

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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Two-qubit States

by

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Preprint no.: 4

2011



Quantum Discord and Geometry for a Class of Two-qubit States

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We study the level surfaces of quantum discord for a class of two-qubit states with parallel nonzero Bloch vectors. The dynamic behavior of quantum discord under decoherence is investigated. It is shown that a class of X states has sudden transition between classical and quantum correlations under decoherence. Our results include the ones in [Phys. Rev. Lett. 105. 150501] as a special case and show new pictures and structures of quantum discord.

PACS numbers: 03.67.Mn, 03.65.Ud, 03.65.Yz

I. INTRODUCTION

The quantum entanglement acts as the most important resources in quantum information [1, 2]. However, entanglement is not the only correlation that is useful for quantum information processing. Recently, it is found that many tasks, e.g. quantum nonlocality without entanglement [1, 3, 5], can be carried out with quantum correlations other than entanglement. It has been shown both theoretically and experimentally [6, 7] that some separable states may speed up certain tasks over their classical counterparts.

One kind of nonlocal correlation called quantum discord, as introduced by Oliver and Zurek [13], has received much attention recently [13–26]. The idea is to measure the discrepancy between two natural yet different quantum analogs of the classical mutual information. Let ρ^{AB} denote the density operator of a composite bipartite system AB, and $\rho^{A(B)} = \text{Tr}_{B(A)}(\rho^{AB})$ the reduced density operator of the partition B(A). The quantum mutual information is defined by

$$\mathcal{I}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}), \quad (1)$$

where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the Von Neuman entropy. It was shown that quantum mutual information is the information-theoretic measure of the total correlation in a bipartite quantum state. In order to determine quantum discord [13, 22], Ollivier and Zurek use a measurement-based conditional density operator to generalize the classical mutual information. Let B_k be a set of one-dimensional project measurement performed on subsystem B , the conditional density operator ρ_k associated with the measurement result k is

$$\rho_k = \frac{1}{p_k} (I \otimes B_k) \rho (I \otimes B_k), \quad (2)$$

where $p_k = \text{tr}(I \otimes B_k) \rho (I \otimes B_k)$, I is the identity operator on the subsystem A . With this conditional density operator, the quantum conditional entropy with respect to this measurement is defined by

$$S(\rho|\{B_k\}) := \sum_k p_k S(\rho_k), \quad (3)$$

and the associated quantum mutual information is given by

$$\mathcal{I}(\rho|\{B_k\}) := S(\rho^A) - S(\rho|\{B_k\}). \quad (4)$$

Classical correlation is defined as the superior of $\mathcal{I}(\rho|\{B_k\})$ over all possible Von Neumann measurement B_k ,

$$\mathcal{C}(\rho) := \sup_{\{B_k\}} \mathcal{I}(\rho|\{B_k\}). \quad (5)$$

Quantum discord is then given by the difference of mutual information $\mathcal{I}(\rho)$ and the classical correlation $\mathcal{C}(\rho)$,

$$\mathcal{Q}(\rho) := \mathcal{I}(\rho) - \mathcal{C}(\rho). \quad (6)$$

The analytical expressions for classical correlation and quantum discord are only available for two-qubit Bell diagonal state and a seven-parameter family of two-qubit X states [22, 24] till now. For the two-qubit Bell-diagonal state:

$$\rho = \frac{1}{4} (I \otimes I + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i), \quad (7)$$

the classical correlation is given by

$$\mathcal{C}(\rho) = \frac{1-c}{2} \log_2(1-c) + \frac{1+c}{2} \log_2(1+c), \quad (8)$$

where $c = \max\{|c_1|, |c_2|, |c_3|\}$. The quantum discord is given by

$$\begin{aligned} \mathcal{Q}(\rho) = & \frac{1-c_1-c_2-c_3}{4} \log_2(1-c_1-c_2-c_3) \\ & + \frac{1-c_1+c_2+c_3}{4} \log_2(1-c_1+c_2+c_3) \\ & + \frac{1+c_1-c_2+c_3}{4} \log_2(1+c_1-c_2+c_3) \\ & + \frac{1+c_1+c_2-c_3}{4} \log_2(1+c_1+c_2-c_3) \\ & - \frac{1-c}{2} \log_2(1-c) - \frac{1+c}{2} \log_2(1+c). \end{aligned} \quad (9)$$

The geometry of Bell-diagonal states was first introduced by Horodecki [9]. From the positivity of the spectral of a Bell-diagonal state ρ in Eq.(7), one can see that ρ

belongs to a tetrahedron \mathcal{T} with vertices $v_1 = (1, -1, 1)$, $v_2 = (-1, 1, 1)$, $v_3 = (1, 1, -1)$, and $v_4 = (-1, -1, -1)$ in the correlation vector space. Similarly, from the positivity of the partial transpose of ρ , it has been shown that the separable states belong to the octahedron \mathcal{O} with vertices $O_1^\pm = (\pm 1, 0, 0)$, $O_2^\pm = (0, \pm 1, 0)$ and $O_3^\pm = (0, 0, \pm 1)$ [9, 12, 23].

Very recently, Matthias D. Lang and Carlton M. Caves [23] depicted the level surfaces of entanglement and quantum discord for Bell-diagonal states, they discovered that the picture and the structure of the quantum entanglement and the quantum discord are very different. There doesn't exist simple relations between them.

In this article, we study the quantum discord for a class of X states that the Bloch vectors are z directional, which including Bell-diagonal states as a special case. We study the level surfaces of quantum discord and dynamic behavior of quantum discord under decoherence. It is demonstrated that the surfaces of constant discord shrinks along with the geometrical deformation of \mathcal{T} in Ref.[12]. Moreover we find that there is a class of X states for which the quantum discord is not destroyed by decoherence in a finite time interval.

We calculate different kinds of correlation such as entanglement, classical correlation and quantum discord for the state we concerned in sec. II. We depict the level

surface of constant discord in four different situations. In sec. III, we discuss the dynamics of quantum discord and show that the quantum discord of a certain class of X states does not decay under decoherence. A brief conclusion is given in sec. IV.

II. GEOMETRICAL DEPICTION OF \mathcal{C} AND \mathcal{D}

Under appropriate local unitary transformations, any two-qubit state ρ can be written as:

$$\rho = \frac{1}{4}[I \otimes I + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i], \quad (10)$$

where \mathbf{r} and \mathbf{s} are Bloch vectors and $\{\sigma_i\}_{i=1}^3$ are the standard Pauli matrices. When $\mathbf{r}=\mathbf{s}=\mathbf{0}$, ρ reduces to the two-qubit Bell-diagonal states. In the following, we assume that the Bloch vectors are z directional, that is, $\mathbf{r} = (0, 0, r)$, $\mathbf{s} = (0, 0, s)$. One can also change them to be x or y directional via an appropriate local unitary transformation without losing its diagonal property of the correlation term [12]. In this case the arbitrary state ρ defined in Eq.(10) has the form

$$\rho = \frac{1}{4} \begin{pmatrix} 1+r+s+c_3 & 0 & 0 & c_1-c_2 \\ 0 & 1+r-s-c_3 & c_1+c_2 & 0 \\ 0 & c_1+c_2 & 1-r+s-c_3 & 0 \\ c_1-c_2 & 0 & 0 & 1-r-s+c_3 \end{pmatrix}. \quad (11)$$

The entanglement of formation [8] is a monotonically increasing function of the Wootters's concurrence. While the concurrence can be calculated in terms of the eigenvalues of $\rho \tilde{\rho}$, where $\tilde{\rho} = \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y$. For the state Eq.(11), the eigenvalues of $\rho \tilde{\rho}$ are

$$\begin{aligned} \lambda_1 &= \frac{1}{16}(c_1 - c_2 - \sqrt{(1+c_3)^2 - (r+s)^2})^2 \\ &= \frac{1}{16}(c_1 - c_2 - \sqrt{(1+r+s+c_3)(1-r-s+c_3)})^2, \end{aligned}$$

$$\begin{aligned} \lambda_2 &= \frac{1}{16}(c_1 - c_2 + \sqrt{(1+c_3)^2 - (r+s)^2})^2 \\ &= \frac{1}{16}(c_1 - c_2 + \sqrt{(1+r+s+c_3)(1-r-s+c_3)})^2, \end{aligned}$$

$$\begin{aligned} \lambda_3 &= \frac{1}{16}(c_1 + c_2 - \sqrt{(1-c_3)^2 - (r-s)^2})^2 \\ &= \frac{1}{16}(c_1 + c_2 - \sqrt{(1+r-s-c_3)(1-r+s-c_3)})^2, \end{aligned}$$

$$\begin{aligned} \lambda_4 &= \frac{1}{16}(c_1 + c_2 + \sqrt{(1-c_3)^2 - (r-s)^2})^2 \\ &= \frac{1}{16}(c_1 + c_2 + \sqrt{(1+r-s-c_3)(1-r+s-c_3)})^2. \end{aligned}$$

The concurrence is given by

$$C(\rho) = \max\{2 \max\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}, \sqrt{\lambda_4}\} - \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\}. \quad (12)$$

If one fixes the parameters r and s , the above states and their concurrence are a three parameters set, with

the Bell-diagonal states belonging to the set with $r =$

$s = 0$. The geometry of such set with nonzero Bloch vectors has been considered by Hungsoo Kim *et al.* recently [12]. The geometrical deformation of the octahedron \mathcal{T} for the set of Bell-diagonal states and the octahedron \mathcal{O} for the separable Bell-diagonal states has been depicted. The deformation of \mathcal{O} can also be obtained from the region where $C(\rho) = 0$ in Eq.(12), as the concurrence of separable state must be zero. The level surfaces of concurrence or entanglement can be plotted correspondingly.

As ρ in Eq.(11) is a two-qubit X state, the discord can be calculated in a way presented in [24]. The eigenvalues of ρ in Eq.(11) is given by

$$u_{\pm} = \frac{1}{4}[1 - c_3 \pm \sqrt{(r-s)^2 + (c_1 + c_2)^2}],$$

$$v_{\pm} = \frac{1}{4}[1 + c_3 \pm \sqrt{(r+s)^2 + (c_1 - c_2)^2}].$$

For convenience, we define $f(t) = -\frac{1+t}{2} \log_2(1-t) - \frac{1-t}{2} \log_2(1+t)$. $f(t)$ is a monotonically decreasing function for $0 \leq t \leq 1$. The quantum mutual information is given by

$$\mathcal{I}(\rho) = S(\rho^A) + S(\rho^B) + u_+ \log_2 u_+ + u_- \log_2 u_- + v_+ \log_2 v_+ + v_- \log_2 v_-, \quad (13)$$

where $S(\rho^A)$ and $S(\rho^B)$ are given by $S(\rho^A) = 1 + f(r)$, $S(\rho^B) = 1 + f(s)$.

We evaluate next the classical correlation $\mathcal{C}(\rho)$. The Von Neumann measurement for subsystem B can be written as $B_i = V \prod_i V^+$, $i = 0, 1$, where $\prod_i = |i\rangle\langle i|$ is the projector associated with the subsystem B and $V = tI + i\vec{y} \cdot \vec{\sigma} \in SU(2)$, $t, y_1, y_2, y_3 \in \mathbb{R}$ and $t^2 + y_1^2 + y_2^2 + y_3^2 = 1$. After the measurement, we have the ensemble $\{\rho_i, p_i\}$. The classical correlation is therefore given by

$$\mathcal{C}(\rho) = \sup_{\{B_i\}} \mathcal{I}(\rho|\{B_i\}) = S(\rho^A) - \min_{\{B_i\}} S(\rho|\{B_i\}), \quad (14)$$

where

$$S(\rho|\{B_i\}) = p_0 S(\rho_0) + p_1 S(\rho_1). \quad (15)$$

By a the parameter transformation

$$m = (ty_1 + y_2y_3)^2, \quad n = (ty_2 - y_1y_3)(ty_1 + y_2y_3),$$

$$k = t^2 + y_3^2, \quad l = y_1^2 + y_2^2,$$

which satisfies $m^2 + n^2 = klm$, $k + l = 1$, $k \in [0, 1]$, $m \in [0, \frac{1}{4}]$ and $n \in [-\frac{1}{8}, \frac{1}{8}]$, according to [24] we observe that the minimum of Eq.(15) can only be obtained in the following cases:

(1) $k = 1, l = 0, m = n = 0$. For state (11), Eqs.(14-17) in Ref.[24] turn out to be $p_0 = \frac{1+s}{2}, p_1 = \frac{1-s}{2}, \theta = |\frac{r+c_3}{1+s}|$,

$\theta' = |\frac{r-c_3}{1-s}|$, $v_{\pm}(\rho_0) = \frac{1\pm\theta}{2}, \omega_{\pm}(\rho_1) = \frac{1\pm\theta'}{2}$. Thus,

$$S_1 = S(\rho|\{B_i\}) = p_0 S(\rho_0) + p_1 S(\rho_1)$$

$$= -\frac{1+r+s+c_3}{4} \log_2 \frac{1+r+s+c_3}{2(1+s)}$$

$$- \frac{1-r+s-c_3}{4} \log_2 \frac{1-r+s-c_3}{2(1+s)}$$

$$- \frac{1+r-s-c_3}{4} \log_2 \frac{1+r-s-c_3}{2(1-s)}$$

$$- \frac{1-r-s+c_3}{4} \log_2 \frac{1-r-s+c_3}{2(1-s)}. \quad (16)$$

(2) $k = 0, l = 1, m = n = 0$. It is easy to find that the minimum is the same as S_1 .

(3) $k = l = \frac{1}{2}$. In this case, we have

$$\theta = \theta' = \sqrt{r^2 + c_1^2 - 4m(c_1^2 - c_2^2)},$$

here θ, θ' are defined by Eqs. (16,17) in [24], $S(\rho_0) = S(\rho_1)$, which is a monotonically function of m . Therefore the minimum is obtained at $m = 0$ or $m = \frac{1}{4}$. We have either $\theta = \theta' = \sqrt{r^2 + c_1^2}$ or $\theta = \theta' = \sqrt{r^2 + c_2^2}$. The quantum conditional entropy is given by

$$S_2 = 1 + f(\sqrt{r^2 + c_1^2}), \quad (17)$$

$$S_3 = 1 + f(\sqrt{r^2 + c_2^2}). \quad (18)$$

Therefore, we have

Theorem 1 For any state ρ of the form Eq.(11), the classical correlation of ρ is given by

$$\mathcal{C}(\rho) = S(\rho^A) - \min\{S_1, S_2, S_3\}, \quad (19)$$

where S_1, S_2, S_3 are defined by Eqs.(16), (17), (18) respectively. The quantum discord is given by

$$\mathcal{Q}(\rho) = \mathcal{I}(\rho) - \mathcal{C}(\rho), \quad (20)$$

with $\mathcal{I}(\rho)$ given by (13).

In Fig.1 we plot the level surface of discord when (a) $r = s = 0.3$, $\mathcal{Q}(\rho) = 0.03$; (b) $r = s = 0.5$, $\mathcal{Q}(\rho) = 0.03$; (c) $r = s = 0.3$, $\mathcal{Q}(\rho) = 0.15$; (d) $r = s = 0.5$, $\mathcal{Q}(\rho) = 0.15$. From Fig.1 one can see that the level surface of discord has a great change from the case $r = s = 0$ studied in Ref. [23]. The surface shrinks with the effect of r and s and the shrinking rate becomes larger with the increasing $|r|$ and $|s|$. What is more, when the discord is small (such as $\mathcal{Q}(\rho) = 0.03$), the horizontal "tubes" are closed! see Fig (a). For larger r and s , the picture is moved up the plane $c_3 = 0$, see Fig. (b). For larger discord and small r and s , Fig. (c), the figure is similar to the ones in case of $r = s = 0$. But for larger r and s , Fig. (d), the figure is moved up again and changes dramatically also.

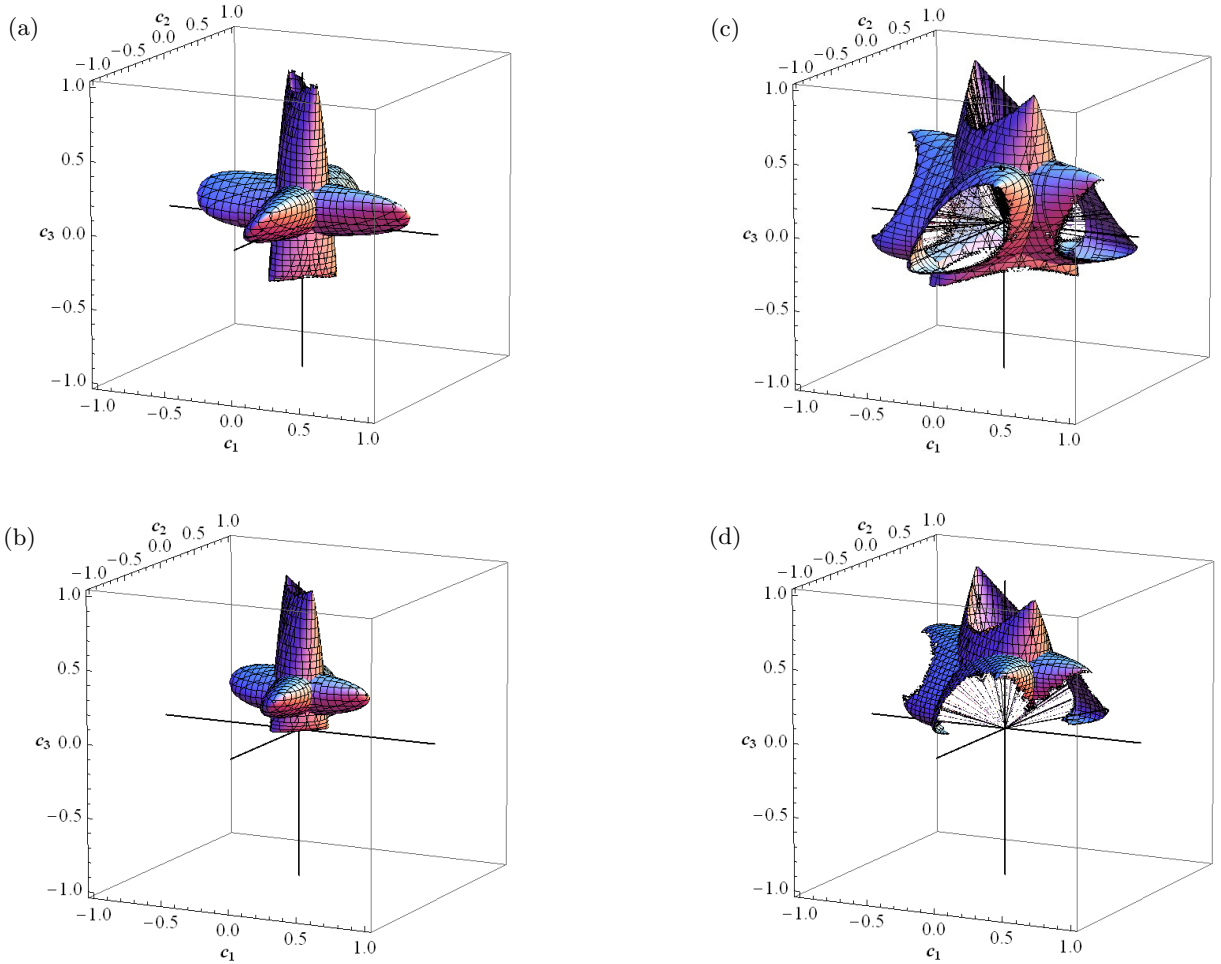


FIG. 1: Surfaces of constant discord: (a) $r = s = 0.3$, $\mathcal{Q}(\rho) = 0.03$; (b) $r = s = 0.5$, $\mathcal{Q}(\rho) = 0.03$; (c) $r = s = 0.3$, $\mathcal{Q}(\rho) = 0.15$; (d) $r = s = 0.5$, $\mathcal{Q}(\rho) = 0.15$.

III. DYNAMICS OF QUANTUM DISCORD UNDER LOCAL NONDISSIPATIVE CHANNELS

It has been recently discovered that for some Bell-diagonal states, their quantum discord are invariant under some decoherence for a finite time interval [25]. An interesting question is if such phenomena exists in other systems. In the following we consider that the state ρ in Eq.(11) undergoes the phase flip channel [26], with the Kraus operators $\Gamma_0^{(A)} = \text{diag}(\sqrt{1-p/2}, \sqrt{1-p/2}) \otimes I$, $\Gamma_1^{(A)} = \text{diag}(\sqrt{p/2}, -\sqrt{p/2}) \otimes I$, $\Gamma_0^{(B)} = I \otimes \text{diag}(\sqrt{1-p/2}, \sqrt{1-p/2})$, $\Gamma_1^{(B)} = I \otimes \text{diag}(\sqrt{p/2}, -\sqrt{p/2})$, where $p = 1 - \exp(-\gamma t)$, γ is the phase damping rate [10, 26].

Let $\varepsilon(\cdot)$ represent the operator of decoherence. Then under the phase flip channel, we have

$$\varepsilon(\rho) = \frac{1}{4}(I \otimes I + r\sigma_3 \otimes I + I \otimes s\sigma_3 + (1-p)^2 c_1 \sigma_1 \otimes \sigma_1 + (1-p)^2 c_2 \sigma_2 \otimes \sigma_2 + c_3 \sigma_3 \otimes \sigma_3). \quad (21)$$

Noting that r, s, c_3 are independent of time, we consider the case that

$$c_2 = -c_3 c_1, \quad s = c_3 r, \quad -1 \leq c_3 \leq 1, \quad -1 \leq r \leq 1. \quad (22)$$

Then the eigenvalues of $\varepsilon(\rho)$ are given by

$$u_{\pm} = \frac{1-c_3}{4}(1 \pm \sqrt{r^2 + (1-p)^4 c_1^2}),$$

$$v_{\pm} = \frac{1+c_3}{4}(1 \pm \sqrt{r^2 + (1-p)^4 c_1^2}).$$

From (13) we have the quantum mutual information

$$\mathcal{I}(\varepsilon(\rho)) = f(r) + f(c_3 r) - f(c_3) - f(\sqrt{r^2 + (1-p)^4 c_1^2}). \quad (23)$$

To calculate the classical correlation, we need to determine S_1 , S_2 and S_3 defined by (16), (17) and (18) respectively, which are given by

$$S_1(p) = 1 + f(r) + f(c_3) - f(c_3 r), \quad (24)$$

$$S_2(p) = 1 + f(\sqrt{r^2 + (1-p)^4 c_1^2}), \quad (25)$$

$$S_3(p) = 1 + f(\sqrt{r^2 + (1-p)^4 c_2^2}). \quad (26)$$

From the condition (22), we have $S_3(p) \geq S_2(p)$ for any p , while $S_2(p)$ increases under decoherence, and $S_1(p)$ is constant under decoherence. If we select appropriate r, c_1, c_3 then the initial state $S_2(0) < S_1(0)$. On the other hand, since $f(c_3) \leq f(c_3 r)$, we always have $S_2(1) \geq S_1(1)$. Therefore there exist $0 \leq p_0 \leq 1$ such that $\min\{S_1, S_2, S_3\} = S_2$ for $0 \leq p \leq p_0$, and $\min\{S_1, S_2, S_3\} = S_1$ for $p_0 \leq p \leq 1$. In this case $\mathcal{Q}(\varepsilon(\rho))$ monotonically decreases to zero.

When $\min\{S_1, S_2, S_3\} = S_2$, we have

$$\begin{aligned} \mathcal{Q}(\varepsilon(\rho)) &= \mathcal{I}(\varepsilon(\rho)) - \mathcal{C}(\varepsilon(\rho)) \\ &= f(c_3 r) - f(c_3), \end{aligned} \quad (27)$$

$\mathcal{Q}(\varepsilon(\rho))$ is constant under decoherence during the time interval such that the condition $\min\{S_1, S_2, S_3\} = S_2$ is satisfied.

As an example, for $r = s = 0$, $c_1 = 1$, $-1 \leq c_2 = -c_3 \leq 1$, we have that $S_1(0) = 1 + f(c_3)$, $S_2(0) = 1 + f(1) < S_1(0)$. Therefore the state has constant discord under decoherence, which recovers the results in [23, 25]. For an example with nonzero r and s , we set $r = \frac{3}{10}$, $s = \frac{3}{20}$, $c_1^2 = \frac{4}{5}$, $c_2 = -\frac{c_1}{2}$, $c_3 = \frac{1}{2}$. It is direct to verify that $S_1(0) = 0.762$, $S_2(0) = 0.186$. Therefore we have $\min\{S_1, S_2, S_3\} = S_2$ and the state has a constant discord. The dynamic behavior of correlation of the state under the phase flip channel is depicted in Fig.2. We find that the concurrence \mathcal{C} is greater than the quantum discord \mathcal{Q} for $0 \leq p \leq 0.217$. A sudden transition of classical and quantum correlation happens at $p = 0.274$, and a sudden death of entanglement [11] appears at $p = 0.4$. Moreover, different from the case of zero r and s in [25], where the entanglement disappears before the sudden transition of classical and quantum correlation, here one sees that the concurrence keeps non-zero after the transition. Therefore for these states the entanglement is more robust against the decoherence than the discord.

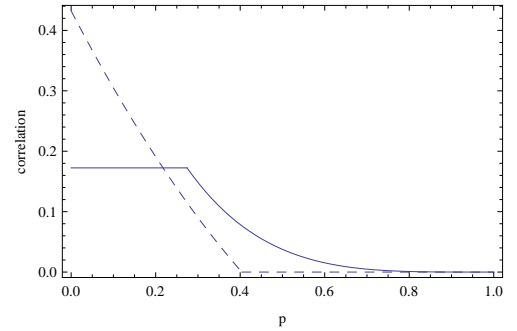


FIG. 2: Concurrence(dashed line) and quantum discord(solid line) under phase flip channel for $r = \frac{3}{10}$, $s = \frac{3}{20}$, $c_1^2 = \frac{4}{5}$, $c_2 = -\frac{c_1}{2}$ and $c_3 = \frac{1}{2}$.

IV. SUMMARY

We have studied the correlation for a class of X states. The level surfaces of quantum discord have been depicted. For $r = s = 0$ our results reduce to the ones for Bell-diagonal states. For nonzero r and s , it has been shown that the level surfaces of quantum discord may have quite different geometry and topology. While the quantum discord could still keep constant under decoherence in certain time interval for some initial states, the order of sudden transition of classical and quantum correlation and the sudden death of entanglement can be exchanged.

Acknowledgments We thank Matthias D. Lang for very helpful discussions. This work is supported by the NSFC10875081, NSFC10871227, KZ200810028013 and PHR201007107 and NSFBJ1092008.

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