = \sum_{j=-1}^{d}(-1)^i \dim C^i = \sum_{j=-1}^{d}(-1)^i \dim \tilde{H}^i$. As for the $\mathcal{L}_i^{down}$, the following holds

$$\begin{aligned}\dim\ker\mathcal{L}_i^{down} &= \dim\ker\delta_{i-1}^* = \dim C^i - \dim\ker\delta_{i-1}\\ &= \dim\tilde{H}^i - \sum_{j=0}^{i-1}(-1)^{j+i-1}(\dim C^j - \dim\tilde{H}^j).\end{aligned}$$

□

The number of zero eigenvalues in spectra of various Laplace operators, as expected, does not depend on a choice of the scalar products on the cochain vector spaces.

The upper bound on $s(\mathcal{L}_i^{up})$ follows from the subsequent discussion.

$$(\mathcal{L}_i^{up} f, f) = (\delta_i f, \delta_i f) \tag{3.3a}$$

$$= \Big(\sum_{\bar{F}\in S_{i+1}(K)} f(\partial[\bar{F}])\, \mathrm{e}_{[\bar{F}]}, \sum_{\bar{F}\in S_{i+1}(K)} f(\partial[\bar{F}])\, \mathrm{e}_{[\bar{F}]}\Big) \tag{3.3b}$$

$$= \sum_{\bar{F}\in S_{i+1}(K)} f(\partial[\bar{F}])^2 w(\bar{F}) \tag{3.3c}$$

$$\leq (i+2)\sum_{F\in S_i(K)} f([F])^2 \sum_{\bar{F}\in S_{i+1}(K): F\in\partial\bar{F}} w(\bar{F}), \tag{3.3d}$$

where (3.3d) is obtained by using the Cauchy-Schwartz inequality. Next we introduce the degree of a simplex F.

Definition 3.1. *The degree of an i-face F* of K is equal to the sum of weights of all simplices which contain F in its boundary, i.e.

$$\deg F = \sum_{\bar{F}\in S_{i+1}(K): F\in\partial\bar{F}} w(\bar{F}).$$

The inequality (3.3d) can be restated in terms of degrees as

$$(\mathcal{L}_i^{up} f, f) \leq (i+2)\sum_{F\in S_i(K)} f([F])^2 \deg F. \tag{3.4}$$

By dividing (3.4) by(f, f) we get

$$\frac{(\mathcal{L}_i^{up} f, f)}{(f, f)} \leq (i+2)\frac{\sum_{F\in S_i(K)} f([F])^2 \deg F}{\sum_{F\in S_i(K)} f([F])^2 w(F)}. \tag{3.5}$$

Replacing f in (3.5) with the eigenfunction f_m, corresponding to the largest eigenvalue λ_m of $\mathcal{L}_i^{up}$ gives

$$\lambda_m \leq (i+2)\frac{\sum_{F\in S_i(K)} f_m([F])^2 \deg F}{\sum_{F\in S_i(K)} f_m([F])^2 w(F)}. \tag{3.6}$$

Therefore, if

$$w(F) = \deg F \tag{3.7}$$

holds, then $\lambda_m \leq i+2$ and the eigenvalues of $\mathcal{L}_i^{up}$ are in the interval $[0, i+2]$.

Definition 3.2. Let w be a weight function on K which satisfies (3.7), then the Laplace operator defined on the cochain complex of K is called the *weighted normalized combinatorial Laplace operator*. If additionally, the weights of facets of K are equal to 1, then the obtained operator is called the *normalized combinatorial Laplace operator* and is denoted by Δ_i^{up}.

However, if (3.7) does not hold, we derive a bound on the maximum eigenvalue of the Laplacian $\mathcal{L}_i^{up}$ from the inequality (3.6), i.e.

$$\lambda_m \leq (i+2) \frac{\max_{F \in S_i(K)} \deg F}{\min_{F \in S_i(K)} w(F)}. \tag{3.8}$$

Here $\min_{F \in S_i(K)} w(F)$ stands for the minimum *non-zero* weight over all i-faces F of K. The inequality (3.8) in case of the combinatorial Laplacian L_i^{up} reduces to

$$\lambda_m \leq (i+2) \max_{F \in S_i(K)} \deg F, \tag{3.9}$$

which for $i = 0$ becomes exactly

$$\lambda_m \leq 2 \max_{v \in S_0(G)} \deg v.$$

This is the well-known bound on the maximum eigenvalue of L_0^{up} (see [1]). Another upper bound of the spectrum of L_i^{up} was obtained by Duval and Reiner in [10] as a part of more general study, i.e.

$$\lambda_m \leq n, \tag{3.10}$$

where n is the number of vertices of the complex K. The inequality (3.9) is sharper than (3.10) for large values of n and small values of i. In particular, if $\max_{F \in S_i} \deg F < \frac{n}{i+2}$, then the estimate (3.9) is sharper, otherwise it is (3.10). We sum up our results in the following theorem.

Theorem 3.2. *The spectrum of $\mathcal{L}_i^{up}$ is bounded from above by:*

(i) $i+2$, *if* $\mathcal{L}_i^{up} = \Delta_i^{up}$,
(ii) $(i+2)\max_{F \in S_i(K)} \deg F$, *if* $\mathcal{L}_i^{up} = L_i^{up}$,
(iii) $(i+2)\frac{\max_{F \in S_i(K)} \deg F}{\min_{F \in S_i(K)} w(F)}$, *for all other choices of scalar products.*

Remark 3.1 (Negative Weights). If negative weights are allowed in the definition of the weight function , then instead of positive definite Hermitian sesquilinear forms (inner products) on cochain vector spaces we observe Hermitian sesquilinear forms. Therefore $\mathcal{L}_i^{up}$ acts on functions on i-simplices

$$\Delta_i^{up} f([F]) = \frac{1}{w(F)} \sum_{\substack{\bar{F} \in S_{i+1} \\ F \in \partial \bar{F}}} \operatorname{sgn}([F], \partial[\bar{F}]) f(\partial[\bar{F}]),$$

where the weights are chosen arbitrary. This approach enables us to use negative weights, but it also deprives us of the possibility to benefit from the structure coming from cohomology of simplicial complex. Furthermore, the eigenvalues of newly obtained operator will be neither real nor non-negative. The Laplacian with neagtive weights will not be a subject of a further investigation.

4. Circuits, paths, stars and their spectrum

In this section we calculate the spectrum of the up (down) normalized Laplace operator for some classes of simplicial complexes.

Theorem 4.1. *Let K be a $(n-1)$-dimensional simplex. Then $\mathbf{s}(\Delta_i^{up}(K))$ consists of eigenvalue $\frac{n}{n-i-1}$ with multiplicity $\binom{n-1}{i+1}$ and eigenvalue zero with multiplicity $\binom{n-1}{i}$.*

Proof. We will prove that a function $f \in C^i(K, \mathbb{R})$, given by

$$f_{[\bar{F}]}([F]) = \begin{cases} \operatorname{sgn}([F], \partial[\bar{F}]) & \text{if } F \text{ is facet of } (i+1)\text{-face } \bar{F} \\ 0 & \text{otherwise} \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K)$ corresponding to the eigenvalue $\frac{n}{n-i-1}$.
It is not difficult to see that there are exactly $\binom{n-1}{i+1}$ linearly independent functions of this form. Next we check if the equality

$$(\Delta_i^{up} f_{[\bar{F}]})[F] = \frac{n}{n-i-1} f([F])$$

holds for every i-dimensional face F of K. Here, we distinguish three cases:
(i) F is an arbitrary facet of $\bar{F}$. Therefore,

$$\begin{aligned}
(\Delta_i^{up} f_{[\bar{F}]})([F]) = & \sum_{\substack{\bar{E} \in S_{i+1}: \\ F \in \partial \bar{E}}} \frac{w(\bar{E})}{w(F)} f_{[\bar{F}]}([F]) \\
& + \sum_{\substack{F' \in S_i(L): \\ (\exists \bar{E} \in S_{i+1}(L)) F, F' \in \partial \bar{E}}} \frac{w(\bar{E})}{w(F)} \operatorname{sgn}([F], \partial[\bar{E}]) \operatorname{sgn}([F'], \partial[\bar{E}]) f_{[\bar{F}]}([F']) \\
= & \frac{1}{n-i-1} \sum_{\substack{\bar{E} \in S_{i+1}: \\ F \in \partial \bar{E}}} f_{[\bar{F}]}(F) \\
& + \frac{1}{n-i-1} \sum_{\substack{F' \in S_i(L): \\ (\exists \bar{E} \in S_{i+1}(L)) F, F' \in \partial \bar{E}}} \operatorname{sgn}([F], \partial[\bar{E}]) \operatorname{sgn}([F'], \partial[\bar{E}]) f_{[\bar{F}]}([F']) \\
= & f_{[\bar{F}]}([F]) + \frac{i+1}{n-i-1} \operatorname{sgn}([F], \partial[\bar{F}]) \\
= & \frac{n}{n-i-1} f([F]).
\end{aligned}$$

(ii) $\dim(F \cap \bar{F}) = i$, i.e. F and $\bar{F}$ have i vertices in common.
Then by the definition $f([F]) = 0$. Let $v_0, v_1, \dots, v_{i+2} \in [n]$ be arbitrary vertices of L ordered increasingly. Without a loss of generality assume $0 \leq j < k < l \leq i+2$ and $\bar{F} = [v_0, \dots, \hat{v_l}, \dots, v_{i+2}]$ and $[F] = [v_0, \dots, \hat{v_j}, \dots, \hat{v_k}, \dots, v_{i+2}]$. Then there exist exactly two i-faces F_1 and F_2 in the boundary of $\bar{F}$ and two $(i+1)$-simplices $\bar{F}_1$ and $\bar{F}_2$ of L, such that $F, F_1 \in \partial \bar{F}_1$ and $F, F_2 \in \partial \bar{F}_2$. In particular, $F_1 = [v_0, \dots, \hat{v_k}, \dots, \hat{v_l}, \dots, v_{i+2}]$, $F_2 = [v_0, \dots, \hat{v_j}, \dots, \hat{v_l}, \dots, v_{i+2}]$ and $\bar{F}_1 = [v_0, \dots, \hat{v_k}, \dots, v_{i+2}]$, $\bar{F}_2 = [v_0, \dots, \hat{v_j}, \dots, v_{i+2}]$. Now it is straightforward to

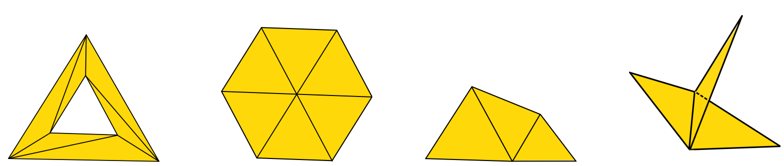
calculate

$$\begin{aligned}(\Delta_i^{up} f_{[\bar{F}]})([F]) &= 0 + \operatorname{sgn}([F], \partial[\bar{F}_1]) \operatorname{sgn}([F_1], \partial[\bar{F}_1]) f_{[\bar{F}]})([F_1]) \\ &\quad + \operatorname{sgn}([F], \partial[\bar{F}_2]) \operatorname{sgn}([F_2], \partial[\bar{F}_2]) f_{[\bar{F}]})([F_2]) \\ &= \operatorname{sgn}([F], \partial[\bar{F}_1]) \operatorname{sgn}([F_1], \partial[\bar{F}_1]) \operatorname{sgn}([F_1], \partial[\bar{F}]) \\ &\quad + \operatorname{sgn}([F], \partial[\bar{F}_2]) \operatorname{sgn}([F_2], \partial[\bar{F}_2]) \operatorname{sgn}([F_2], \partial[\bar{F}]) \\ &= (-1)^j(-1)^{l-1}(-1)^k + (-1)^{k-1}(-1)^{l-1}(-1)^j \\ &= 0.\end{aligned}$$

(iii) $\dim(F \cap \bar{F}) < i$, i.e. F and $\bar{F}$ have less than i vertices in common.
Then there are no faces in the boundary of $\bar{F}$ which are $(i+1)$-up neighbours of F. This implies that $\Delta_i^{up} f([F]) = 0$, which completes the proof. □

In the remainder of this section, we calculate the spectrum of circuits, paths and stars.

Definition 4.1. A pure simplicial complex L of dimension i, is called an *i-path of length m* iff there is an ordering of its i-simplices $F_1 < F_2 < \ldots < F_m$, such that $\dim(F_j \cap F_l) < i - 1$ for $\mid j - l \mid > 1$ and $\dim(F_j \cap F_l) = i - 1$ for $\mid j - l \mid = 1$ for every $1 \leq j, l \leq m$.
When F_m coincides with F_1, we say that L is an *i-circuit* of length $(m-1)$. The vertices in the intersection $\bigcap_{j=1}^{m-1} F_j$ are called *centers* of L.



(a) 2-circuit of length 6 with an empty center

(b) 2-circuit of length 6 with one vertex in a center

(c) 2-path of length 3

(d) 2-star of length 3

Figure 1. Examples of circuits, paths and stars

Note that simplicial complexes in Figures 1(b) and 1(c) have one central vertex, i.e. a center. Before we proceed to calculate $\mathbf{s}(\Delta_i^{up})$ of the above defined complexes, we recall the definition of orientability.

Definition 4.2. Let K be a pure $(i+1)$-dimensional simplicial complex. We say that K is *orientable* iff it is possible to assign an orientation to all $(i+1)$-faces of K in such a way that any two simplices which intersect by an i-face induce a different orientation on that face. We say that such simplices are oriented *coherently.*

Therefore, choosing an orientation on $(i+1)$-faces of orientable simplicial complex K is equivalent to choosing a basis $B_{i+1}(K)$ of the vector space $C_{i+1}(K, \mathbb{R})$ consisting of elementary $(i+1)$- chains $[\bar{F}]$ which are oriented *coherently.* The following theorem from [14] will be extensively used in the subsequent calculations.

Theorem 4.2 (Greenman, 1977)**.** *If two matrices M and P commute, i.e. $MP = PM$, and if λ is a simple eigenvalue of matrix P, then its corresponding eigenvector v is also an eigenvector of M.*

We state the proof for the sake of completeness.

Proof. Since $P(Mv) = MPv = \lambda Mv$, Mv is an eigenvector of P corresponding to eigenvalue λ. Since λ is a simple eigenvalue, then $Mv = v$. □

Let $\tilde{p}$ be a permutation on the elements of a basis $B_i(K)$ of $C_i(K,\mathbb{R})$, for an arbitrary simplicial complex K, and let $\bar{p}$ be a permutation on elementary cochains of dimension i induced by $\tilde{p}$. Denote the linear extension of $\bar{p}$ on $C^i(K,\mathbb{R})$ by p. Then we have the following equivalences

$$\tilde{p}([F]) = [F] \Leftrightarrow \bar{p}(e_{[F]}) = e_{[F]} \Leftrightarrow p(e_{[F]}) = e_{[F]}.$$

To simplify the notation, we will designate any of the maps $\tilde{p}$, $\bar{p}$, p by p. It will be clear from the argument of p, which one is used. Furthermore, we will write $p(F)$ to denote the i-face which is uniquely determined by the mapping $p([F])$. To prove that p and Δ_i^{down} commute, it is necessary to check if $p\Delta_i^{down}e_{[F]} = \Delta_i^{down}pe_{[F]}$ holds for every i-face F. Since

$$\begin{aligned} p\Delta_i^{down}e_{[F]} = &\sum_{E\in\partial F}\frac{w(F)}{w(E)}p(e_{[F]}) \\ &+ \sum_{\substack{F'\in S_i(K): \\ (\exists E\in S_{i-1}(K))F\cap F'=E}} \frac{w(F)}{w(E)}\operatorname{sgn}([E],\partial[F])\operatorname{sgn}([E],\partial[F'])p(e_{[F']}), \end{aligned}$$

and

$$\begin{aligned} \Delta_i^{down}pe_{[F]} = &\sum_{p(E)\in\partial p(F)}\frac{w(p(F))}{w(p(E))}e_{p([E])} \\ &+ \sum_{\substack{p(F')\in S_i(K): \\ (\exists p(E)\in S_{i-1}(K)) \\ p(F)\cap p(F')=p(E)}} \frac{w(p(F))}{w(p(E))}\operatorname{sgn}(p([E]),\partial p([F]))\operatorname{sgn}(p([E]),\partial p([F']))e_{p([F'])}, \end{aligned}$$

it suffices to show

$$\sum_{E\in\partial F}\frac{w(F)}{w(E)} = \sum_{p(E)\in\partial p(F)}\frac{w(p(F))}{w(p(E))} \tag{4.1}$$

and

$$\frac{w(p(F))}{w(p(E))}\operatorname{sgn}(p([E]),\partial p([F]))\operatorname{sgn}(p([E]),\partial p([F'])) = \frac{w(F)}{w(E)}\operatorname{sgn}([E],\partial[F])\operatorname{sgn}([E],\partial[F']) \tag{4.2}$$

for every F and F' which are $(i-1)$-down neighbours in K and every elementary i-cochain $e_{[F]}$. Now we are ready to prove the following theorem.

Theorem 4.3. *Let K be an orientable i-circuit of length m. Then the eigenvalues of $\Delta_i^{down}(K)$ are $i - \cos(\frac{2\pi j}{m})$, $j = 0, 1, \ldots m-1$.*

Proof. Let $F_1 < F_2 < \ldots < F_m$ be the ordering of the i-simplices of K satisfying the conditions of Definition 4.1. Moreover, let $[F_1], [F_2], \ldots, [F_m]$ be a choice of coherent orientation on them. The map $p : C^i(K,\mathbb{R}) \to C^i(K,\mathbb{R})$ is given by $p([F_k]) = [F_{k+1}]$, for $1 \le k < m$ and $p([F_m]) = [F_1]$. It is not difficult to check that

$$p\Delta_i^{down} = \Delta_i^{down}p \tag{4.3}$$

In particular, equality (4.1) is satisfied since the weights of all i-faces are equal to 1 and $\frac{w(F)}{w(E)} = \frac{w(pF)}{w(pE)}$. Equality (4.2) holds because i-faces of K are coherently oriented, which gives the equalities

$$\operatorname{sgn}([E], \partial[F]) \operatorname{sgn}([E], \partial[F']) = -1$$

and

$$\operatorname{sgn}([pE], \partial[pF]) \operatorname{sgn}([pE], \partial[pF']) = -1,$$

where F and F' are $(i-1)$-down neighbours of K and E is their intersecting face. Hence (4.3) is true.

Denote P to be the matrix associated to mapping p. P is a permutation matrix and its characteristic polynomial is $\lambda^m - 1 = 0$. The eigenvectors of P are $U_\theta = (1, \theta, \theta^2, \dots \theta^{m-1})^T$, where θ is the m-th root of unity. Thus, the eigenfunctions of the map p are

$$u_\theta([F_k]) = \theta^{k-1}.$$

Following Theroem 4.2, we can now easily calculate the eigenvalues of Δ_i^{down}. Let $E_k := F_{k-1} \cap F_k$ for $2 \le k \le m-1$ and let $E_m := F_m \cap F_1$. We have

$$\begin{aligned}
\Delta_i^{down} u_\theta([F_k]) &= \sum_{\substack{E \in S_{i-1}(L): \\ E \in \partial F_k}} \frac{w(F_k)}{w(E)} \theta^{k-1} + \frac{w(F_k)}{w(E_k)} \operatorname{sgn}([E_k], \partial[F_k]) \operatorname{sgn}([E_k], \partial F_{k-1}) \theta^{k-2} \\
&\quad + \frac{w(F_k)}{w(E_{k+1})} \operatorname{sgn}([E_{k+1}], \partial[F_k]) \operatorname{sgn}([E_{k+1}], \partial[F_{k+1}]) \theta^k \\
&= (\frac{2}{2} + i - 1)\theta^{k-1} - \frac{1}{2}\theta^{k-2} - \frac{1}{2}\theta^k \\
&= \theta^{k-1}(i - \frac{\theta^{-1} + \theta}{2}) \\
&= \theta^{k-1}(i - \cos(\frac{2\pi j}{m})).
\end{aligned}$$

It is straightforward to check that a similar equality holds for $k = 1$ and $k = m$. Thus, $\lambda_j = i - \cos(\frac{2\pi j}{n})$, where $j = 0, 1, \dots m-1$ are the eigenvalues of $\Delta_i^{down}(K)$. □

Remark 4.1. The eigenvalues of an orientable i-circuit depend only on its length, thus there are different combinatorial structures which give the same eigenvalues of Δ_i^{down}. For example, $1, 1.5, 1.5, 2.5, 2.5, 3$ are the eigenvalues of Δ_2^{down} of the simplicial complex in Figure 1(b) and the simplicial complex in Figure 1(a).

Remark 4.2. A similar analysis can be done for a non-orientable i-circuit of length m. In that case we define p to be $p([F_k]) = [F_{k+1}]$, for $1 \le k < m$ and $p([F_m]) = -[F_1]$. The remaining calculations are done as in Theorem 4.3.

Theorem 4.4. *Let K be a non-orientable i-circuit of length m. Then the eigenvalues of $\Delta_i^{down}(K)$ are $i - \sin(\frac{2\pi j}{m})$ for m even and $i + \cos(\frac{2\pi j}{m})$ for m odd, where $j = 0, 1, \dots m-1$.*

Corollary 4.5. *Eigenvalues of $\Delta_i^{down}(K)$ of an i-path K of length m are $\lambda_k = i - \cos(\frac{\pi k}{m})$, for $k = 0, \dots, m-1$*

Proof. Since there are no self-intersections of dimension $(i-1)$ in an i-path, every path is orientable. From Theorem 4.3, we conclude that in the spectrum of the i-th down Laplacian of an i-circuit of length $2m$, all eigenvalues appear twice, except $(i-1)$ and $(i+1)$. In particular, $\lambda_k = i - \cos(\frac{2k\pi}{2m}) = i - \cos(\frac{2(2m-k)\pi}{2m}) = \lambda_{2m-k}$, for $k \neq 0$ and $k \neq m$. Let $\phi = \mathrm{e}^{i\frac{k2\pi}{2m}}$ [4], then the eigenvector corresponding to λ_k is $u_k = (1, e^{i\frac{k\pi}{m}}, \ldots, e^{i\frac{(2m-1)k\pi}{m}})^T$.

The function $v_k = u_k + u_{2m-k}$ is the eigenvector for the eigenvalue λ_k as well

$$v_k(m) = \mathrm{e}^{i\frac{\pi k}{m}} + \mathrm{e}^{i\frac{\pi(2m-k)}{m}} = \mathrm{e}^{i\frac{\pi k}{m}} + \mathrm{e}^{-i\frac{\pi k}{m}}.$$

It is now a straightforward calculation to see that the first m-entries of v_k, for every $k = 0, 1, \ldots m-1$ constitute the eigenvectors of K, whose corresponding eigenvalue is $i - \cos(\frac{\pi k}{m})$. □

This idea generalizes to the paths with self-intersections of dimension $(i-1)$, but then it is necessary to distinguish among orientable and non-orientable paths. The eigenvalues of a star are described in the following theorem.

Theorem 4.6. *Let K be a simplicial complex which consist of m i-dimensional simplices assembled in a star like formation, i.e. all simplices have in common one $(i-1)$-dimensional face. Then non-zero eigenvalues of $\Delta_i^{down}(K)$ are: i with multiplicity $(m-1)$ and $(i+1)$ with multiplicity* 1.

Proof. Let F_k, $k \in \{1, \ldots, m\}$, be an i-dimensional face of K and let $\bigcap_k F_k = E$. Denote $p : B_i(K, \mathbb{R}) \to B_i(K, \mathbb{R})$ to be a permutation, such that $p([F_k]) = [F_{k+1}]$. Since $F_k \cap F_j = E$, for any two i-faces of K, then we can fix the orientation on F_k's such that they induce the same orientation on E. Now it is easy to check that

$$p\Delta_i^{down} = \Delta_i^{down} p.$$

Let θ denote m-th root of unity different from 1 and u the eigenvector of p corresponding to it. Then we obtain

$$\begin{aligned}\Delta_i^{down} u_\theta([F_k]) &= \sum_{E, E\in\partial F_k} \frac{w(F_k)}{w(E)}\theta^{k-1} + \sum_{F, F\neq F_k} \frac{w(F)}{w(E)} u_\theta([F]) \\ &= i\theta^{k-1} + \frac{1}{m}(1 + \theta + \ldots + \theta^{m-1}) \\ &= i\theta^{k-1}.\end{aligned}$$

Thus, u_θ is an eigenfunction of $\Delta_i^{down}(K)$ corresponding to the eigenvalue i. The case when $\theta = 1$ results in the eigenvalue $k+1$. □

5. Constructions and their effect on the spectrum: wedges, joins and duplication of motifs

5.1. **Wedges.** Let $(X_i)_{i\in I}$ be a family of topological spaces and $x_i \in X_i$, then the wedge sum $\bigvee_i X_i$ is a quotient of their disjoint union by the identification $x_i \sim x_j$, for all $i, j \in I$, i.e.

$$\bigvee_i X_i := \bigsqcup_i X_i \;/\; \{x_i \sim x_j \mid i, j \in I\}.$$

For the purposes of this paper we define a combinatorial wedge sum, which is in many ways similar to the above defined wedge sum.

[4] i appearing in the exponent of $\mathrm{e}^{i\frac{k2\pi}{2m}}$ is the imaginary unit and has no relation to i which denotes the order of the Laplace operator.

Definition 5.1. Given two simplicial complexes K_1 and K_2 on vertex set $[n]$ and $[m]$, respectively, and two k-simplices $F_1 = \{v_0, \dots, v_k\}$ in $S_k(K_1)$ and $F_2 = \{u_0, \dots, u_k\}$ in $S_k(K_2)$, then the *combinatorial k-wedge sum* of K_1 and K_2 is an abstract simplicial complex on a vertex set $[m+n-k-1]$, such that

$$K_1 \vee_k K_2 := \{\{v_{i_0}, \dots, v_{i_k}\} \mid \{v_{i_0}, \dots, v_{i_k}\} \in K_1 \text{ or if } \{u_{i_0}, \dots, u_{i_k}\} \in K_2\},$$

where $u_{i_j} := u_l$ if $v_{i_j} = v_l$, $u_{i_j} := v_{i_j} + k + 1$ if $v_{i_j} > n$ and $u_{i_j} := v_{i_j}$ for the other values of v_{i_j}.

Remark 5.1. The combinatorial wedge sum $K_1 \vee_k K_2$ can also be viewed as

$$K_1 \sqcup K_2 \;/\; \{F_1 \sim F_2\},$$

where $\sim$ is an equivalence relation which identifies the faces F_1 and F_2.

Remark 5.2. It is not difficult to check that $K_1 \vee_k K_2$ is a simplicial complex, too.

Remark 5.3. Definition 5.1 can be generalized in the obvious way to the k-wedge sum of arbitrary many simplicial complexes.

Note that $K_1 \vee_k K_2$, for arbitrary k has the same homology as the wedge sum of K_1 and K_2 as defined for general topological spaces. From the homological point of view it is impossible to distinguish among k-wedge sums for different values of k as well as among different choices of the base points. However, combinatorially, the distinction among them is notable, e.g. two wedge sums in Figure 2. Consequently,

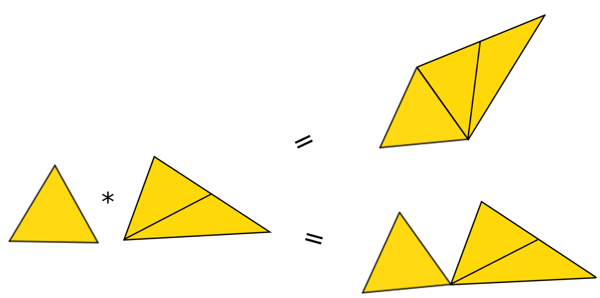


Figure 2. The homology groups of two spaces on the right are isomorphic, nonetheless these complexes are combinatorially different.

in a combinatorial k wedge sum of simplicial complexes, it is important which complexes are identified as well as the dimension of these complexes. The following theorem gives the first characterization of the effect of the wedge sum on the spectrum of the Laplacian.

Theorem 5.1.

$$\mathbf{s}(\Delta_i^{up}(K_1 \vee_k K_2)) \overset{\circ}{=} \mathbf{s}(\Delta_i^{up}(K_1)) \overset{\circ}{\cup} \mathbf{s}(\Delta_i^{up}(K_2))$$

for all ,i, k, such that $0 \leq k < i$.

Proof. Since we identify K_1 and K_2 by a face of dimension k, then obviously, $C^i(K_1 \vee_k K_2, \mathbb{R}) = C^i(K_1, \mathbb{R}) \oplus C^i(K_2, \mathbb{R})$ for every $i > k$. Thus, the coboundary mapping $\delta_i : C^i(K_1 \vee_k K_2, \mathbb{R}) \to C^{i+1}(K_1 \vee_k K_2, \mathbb{R})$ will map $C^i(K_j, \mathbb{R})$ to $C^{i+1}(K_j, \mathbb{R})$, $j = 1, 2$ and the same stands for the adjoint δ_i^*. □

The operator Δ_i^{up} is uniquely determined by i- and $(i+1)$-simplices of K. Hence its non-zero eigenvalues depend only on the structure of $(i+1)$-faces of K. By abuse of notation, let $S_{i+1}(K)$ determine a pure $(i+1)$-dimensional subcomplex of K, whose facet set is $S_{i+1}(K)$. If

$$S_{i+1}(K) = K_1 \vee_{k_1} K_2 \vee_{k_2} \ldots \vee_{k_{m-1}} K_m \tag{5.1}$$

for some $k_1, \ldots, k_{m-1} < i$, then

$$\mathbf{s}(\Delta_i^{up}(K)) \overset{\circ}{=} \mathbf{s}(\Delta_i^{up}(K_1)) \mathring{\cup} \ldots \mathring{\cup} \mathbf{s}(\Delta_i^{up}(K_m)).$$

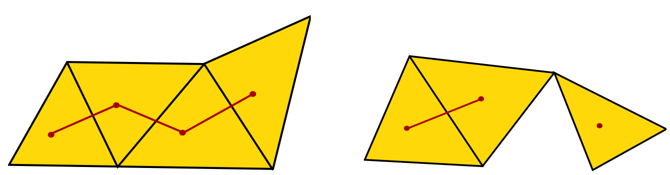
Therefore, when studying Δ_i^{up}, it is useful to determine if K can be represented as a combinatorial k-wedge sum of simplicial complexes and if so, how many of them are there. One possible way to answer this question is via observing $(i+1)$-dual graph of K.

Definition 5.2. Let K be a simplicial complex. Then a graph G_K with the vertex set $V = \{F_j \mid F_j \in S_i(K)\}$ and the edge set $E = \{(F_j, F_l) \mid F_j \cap F_l \in S_{i-1}(K)\}$ is called an *i-dual graph* of K.

It is not difficult to see that the number of complexes in the wedge sum (5.1) is exactly the number of connected components of $(i+1)$-dual graph of K. To explain this concept further, we will use the term $(i+1)$-path connected simplicial complex.

Definition 5.3. A simplicial complex K is i-path connected iff for arbitrary two i-faces F_1, F_2 of K there exists an i-path connecting them.

Remark 5.4. The definition of i-path connectedness differs from the standard definition of i-connected simplicial complexes that can be found in [22].



(a) 2-path connected simplicial complex and its 1-dual graph

(b) Simplicial complex which is not 2-connected and its 1-dual graph

Figure 3. Examples of i-path connected simplicial complexes and their dual graphs

Remark 5.5. If K is an $(i+1)$- path connected simplicial complex, it cannot be decomposed into a combinatorial k-wedge $(k < i)$ of simplicial complexes.

We assemble the observations above into the following proposition.

Proposition 5.2. *The following statements are equivalent.*

(i) $S_{i+1}(K) \cong K_1 \vee_{k_1} K_2 \vee_{k_2} \ldots \vee_{k_{m-1}} K_m$, *where* $k_1, \ldots, k_{m-1} < i$.
(ii) $(i+1)$*-dual graph* G_K *of* K *has* m *connected components.*
(iii) *the number of* $(i+1)$*-path connected components in simplicial complex* K *is equal to* m.

The analysis on the combinatorial wedge sum above does not depend on the choice of the scalar products. Hence Theorem 5.1 and Proposition 5.2 hold for the general Laplace operator $\mathcal{L}$ as well. In the remainder of this section we investigate the effect of the k-wedge sum for $i = k$ on the spectrum of the (weighted) normalized combinatorial Laplacian Δ_i^{up}.

Theorem 5.3. *Let* K_1 *and* K_2 *be simplicial complexes, such that eigenvalue* λ *is contained in the spectrum of* $\Delta_i^{up}(K_1)$ *and* $\Delta_i^{up}(K_2)$ *and let* f_1, f_2 *be their corresponding eigenfunctions. If an* i*-wedge* $K := (K_1 \vee_i K_2)$ *is obtained by identifying* i*-faces* F_1 *and* F_2, *for which* $f_1([F_1]) = f_2([F_2])$, *then the spectrum of* $\Delta_i^{up}(K)$ *contains eigenvalue* λ, *too.*

Proof. We will prove that

$$g([F]) = \begin{cases} f_1([F]) & \text{for every } F \text{ which is an } i\text{-face of } K_1 \text{ different from } F_1 \\ f_2([F]) & \text{for every } F \text{ which is an } i\text{-face of } K_2 \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K)$ corresponding to the eigenvalue λ. For an i-dimensional face F of K_1 different than F_1, the following equality holds

$$\Delta_i^{up}(K)\mid_{K_1 - F_1} f_1([F]) = \lambda f_1([F]).$$

Similar is true when $F \in S_i(K_2)$, $F \neq F_2$, i.e.

$$\Delta_i^{up}(K)\mid_{K_2 - F_2} f_2([F]) = \lambda f_2([F]).$$

Let w_{K_1} and w_{K_2} denote the weight functions on complexes K_1, K_2 respectively. Since we investigate Δ_i^{up}, then the weights on i-simplicies are uniquely determined by the weighs on $(i+1)$-simplices and the incidence relation among them. Thus, the weight of any i-simplex in K_1 or K_2 , different from F_1 and F_2, will remain the same in K. As for the weight of the simplex $F = F_1 = F_2$, it will be equal to the sum of the weights of F_1 and F_2 in K_1 and K_2, respectively, i.e. $w(F)_{K_1 * K_2} = w_{K_1}(F_1) + w_{K_1}(F_2)$. Hence

$$\begin{aligned}
\Delta_i^{up}(K) f([F]) = &\frac{1}{w_{K_1}(F_1) + w_{K_2}(F_2)} \sum_{\bar{F} \in S_{i+1}(K_1)} w_{K_1}(\bar{F}) \operatorname{sgn}([F], \partial[\bar{F}]) f(\partial[\bar{F}]) \\
&+ \frac{1}{w_{K_1}(F_1) + w_{K_2}(F_2)} \sum_{\bar{F} \in S_{i+1}(K_2)} w_{K_2}(\bar{F}) \operatorname{sgn}([F], \partial[\bar{F}]) f_2(\partial[\bar{F}]) \\
= &\frac{w_{K_1}(F_1)}{w_{K_1}(F_1) + w_{K_2}(F_2)} \frac{1}{w_{K_1}(F_1)} \sum_{\bar{F} \in S_{i+1}(K_1)} w_{K_1}(\bar{F}) \operatorname{sgn}([F], \partial[\bar{F}]) f(\partial[\bar{F}]) \\
&+ \frac{w_{K_2}(F_2)}{w_{K_1}(F_1) + w_{K_2}(F_2)} \frac{1}{w_{K_2}(F_2)} \sum_{\bar{F} \in S_{i+1}(K_2)} w_{K_2}(\bar{F}) \operatorname{sgn}([F], \partial[\bar{F}]) f_2(\partial[\bar{F}]) \\
= &\frac{w_{K_1}(F_1)}{w_{K_1}(F_1) + w_{K_2}(F_2)} \lambda f_1([F]) + \frac{w_{K_2}(F_2)}{w_{K_1}(F_1) + w_{K_2}(F_2)} \lambda f_2([F]) \\
= &\lambda f([F]).
\end{aligned}$$

□

This also includes the case when either f_1 or f_2 is identically equal to zero.

Remark 5.6. The previous theorem will hold for the weighted normalized Laplacian if the weight function $w_K : \bigcup_k S_k(K) \to \mathbb{R}^+$ is given as follows

$$w_K(F) = \begin{cases} w_{K_1}(F) & \text{if } F \text{ is a face of } K_1 \text{ and } \dim F > i \\ w_{K_2}(F) & \text{if } F \text{ is a face of } K_2 \text{ and } \dim F > i \\ \sum\limits_{\substack{\bar{F_1} \in K_1: \\ F \in \partial \bar{F_1}}} w_{K_1}(\bar{F_1}) + \sum\limits_{\substack{\bar{F_2} \in K_2 \\ :F \in \partial \bar{F_2}}} w_{K_2}(\bar{F_2}) & \text{if } F \text{ is a face of } K \text{ and } \dim F \leq i \end{cases}$$

Example 5.1. Let σ_1 be an i-simplex, then $\mathbf{s}(\Delta_i^{down}(\sigma)) \overset{\circ}{=} \mathbf{s}(\Delta_{i-1}^{up}(\sigma)) \overset{\circ}{=} \{i+1\}$. A function which is equal to 1 on every oriented simplex in the boundary of $[\sigma]$ will be an eigenfunction of Δ_i^{down} corresponding to $(i+1)$.
According to Theorem 5.3, an $(i-1)$-wedge of any number of i-simplices, will contain eigenvalue $(i+1)$, as long as we are able to orient them such that any two simplices whose intersection is of dimension i, induce the same orientation on their intersecting face. For an alternative proof of this claim see Theorem 6.2.

Theorem 5.3 provides a way to identify some eigenvalues of the combinatorial wedge sum. However, the results obtained by using the interlacing theorem for simplicial maps, as shown in the next theorem, are more comprehensive.

Theorem 5.4. *Let $\mu_1, \dots, \mu_m$ be eigenvalues of $\Delta_i^{up}(K_1 \cup K_2)$ and $\lambda_1, \dots, \lambda_{m-1}$ eigenvalues of $\Delta_i^{up}(K)$, where $K := (K_1 \vee_i K_2)$, then*

$$\mu_i \leq \lambda_i \leq \mu_{i+1}$$

for every $0 \leq i \leq m-1$.

Proof. Let F_1 and F_2 be i-faces which are identified in an i-wedge sum K. The map $f : K_1 \cup K_2 \to_1 \vee_{F_1 \tilde{F}_2} K_2$ identifies vertices of F_1 with the vertices of F_2, and is identity on the remaining vertices of $K_1 \cup K_2$. Furthermore, f is a simplicial map. The interlacing theorem for simplicial maps (see [20]) gives

$$\mu_i \leq \lambda_i \leq \mu_{i+k},$$

where $k = \mid S_i(K_1 \cup K_2) \mid - \mid S_i(K) \mid$. □

Thus the spectrum of Δ_i^{up} of the union of two simlicial complexes majorizes the spectrum of their i-wedge sum.

Remark 5.7. The wedge sums of graphs and its effect on the spectrum of the normalized graph Laplacian has already been analysed in [2], and the spectrum of the combinatorial graph Laplacian was analysed in [16]. These are the special cases of the general theory presented here.

5.2. **Joins.** Let K_1 and K_2 be simplicial complexes on vertex sets $[n]$ and $[m]$, respectively. The *join* $K_1 * K_2$ is a simplicial complex on a vertex set $[m+n]$, whose faces are $F_1 * F_2 := \{v_0, \dots, v_k, n+u_0, \dots, n+u_l\}$, where $F_1 = \{v_0, \dots, v_{i_1}\}$ is a simplex in K_1 and $F_2 = \{u_0, \dots, u_{i_2}\}$ a simplex in K_2. The cochain groups of $K_1 * K_2$ are

$$C^i(K_1 * K_2, \mathbb{R}) = \bigoplus_{i_1+i_2+1=i} C^{i_1}(K_1, \mathbb{R}) \otimes C^{i_2}(K_2, \mathbb{R}),$$

and the coboundary map $\delta_i : C^i(K_1 * K_2, \mathbb{R}) \to C^{i+1}(K_1 * K_2, \mathbb{R})$ is

$$\delta_i(f \otimes g) = \delta_{i_1} f \otimes g + (-1)^{i_1} f \otimes \delta_{i_2} g,$$

where $f \in C^{i_1}(K_1)$ and $g \in C^{i_2}(K_2)$. The cochain groups of $K_1 * K_2$ are the sums of tensor product of Hilbert spaces, hence a naturally defined scalar product on them is

$$(f_1 \otimes g_1, f_2 \otimes g_2) = (f_1, f_2)_{C^{i_1}(K_1)}(g_1, g_2)_{C^{i_2}(K_2)}, \tag{5.2}$$

where $f_1, f_2 \in C^{i_1}(K_1)$, $g_1, g_2 \in C^{i_2}(K_2)$ and $i_1 + i_2 + 1 = i$. In terms of the weight functions the latter equality is

$$w_{K_1 * K_2}(F_1 \otimes F_2) = w_{K_1}(F_1)w_{K_2}(F_2). \tag{5.3}$$

Then the following proposition[5] holds.

Proposition 5.5 (Duval, Reiner)**.**

$$\begin{aligned}(\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*)(f \otimes g) = {} & (\delta_{i_1}^*\delta_{i_1} + \delta_{i_1-1}\delta_{i_1-1}^*) \otimes id(f \otimes g) \\ & + id \otimes (\delta_{i_2}^*\delta_{i_2} + \delta_{i_2-1}\delta_{i_2-1}^*)(f \otimes g)\end{aligned} \tag{5.4}$$

From the equality above follows

$$\mathbf{s}((\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*)(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}((\delta_{i_1}^*\delta_{i_1} + \delta_{i_1-1}\delta_{i_1-1}^*)(K_1)) \\ \mu_j \in \mathbf{s}((\delta_{i_2}^*\delta_{i_2} + \delta_{i_2-1}\delta_{i_2-1}^*)(K_2))}} \lambda_i + \mu_j. \tag{5.5}$$

The adjoint δ_i^* of δ_i in Proposition 5.5 is calculated with respect to scalar products as defined in (5.2). Furthermore, Proposition 5.5 holds, regardless of the choice of the scalar products on cochain groups K_1 and K_2. However, the problem occurring here is to decide on nature (type) of the Laplace operator $\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*$ obtained this way, i.e. is it normalized, combinatorial or some other type of Laplacian.

Duval and Reiner analysed the combinatorial Laplace operator L. In this case, the weight functions on complexes K_1 and K_2 are constant, that is equal to 1. Thus, due to (5.3), the weight function on $K_1 * K_2$ is 1, as well. In other words,

$$(\delta_i^*\delta_i + \delta_{i-1}\delta_{i-1}^*)(K_1 * K_2) = L_i(K_1 * K_2),$$

and

$$\mathbf{s}(L_i(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}(L_{i_1}(K_1)) \\ \mu_j \in \mathbf{s}(L_{i_2}(K_2))}} \lambda_i + \mu_j, \tag{5.6}$$

where $i = i_1 + i_2 + 1$.

The following theorem gives a characterization of $\mathbf{s}(\Delta_i(K_1 * K_2))$ in terms of $\mathbf{s}(\Delta_i(K_1))$ and $\mathbf{s}(\Delta_i(K_2))$. Note that due to the nature of the weight functions which determines the normalized Laplacian, this characterization will not be complete as in the case of the combinatorial Laplacian L and (5.7).

Theorem 5.6. *Let* $\dim K_1 = d_1$ *and* $\dim K_2 = d_2$*. Then*

$$\mathbf{s}(\Delta_{d_1+d_2+1}^{down}(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\Delta_{d_1}^{down}(K_1)) \\ \mu_j \in \mathbf{s}(\Delta_{d_2}^{down}(K_2))}} \lambda_i + \mu_j, \tag{5.7}$$

[5] This is Proposition 4.9. in [10]

or equivalently

$$\mathbf{s}(\Delta^{up}_{d_1+d_2}(K_1 * K_2)) \overset{\circ}{=} \bigcup_{\substack{\lambda_i \in \mathbf{s}(\Delta^{up}_{d_1-1}(K_1)) \\ \mu_j \in \mathbf{s}(\Delta^{up}_{d_2-1}(K_2))}} \lambda_i + \mu_j.$$

Proof. Let w_{K_1} and w_{K_2} be the weight functions corresponding to the normalized Laplacian on K_1 and K_2, and let F_1, F_2 be faces of K_1 and K_2, respectively. The weight function $w_{K_1 * K_2}$ on the join $K_1 * K_2$ is determined by (5.3). Note that in this case (5.5) holds, as well. In the following, we check if $w_{K_1 * K_2}$ determines the normalized combinatorial Laplacian on $K_1 * K_2$, i.e.

$$\begin{aligned}\deg F_1 \otimes F_2 &= \sum_{\substack{F \in S_{i+1}(K_1 * K_2): \\ F_1 \otimes F_2 \in \partial F}} w(F) \\ &= \sum_{\bar{F}_1 : F_1 \in \partial \bar{F}_1} w_{K_1 * K_2}(\bar{F}_1 \otimes F_2) + \sum_{\bar{F}_2 : F_2 \in \partial \bar{F}_2} w_{K_1 * K_2}(F_1 \otimes \bar{F}_2) \\ &= \sum_{\bar{F}_1 : F_1 \in \partial \bar{F}_1} w_{K_1}(\bar{F}_1) w_{K_2}(F_2) + \sum_{\bar{F}_2 : F_2 \in \partial \bar{F}_2} w_{K_1}(F_1) w_{K_2}(\bar{F}_2).\end{aligned}$$

If neither F_1 nor F_2 is a facet of K_1, K_2, then the degree of $F_1 \otimes F_2$ is equal to

$$2 w_{K_1}(F_1) w_{K_2}(F_2).$$

Therefore, (3.7) does not hold. Consequently, the Laplace operator determined by this function will not be the normalized Laplace operator of join $K_1 * K_2$. However, if F_1 or F_2 is a facet, then

$$\deg F_1 \otimes F_2 = w_{K_1}(F_1) w_{K_2}(F_2).$$

Thus, $w_{K_1 * K_2}$ coincides with the weight function which determines $\Delta^{up}_i(K_1 * K_2)$, for $i = d_1 + d_2 + 1$. Together with (5.6), this gives equivalence (5.8). □

5.3. **Duplication of motifs.** Let K be a simplicial complex on a vertex set $[n]$. If Σ is its subcomplex on the vertices $v_0, \ldots, v_k$, containing all of K's faces on those vertices, then it is called a *motif*.

Definition 5.4. A subcomplex Σ of a given simplicial complex K is its k-motif iff:

(i) $(\forall F_1, F_2 \in \Sigma) \quad F_1 * F_2 \in K \Rightarrow F_1 * F_2 \in \Sigma$
(ii) $\dim \operatorname{lk} \Sigma = k$, where $\operatorname{lk} \Sigma$ denotes the link of Σ.

In fact, as a consequence of Theorem 5.1 for $i < k$ we obtain

$$\mathbf{s}(\Delta^{up}_i(K)) \overset{\circ}{=} \mathbf{s}(\Delta^{up}_i(K - \operatorname{St} \Sigma)) \overset{\circ}{\cup} \mathbf{s}(\operatorname{Cl} \operatorname{St} \Sigma).$$

Therefore, it is meaningful to observe the effect of duplication of k-motif to the spectrum of Δ^{up}_i only if $i = k$. For definitions of link lk, star St and closure Cl, the reader is invited to consult [22].

Remark 5.8. If K is an $(i+1)$-path connected simplicial complex, then any motif satisfying (i) in Definition 5.4 will have a link of dimension i.

Let $u_0, \ldots, u_m$ be vertices of $\operatorname{lk} \Sigma$. Due to the definition of a link, these vertices are different than the one in the motif Σ ($u_i \neq v_j$, for every $0 \leq i \leq m$ and $0 \leq j \leq k$). Let Σ' denote a simplicial complex with vertices $v'_0, \ldots, v'_k$, which is isomorphic to Σ. And let $f : v'_i \mapsto v_i$ be a simplicial isomorphism among these complexes. Then $K^{\Sigma} := K \cup \{\{v'_{i_0}, \dots, v'_{i_l}, u_{j_1}, \dots, u_{j_s}\} \mid \{v_{i_0}, \dots, v_{i_l}, u_{j_1}, \dots, u_{j_s}\} \in K\}$.

Proposition 5.7. *K^{Σ} is a simplicial complex and $\operatorname{Cl}\operatorname{St}\Sigma$ is isomorphic to $\operatorname{Cl}\operatorname{St}\Sigma'$.*

Proof. Elementary. □

Definition 5.5. We say that a simplicial complex K^{Σ} is obtained from a simplicial complex K by the *duplication of i-motif* Σ.

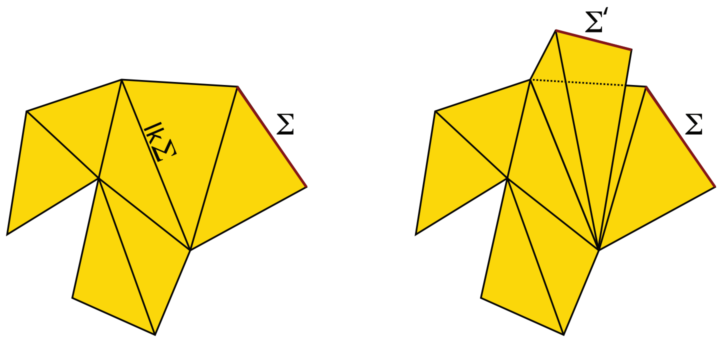


Figure 4. Duplication of motif Σ

Remark 5.9. It could be argued, that it is $\operatorname{Cl}\operatorname{St}\Sigma$ that we duplicate rather than Σ alone. This point of view will be very helpful in the subsequent work, but we will refer to duplication as the duplication of motif Σ, since this terminology is consistent with the previous work on the duplication of motifs of graphs (see [2]).

Theorem 5.8. *Let n be the number of i-simplices in $\operatorname{St}\Sigma$. Then there exist n linearly independent functions $f_1, \dots, f_n$, satisfying*

$$\Delta_i^{up}(K) f_j([F]) = \lambda_j f_j([F]),$$

for every $F \in S_i(\operatorname{St}\Sigma)$ and some real values λ_j. The doubling of the motif Σ produces a simplicial complex K^{Σ} with the eigenvalues λ_j and the eigenfunctions g_j which agree with f_j on $\operatorname{St}\Sigma$ and $-f_j$ on $\operatorname{St}\Sigma'$ and are zero elsewhere.

Proof. It is trivial to check that $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma)$ and $\Delta_i^{up}(K^{\Sigma})$ coincide on $\operatorname{St}\Sigma$. Let $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma}$ be a restriction of the operator $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma)$ on $\operatorname{St}\Sigma$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma}$ and $f_1, \dots f_n$ the corresponding eigenfunctions. Then

$$g_j([F]) = \begin{cases} f_j([F]) & \text{for every } F \text{ in} \operatorname{St}\Sigma \\ -f_j([F]) & \text{for every } F \text{ in} \operatorname{St}\Sigma' \\ 0 & \text{otherwise} \end{cases}$$

is an eigenfunction of $\Delta_i^{up}(K^{\Sigma})$ with associated eigenvalue λ_j. Without a loss of generality assume that the labelling of the vertices of Σ is $v_0, \dots, v_k$ and the vertices of Σ' is $v'_0, \dots, v'_k$, and they are choosen such that $v_0 < \dots < v_k < v'_0, < \dots < v'_k$.

Enumerate vertices of $\operatorname{lk}\Sigma$ with $u_1, \ldots, u_m$ [6] such that $v_0 < \ldots < v_k < v'_0, < \ldots < v'_k < u_1 < \ldots < u_m$. Then it is trivial to check that

$$\Delta_i^{up} f_j([F]) = \Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma} f_j([F]) = \lambda_j f_j([F])$$

and

$$\Delta_i^{up}(-f_j)([F']) = \Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma} -f_j([F']) = -\lambda_j f_j([F'])$$

for all $F \in S_i(\Sigma)$ and $F' \in S_i(\Sigma')$.
Furthermore, assume that $[u_1, \ldots u_{i+1}]$ is a face of $\operatorname{lk}\Sigma$ [7], then

$$\begin{aligned}\Delta_i^{up} f_j([u_1, \ldots u_{i+1}]) = & \sum_{v_j, [v_j, u_1, \ldots, u_{i+1}] \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)} (-1)^1 f_j(\partial[v_j, u_1, \ldots, u_{i+1}]) \\ & + \sum_{v'_j, [v'_j, u_1, \ldots, u_{i+1}] \in S_{i+1} \operatorname{Cl}\operatorname{St}\Sigma'} (-1)^1 (-f_j)(\partial[v'_j, u_1, \ldots, u_{i+1}]) \\ & = 0.\end{aligned}$$

Since the value of the functions f_j on the boundary of $(i+1)$-simplices, which are neither in $\operatorname{Cl}\operatorname{St}\Sigma$ nor in $\operatorname{Cl}\operatorname{St}\Sigma'$ is zero, we omit them. Hence λ_j's are eigenvalues of $\Delta_i^{up}(K^\Sigma)$. □

As a simple consequence of Theorem 5.8 we have the following corollary.

Corollary 5.9. *If the spectrum of simplicial complex* $\operatorname{Cl}\operatorname{St}\Sigma$ *possesses eigenvalue* λ*, with eigenfunction* f *which is identically equal to zero on* $\operatorname{lk}\Sigma$*, then the spectrum of* K^Σ *will contain eigenvalue* λ *as well.*

Theorem 5.8 is an improved and generalized version of Theorem 2.3 from [2], which was stated for the case of the normalized graph Laplacian Δ_0^{up}. The duplication of the motif Σ will leave a specific trace in a spectrum of newly obtained simplicial complex K^Σ. In particular, if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma}$, then after duplicating motif Σ m times, the spectrum of newly obtained complex will contain $(m-1)$ instances of every eigenvalue λ_j.

Since it is not always straightforward to calculate eigenvalues of $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma}$, we prove a theorem about interlacing of λ_j's and eigenvalues of $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma)$, μ_j's. The notation in the following theorem is as in Theorem 5.8.

Theorem 5.10. *The following inequality holds*

$$\mu_i \leq \lambda_i \leq \mu_{i+|S_i(\operatorname{lk}\Sigma)|},$$

where $\mid S_i(\operatorname{lk}\Sigma) \mid$ *denotes the number of* i*-simplices in the link of a motif* Σ.

Proof. Matrix $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma) \mid_{\operatorname{St}\Sigma}$ is obtained from matrix $\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma)$ by deleting $\mid S_i(\operatorname{lk}\Sigma) \mid$ rows and columns. Thus, the interlacing inequality follows directly from the Cauchy interlacing theorem. □

Remark 5.10. Theorem 5.8 and Corollary 5.9 will hold for any choice of the weight function satisfying (3.7).

[6] From the definition of lk, follows that zero simplices of $\operatorname{lk}\Sigma$ and Σ are different.

[7] Faces of $\operatorname{lk}\Sigma$ which are of dimension less than i are not relevant for the subsequent discussion.

6. Eigenvalues in the spectrum of Δ_i^{up} and the combinatorial properties they encode

One of the main advantages of the *normalized* combinatorial Laplace operator is the fact that the spectrum of any simplicial complex K is bounded from above by a constant. The eigenvalues of $\Delta_i^{up}(K)$ are in the interval $[0, i+2]$. However, this is not the case for the spectrum of the combinatorial Laplacian L or for any other known type of the combinatorial Laplace operator $\mathcal{L}$. Therefore, it impossible to assign combinatorial properties to the existence of particular eigenvalue in the spectrum of L and $\mathcal{L}$. Nonetheless, it is worthwhile to mention that the global properties of spectrum of L_i relate to combinatorial properties of complex. For instance, the spectrum of certain combinatorially suitable complexes is proved to be integer (see [8],[10]).

The appearance of eigenvalue 2 in the spectrum of the normalized graph Laplacian Δ_0^{up} means that the underlying graph is bipartite (see [6]), while some occurrences of eigenvalue 1 appear due to duplication of motifs (see [2]). In the following, we characterize some of the integer eigenvalues, which appear in the spectrum of Δ_i^{up}.

6.1. **Eigenvalue $i+2$.** Without a loss of generality assume K is an $(i+1)$-path connected simplicial complex on a vertex set $[n]$. As shown earlier, the following inequality holds

$$(\Delta_i^{up}(K)f, f) = \sum_{\bar{F}\in S_{i+1}(K)} f(\partial[\bar{F}])^2 w(\bar{F}) \tag{6.1a}$$

$$\leq (i+2) \sum_{F\in S_i(K)} f([F])^2 w(F). \tag{6.1b}$$

The equality in (6.1b) is reached iff there exists a function $f \in C^i(K, \mathbb{R})$, which satisfies

$$\operatorname{sgn}([F_j], \partial[\bar{F}]) f([F_j]) = \operatorname{sgn}([F_k], \partial[\bar{F}]) f([F_k]),$$

for every $\bar{F}$ in S_{i+1} and $F_j, F_k \in \partial\bar{F}$. Thus $\mid f([F]) \mid$ must be constant for every $F \in S_i(K)$. Assume further that $\mid f([F]) \mid = 1$, then for every $F \in \partial\bar{F}$, $f([F])$ is equal either to $\operatorname{sgn}([F], \partial[\bar{F}])$ or to $-\operatorname{sgn}([F], \partial[\bar{F}])$. Now it is possible to consider f as a choice of orientation on $(i+1)$-faces of K.

Theorem 6.1. *The existence of a function f satisfying the equality in (6.1b) is equivalent to the existence of the orientation on $(i+1)$-simplices of K, such that every two $(i+1)$-simplices intersecting by a common i-face induce the same orientation on the intersecting simplex*[8].

Theorem 6.2. *For an i-connected simplicial complex K the following statements are equivalent*

(1) *Spectrum $\Delta_i^{up}(K)$ has eigenvalue $i+2$,*
(2) *There are no $(i+1)$-orientable circuits of odd length nor $(i+1)$-non orientable circuits of even length in K.*

Proof. (1) $\Rightarrow$ (2) This part of the proof is by contradiction. Assume that there exists an $(i+1)$-orientable circuit of odd length, whose i-simplices $F_1, \ldots, F_{2n+1}$ are ordered increasingly, as suggested in Definition 4.1. Then it is possible to

[8]This condition is opposite to the condition of coherently oriented simplices.

orient these simplices in such a way that every two neighbouring simplices induce different orientation on their intersecting face. Denote these oriented simplices by $[F_1], \dots, [F_{2n+1}]$. In order to have the same orientation induced on the intersecting face, we reverse the orientation of every simplex $[F_k]$, for k even[9]. Thus, $[F_l]$ and $-[F_{l+1}]$ induce the same orientation on $[F_l \cap F_{l+1}]$, for every $1 \leq l \leq 2n$. However, $[F_1]$ and $[F_{2n+1}]$ remain coherently oriented, which contradicts Theorem 6.1. The analysis for the case of $(i+1)$-non-orientable circuits is analogous.
(2) $\Rightarrow$ (1) Let F_1 be an arbitrary $(i+1)$-face of K. Consider its positive orientation $[F_1]$ and call it *initial* oriented face . Let $[F_{i_1 i_2 \dots i_n}]$ be an $(i+1)$-face of K which shares an i-face with $[F_{i_1 i_2 \dots i_{n-1}}]$ and both faces induce the same orientation on their intersecting face. Now, assume opposite: it is not possible to choose an orientation on $(i+1)$-faces of K, which satisfies the conditions of Theorem 6.1. This means that after some number of steps in the construction above, two faces $[F_{i_1 i_2 \dots i_n}]$, $[F_{i_1 i_2 \dots i_m}]$ which are the same, but differently oriented are obtained. Obviously, there exists a circuit containing $[F_{i_1 i_2 \dots i_n}]$, which does not admit an orientation as in Theorem 6.1. This is possible only in the case when a circuit is orientable and odd or even and non-orientable. This is a contradiction, hence $i+2$ is contained in the spectrum of Δ_i^{up}.

□

The spectrum of the normalized graph Laplacian contains eigenvalue 2 iff the chromatic number of the underlying graph is 2. However, the connection of chromatic number and the boundary eigenvalue in the spectrum of the normalized combinatorial Laplace operator is one directional.

Theorem 6.3. *If the chromatic number of* 1*-skeleton of simplicial complex K is $i+2$, then $i+2$ is contained in* $\mathbf{s}(\Delta_i^{up}(K))$.

Proof. Let $I_0, \dots, I_{i+1}$ be disjoint sets of the vertices of K, such that every simplex of K contains at most one point of each set. Thus, there are no vertices of $\bar{F} \in S_{i+1}(K)$ which are contained in the same I_j. To avoid notational complications we relabel the vertices of K: instead of $v \in I_j$ $(v \in \{1, \dots, n\})$ we write $in + v$. Therefore, we have

$$v \in I_j, u \in I_k \text{ and } j < k \Rightarrow v < u.$$

Function f, defined as $f([v_0, \dots, \hat{v_j}, \dots, v_{i+1}]) = (-1)^j$ $([v_0, \dots, v_{i+1}]$, is an $(i+1)$-simplex of K whose vertices are ordered increasingly, i.e. $v_0 < \dots < v_{i+1})$ is the eigenfunction of $\Delta_i^{up}(K)$ corresponding to the eigenvalue $i+2$, i.e.

$$\begin{aligned}\Delta_i^{up} f([F]) =& \frac{\sum_{\bar{F}: F \in \partial \bar{F}} f(\partial [\bar{F}])}{\deg F} \\ =& (i+2) f([F]).\end{aligned}$$

□

Remark 6.1. The opposite claim is not true. A counter example is given in Figure 5 left. However, even if we exclude simplicial complexes of this type, i.e. if we

[9]One can also consider the option to reverse the orientation for every odd k, but this does not make any difference in the remainder of the proof.

assume

(6.2)

$$(\forall \bar{F}_1, \bar{F}_2 \in S_{i+1}(K)) \bar{F}_1 \cap \bar{F}_2 = F \neq \emptyset \Rightarrow (\exists F_1 = F_{11}, F_{12}, \dots F_{1m} = F_2)(\exists 1 < k < m) F \in F_{1k},$$

it is possible to construct a counterexample, see Figure 5 right.

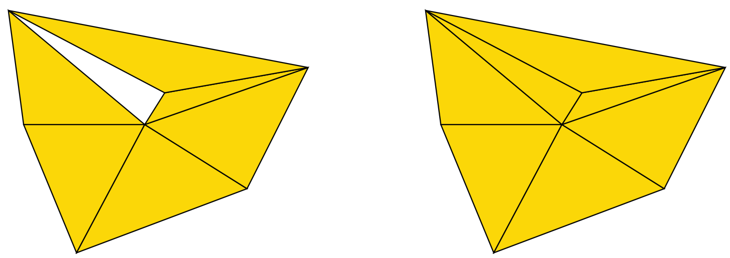


Figure 5. Counterexamples for the equivalence in Theorem 6.2

6.2. **Eigenvalues** $(i+1)$ **and** 1. As a special case of Theorem 5.8 we consider a motif Σ consisting of only one vertex.

Corollary 6.4. *Duplication of an* i*- motif* Σ *consisting of one vertex which is the center of neither an* $(i+1)$*-orientable odd circuit nor an* $(i+1)$*-non-orientable even circuit, results in appearance of the eigenvalue* $(i+1)$ *in the spectrum of* K^Σ*.*

Proof. Let $v_0 = \Sigma$ and let 0-simplices of $\operatorname{lk}\Sigma$ be $u_1, \dots, u_k$. In $\operatorname{Cl}\operatorname{St}\Sigma$ all $(i+1)$-simplices must contain v_1. Since v_1 is neither a center of an $(i+1)$-orientable odd circuit nor a center of an $(i+1)$-non-orientable even circuit, then by Theorem 6.2, $i+2 \in \mathbf{s}(\operatorname{Cl}\operatorname{St}\Sigma)$. From Theorem 6.1 follows that there is a function $f \in C^i(\operatorname{Cl}\operatorname{St}\Sigma, \mathbb{R})$, s.t.

$$\operatorname{sgn}([F_1], \partial[\bar{F}]) f([F_1]) = \dots = \operatorname{sgn}([F_{i+2}], \partial[\bar{F}]) f([F_{i+2}])$$

for every $\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)$ and each of its i-faces. Let g be a function which coincides with f on oriented i-faces of $\operatorname{St}\Sigma$, with $-f$ on oriented i-faces of $\operatorname{St}\Sigma'$ and is zero elsewhere. We will now show that g is the eigenfunction of $\Delta_i^{up}(K^\Sigma)$ associated to the eigenvalue $(i+1)$. Let F be an arbitrary i-face of $\operatorname{St}\Sigma$, then

$$\begin{aligned}
\Delta_i^{up}(\operatorname{Cl}\operatorname{St}\Sigma)\mid_{\operatorname{St}\Sigma} g([F]) &= \frac{1}{w(F)} \sum_{\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)} sgn(F, \partial\bar{F}) g(\partial[\bar{F}]) \\
&= \frac{1}{w(F)} \sum_{\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)} \operatorname{sgn}([F], \partial[\bar{F}]) \sum_{\substack{F_j \in \partial\bar{F} \\ F_j \notin \operatorname{lk}\Sigma}} \operatorname{sgn}([F_j], \partial[\bar{F}]) f([F_j]) \\
&= \frac{1}{w(F)} \sum_{\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)} \operatorname{sgn}([F], \partial[\bar{F}])(i+1) \operatorname{sgn}([F], \partial[\bar{F}]) f([F]) \\
&= (i+1) \frac{1}{w(F)} \sum_{\bar{F} \in S_{i+1}(\operatorname{Cl}\operatorname{St}\Sigma)} f([F]) \\
&= (i+1).
\end{aligned}$$

The same analysis holds for i-faces of $\mathrm{St}\,\Sigma'$. Let F be an i-faces of $\mathrm{Cl}\,\mathrm{St}\,\Sigma - \mathrm{St}\,\Sigma$, then

$$\begin{aligned}\Delta_i^{up}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)\mid_{\mathrm{St}\,\Sigma} f([F]) &= \frac{1}{w(F)}\left(\sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} \mathrm{sgn}([F],\partial[\bar{F}])g(\partial[\bar{F}])\right.\\ &\quad \left. + \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma')} \mathrm{sgn}([F],\partial[\bar{F}])g(\partial[\bar{F}])\right)\\ &= \frac{1}{w(F)}(i+1)\left(\sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma)} g([F_j]) + \sum_{\bar{F}\in S_{i+1}(\mathrm{Cl}\,\mathrm{St}\,\Sigma')} g([F_j])\right)\\ &= \frac{1}{w(F)}(i+1)\left(\sum_{F_j\in S_i(\mathrm{St}\,\Sigma)} f([F_j]) + \sum_{F'_j\in S_i(\mathrm{St}\,\Sigma')} -f([F_j])\right)\\ &= 0,\end{aligned}$$

where F_j is a face of $\bar{F}$. □

Note that, this theorem is a generalization of the vertex doubling effect on the normalized graph Laplacian Δ_0^{up} discussed in [2].

In a graph case eigenvalue 1 plays a very important role, since its multiplicity is significantly higher that other eigenvalues when it comes to graphs obtained by modelling the real world processes, for details and examples see [3]. However, when it comes to the Laplace operator defined on simplicial complexes, eigenvalue 1 looses some of its importance. This is due to the fact that the role of the eigenvalue one is partially transferred to the eigenvalue $(i+1)$ in higher dimensions, as shown previously. The next theorem gives one characterization of eigenvalue 1 in the spectrum of Δ_i^{up}.

Theorem 6.5. *Let K be a simplicial complex with an eigenvalue $i+2$ in the spectrum of Δ_i^{up} and let G_K^i be its i-dual graph. Then,*

$$1 \in \mathbf{s}(\Delta_0^{up}(G_K^i)) \Leftrightarrow 1 \in \mathbf{s}(\Delta_i^{up}(K)).$$

Proof. The multiplicity of eigenvalue 1 in the spectrum of $\Delta_i^{up}(K)$ is equal to the dimension of a kernel of adjacency matrix A_i^{up} of i-faces of K. Its entries are

$$(A_i^{up})_{[F],[F']} = \begin{cases} \mathrm{sgn}([F],\partial[\bar{F}])\,\mathrm{sgn}([F'],\partial[\bar{F}]) & F,\ F' \text{ are } (i+1)\text{-up neighbours}\\ 0 & \text{otherwise}\end{cases}$$

Due to Theorem 6.1, it is possible to orient $(i+1)$-simplices of K such that $\mathrm{sgn}([F],\partial[\bar{F}])\,\mathrm{sgn}([F'],\partial[\bar{F}])$ is always positive. Consequently, all entries of matrix A_i^{up} will be positive. The adjacency matrix of G_K^i and A_i^{up} are equal, thus the dimension of the kernel of A_i^{up} is equal to the multiplicity of eigenvalue 1 in the spectrum of the normalized graph Laplacian of the graph G_K^i. □

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Max Planck Institute for Mathematics in the Sciences, Inselstrasse 22, D-04103 Leipzig, Germany